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Asymptotically optimal sampling schemes for periodic functions

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1. *Introduction*

In this paper we are concerned with the following question: Given a function class $\mathscr{F}$ in $L^2_*[0,1)$, the space of L^2-one periodic complex valued functions, when does sampling a function $f \in \mathscr{F}$ give optimal information for approximation? To make this question precise, we introduce the quantity

$$i_N(\mathscr{F}) = \inf_{A,I} \sup_{f \in \mathscr{F}} \|f - A(If)\|, \tag{1·1}$$

$\|f\|^2 = \int_0^1 |f(t)|^2 dt$, where A is any mapping from $\mathbb{R}^N$ into $L^2_*[0,1)$ and I is any continuous linear mapping from $L^2_*[0,1)$ into $\mathbb{R}^N$. Therefore the number $i_N(\mathscr{F})$ is the error in approximating the class $\mathscr{F}$ with any N linear observations If. Likewise the error in sampling is given by

$$s_N(\mathscr{F}) = \inf_{\Delta} E(\mathscr{F}; \Delta) \tag{1·2}$$

where $\Delta = \{x_1, \ldots, x_N\}$ forms the partition $0 \leqslant x_1 < \ldots < x_N < 1$ of $[0,1]$ and

$$E(\mathscr{F}; \Delta) = \inf_{A} \sup_{f \in \mathscr{F}} \|f - A(f(x_1), \ldots, f(x_N))\| \tag{1·3}$$

so that $i_N(\mathscr{F}) \leqslant s_N(\mathscr{F})$.

We are interested in function classes for which

$$\overline{\lim_{N \to \infty}} \frac{s_N(\mathscr{F})}{i_N(\mathscr{F})} < \infty, \tag{1·4}$$

i.e. the error in sampling is comparable to any scalar observations. Even more we wish to identify $\mathscr{F}$ so that

$$\overline{\lim_{N \to \infty}} \frac{E(\mathscr{F}; \Delta_N)}{i_N(\mathscr{F})} < \infty \tag{1·5}$$

where $\Delta_N = \{i/N : i = 0, 1, 2, \ldots, N-1\}$ corresponds to an equally spaced partition of $[0,1]$.

Our function class always has the form

$$\mathscr{F}_\phi = \{\phi * h : \|h\| \leqslant 1\} \tag{1·6}$$

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(* denotes convolution). A fundamental difference between $i_N(\mathscr{F}_\phi)$ and $s_N(\mathscr{F}_\phi)$ is that the former is invariant under any rearrangement of the Fourier coefficients

$$\hat{\phi}(k) = \int_0^1 \phi(t)\, e^{-2\pi i k t}\, dt, \tag{1·7}$$

while this cannot be expected to be true for $s_N(\mathscr{F}_\phi)$. In fact, we use this property of $i_N(\mathscr{F}_\phi)$ and our evaluation of $E(\mathscr{F}_\phi; \Delta_N)$ in Section 2 to give in Section 4 functions ϕ for which (1·5) fails. In Section 3, we give an upper bound on $E(\mathscr{F}_\phi; \Delta_N)$ (by using discrete Fourier transform with judiciously chosen attenuation factors) which is compatible (in the sense of (1·4)) with $i_N(\mathscr{F}_\phi)$ for a wide class of functions ϕ.

The problem studied here is closely related to various notions of N-widths. However, we have made an effort to provide all necessary background material so that this paper is self-contained. For the reader who wishes elaboration on the connection to N-widths we recommend [2] and references therein.

2. *Error for equally spaced sample points*

We require the following notation. For any $\phi \in L_*^2[0, 1)$ we set

$$\phi_{N,l}(t) = \frac{1}{N} \sum_{j=0}^{N-1} \phi\left(t + \frac{j}{N}\right) e^{-2\pi i j l/N}$$

so that

$$\hat{\phi}_{N,l}(n) = \begin{cases} \hat{\phi}(n) & \text{if} \quad n \equiv l \,(\mathrm{mod}\, N), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we let

$$H(\phi) = \max\{(\sum_{k \in Z} |a_k \hat{\phi}(k)|^2)^{\frac{1}{2}} : \sum_{k \in Z} |a_k|^2 \leqslant 1, \sum_{k \in Z} a_k \hat{\phi}(k) = 0\}$$

and also $G_{N,l} = \{n : n \equiv l \,(\mathrm{mod}\, N)\}$.

THEOREM 2·1.

$$E(\mathscr{F}_\phi; \Delta_N) = \max_{0 \leqslant l < N} H(\phi_{N,l}).$$

Proof. It is known from [1] that

$$E(\mathscr{F}, \Delta_N) = \sup\left\{\|f\| : f \in \mathscr{F}, f\left(\frac{j}{N}\right) = 0, \quad j = 0, 1, \ldots, N-1\right\}. \tag{2·1}$$

Moreover, we recall that the discrete Fourier transform

$$\hat{f}_N(k) = \frac{1}{N} \sum_{j=0}^{N-1} f\left(\frac{j}{N}\right) e^{-2\pi i k j/N}$$

satisfies

$$\hat{f}_N(l) = \sum_{k \in G_{N,l}} \hat{f}(k) \tag{2·2}$$

and that

$$\|f\|^2 = \sum_{k \in Z} |\hat{f}(k)|^2. \tag{2·3}$$

Therefore for any function $f = \phi * h$, $\|h\| \leqslant 1$, with $f(j/N) = 0, j = 0, 1, \ldots, N-1$, and since

$$0 = \sum_{k \in G_{N,l}} \hat{f}(k) = \sum_{k \in G_{N,l}} \hat{\phi}(k)\, \hat{h}(k) = \sum_{k \in G_{N,l}} \hat{\phi}_{N,l}(k)\, \hat{h}(k),$$

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we have

$$\begin{aligned}\|f\|^2 &= \sum_{l=0}^{N-1} \sum_{k\in G_{N,l}} |\hat{\phi}(k)\hat{h}(k)|^2 \\ &= \sum_{l=0}^{N-1} \sum_{k\in G_{N,l}} |\hat{\phi}_{N,l}(k)\hat{h}(k)|^2 \\ &\leqslant \sum_{l=0}^{N-1} (H(\phi_{N,l}))^2 \sum_{k\in G_{N,l}} |\hat{h}(k)|^2 \\ &\leqslant \max_{0\leqslant l<N} (H(\phi_{N,l}))^2,\end{aligned}$$

which proves

$$E(\mathscr{F}_\phi; \Delta_N) \leqslant \max_{0\leqslant l<N} H(\phi_{N,l}).$$

To prove the reverse inequality, we choose m such that

$$H(\phi_{N,m}) = \max_{0\leqslant l<N} H(\phi_{N,l})$$

and $\{a_k\}$ satisfying

$$(H(\phi_{N,m}))^2 = \sum_{k\in Z} |a_k \hat{\phi}_{N,m}(k)|^2,$$

where $\sum_{k\in Z} |a_k|^2 \leqslant 1$ and $\sum_{k\in Z} a_k \hat{\phi}_{N,m}(k) = 0$. The function $f_0 = \phi * h_0$ defined by

$$\hat{h}_0(k) = \begin{cases} a_k & (k\in G_{N,m}) \\ 0 & \text{otherwise} \end{cases}$$

satisfies $f_0 \in \mathscr{F}_\phi$ and, in view of (2·2), $f_0(j/N) = 0, j = 0, 1, \ldots, N-1$. Thus

$$\begin{aligned}(E(\mathscr{F}_\phi; \Delta_N))^2 \geqslant \|f_0\|^2 &= \sum_{k\in Z} |a_k \hat{\phi}_{N,m}(k)|^2 \\ &= (H(\phi_{N,m}))^2,\end{aligned}$$

which proves the theorem.

To complete our evaluation of $E(\mathscr{F}_\phi; \Delta_N)$ we show how to compute $H(\phi)$. For this purpose we set

$$d(\phi) = \max_{k\in Z} |\hat{\phi}(k)|.$$

PROPOSITION 2·1. *Let $\phi \in L^2_*[0, 1)$. Then $H(\phi) = d(\phi)$ when $|\{k: |\hat{\phi}(k)| = d(\phi)\}| \geqslant 2$ (where $|S|$ denotes the cardinality of a given set S). Otherwise $(H(\phi))^2$ is the largest root of the function*

$$F(y) = \sum_{k\in Z} \frac{|\hat{\phi}(k)|^2}{|\hat{\phi}(k)|^2 - y}.$$

Proof. First suppose that there are two integers $n_1 \neq n_2$ with $d(\phi) = |\hat{\phi}(n_1)| = |\hat{\phi}(n_2)|$. Define $h \in L^2_*[0, 1)$ by setting

$$\hat{h}(n) = \begin{cases} \dfrac{\bar{\hat{\phi}}(n_1)}{\sqrt{2}\, d(\phi)} & (n = n_1) \\ \dfrac{-\bar{\hat{\phi}}(n_2)}{\sqrt{2}\, d(\phi)} & (n = n_2) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum_{n\in Z} \hat{h}(n)\hat{\phi}(n) = 0$, $\sum_{n\in Z} |\hat{h}(n)|^2 = 1$ and $(d(\phi))^2 = \sum_{n\in Z} |\hat{\phi}(n)\hat{h}(n)|^2$, which gives $H(\phi) \geqslant d(\phi)$. However, trivially we have $d(\phi) \geqslant H(\phi)$.

Next, suppose $|\{n: |\hat{\phi}(n)| = d(\phi)\}| = 1$ and observe that $F(y)$ is increasing for all real y and has poles at $\{|\hat{\phi}(k)|^2: \phi(k) \neq 0\}$ as well as an additional singularity at $y = 0$ if ϕ is not a trigonometric polynomial. Consequently, it has a unique positive zero between

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each of its poles. Let λ be any zero of F. Then clearly $\lambda \neq |\hat{\phi}(n)|^2$, $n \in Z$, and so we define $h_\lambda \in L^2_*[0,1)$ by

$$\hat{h}_\lambda(n) = \frac{\bar{\hat{\phi}}(n)}{|\hat{\phi}(n)|^2 - \lambda}.$$

Now in general we have

$$F(y) = \sum_{k \in Z} \frac{|\hat{\phi}(k)|^2}{|\hat{\phi}(k)|^2 - y} = \sum_{k \in Z} \frac{|\hat{\phi}(k)|^4}{(|\hat{\phi}(k)|^2 - y)^2} - y \sum_{k \in Z} \frac{|\hat{\phi}(k)|^2}{(|\hat{\phi}(k)|^2 - y)^2}$$

and consequently

$$\lambda = \sum_{k \in Z} |\hat{h}_\lambda(k)\, \hat{\phi}(k)|^2 / \sum_{k \in Z} |\hat{h}_\lambda(k)|^2.$$

Since we also have

$$\sum_{k \in Z} \hat{h}_\lambda(k)\, \hat{\phi}(k) = F(\lambda) = 0$$

we conclude

$$\lambda \leqslant (H(\phi))^2.$$

Next, we prove that $(H(\phi))^2$ is in fact a zero of F. For this part of the argument, we suppose

$$(H(\phi))^2 = \sum_{k \in Z} |\hat{\phi}(k)\, \hat{h}(k)|^2,$$

where

$$\sum_{k \in Z} |\hat{h}(k)|^2 = 1, \quad \sum_{k \in Z} \hat{h}(k)\, \hat{\phi}(k) = 0.$$

Then using Lagrange multipliers there exist $\tau \in \mathbb{R}$, $\mu \in \mathbb{C}$ such that

$$\bar{\hat{h}}(k)\, (|\hat{\phi}(k)|^2 - \tau) = \mu \hat{\phi}(k) \quad (k \in Z).$$

From this equation we get

$$0 = (H(\phi))^2 - \tau - \mu \sum_{k \in Z} \hat{h}(k)\, \hat{\phi}(k) = (H(\phi))^2 - \tau. \tag{2·4}$$

If $\mu \neq 0$ then we would have $\tau \neq |\hat{\phi}(k)|^2$ for all $k \in Z$ such that $\hat{\phi}(k) \neq 0$. Thus the equation

$$\mu F(\tau) = \sum_{k \in Z} \overline{\hat{h}(k)\, \hat{\phi}(k)} = 0$$

with (2·4) proves the proposition. However, if $\mu = 0$ then $|\hat{\phi}(n)|^2 = \tau$ for at least two values, because

$$\sum_{k \in Z} \hat{h}(k)\, \hat{\phi}(k) = 0$$

and $\hat{h}(k) = 0$ when $|\hat{\phi}(k)|^2 \neq \tau$. Therefore $H(\phi) = d(\phi)$ (because

$$d(\phi) \geqslant H(\phi) \geqslant \min_{n_1 \neq n_2} (|\hat{\phi}(n_1)|, |\hat{\phi}(n_2)|))$$

which contradicts our assumption that $|\{k: |\hat{\phi}(k)| = d(\phi)\}| = 1$.

3. *An upper bound for* $E(\mathscr{F}_\phi; \Delta_N)$

In this section, we prove

THEOREM 3·1. *For any set* $\Lambda \subset Z$ *with* $|\Lambda| = N$ *one has*

$$(E(\mathscr{F}_\phi; \Delta_N))^2 \leqslant \max_{k \in \Lambda} \sum_{l \in G_{N,k} \setminus \{k\}} |\hat{\phi}(l)|^2 + \max_{k \notin \Lambda} |\hat{\phi}(k)|^2. \tag{3·1}$$

Proof. We define

$$(T_\Lambda f)(x) = \sum_{k \in \Lambda} \gamma_k \hat{f}_N(k)\, e^{2\pi i k x}$$

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where $$\gamma_k = |\hat{\phi}(k)|^2 / \sum_{n \in G_{N,k}} |\hat{\phi}(n)|^2.$$

Then $$(E(\mathscr{F}_\phi; \Delta_N))^2 \leqslant \sup_{f \in \mathscr{F}_\phi} \|f - T_\Lambda f\|^2$$

and $$\|f - T_\Lambda f\|^2 = \sum_{k \in \Lambda} |\hat{\phi}(k)\hat{h}(k) - \gamma_k \sum_{n \in G_{N,k}} \hat{\phi}(n)\hat{h}(n)|^2 + \sum_{k \notin \Lambda} |\hat{\phi}(k)\hat{h}(k)|^2$$
$$= \sum_{k \in \Lambda} |(1-\gamma_k)\hat{\phi}(k)\hat{h}(k) - \gamma_k \sum_{n \in G_{N,k} \setminus \{k\}} \hat{\phi}(n)\hat{h}(n)|^2 + \sum_{k \notin \Lambda} |\hat{\phi}(k)\hat{h}(k)|^2.$$

Thus for any $f \in \mathscr{F}_\phi$

$$\|f - T_\Lambda f\|^2 \leqslant \max_{k \in \Lambda} ((1-\gamma_k)^2 |\hat{\phi}(k)|^2 + \gamma_k^2 \sum_{n \in G_{N,k} \setminus \{k\}} |\hat{\phi}(n)|^2) + \max_{k \notin \Lambda} |\hat{\phi}(k)|^2$$
$$= \max_{k \in \Lambda} \frac{|\hat{\phi}(k)|^2}{\sum_{n \in G_{N,k}} |\hat{\phi}(n)|^2} \sum_{n \in G_{N,k} \setminus \{k\}} |\hat{\phi}(n)|^2 + \max_{k \notin \Lambda} |\hat{\phi}(k)|^2$$

which completes the proof.

Next we prove

THEOREM 3·2. *Suppose $\phi \in L^2_*[0,1)$ and that there is a positive increasing sequence $\psi(n)$, $n \in Z_+$ and constants L, U, V such that*

$$0 < L \leqslant |\hat{\phi}(\pm n)|^2 \Psi(n) \leqslant U \quad (n \in Z_+),$$

and

$$\overline{\lim_{N \to \infty}} \Psi(N) \sum_{j=1}^{\infty} \frac{1}{\Psi(jN)} = V$$

then

$$\overline{\lim_{N \to \infty}} \frac{E(\mathscr{F}_\phi; \Delta_N)}{i_N(\mathscr{F}_\phi)} < \infty.$$

Remark. In particular, Theorem 3·2 implies that under the above hypotheses

$$\overline{\lim_{N \to \infty}} \frac{E(\mathscr{F}_\phi; \Delta_N)}{s_N(\mathscr{F}_\phi)} < \infty,$$

i.e. equidistant sampling is asymptotically comparable to optimal sampling.

Proof. Suppose N is odd, $N = 2M+1$, (the even case can be argued similarly) and choose $\Lambda = \{0, \pm 1, \ldots, \pm M\}$. Then

$$\max_{l \in \Lambda} \sum_{k \neq 0} |\hat{\phi}(l+kN)|^2 \leqslant 2U \sum_{k=1}^{\infty} \frac{1}{\Psi(k(M+1))}$$

and

$$\max_{k \notin \Lambda} |\hat{\phi}(k)|^2 \leqslant \frac{U}{\Psi(M+1)}.$$

By Theorem 3·1 we therefore have

$$\overline{\lim_{N \to \infty}} \Psi(M+1)(E(\mathscr{F}_\phi; \Delta_N))^2 \leqslant (2UV + U).$$

To derive a comparable lower bound for $i_N(\mathscr{F}_\phi)$, we need

LEMMA 3·1.

$$i_N(\mathscr{F}_\phi) = \max_{|\Lambda| = N+1} \min_{k \in \Lambda} |\hat{\phi}(k)|.$$

Proof. Pick any $\Lambda \subset Z$ such that $|\Lambda| = N+1$. Then for any $I: L^2[0,1) \to \mathbb{R}^N$ we can

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find an $h(t) = \sum_{k\in\Lambda} c_k e^{2\pi ikt}$ such that $\|h\|^2 = \sum_{k\in\Lambda} |c_k|^2 = 1$ and $I(\phi * h) = 0$. Hence for any $A: \mathbb{R}^N \to L^2_*[0,1)$ we have

$$\begin{aligned}\|\phi * h\| &\leqslant \tfrac{1}{2}\{\|\phi * h - A(0)\| + \|\phi * h + A(0)\|\} \\ &\leqslant \sup_{f\in\mathscr{F}_\phi} \|f - AIf\|,\end{aligned}$$

and also

$$\|\phi * h\|^2 = \sum_{k\in\Lambda} |\hat{\phi}(k)\, c_k|^2 \geqslant \min_{k\in\Lambda} |\hat{\phi}(k)|^2.$$

Therefore we have shown that

$$\max_{|\Lambda|=N+1} \min_{k\in\Lambda} |\hat{\phi}(k)| \leqslant i_N(\mathscr{F}_\phi).$$

Conversely, define for any $\Lambda \subset Z$, $|\Lambda| = N$,

$$I_\Lambda: f \to (\hat{f}(k): k\in\Lambda)$$

as well as

$$A_\Lambda: (c_k: k\in\Lambda) \to \sum_{k\in\Lambda} c_k e^{2\pi ikt}.$$

Then we obtain, for any $f\in\mathscr{F}_\phi$,

$$\|f - A_\Lambda I_\Lambda f\|^2 = \sum_{k\notin\Lambda} |\hat{f}(k)|^2 \leqslant \max_{k\notin\Lambda} |\hat{\phi}(k)|^2$$

and so

$$i_N(\mathscr{F}_\phi) \leqslant \min_{|\Lambda|=N} \max_{k\notin\Lambda} |\hat{\phi}(k)|.$$

To complete the proof, we observe that

$$\min_{|\Lambda|=N} \max_{k\notin\Lambda} |\hat{\phi}(k)| = |\hat{\phi}(k_{N+1})| = \max_{|\Lambda|=N+1} \min_{k\in\Lambda} |\hat{\phi}(k)|$$

where $|\hat{\phi}(k_l)|$ is the lth largest element of $\{|\hat{\phi}(k)|: k\in Z\}$.

Notice that we have identified an optimal recovery scheme in the sense of [1] and the Fourier coefficients are optimal linear functionals. Moreover, the $|\hat{\phi}(k)|$, $k\in Z$, are the singular values of the (compact) convolution operator $T_\phi h = \phi * h$, so that Lemma 3·1 could have been derived by appealing to the theory of n-widths for Hilbert spaces.

Returning to the proof of Theorem 3·2, Lemma 3·1 gives

$$i_N(\mathscr{F}_\phi) \geqslant \min_{|k|\leqslant M+1} |\hat{\phi}(k)| \geqslant \left(\frac{L}{\Psi(M+1)}\right)^{\frac{1}{2}},$$

so that

$$\overline{\lim_{N\to\infty}} \left(\frac{E(\mathscr{F}_\phi; \Delta_N)}{i_N(\mathscr{F}_\phi)}\right)^2 \leqslant \frac{U}{L}(1+2V)$$

which proves the theorem.

Remark. Note that the function $(1+x)^\beta$, $\beta > 1$, or $e^{\alpha x^\tau}$, $\alpha > 0$, $0 < \tau \leqslant 1$ are admissible choices for $\Psi(x)$, as well as any function for which

$$\overline{\lim_{X\to\infty}} \frac{\Psi(x+1)}{\Psi(x)} > 1.$$

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4. *Equally spaced sampling is not optimal*

Consider the following construction of a $\phi \in L^2_*[0, 1)$. For any decreasing sequence $\{a_k\}_{k \in Z_+}$ and any increasing sequence $\{m_k\}_{k \in Z_+}$ of positive integers, we define a new sequence $\{b_k\}_{k \in Z_+}$ as follows:

$$b_k = \begin{cases} a_k & (k \neq m_j, j \in Z_+) \\ 0 & (k = m_0) \\ a_{m_{j-1}} & (k = m_j, j \in \mathbb{N}). \end{cases}$$

For this sequence, we define

$$\phi(x) = \sum_{k=0}^{\infty} b_k e^{2\pi i k x}.$$

According to Theorem 2·1 we have

$$E(\mathscr{F}_\phi; \Delta_N) = \max_{0 \leqslant l < N} H(\phi_{N,l}) \geqslant H(\phi_{N,0}).$$

For $N = m_i$, $i \geqslant 1$, the two largest Fourier coefficients of $\phi_{N,0}$ are a_0 and $a_{m_{i-1}}$. In view of the definition of $H(\phi)$ we therefore have

$$E(\mathscr{F}_\phi; \Delta_{m_i}) \geqslant b_{m_i} = a_{m_{i-1}}.$$

However, by Lemma 3·1, $i_{m_i}(\mathscr{F}_\phi) = a_{m_i}$. Thus for sampling to be optimal we must have a constant $M > 0$ such that for all i

$$a_{m_{i-1}} < M a_{m_i}.$$

This is certainly false for $a_k = \lambda^k$, $\lambda < 1$, and $m_i = 4^i$.

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