Appendix A
Cross-sections

We begin this appendix by considering neutron cross-sections. There is some simplification in the case of neutrons, since they are electrically neutral and do not interact through the long-range Coulomb force. To a good approximation they can be considered to interact only through the short-range nuclear force. The concepts developed for neutrons may be applied almost immediately to other electrical neutral particles, such as photons. We then turn to the case of charged particles.

A.1 Neutron and photon cross-sections

We consider a neutron approaching from a distance a nucleus which is at rest. (Any interactions between the neutron and the atomic electrons will be neglected.) We suppose that, if the nucleus were not present, the probability of the neutron passing anywhere through a circle of radius a, centred on the nucleus and perpendicular to the direction of the neutron's motion, would be uniform (Fig. A.1), i.e. the probability of it passing through an area δA would be $\delta A/(\pi a^2)$. We can think of the neutron as a classical particle, or better, as a quantum-mechanical wave-packet. The radius a must be large compared with both the size of the wave-packet and the size of the nucleus. With the nucleus present, an interaction can take place, for example scattering, induced fission, or radiative capture. It is found that, provided a is large enough, the probability of an interaction is inversely proportional to the area πa^2, i.e.

$$\text{probability of interaction} = \frac{\sigma_{\text{tot}}}{\pi a^2}. \tag{A.1}$$

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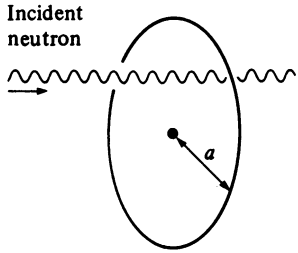


Fig. A1 Neutron incident on area centred on nucleus at rest.

The constant of proportionality introduced here, σ_{tot}, is called the *total cross-section*. Clearly σ_{tot} has the dimensions of area. It can be regarded as the effective area presented to the neutron by the nucleus, but it must be realised that the cross-section is a *joint* property of the neutron and nucleus, and for a given nucleus it is a sensitive function of neutron energy. The probability of interaction is a quantum-mechanical property: σ_{tot} can be very much larger than the geometrical cross-section of the nucleus.

The system of a moving particle incident on a nucleus at rest is called the *laboratory system*. With fixed targets, it is the situation most easy to simulate in the laboratory. It is also useful to consider interactions in the frame of reference in which the nucleus has momentum equal in magnitude but opposite in direction to the neutron. It is clear from the definition that the total cross-section is the same viewed in this '*centre-of-mass*' system as in the laboratory system.

There will usually be several possible *reaction channels*, i.e. types of interaction that can occur. Examples are:

Elastic scattering: the incoming neutron changes direction but, in the centre-of-mass system, loses no energy.

Inelastic scattering: the incoming neutron changes direction and, even in the centre-of-mass system, loses energy in exciting the nucleus.

Radiative capture: the incoming neutron is captured by the nucleus. The resulting nucleus is formed in an excited state, which eventually decays by photon emission.

Given that a reaction occurs, each reaction channel 'i' has a definite probability p_i, where

$$\sum_i p_i = 1.$$

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The *partial cross-section*, σ_i for the ith channel, is defined to be $\sigma_i = p_i \sigma_{\text{tot}}$, and may be regarded as the effective area presented by the target nucleus to the neutron for that particular reaction. We have

$$\sigma_{\text{tot}} = \sum_i \sigma_i.$$

Photon cross-sections can be defined in direct analogy with neutron cross-sections, but since photons interact with atomic electrons as well as with nuclei it is more appropriate to consider the target to be an atom. Various contributions to the total cross-section for a photon to interact with an atom are discussed in Chapter 14.

A.2 Differential cross-sections

In considering a particular reaction channel it is often useful to subdivide it further. For example, in the elastic scattering of neutrons it can be of interest to know the probability distribution of the angle at which the neutron emerges from the interaction. Given that an elastic scattering occurs, if $p_{\text{e}}(\theta, \phi)\mathrm{d}\Omega$ is the probability that the neutron is scattered into a small solid angle $\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$, at a polar angle θ and azimuthal angle ϕ with respect to its incident direction, we write (Fig. A2)

$$p_{\text{e}}(\theta, \phi) \, \mathrm{d}\Omega = \frac{1}{\sigma_{\text{e}}} \left(\frac{\mathrm{d}\sigma_{\text{e}}}{\mathrm{d}\Omega} \right) \mathrm{d}\Omega. \tag{A.2}$$

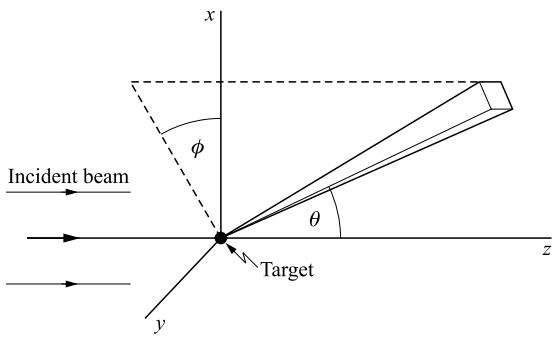


Fig. A2 Geometry of elastic scattering from a fixed target into a small solid angle $\mathrm{d}\Omega = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$ about a polar angle θ and azimuthal angle ϕ.

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This defines the *elastic differential cross-section* $\mathrm{d}\sigma_{\mathrm{el}}/\mathrm{d}\Omega$, and

$$\int \frac{\mathrm{d}\sigma_{\mathrm{e}}}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = \sigma_{\mathrm{e}}. \tag{A.3}$$

Differential cross-sections are usually measured in the laboratory with respect to a fixed target. In the centre-of-mass frame the direction of a scattered neutron, and hence the angular dependence of the cross-section, will be different. The kinematic transformation between the frames is straightforward, and experimental data is often presented in the centre-of-mass frame to facilitate comparison with theory.

A.3 Reaction rates

Consider a broad collimated beam of mono-energetic neutrons. Let ρ_{n} be the number density of neutrons in the beam, and v the neutron velocity. The *neutron flux*, i.e. the number of neutrons crossing a unit area normal to the beam per unit time, is $\rho_{\mathrm{n}}v$. Hence in time $\mathrm{d}t$, the number of neutrons passing through a circle of radius a centred on a nucleus is $\rho_{\mathrm{n}}v\,\mathrm{d}t\cdot\pi a^2$. From (A.1), the probability of a reaction with the nucleus taking place in the time interval t, $t+\mathrm{d}t$, given the nucleus is in its ground state at time t, is $\rho_{\mathrm{n}}v\sigma_{\mathrm{tot}}\,\mathrm{d}t$. Thus the *reaction rate* per nucleus is $\rho_{\mathrm{n}}v\sigma_{\mathrm{tot}}$ or

$$\text{reaction rate} = \text{flux} \times \text{cross-section}.$$

We may also consider a single neutron, moving with velocity v in a random array of nuclei of number density ρ_{nuc}. By the same argument (in the frame in which the neutron is at rest and the nuclei are regarded as a beam) the reaction rate is $\rho_{\mathrm{nuc}}v\sigma_{\mathrm{tot}}$. Given that the neutron exists at $t=0$, the mean time τ before an interaction takes place is, therefore, $\tau=(\sigma_{\mathrm{nuc}}v\sigma_{\mathrm{tot}})^{-1}$, and the *mean free path* l, the distance it travels in this time, is

$$l = v\tau = 1/(\rho_{\mathrm{nuc}}\sigma_{\mathrm{tot}}). \tag{A.4}$$

(We have assumed that τ is very much shorter than the quarter of an hour intrinsic mean life of the neutron.)

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A.4 Charged particle cross-sections: Rutherford scattering

The difficulties that arise in charged particle scattering stem from the long range of the Coulomb force. In §14.2 it is shown that a charged particle, say a proton, passing a fixed target, say a nucleus of charge Ze, is deflected through an angle θ given approximately by

$$\theta = \left(\frac{Ze^2}{4\pi\varepsilon_0}\right)\frac{2}{bpv}, \tag{A.5}$$

where p is the momentum and v the velocity of the proton, and b is the impact parameter, the distance at which the proton would pass the nucleus if there were no interaction (Fig. 14.1).

Impact parameters between b and $b + \mathrm{d}b$ correspond to scattering angles between θ and $\theta - \mathrm{d}\theta$ where

$$\mathrm{d}\theta = \frac{\mathrm{d}b}{b^2}\left(\frac{2Ze^2}{4\pi\varepsilon_0 pv}\right).$$

The effective area presented to the proton which corresponds to this range $\mathrm{d}b$ of impact parameters is $2\pi b\,\mathrm{d}b$, and we can interpret this as a contribution to the elastic scattering cross-section,

$$\begin{aligned}\mathrm{d}\sigma_\mathrm{e} &= 2\pi b\,\mathrm{d}b \\ &= 2\pi\left(\frac{2Ze^2}{4\pi\varepsilon_0 pv}\right)^2\frac{\mathrm{d}\theta}{\theta^3}, \quad \text{using (A.5).}\end{aligned}$$

The differential scattering cross-section for small angles is therefore

$$\frac{\mathrm{d}\sigma_\mathrm{e}}{\mathrm{d}\Omega} = \frac{1}{2\pi\sin\theta}\frac{\mathrm{d}\sigma_\mathrm{e}}{\mathrm{d}\theta} = \left(\frac{2Ze^2}{4\pi\varepsilon_0 pv}\right)^2\frac{1}{\theta^4},$$

where we have replaced $\sin\theta$ by θ. This is the small-angle limit of the famous Rutherford scattering formula. The same expression is obtained from a quantum-mechanical calculation.

The differential cross-section becomes very large when θ is very small, and the total elastic cross-section, defined by the integral (A.3), and hence the total cross-section, is infinite. Physically this is because the Coulomb force is still felt by the proton no matter how large the impact parameter.

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In practice this formally infinite result is not a serious difficulty, since there is always a limit to the experimentalist's ability to measure small-angle scattering, and if one is interested only in elastic scattering through angles greater than some minimum angle the cross-section is finite.

At large impact parameters the Coulomb force is weak, and can only give rise to small-angle elastic scattering. The cross-sections for other possible processes are all finite.

Appendix B
Density of states

Consider a particle moving freely inside a cubic box of side L, volume $V = L^3$. We take the potential to be zero inside the box, and represent the walls by infinite potential barriers. The Schrödinger equation for the particle,

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi, \tag{B.1}$$

is separable in (x, y, z) coordinates, and the solutions must vanish at the walls which we can take to be the planes $x = 0$ and $x = L$, $y = 0$ and $y = L$, $z = 0$ and $z = L$. These solutions are easily seen to be standing waves of the form

$$\psi(x, y, z) = (\text{constant})\sin(k_x x)\sin(k_y y)\sin(k_z z), \tag{B.2}$$

provided that we choose $\mathbf{k} = (k_x, k_y, k_z)$ from the values

$$k_x = \frac{n_x\pi}{L}, \quad n_x = 1, 2, 3, \ldots; \quad k_y = \frac{n_y\pi}{L}, \quad n_y = 1, 2, 3, \ldots;$$
$$k_z = \frac{n_z\pi}{L}, \quad n_z = 1, 2, 3, \ldots$$

to satisfy the boundary conditions. Negative integer values of n_x, n_y, n_z do not give new states, since they merely change the sign of the wave-function, and such a phase factor has no physical significance.

Thus the allowed values of $\mathbf{k}$ form a cubic lattice of points in the $(+, +, +)$ quadrant of '$\mathbf{k}$-space'. Each eigenstate (B.2) corresponds to one point of the lattice, and counting states is equivalent to counting lattice points. The spacing between these lattice points is (π/L), so that the

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