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On topological Stokes phenomena on the two-dimensional cylinder

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Abstract. On the cylinder $\mathbb{T} \times \mathbb{R}^1$, smooth dynamical systems are constructed that, in spite of being smoothly conjugate on an invariant covering, are not topologically conjugate globally.

1. *Introduction*

The classical (*linear*) Stokes phenomenon consists of the appearance of additional invariants (*Stokes operators*) in the local classification of analytical linear differential equations in a neighborhood of the origin that are equivalent on elements of an open local covering.

Similar nonlinear phenomena (*Ecalle–Voronin moduli*) arise in the local classification of germs of one-dimensional non-hyperbolic analytic diffeomorphisms (see [**N**]).

The aforementioned phenomena are local. Mather [**M**] was the first to find a similar 'global' phenomenon for diffeomorphisms of the unit segment with hyperbolic fixed endpoints. Two such diffeomorphisms with the same multipliers at fixed points that are smoothly conjugate on half-intervals are not as a rule smoothly conjugate globally. A functional invariant arises as an obstacle to the conjugacy. In [**B1, B2, K**], functional moduli of global smooth conjugacy for diffeomorphisms of the unit circle and of the real line with hyperbolic fixed points were constructed.

In [**B3, B4**] (see also [**A**]) similar moduli for dynamical systems on the unit sphere with sink-source were constructed.

In the above cases, dynamical systems have no obstacle to the topological conjugacy.

In the present paper we construct diffeomorphisms and flows on the standard two-dimensional cylinder $M = \mathbb{T} \times \mathbb{R}^1$ that, in spite of being smoothly conjugate on elements of an invariant covering, are not topologically conjugate on the whole. These dynamical systems are *glued* from two simple systems determined on the same cylinder. In [**B3**], a general notion of glued dynamical systems was introduced.

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2. *Diffeomorphisms*

Consider two diffeomorphisms of the standard cylinder

$$N_1(z, y) = (r_\alpha(z), \lambda y), \quad N_2(z, y) = (r_\alpha(z), \mu y) \quad (z, y) \in \mathbb{T} \times \mathbb{R}^1$$

where $0 < \lambda < 1 < \mu$ and r_α is an irrational rotation of the unit circle,

$$r_\alpha(z) = e^{2\pi i \alpha} z, \quad z \in \mathbb{T} \subset \mathbb{R}^2.$$

A diffeomorphism $F : M \to M$ is said to be C^k-glued from the diffeomorphisms $\{N_i\}$ if there exists an open invariant covering $M = U_1 \cup U_2$ such that the restriction $F|U_i$ is C^k-conjugate with $N_i, i = 1, 2$, respectively. Denote by $C^k(M, \{N_i\})$ the set of all C^k-glued diffeomorphisms.

The smoothness k may take all values $0, 1, \ldots, \infty$ and $\aleph$, i.e. we also consider real-analytic mappings. By definition, homeomorphisms are C^0-diffeomorphisms.

The set $C^k(M, \{N_i\})$ is non-empty for every k.

Example 1. Let f be a C^k-diffeomorphism of the real line $\mathbb{R}^1$ glued from the linear mappings λx and μx. This means (see [**B2**]) that f has two fixed points. Fix $f = \{z_1, z_2\}$ and the restrictions $f|(-\infty, z_2)$, $f|(z_1, +\infty)$ are C^k-conjugate to $\lambda x, \mu x$, respectively. Let

$$\varphi_1 : (-\infty, z_2) \to \mathbb{R}^1, \quad \varphi_2 : (z_1, +\infty) \to \mathbb{R}^1$$

be *normalizing C^k-diffeomorphisms* for f:

$$\varphi_1 f \varphi_1^{-1}(x) = \lambda x, \quad \varphi_2 f \varphi_2^{-1}(x) = \mu x, \quad \varphi_i(z_i) = 0.$$

The *gluing diffeomorphism* $g = \varphi_2 \varphi_1^{-1}$ maps $\mathbb{R}^\pm$ onto $\mathbb{R}^\pm$. The choice of signs depends on the orientations of φ_1 and φ_2. Namely, if both φ_1 and φ_2 preserve the orientation, then g maps $\mathbb{R}^+$ onto $\mathbb{R}^-$. If both of them invert the orientation, then g maps $\mathbb{R}^-$ onto $\mathbb{R}^+$. If φ_1 preserves the orientation while φ_2 inverts it, then g maps $\mathbb{R}^+$ onto itself. Finally, in the opposite case g maps $\mathbb{R}^-$ onto itself.

According to [**B2**], the diffeomorphism g up to a simple linear equivalence is a functional invariant of C^1-conjugacy. On the other hand, all diffeomorphisms $f : \mathbb{T} \to \mathbb{T}$ glued from λx and μx are topologically conjugate.

The direct product $P(u, v) = (r_\alpha(u), f(v))$ belongs to $C^k(M, \{N_i\})$. Indeed, set

$$U_1 = \mathbb{T} \times (-\infty, z_2), \quad U_2 = \mathbb{T} \times (z_1, +\infty)$$

and

$$\Phi_i(u, v) = (u, \varphi_i(v)), \quad i = 1, 2.$$

Then $M = U_1 \cup U_2$ and

$$P|U_i = \Phi_i^{-1} N_i \Phi_i, \quad i = 1, 2.$$

Given $F \in C^k(M, \{N_i\})$, let

$$\Phi_i : U_i \to \mathbb{T} \times \mathbb{R}^1, \quad i = 1, 2$$

be its *normalizing diffeomorphisms*:

$$F|U_i = \Phi_i^{-1} N_i \Phi_i, \quad i = 1, 2.$$

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The curves $l_i = \Phi_i^{-1}(\mathbb{T} \times \{0\})$ are F-invariant. If $(u, v) \notin l_2$, then $F^n(u, v) \to l_1$ as $n \to +\infty$, while $(u, v) \notin l_1$ implies that $F^n(u, v) \to l_2$ as $n \to -\infty$. In this sense l_1 is an attractor for F and l_2 is a repeller. In addition,

$$l_2 = \partial U_1, \quad l_1 = \partial U_2.$$

The normalizing diffeomorphisms carry over the boundary curves onto either $\mathbb{T} \times (+\infty)$ or $\mathbb{T} \times (-\infty)$. For example, let Φ_1 carry over l_2 onto $\mathbb{T} \times (+\infty)$. Then the diffeomorphism $\tilde{\Phi}_1 = I\Phi_1$, $I(z, y) = (z, -y)$ conjugates $F|U_1$ with N_1 too and carries over l_2 onto $\mathbb{T} \times (-\infty)$.

The diffeomorphism $G = \Phi_2\Phi_1^{-1}$ maps $\mathbb{T} \times \mathbb{R}^{\pm}$ onto $\mathbb{T} \times \mathbb{R}^{\pm}$ depending on the above-mentioned boundary property of Φ_i. We call G a *gluing diffeomorphism for* F and denote it by $c(F, \{\Phi_i\})$. The diffeomorphism G satisfies

$$GN_1 = N_2G. \tag{1}$$

Example 2. Let $F \in C^k(M, \{N_i\})$ be a direct product, $F(u, v) = (h(u), f(v))$, where $h : \mathbb{T} \to \mathbb{T}$ is C^k-conjugate with the rotation r_α and f is as in Example 1. If $\psi : \mathbb{T} \to \mathbb{T}$ conjugates h with the rotation and $\{\varphi_i\}$ are normalizing diffeomorphisms for f, then

$$\Phi_i(u, v) = (\psi(u), \varphi_i(v)), \quad i = 1, 2$$

are normalizing diffeomorphisms for F and

$$G(z, y) = (z, g(y))$$

where $g = \varphi_2\varphi_1^{-1}$ is the gluing diffeomorphism for f.

Let $(\tilde{\Phi}_i, \tilde{U}_i)$ be other normalizing diffeomorphisms for the same $F \in C^k(M, \{N_i\})$. Then $\tilde{U}_i = U_i$, $i = 1, 2$, and the diffeomorphisms $A_i = \tilde{\Phi}_i\Phi_i^{-1}$ commute with N_i, $i = 1, 2$, respectively. Hence,

$$\tilde{G} = \tilde{\Phi}_2\tilde{\Phi}_1^{-1} = A_2GA_1^{-1}. \tag{2}$$

In other words, gluing diffeomorphisms corresponding to the same $F \in C^k(M, \{N_i\})$ are equivalent in the sense of (2).

Let us describe homeomorphisms commuting with N_i.

Let $A : \mathbb{T} \times \mathbb{R}^1 \to \mathbb{T} \times \mathbb{R}^1$ be a homeomorphism commuting with N_1. Then its coordinates $A^1 : \mathbb{T} \times \mathbb{R}^1 \to \mathbb{T}$ and $A^2 : \mathbb{T} \times \mathbb{R}^1 \to \mathbb{R}^1$ satisfy the equalities

$$A^1(e^{2\pi i\alpha}z, \lambda y) = e^{2\pi i\alpha}A^1(z, y), \quad A^2(e^{2\pi i\alpha}z, \lambda y) = \lambda A^2(z, y).$$

Since α is irrational, an immediate calculation shows that

$$A^1(z, y) = r_a(z), \quad a \in \mathbb{R}, \tag{3}$$

while $A^2(z, y)$ is a monotonic by y for every $z \in \mathbb{T}$ having the form

$$A^2(z, y) = \begin{cases} y\tau^+\left(e^{-2\pi i\alpha(\ln y/\ln\lambda)}z, \dfrac{\ln y}{\ln\lambda}\right), & y > 0 \\ y\tau^-\left(e^{-2\pi i\alpha(\ln|y|/\ln\lambda)}z, \dfrac{\ln|y|}{\ln\lambda}\right), & y < 0 \end{cases} \tag{4}$$

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with 1-periodic by the second variable continuous functions $\tau^{\pm} : \mathbb{T} \times \mathbb{R}^1 \to \mathbb{R}^1$ such that $\tau^+(u, v)\tau^-(u, v) > 0$.

Homeomorphisms commuting with N_2 have the same form if we replace λ by μ.

As a consequence, C^1-diffeomorphisms commuting with N_i are linear

$$A(z, y) = (r_a(z), by), \quad a, b \in \mathbb{R}. \tag{5}$$

Let us call diffeomorphisms G and $\tilde{G}$ satisfying (1) *linearly equivalent* if there are $a_i, b_i \in \mathbb{R}$ such that $GA_1 = A_2\tilde{G}$ with $A_1(z, y) = (r_{a_1}(z), b_1 y)$ and $A_2(z, y) = (r_{a_2}(z), b_2 y)$.

THEOREM 1. *Given a C^k-diffeomorphism G, satisfying (1), there exists a diffeomorphism $F \in C^k(M, \{N_i\})$ and its normalizing diffeomorphisms $\{\Phi_i\}$ such that $G = c(F, \{\Phi_i\})$.*

Diffeomorphisms $F, \tilde{F} \in C^k(M, \{N_i\})$ are topologically conjugate if and only if there are homeomorphisms A_i commuting with N_i, $i = 1, 2$, respectively, such that

$$GA_1 = A_2\tilde{G}.$$

If F and $\tilde{F}$ are C^1-conjugate, then G and $\tilde{G}$ are linearly equivalent. Conversely, if G and $\tilde{G}$ are linearly equivalent, then F and $\tilde{F}$ are C^k-conjugate.

Here $\tilde{G} = c(\tilde{F}, \{\tilde{\Phi}_i\})$.

Proof. We can assume that G maps $\mathbb{T} \times \mathbb{R}^+$ onto $\mathbb{T} \times \mathbb{R}^-$. Set

$$U_1 = \mathbb{T} \times \{y : y < 1\}, \quad U_2 = \mathbb{T} \times \{y : y > -1\}$$

and choose a homeomorphism

$$\hat{\Phi}_1 : U_1 \to \mathbb{T} \times \mathbb{R}^1, \quad \hat{\Phi}_1(\mathbb{T} \times \{-1\}) = \mathbb{T} \times \{0\}$$

mapping the subset $\mathbb{T} \times (-1, 1)$ onto $\mathbb{T} \times \mathbb{R}^+$. The homeomorphism $\hat{\Phi}_2 = G\hat{\Phi}_1$ is determined on the set $\mathbb{T} \times (-1, 1)$ and maps it onto $\mathbb{T} \times \mathbb{R}^-$. Choose a point y_0 from the interval $(-1, 1)$, say $y_0 = 0$, and let $\Phi_2 : U_2 \to \mathbb{T} \times \mathbb{R}^1$ be a homeomorphism coinciding with $\hat{\Phi}_2$ on the subset $\mathbb{T} \times \{-1 < y \le y_0\}$. The homeomorphism $\Phi = G^{-1}\Phi_2$ is determined on the set $\mathbb{T} \times (-1, 1)$, maps it onto $\mathbb{T} \times \mathbb{R}^+$ and coincides with $\hat{\Phi}_1$ on the subset $\mathbb{T} \times \{-1 < y \le y_0\}$. Hence, the mapping

$$\Phi_1(z, y) = \begin{cases} \hat{\Phi}_1(z, y), & y \le y_0 \\ \Phi(z, y), & y > y_0 \end{cases}$$

is a well-defined homeomorphism from U_1 onto $\mathbb{T} \times \mathbb{R}^1$.

As a result, we arrive at the decomposition $G = \Phi_2\Phi_1^{-1}$. The maps $(U_i, \Phi_i)_{i=1,2}$ form an atlas of a smooth structure on M with the transaction function G. It is well known that all smooth structures on two-dimensional manifolds are diffeomorphic. Hence, we can assume that Φ_i is of the same smoothness class as G itself, including the analytic case.

Now let F and $\tilde{F}$ be diffeomorphisms from $C^k(M, \{N_i\})$ with $G = c(F, \{\Phi_i\})$ and $\tilde{G} = c(\tilde{F}, \{\tilde{\Phi}_i\})$, respectively. Assume that they are C^p-conjugate:

$$\tilde{F} = \Phi^{-1}F\Phi, \quad \Phi \in C^p.$$

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Then $\Phi(\tilde{U}_i) = U_i$, where $\tilde{U}_i$ and U_i are corresponding open invariant subsets for $\tilde{F}$ and F, respectively. Hence,

$$\tilde{\Phi}_i^{-1} N_i \tilde{\Phi}_i | \tilde{U}_i = \Phi^{-1} \Phi_i^{-1} N_i \Phi_i \Phi | \tilde{U}_i$$

and the C^p-diffeomorphisms $A_i = \Phi_i \Phi \tilde{\Phi}_i^{-1}$ commute with N_i. As a consequence,

$$\Phi | \tilde{U}_i = \Phi_i^{-1} A_i \tilde{\Phi}_i | \tilde{U}_i, \quad i = 1, 2 \tag{6}$$

and

$$\Phi_1^{-1} A_1 \tilde{\Phi}_1 | \tilde{U}_1 \cap \tilde{U}_2 = \Phi_2^{-1} A_2 \tilde{\Phi}_2 | \tilde{U}_1 \cap \tilde{U}_2,$$

implying $GA_1 = \tilde{G} A_2$.

Conversely, assume that the latter holds. Then the mapping Φ determined by (6) is a well-defined C^p-diffeomorphism of M, conjugating F and $\tilde{F}$.

Taking into account the description of C^1-diffeomorphisms commuting with N_i, we prove the last part of Theorem 1. □

As we have noted, all direct products $(h(u), f(v)) \in C^k(M, \{N_i\})$ are topologically conjugate. However, they may not be smoothly conjugate. Indeed, the smooth conjugacy means that

$$\tilde{g}(y) = b_2 g(b_1 y), \quad b_1, b_2 \in \mathbb{R}$$

for gluing diffeomorphisms g and $\tilde{g}$ of f and $\tilde{f}$, respectively. The latter does not hold, for instance, if

$$g(y) = y^\theta, \quad \tilde{g}(y) = y^\theta \left(1 + \varepsilon \sin 2\pi \frac{\ln y}{\ln \lambda}\right)$$

with small $\varepsilon > 0$ and $\theta = \ln \lambda / \ln \mu$. At the same time Theorem 1 shows that there are obstacles to topological conjugacy in $C^k(M, \{N_i\})$.

COROLLARY 1. *The set $C^k(M, \{N_i\})$ contains topologically non-conjugating diffeomorphisms and even diffeomorphisms topologically non-conjugating with the direct product $P(u, v) = (r_\alpha(u), f(v))$ from Example 1.*

Proof. Let $y > 0$ and

$$G(z, y) = (e^{-2\pi i (\ln y / \ln \lambda)} z, y^\theta), \quad \theta = \frac{\ln \mu}{\ln \lambda}.$$

Then G satisfies (1) and is a gluing diffeomorphism for $F \in C^k(M, \{N_i\})$. We show that F is topologically non-conjugate to P. Indeed, otherwise there are homeomorphisms

$$A_1(z, y) = (r_{a_1}(z), A_1^2(z, y)), \quad A_2(z, y) = (r_{a_2}(z), A_2^2(z, y))$$

where A_i^2 have the form (4) and such that $GA_1 = A_2 \tilde{G}$, where $\tilde{G}(z, y) = (z, g(y))$ is a gluing diffeomorphism for P. In particular.

$$e^{-2\pi i (\ln y / \ln \lambda - a_1)} z = e^{2\pi i a_2} z,$$

which is a contradiction. □

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In order to obtain an explicit description of the set of diffeomorphisms satisfying (1), consider the mappings

$$\Psi_1^+(z, y) = \left(e^{-2\pi i\alpha(\ln y/\ln\lambda)}z, \frac{\ln y}{\ln\lambda}\right), \quad y > 0$$

and

$$\Psi_1^-(z, y) = \left(e^{-2\pi i\alpha(\ln|y|/\ln\lambda)}z, \frac{\ln|y|}{\ln\lambda}\right), \quad y < 0.$$

They are diffeomorphisms from $\mathbb{T}\times\mathbb{R}^{\pm}$ onto the standard cylinder $\mathbb{T}\times\mathbb{R}^1$ and *straighten* N_1 in the sense

$$N_1 = (\Psi_1^{\pm})^{-1}S\Psi_1^{\pm},$$

where $S(z, y) = (z, y + 1)$ is the *unit shift* of $\mathbb{T}\times\mathbb{R}^1$. Respectively, the mappings

$$\Psi_2^+(z, y) = \left(e^{-2\pi i\alpha(\ln y/\ln\mu)}z, \frac{\ln y}{\ln\mu}\right), \quad y > 0$$

and

$$\Psi_2^-(z, y) = \left(e^{-2\pi i\alpha(\ln|y|/\ln\mu)}z, \frac{\ln|y|}{\ln\mu}\right), \quad y > 0$$

straighten N_2 in the same sense.

Given a gluing diffeomorphism $G : \mathbb{T}\times\mathbb{R}^+ \to \mathbb{T}\times\mathbb{R}^+$, the mapping

$$H = \Psi_2^+ G(\Psi_1^+)^{-1}$$

is a diffeomorphism of the standard cylinder commuting with the unit shift. Similarly, if G maps $\mathbb{T}\times\mathbb{R}^-$ onto itself, then the diffeomorphism

$$H = \Psi_2^- G(\Psi_1^-)^{-1}$$

possesses the same properties. If G maps $\mathbb{T}\times\mathbb{R}^+$ onto $\mathbb{T}\times\mathbb{R}^-$, then

$$H = \Psi_2^- G(\Psi_1^+)^{-1}$$

is a well-defined diffeomorphism of $\mathbb{T}\times\mathbb{R}^1$ commuting with S. Finally, if G maps $\mathbb{T}\times\mathbb{R}^-$ onto $\mathbb{T}\times\mathbb{R}^+$, then

$$H = \Psi_2^+ G(\Psi_1^-)^{-1}$$

has the mentioned properties.

Example 3. Let $P(z, y) = (h(z), f(y))$ as in Example 2. Then $G(z, y) = (z, g(y))$ where g is a gluing diffeomorphism for f. It either has the form

$$g(y) = y^{\theta}\tau\left(\frac{\ln y}{\ln\lambda}\right), \quad y > 0, \quad \theta = \frac{\ln\mu}{\ln\lambda}$$

or the form

$$g(y) = |y|^{\theta}\tau\left(\frac{\ln|y|}{\ln\lambda}\right), \quad y < 0$$

with non-vanishing 1-periodic C^k-function $\tau(t)$ satisfying

$$\tau(t)\ln\mu + \tau'(t) \neq 0.$$

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If $G : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{T} \times \mathbb{R}^+$, then $y > 0$ and $\tau > 0$. Hence,

$$H(z, y) = \Psi_2^+ G (\Psi_1^+)^{-1}(z, y) = (e^{-2\pi i \alpha b(y)} z, y + b(y))$$

with

$$b(y) = \frac{\ln \tau(y)}{\ln \mu}.$$

Conversely, given a diffeomorphism $H : \mathbb{T} \times \mathbb{R}^1 \to \mathbb{T} \times \mathbb{R}^1$, commuting with the unit shift, all four diffeomorphisms $G = (\Psi_2^{\pm})^{-1} H \Psi_1^{\pm}$ satisfy (1).

Example 4. Let $H(z, y) = (z, y)$ be identical. Then

$$G(z, y) = (\Psi_2^{\pm})^{-1} \Psi_1^{\pm}(z, y) = (z, \pm |y|^{\theta}).$$

If $H(z, y) = (1/z, y)$, then

$$G(z, y) = (\Psi_2^{\pm})^{-1} H \Psi_1^{\pm}(z, y) = \left(\frac{1}{z} e^{4\pi i \alpha (\ln |y| / \ln \lambda)}, \pm |y|^{\theta} \right).$$

In general, a diffeomorphism of the standard cylinder commuting with the unit shift possesses the form

$$H(z, y) = (e^{2\pi i (my + h^1(z,y))} z^{\pm 1}, y + h^2(z, y)) \tag{7}$$

with an integer m and functions

$$h^i : \mathbb{T} \times \mathbb{R}^1 \to \mathbb{R}^1, \quad i = 1, 2$$

1-periodic by y.

The correspondence $G \leftrightarrow H$ is determined by a choice of straightening diffeomorphisms $\Psi_i^{\pm}$. Fixing them, we arrive at a parametrization of the set gluing diffeomorphisms by diffeomorphisms (7).

In turn, every diffeomorphism (7) covers a diffeomorphism of the torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$ homotopic to the automorphism $(u, v) \mapsto (u^m v^{\pm 1}, v)$, $(u, v) \in \mathbb{T}^2$. Hence, the set of gluing diffeomorphisms G also possesses a parametrization by diffeomorphisms of the standard torus homotopic to the above-mentioned automorphisms. However, we prefer to use the previous parametrization by diffeomorphisms of the standard cylinder.

Now we compute the equivalence (2) in terms of the parametrization. Let G and $\tilde{G}$ be gluing diffeomorphisms equivalent in the sense of (2), and let H and $\tilde{H}$ be corresponding diffeomorphisms of the cylinder. Then $\tilde{H} = B_2 H B_1^{-1}$, where

$$B_i = \Psi_i^{\pm} A_i (\Psi_i^{\pm})^{-1}, \quad i = 1, 2,$$

depending on a distribution of the signs $\pm$ for diffeomorphisms G and $\tilde{G}$. Using the representation (3)–(4) for homeomorphisms commuting with N_1, we arrive at the formula

$$\Psi_1^+ A (\Psi_1^+)^{-1}(z, y) = (e^{-2\pi i (\alpha b(z,y) - a)} z, y + b(z, y)) \tag{8}$$

where

$$b(z, y) = \frac{\ln \tau^+(z, y)}{\ln \lambda}.$$

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The mapping $\Psi_1^- A(\Psi_1^-)^{-1}$ has the same form with

$$b(z, y) = \frac{\ln \tau^-(z, y)}{\ln \lambda}.$$

Similarly, $\Psi_1^+ A(\Psi_1^-)^{-1}$ has the same form with

$$b(z, y) = \frac{\ln |\tau^-(z, y)|}{\ln \lambda}$$

and, finally,

$$b(z, y) = \frac{\ln |\tau^+(z, y)|}{\ln \lambda}$$

for $\Psi_1^- A(\Psi_1^+)^{-1}$. 'Rectifications' $\Psi_2^\pm A(\Psi_2^\pm)^{-1}$ have the same form (8).

Conversely, given a homeomorphism B of the form (8), all four homeomorphisms $(\Psi_1^\pm)^{-1} B \Psi_1^\pm$ may be extended by continuity to homeomorphisms of the cylinder $\mathbb{T} \times \mathbb{R}^1$ commuting with the unit shift.

If A is C^1-diffeomorphism, i.e. is a linear mapping (5), then all of its rectifications coincide:

$$\Psi_1^\pm A(\Psi_1^\pm)^{-1}(z, y) = (r_d(z), y + c)$$

with $d = a - \alpha b$, $c = \ln |b| / \ln \lambda$.

Given $F \in C^k(M, \{N_i\})$ with normalizing diffeomorphisms $\{\Phi_i\}$, denote by $\hat{c}(F, \{\Phi_i\})$ the diffeomorphism (7) corresponding to $G = c(F, \{\Phi_i\})$.

COROLLARY 2. *Given a C^k-diffeomorphism $H : \mathbb{T} \times \mathbb{R}^1 \to \mathbb{T} \times \mathbb{R}^1$ commuting with the unit shift of the cylinder, there is a diffeomorphism $F \in C^k(M, \{N_i\})$ and its normalizing diffeomorphisms $\{\Phi_i\}$ such that $H = \hat{c}(F, \{\Phi_i\})$.*

Diffeomorphisms $F, \tilde{F} \in C^k(M, \{N_i\})$ are topologically conjugate if and only if there are homeomorphisms $B_i : \mathbb{T} \times \mathbb{R}^1 \to \mathbb{T} \times \mathbb{R}^1$ having the form (8) and commuting with the unit shift such that $\tilde{H} = B_2 H B_1^{-1}$.

If F and $\tilde{F}$ are C^1-conjugate, then H and $\tilde{H}$ are equivalent in the same sense with linear B_i. Conversely, if the latter is fulfilled, then F and $\tilde{F}$ are C^k-conjugate.

In order to prove the corollary it suffices to apply Theorem 1 to diffeomorphisms $G = \Psi_2^\pm H (\Psi_1^\pm)^{-1}$.

Homeomorphisms (8) are homotopic to the identity mapping of the cylinder. Therefore, if F and $\tilde{F}$ are topologically conjugate, then corresponding diffeomorphisms H and $\tilde{H}$ must be homotopic. In particular, the integer exponents ± 1 and m in (7) are invariant with respect to the topological conjugacy.

We now present the conjugacy conditions in terms of covering mappings. Covering mappings for the diffeomorphisms (7) are diffeomorphisms of the plane $\mathbb{R}^2$ having the form

$$C(x, y) = (\pm x + my + c^1(x, y), y + c^2(x, y)) \tag{9}$$

where c^1 and c^2 are twice-periodic functions. The function c^1 is determined up to an integer number, while c^2 is determined uniquely by the diffeomorphism H.

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Covering mappings for diffeomorphisms (8) are homeomorphisms of the plane having the special form

$$\Gamma(x, y) = (x - \alpha d(x, y) + a + n, y + d(x, y)) \tag{10}$$

with a twice-periodic function d, real a and integer n. Therefore, the topological conjugacy condition of Corollary 2 in terms of covering mappings C and $\tilde{C}$ for H and $\tilde{H}$, respectively, takes on the following form: *there are homeomorphisms* Γ_1 *and* Γ_2 *having the form (10) such that*

$$C(\Gamma_1(x, y)) = \Gamma_2(\tilde{C}(x, y)). \tag{11}$$

Respectively, the smooth conjugacy condition has the same form with

$$d_i(x, y) = \text{constant}, \quad i = 1, 2.$$

In fact, one of the homeomorphisms Γ_1, Γ_2 in (11) may be eliminated. In order to arrive at a more simple form of conjugacy conditions, we set

$$\rho_C(x, y) = \pm x + my + c^1(x, y) + \alpha c^2(x, y)$$

for the diffeomorphism (9). Since the Jacobi matrix $C'(x, y)$ is non-degenerate for all x, y, the functions ρ_C has no critical points.

To every $F \in C^k(M, \{N_i\})$, various functions ρ_C correspond depending on a choice of normalizing diffeomorphisms $\{\Phi_i\}$ and a covering (9) for $H = \hat{c}(F, \{\Phi_i\})$. However, if $k \geq 1$, then functions ρ_C corresponding to the same $F \in C^k(M, \{N_i\})$ differ by linear transformations. Namely, let H and $\tilde{H}$ correspond to the same F. Then, by Corollary 2, the mappings Γ_i in (11) are linear:

$$\Gamma_i(x, y) = (x + q_i^1, y + q_i^2), \quad i = 1, 2$$

with constant q_i^1 and q_i^2. It follows that

$$\rho_{\tilde{C}}(x, y) = q + \rho_C(\Gamma(x, y)) \tag{12}$$

where $\Gamma(x, y) = (x + q_1^1, y + q_1^2)$ and $q = -(q_2^1 + \alpha q_2^2)$. Conditions of the topological conjugacy have the same form (12) with a homeomorphism Γ.

THEOREM 2. *Let* $F, \tilde{F} \in C^k(M, \{N_i\})$ *be diffeomorphisms with functions* $\rho_C, \rho_{\tilde{C}}$, *respectively. Then* F *and* $\tilde{F}$ *are topologically conjugate if and only if there are a real* q *and a homeomorphism (10) such that (12) is valid.*

Proof. The proof consists of an immediate calculation. Let $H = \hat{c}(F, \{\Phi_i\})$ and $\tilde{H} = \hat{c}(\tilde{F}, \{\tilde{\Phi}_i\})$. If F and $\tilde{F}$ are topologically conjugate, then $HB_1 = B_2\tilde{H}$ with homeomorphisms B_i having the form (9). Rewriting this equality in terms of covering mappings, we obtain

$$\begin{aligned} &\pm (x - \alpha d_2(x, y) + a_1) + m(y + d_1(x, y)) + c^1(\Gamma_1(x, y)) \\ &\quad = \pm x + \tilde{m}y + \tilde{c}^1(x, y) - \alpha d_2(\tilde{C}(x, y)) + a_2 + n \end{aligned}$$

and

$$y + d_1(x, y) + c^2(\Gamma_1(x, y)) = y + \tilde{c}^2(x, y) + d_2(\tilde{C}(x, y)).$$

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Here

$$\Gamma_i(x, y) = (x - \alpha d_i(x, y), y + d_i(x, y)), \quad i = 1, 2$$

are covering for B_i and n is an integer number. It follows that $m = \tilde{m}$ and

$$\rho_C(x, y) = q + \rho_C(\Gamma_1(x, y))$$

with $q = a_2 + n$.

Conversely, let (12) hold. Then, setting $\Gamma_1 = \Gamma$ and

$$d_2(\tilde{C}(x, y)) = c^2(\Gamma_1(x, y)) + d_1(x, y) - \tilde{c}^2(x, y),$$

we arrive at (11) with

$$\Gamma_2(x, y) = (x - \alpha d_2(x, y) - q, y + d_2(x, y)).$$

Since C, Γ_1 and $\tilde{C}$ are homeomorphisms, the mapping Γ_2 is also a homeomorphism. Both Γ_1 and Γ_2 cover homeomorphisms B_1 and B_2, respectively, having the form (9). Hence, $HB_1 = \tilde{H}B_2$, implying the topological conjugacy of F and $\tilde{F}$. □

If F and $\tilde{F}$ are smoothly conjugate, then (12) is valid with a linear $\Gamma(x, y)$. However, the inverse assertion is not true: the equality with linear $\Gamma(x, y)$ does not imply smooth conjugacy. Indeed, let

$$H(z, y) = (z, y), \quad \tilde{H}(z, y) = (e^{-2\pi i\alpha b(y)}z, y + b(y)).$$

Then (12) is valid with $q = 0$ and $\Gamma = \mathrm{id}$, implying the topological conjugacy of F and $\tilde{F}$ with $H = \hat{c}(F, \{\Phi_i\})$ and $\tilde{H} = \hat{c}(\tilde{F}, \{\tilde{\Phi}_i\})$. Nevertheless, F and $\tilde{F}$ are not smoothly conjugate if $b(y) \neq$ constant.

3. *Flows*

The diffeomorphisms N_1 and N_2 are embedded in the linear flows

$$N_1^t(z, y) = (e^{2\pi it\alpha}z, e^{t\ln\lambda}y), \quad N_2^t(z, y) = (e^{2\pi it\alpha}z, e^{t\ln\mu}y).$$

The set $C^k(M, \{N_i^t\})$ of flows glued from N_1^t and N_2^t is not empty. For instance, it contains direct products $P^t(u, v) = (e^{2\pi it\alpha}u, f^t(v))$ where f^t is a flow on $\mathbb{R}^1$ glued from linear flows $\lambda^t x$ and $\mu^t x$ of the real line.

Let $F^t \in C^k(M, \{N_i^t\})$ with normalizing diffeomorphisms Φ_1 and Φ_2. Then the gluing diffeomorphism $G = \Phi_2\Phi_1^{-1}$ satisfies

$$GN_1^t = N_2^tG, \quad t \in \mathbb{R}. \tag{13}$$

We denote $G = c(F^t, \{\Phi_1\})$.

A homeomorphism A commuting with the flow N_1^t has the form (3)–(4) with the additional condition: the functions $\tau^{\pm}(u, v)$ do not depend on the variable v. C^1-diffeomorphisms commuting with N_1^t are linear and are given by (5).

THEOREM 3. *Given a C^k-diffeomorphism G satisfying (13), there exists a flow $F^t \in C^k(M, \{N_i^t\})$ and its normalizing diffeomorphisms $\{\Phi_i\}$ such that $G = c(F^t, \{\Phi_i\})$.*

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Flows $F^t, \tilde{F}^t \in C^k(M, \{N_i^t\})$ are topologically conjugate if and only if there are homeomorphisms A_i commuting with the flows N_i^t, $i = 1, 2$, respectively, such that $GA_1 = A_2\tilde{G}$.

If F^t and $\tilde{F}^t$ are C^1-conjugate, then G and $\tilde{G}$ are linearly equivalent. Conversely, if G and $\tilde{G}$ are linearly equivalent, then the flows F^t and $\tilde{F}^t$ are C^k-conjugate.

Here $\tilde{G} = c(\tilde{F}^t, \{\tilde{\Phi}_i\})$.

The proof of Theorem 3 follows the same arguments as the proof of Theorem 1.

COROLLARY 3. *The set $C^k(M, \{N_i^t\})$ contains topologically non-conjugating flows and even flows topologically non-conjugating with the direct product $P^t(u, v) = (e^{2\pi i\alpha t}u, f^t(v))$.*

Proof. The gluing diffeomorphism G introduced in the proof of Corollary 1 satisfies (13). Hence, there is a flow $F^t \in C^k(M, \{N_i^t\})$ and its normalizing diffeomorphisms $\{\Phi_i\}$ such that $G = \Phi_2\Phi_1^{-1}$. The flow F^t is topologically non-conjugate with P^t. □

The diffeomorphisms $\Psi_i^{\pm}$ introduced in the previous section also straighten the flows N_i^t. Namely,

$$N_i^t = (\Psi_i^{\pm})^{-1} S^t \Psi_i^{\pm}$$

where $S^t(z, y) = (z, y + t)$ is the *flow of shifts*. Respectively, the diffeomorphisms

$$H = \Psi_2^{\pm} G (\Psi_1^{\pm})^{-1}$$

commute with the flow S^t and have the form

$$H(z, y) = (h^1(z), y + h^2(z)) \tag{14}$$

where $h^1 : \mathbb{T} \to \mathbb{T}$ is a diffeomorphism and $h^2 : \mathbb{T} \to \mathbb{R}^1$ is a function.

If A commutes with the flow N_1^t, then its 'rectifications'

$$B = \Psi_1^{\pm} A (\Psi_1^{\pm})^{-1}$$

have the same form (8) with $b = b(z)$ not depending on y.

As above, we denote by $\hat{c}(F^t, \{\Phi_i\})$ the diffeomorphism $H = \Psi_2^{\pm} G (\Psi_1^{\pm})^{-1}$ where $G = c(F^t, \{\Phi_i\})$.

COROLLARY 4. *Given a C^k-diffeomorphism (14), there exists a flow $F^t \in C^k(M, \{N_i^t\})$ and its normalizing diffeomorphisms $\{\Phi_i\}$ such that $H = \hat{c}(F^t, \{\Phi_i\})$. Flows $F^t, \tilde{F}^t \in C^k(M, \{N_i\})$ are topologically conjugate if and only if there are homeomorphisms*

$$B_1(z, y) = (e^{-2\pi i(\alpha b_1(z) - a_1)}z, y + b_1(z)), \quad B_2(z, y) = (e^{-2\pi i(\alpha b_2(z) - a_2)}z, y + b_2(z))$$

such that $HB_1 = B_2H$.

If F^t and $\tilde{F}^t$ are C^1-conjugate, then the same is valid with $b_i =$ constant. Conversely, if the latter is valid, then F^t and $\tilde{F}^t$ are C^k-conjugate.

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Here $\tilde{H} = \Psi_2^{\pm}\tilde{G}(\Psi_1^{\pm})^{-1}$, where $\tilde{G} = c(\tilde{F}^t, \{\tilde{\Phi}_i\})$.

Covering mappings for diffeomorphisms (14) have the form

$$C(x, y) = (\pm x + c^1(x), y + c^2(x)), \tag{15}$$

where c^1 and c^2 are 1-periodic functions.

Covering mappings for homeomorphisms (8) with $b = b(z)$ have the form

$$\Gamma(x, y) = (x - \alpha d(x) + n, y + d(x)). \tag{16}$$

Respectively, given a flow $F^t \in C^k(M, \{N_i^t\})$ with $H = \hat{c}(F^t, \{N_i^t\})$ having the covering (15), we set

$$\rho_C(x, y) = \pm x + \alpha y + \tau(x) \tag{17}$$

where $\tau(x) = c^1(x) + \alpha c^2(x)$.

Repeating the calculations proving Theorem 2, we arrive at the following corollary.

COROLLARY 5. *Let $F^t, \tilde{F}^t \in C^k(M, \{N_i^t\})$ be flows with functions $\rho_C, \rho_{\tilde{C}}$, respectively. Then F^t and $\tilde{F}^t$ are topologically conjugate if and only if there are a real q and a homeomorphism (16) such that (12) is valid.*

4. *Embedding problem*

We show here that there are diffeomorphisms $F \in C^k(M, \{N_i\})$, non-embeddable in topological flows.

THEOREM 4. *A diffeomorphism $F \in C^k(M, \{N_i\})$ with $H = \hat{c}(F, \{\Phi_i\})$ is embeddable in a topological flow if and only if there are homeomorphisms B_i of the cylinder $\mathbb{T} \times \mathbb{R}^1$ commuting with its unit shift and having the form (8) such that the homeomorphism $\tilde{H} = (B_2)^{-1} H B_1$ has the form (14).*

The diffeomorphism F is embeddable in a C^1-flow if and only if the diffeomorphism H itself has the form (14); if $F \in C^k(M, \{N_i\})$ is C^1-embeddable, then it is C^k-embeddable.

Proof. Let $\Phi_i : U_i \to \mathbb{T} \times \mathbb{R}^1$ be normalizing diffeomorphisms for $F \in C^k(M, \{N_i\})$. Let F^t be a C^p-flow such that $F = F^t|_{t=1}$. Then the covering $\{U_i\}$ of M is F^t-invariant. Hence, $F^t \in C^p(M, \{P_i^t\})$ where

$$P_i^t = \Phi_i^{-1} F^t \Phi_i | U_i, \quad i = 1, 2.$$

Let us show that P_i^t is C^p-conjugate with N_i^t, $i = 1, 2$, respectively. Indeed, since $N_1 = P_1^t|_{t=1}$, the flow P_1^t commutes with N_1 and has the form

$$P_1^t(z, y) = (e^{2\pi i a(t)} z, yh(t, z, y)),$$

where $a(t+s) = a(t) + a(s)$, $a(1) = \alpha$ and the function $h > 0$ is 1-periodic in y and such that $yh(t, z, y)$ is an increasing function in y for every $t \in \mathbb{R}$, $z \in \mathbb{T}$. Since the function $a(t)$ is continuous, $a(t) = \alpha t$. The mapping

$$\Psi(z, y) = \int_0^1 N_1^s P_1^{-s}(z, y)\, ds = (z, yg(z, y)),$$

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where

$$g(z, y) = \int_0^1 e^{s \ln \lambda} h(-s, z, y)\, ds,$$

is a C^p-diffeomorphism of the standard cylinder conjugating P_1^t with N_1^t. Similarly, P_2^t is C^p-conjugate to N_2^t. Therefore, $F^t \in C^p(M, \{N_i^t\})$.

Let $\{\tilde{\Phi}_i\}$ be normalizing C^p-diffeomorphisms for the flow F^t. Then they are normalizing C^p-diffeomorphisms for F as well. Let $\tilde{H} = \hat{c}(F^t, \{\tilde{\Phi}_i\})$. Then $\tilde{H}$ has the form (14) and

$$(B_2)^{-1} H B_1 = \tilde{H},$$

where B_i are C^p-diffeomorphisms commuting with the unit shift and having the form (8).

Conversely, let H satisfy the above-mentioned condition. By Theorem 3, there is a flow $F^t \in C^p(\mathbb{T}, \{N_i^t\})$ and its normalizing C^p-diffeomorphisms $\{\tilde{\Phi}_i\}$ such that $\tilde{H} = \hat{c}(F^t, \{\tilde{\Phi}_i\})$. Then F is C^p-conjugate with $F^t|_{t=1}$. Therefore, F is C^p-embeddable.

It remains to note that C^p-diffeomorphisms B_i with $p \geq 1$ are linear and keep the form (14). □

It follows from Theorem 4 that, if H has the form (7) with $m \neq 0$, then a corresponding diffeomorphism $F \in C^k(M, \{N_i\})$ is not topologically embeddable. More generally, in terms of the functions ρ_C, the embeddability condition takes on the following form.

COROLLARY 6. *A diffeomorphism $F \in C^k(M, \{N_i\})$ with $H = \hat{c}(F, \{\Phi_i\})$ having the covering (9) is topologically embeddable if and only if there is a homeomorphism (10) such that the function $\rho_C(\Gamma(x, y))$ has the form (17) with a 1-periodic function τ.*

Proof. Assume that F is embeddable. Then, by Theorem 4, the homeomorphism $\tilde{H} = B_2^{-1} H B_1$ has the form (14), implying (11) with covering mappings C, Γ_1, Γ_2 and $\tilde{C}$ for H, B_1, B_2 and $\tilde{H}$, respectively. In turn, (11) implies (12). Hence,

$$\rho_C(\Gamma_1(x, y)) = -q + (\pm x + \alpha y + \tilde{c}^1(x) + \alpha \tilde{c}^2(x)).$$

Conversely, assume that

$$\rho_C(\Gamma(x, y)) = \pm x + \alpha y + \tau(x)$$

with a homeomorphism Γ having the form (16) and a continuous 1-periodic function τ. Then

$$\rho_{\tilde{C}}(x, y) = \rho_c(\Gamma_1(x, y))$$

with $\Gamma_1 = \Gamma$ and $\tilde{C} = (\pm x, y + \alpha^{-1}\tau(x))$. Hence, (12) holds with $\Gamma_2(x, y) = (x - \alpha d_2(x, y), y + d_2(x, y))$ and $d_2(\tilde{C}(x, y)) = c^2(\Gamma_1(x, y)) + d_1(x, y) - \alpha^{-1}\tau(x)$. Therefore, $\tilde{H} B_1 = B_2 H$ for corresponding homeomorphisms of the cylinder, implying the embeddability of F. □

In particular, if H is a diffeomorphism with the covering mapping

$$C(x, y) = (x, y + c(y)), \quad c \neq \text{constant},$$

then F with $H = \hat{c}(F, \{\Phi_i\})$ is not embeddable in any topological flow.

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Of course, there are C^0-embeddable diffeomorphisms $F \in C^k(M, \{N_i\})$ which are not smoothly embeddable. In order to construct such a diffeomorphism, let us set

$$H(z, y) = (e^{-2\pi i \alpha b(y)} z, y + b(y)), \quad b(y) = \frac{1}{4\pi} \sin 2\pi y.$$

Let $F \in C^k(M, \{N_i\})$ be a diffeomorphism with $\hat{c}(F, \{\Phi_i\}) = H$. Then F is not smoothly embeddable, since H is not in the form (14). On the other hand,

$$H(B(z, y)) = (z, y)$$

where $B = H^{-1}$ have the form (7). Hence, F is topologically embeddable.

References

[A] V. Afraimovich and T. Young. Mather invariants and smooth conjugacy on S^2. *J. Dynam. Control. Sys.* **6**(3) (2000), 341–352.

[B1] G. Belitskii. Functional moduli of diffeomorphisms of the circle. *Ukrainian Math. J.* **38**(3) (1986), 369–370.

[B2] G. Belitskii. Smoothness classification of one-dimensional diffeomorphisms. *Siberian Math. J.* **27**(6) (1986), 21–24.

[B3] G. Belitskii. Functional invariants of diffeomorphisms of smooth manifolds. *Dokl. Akad. Nauk Ukrainy* **11** (1985), 6–8.

[B4] G. Belitskii. Invariants of vector fields and linear equations on the unite sphere. *Ukrainian Math. J.* **41**(3) (1989), 269–302.

[K] A. Katok and B. Hasseblat. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 1995.

[M] J. Mather. Commutators of diffeomorphisms. *Commun. Math. Helv.* **48** (1973), 195–233.

[N] Yu. S. Il'yashenko. *Nonlinear Stokes Phenomena (Advances in Soviet Mathematics, 14)*. American Mathematical Society, Providence, RI, 1993.

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