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Partitions by congruent sets and optimal positions

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Abstract. Let X be a metrizable space with a continuous group or semi-group action G. Let D be a non-empty subset of X. Our problem is how to choose a fixed number of sets in $\{g^{-1}D;\ g \in G\}$, say $\sigma^{-1}D$ with $\sigma \in \tau$, to maximize the cardinality of the partition $\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\})$ generated by them. Let

$$p^*_{X,G,D}(k) = \sup_{\tau \subset G,\ \#\tau = k} \#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) \quad (k = 1, 2, \ldots).$$

An infinite subset Σ of G is called an optimal position of the triple (X, G, D) if

$$\#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) = p^*_{X,G,D}(k)$$

holds for any $k = 1, 2, \ldots$ and $\tau \subset \Sigma$ with $\#\tau = k$. In this paper, we discuss examples of the triple (X, G, D) admitting or not admitting an optimal position. Let $X = G = \mathbb{R}^n$ $(n \geq 1)$, where the action $g \in G$ to $x \in X$ is the translation $x - g$. If D is the n-dimensional unit ball, then

$$p^*_{X,G,D}(k) = 2\sum_{i=0}^{n} \binom{k-1}{i}$$

holds and the triple (X, G, D) admits an optimal position. In fact, if $n \geq 2$ and Σ is an infinite subset of G such that for some δ with $0 < \delta < 1$, $\Sigma \subset \{x \in \mathbb{R}^n;\ \|x\| = \delta\}$, and that any subset of Σ with cardinality $n + 1$ is not on a hyperplane, then Σ is an optimal position of the triple (X, G, D). We have determined the primitive factor of the uniform sets coming from these optimal positions. We also show that in the above setting with $n = 2$ and the unit square D' in place of the unit disk D, the maximal pattern complexity is unchanged and $p^*_{X,G,D'}(k) = k^2 - k + 2$, but there is no optimal position.

1. *Introduction*

Let X be a metrizable space with a continuous group or semi-group action G. For a family of subsets $A_1, A_2, \ldots, A_k$ of X, $\mathbb{P}(\{A_i;\ i = 1, 2, \ldots, k\})$ denotes the *partition* of X

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generated by these sets, that is, the family of non-empty sets of the form

$$A_1^{i_1} \cap A_2^{i_2} \cap \cdots \cap A_k^{i_k} \quad (i_1, i_2, \ldots, i_k \in \{0, 1\}),$$

where for a set $A \subset X$, we denote $A^1 = A$ and $A^0 = X \backslash A$.

Let D be a non-empty subset of X. Our problem is how to choose a fixed number of sets in $\{g^{-1}D;\ g \in G\}$ to maximize the cardinality of the partition generated by them. Define the *maximal pattern complexity* function $p^*_{X,G,D}$ of the triple (X, G, D) by

$$p^*_{X,G,D}(k) = \sup_{\tau \subset G, \#\tau = k} \#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) \quad (k = 1, 2, \ldots),$$

where # denotes the number of elements in a set.

An infinite subset Σ of G is called an *optimal position* of the triple (X, G, D) if

$$\#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) = p^*_{X,G,D}(k)$$

holds for any $k = 1, 2, \ldots$ and $\tau \subset \Sigma$ with $\#\tau = k$.

In this paper, we discuss examples of the triple (X, G, D) admitting or not admitting an optimal position. Let $X = G = \mathbb{R}^n$ $(n \geq 1)$, where the action $g \in G$ to $x \in X$ is the translation $x - g$. If D is the n-dimensional unit ball, then

$$p^*_{X,G,D}(k) = 2 \sum_{i=0}^{n} \binom{k-1}{i}$$

holds and the triple (X, G, D) admits an optimal position. In fact, if $n \geq 2$ and Σ is an infinite subset of G such that for some δ with $0 < \delta < 1$, $\Sigma \subset \{x \in \mathbb{R}^n;\ \|x\| = \delta\}$, and that any subset of Σ with cardinality $n + 1$ is not on a hyperplane, then Σ is an optimal position of the triple (X, G, D). Also, if $n = 1$ and $\Sigma \subset (-1, 1)$, then Σ is an optimal position.

On the other hand, if in the above with $n = 2$, D is replaced by the unit square $D' = [0, 1] \times [0, 1]$, then we have the same maximal pattern complexity

$$p^*_{X,G,D'}(k) = p^*_{X,G,D}(k) = k^2 - k + 2,$$

but the optimal position does not exist.

If the optimal position Σ for the triple (X, G, D) exists, then the name set Ω which is the closure of $\{\omega_x;\ x \in X\} \subset \{0, 1\}^\Sigma$, where $\omega_x \in \{0, 1\}^\Sigma$ is such that $\omega_x(\sigma) = 1$ if and only if $x \in \sigma^{-1}D$, satisfies the property that $\#\Omega|_S$ depends only on $\#S$ for any finite set $S \subset \Sigma$, where $\Omega|_S = \{\omega|_S;\ \omega \in \Omega\}$ and $\omega|_S \in \{0, 1\}^S$ is the restriction of $\omega \in \{0, 1\}^\Sigma$ to S. Such a set is called a *uniform set*. In fact, we have $\#\Omega|_S = p^*_{X,G,D}(\#S)$.

For $\Omega \subset \{0, 1\}^{\mathbb{N}}$ and an infinite set $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$, we denote $\Omega[\mathcal{N}] = \{\omega[\mathcal{N}];\ \omega \in \Omega\}$, where $\omega[\mathcal{N}] \in \{0, 1\}^{\mathbb{N}}$ is such that $\omega[\mathcal{N}](n) = \omega(N_n)$ $(n \in \mathbb{N})$. We call a non-empty closed set $\Omega \subset \{0, 1\}^{\mathbb{N}}$ a *super-stationary* set if $\Omega[\mathcal{N}] = \Omega$ holds for any infinite subset $\mathcal{N}$ of $\mathbb{N}$.

It is known [**2**] that for any uniform set $\Omega \subset \{0, 1\}^\Sigma$, there exists an injection $\psi : \mathbb{N} \to \Sigma$ such that $\Omega \circ \psi = \{\omega \circ \psi \in \{0, 1\}^{\mathbb{N}};\ \omega \in \Omega\}$ is a super-stationary set. In this case, we say that Ω has a *primitive factor* $[\Omega \circ \psi]$, where $[\Omega \circ \psi]$ is the *isomorphic* class containing $\Omega \circ \psi$ in the sense that two non-empty closed subsets Ω and Θ of $\{0, 1\}^{\mathbb{N}}$

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are said to be isomorphic if there is an isometric bijection between them. The notion of primitive factor is introduced in [**4**] to distinguish pattern Sturmian words of different types.

The class of super-stationary sets is characterized in [**5**] in terms of super-subwords. For $\xi = \xi_1\xi_2\cdots\xi_k \in \{0,1\}^k$ and $\omega \in \{0,1\}^{\mathbb{N}}$, ξ is called a *super-subword* of ω, denoted by $\xi \ll \omega$, if there exist $l_1, l_2, \ldots, l_k$ such that $0 \le l_1 < l_2 < \cdots < l_k$ and $\omega(l_1)\omega(l_2)\cdots\omega(l_k) = \xi$. The set of infinite words with prohibited super-subword ξ is denoted by $\mathcal{P}(\xi) = \{\omega \in \{0,1\}^{\mathbb{N}};\ \xi \ll \omega \text{ does not hold}\}$. Then the class of super-stationary sets other than $\{0,1\}^{\mathbb{N}}$ coincides with the class of sets of the form $\bigcup_{\xi\in\Xi}\mathcal{P}(\xi)$ with non-empty finite sets $\Xi \subset \bigcup_{k=1}^{\infty}\{0,1\}^k$.

For the name set $\Omega \subset \{0,1\}^{\Sigma}$ with respect to the above optimal position Σ for the triple (X, G, D) with $X = G = \mathbb{R}^n$ and D the n-dimensional unit ball, we prove that Ω has a unique primitive factor $[\mathcal{P}((01)^l0) \cup \mathcal{P}((10)^l1)]$ if $n = 2l$ is even, and $[\mathcal{P}((01)^{l+1}) \cup \mathcal{P}((10)^{l+1})]$ if $n = 2l+1$ is odd.

The basic terminology and notations follow from [**2, 5**]. For further related results, see [**1, 3, 4, 6–9**].

2. *Basic lemmas*

Definition 1. The Euclidean space $\mathbb{R}^{n+1}$ is sometimes considered as a vector space. Let $\mathcal{S}^n = \{x \in \mathbb{R}^{n+1};\ \|x\| = 1\}$ $(n \ge 1)$ be the n-dimensional unit sphere, where for $x = (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$, we denote $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2 + x_{n+1}^2}$. For $v \in \mathbb{R}^{n+1}$ with $\|v\| = 1$ and $c \in \mathbb{R}$ with $-1 < c < 1$, we define a *disk* on $\mathcal{S}^n$ by

$$D_n(v, c) := \{x \in \mathcal{S}^n;\ (x, v) \ge c\},$$

where $(\cdot,\cdot)$ is the inner product. A one-point set in $\mathcal{S}^n$ is considered as a *degenerate disk*. Let $\mathcal{D}_n$ be the family of disks on $\mathcal{S}^n$. Also let $\overline{\mathcal{D}}_n$ be the union of $\mathcal{D}_n$ with the family of degenerate disks.

For $A = D_n(v, c)$, ∂A denotes the boundary. That is, $\partial A = \{x \in \mathcal{S}^n;\ (x, v) = c\}$, which is the $(n-1)$-dimensional sphere with center cv and radius $\sqrt{1-c^2}$. In particular, a 0-dimensional sphere is considered as a two-points set, and a (-1)-dimensional sphere is considered as the empty set.

LEMMA 2. *For $A := \bigcap_{i=1}^{k} \partial D_n(v_i, c_i)$ with $n \ge 1$ and $k \ge 1$, one of the following three cases occurs:*

(1) $A = \emptyset$;

(2) $\#A = 1$; *and*

(3) *A is an $(n-d)$-dimensional sphere with $d = 1, 2, \ldots, \min\{k, n\}$.*

Moreover, this condition (3) is equivalent to the following condition (4):

(4) $\dim\{v_1, v_2, \ldots, v_k\} = d$ *and* $\mathbf{c}\mathbf{A}^{-1}\mathbf{c}' < 1$ *hold for any $h_1, h_2, \ldots, h_d$ such that $1 \le h_1 < h_2 < \cdots < h_d \le k$ and $v_{h_1}, \ldots, v_{h_d}$ are linearly independent, where $\mathbf{A} := ((v_{h_i}, v_{h_j}))_{i,j=1,\ldots,d}$ and $\mathbf{c} := (c_{h_1}, \ldots, c_{h_d})$.*

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Proof. Let $\dim\{v_1, v_2, \ldots, v_k\} = d$. Assume without loss of generality that $v_1, v_2, \ldots, v_d$ are linearly independent so that $h_1 = 1, h_2 = 2, \ldots, h_d = d$. Let

$$W = \{x \in \mathbb{R}^{n+1};\ (x, v_i) = c_i \text{ for } i = 1, 2, \ldots, d\}.$$

Then $\{x \in \mathbb{R}^{n+1};\ (x, v_i) = c_i \text{ for } i = 1, 2, \ldots, k\}$ equals either W or $\emptyset$. In the latter case, we have $A = \emptyset$. Assume the former case.

Let $w \in W$ be such that $\|w\| = \min\{\|x\|;\ x \in W\}$. Then

$$\begin{cases} w = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_d v_d \\ (w, v_i) = c_i \quad (i = 1, 2, \ldots, d) \end{cases}$$

holds. Hence, we have

$$\sum_{j=1}^{d} (v_i, v_j)\lambda_j = c_i \quad (i = 1, 2, \ldots, d)$$

and $\mathbf{A}\lambda' = \mathbf{c}'$, where $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_d)$. Therefore, $\lambda' = \mathbf{A}^{-1}\mathbf{c}'$.

Denoting $\mathbf{A} = (a_{ij})_{i,j=1,\ldots,d}$ and $\mathbf{A}^{-1} = (a^{ij})_{i,j=1,\ldots,d}$, we have

$$w = \sum_j a^{1j} c_j v_1 + \sum_j a^{2j} c_j v_2 + \cdots + \sum_j a^{dj} c_j v_d. \tag{1}$$

Hence,

$$\begin{aligned} \|w\|^2 &= \sum_i \sum_{i'} \sum_j \sum_{j'} a^{ij} c_j a^{i'j'} c_{j'} (v_i, v_{i'}) \\ &= \sum_i \sum_{i'} \sum_j \sum_{j'} a^{ij} c_j a^{i'j'} c_{j'} a_{ii'} \\ &= \sum_{i'} \sum_j \sum_{j'} c_j a^{i'j'} c_{j'} \sum_i a^{ij} a_{ii'} \\ &= \sum_{i'} \sum_j \sum_{j'} c_j a^{i'j'} c_{j'} \delta_{i'j} \\ &= \sum_{i'} \sum_{j'} c_{i'} a^{i'j'} c_{j'} = \mathbf{c}\mathbf{A}^{-1}\mathbf{c}'. \end{aligned}$$

Note that W is the $(n - d + 1)$-dimensional plane with $w \in W$ orthogonal to $\vec{Ow}$, where O is the origin. Moreover, $A = W \cap \{x \in \mathbb{R}^{n+1};\ \|x\| = 1\}$. There are 3 cases.

Case 1. If $\|w\| > 1$, then $A = \emptyset$.

Case 2. If $\|w\| = 1$, then $\#A = 1$.

Case 3. If $\|w\| < 1$, then A is an $(n - d)$-dimensional sphere.

Thus, the proof is complete. □

Definition 3. Let $\mathcal{A} \subset \mathcal{D}_n$ with $\#\mathcal{A} \geq 1$ and $n \geq 1$. We call $\mathcal{A}$ a *general position* if for any subset $\{D_n(v_i, c_i);\ i = 1, 2, \ldots, l\}$ of $\mathcal{A}$ with cardinality l such that $1 \leq l \leq n + 1$, we have that $\bigcap_{i=1}^{l} \partial D_n(v_i, c_i)$ is either an $(n - l)$-dimensional sphere or the empty set, where a 0-dimensional sphere is a two-points set and (-1)-dimensional sphere is the empty set. It is called *intersecting* if the former case always holds.

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Definition 4. For a family $\mathcal{A}$ of subsets of $\mathcal{S}^n$, $\mathbb{P}(\mathcal{A})$ denotes the partition of $\mathcal{S}^n$ generated by the sets in $\mathcal{A}$. That is,

$$\mathbb{P}(\mathcal{A}) = \left\{ \bigcap_{A\in\mathcal{A}} A^{i_A} \neq \emptyset;\ i_A \in \{0, 1\} \text{ is chosen for each } A \in \mathcal{A} \right\},$$

where we denote $A^1 = A$ and $A^0 = A^c = \mathcal{S}^n \backslash A$. We also denote by $\mathbb{Q}(\mathcal{A})$ the set of connected components of $\mathcal{S}^n \backslash \bigcup_{A\in\mathcal{A}} \partial A$.

LEMMA 5. *Let $\mathcal{A} \subset \mathcal{D}_n$ with $n \geq 1$ and $\#\mathcal{A} \geq 1$ be a general position. Then every C in $\mathbb{P}(\mathcal{A})$ satisfies $\overline{C} = \overline{C^I}$, where C^I is the set of interior points of C, and $\overline{C}$ is the closure of C.*

Proof. Let $x_0 \in C \in \mathbb{P}(\mathcal{A})$ and $C = \bigcap_{A\in\mathcal{E}} A \cap \bigcap_{A\in\mathcal{A}\backslash\mathcal{E}} A^c$ for $\mathcal{E} \subset \mathcal{A}$. Let

$$\mathcal{C} := \{A \in \mathcal{E};\ x_0 \in \partial A\} = \{D_n(v_i, c_i);\ i = 1, 2, \ldots, d\}.$$

If $d = 0$, then x_0 is an interior point of C. Assume that $d \geq 1$.

Since $\mathcal{A}$ is a general position, $d \leq n$ holds, and $v_1, v_2, \ldots, v_d$ are linearly independent with $\mathbf{c}\mathbf{A}^{-1}\mathbf{c}' < 1$, where $\mathbf{A} := ((v_i, v_j))_{i,j=1,\ldots,d}$ and $\mathbf{c} := (c_1, \ldots, c_d)$. Since $v_1, v_2, \ldots, v_d$ are linearly independent,

$$\bigcap_{i=1}^{d} \{x \in \mathbb{R}^{n+1};\ (x, v_i) > c_i\}$$

is the interior of an n-dimensional polygon. Moreover, since $\mathbf{c}\mathbf{A}^{-1}\mathbf{c}' < 1$,

$$\bigcap_{A\in\mathcal{C}} A^I = \mathcal{S}^n \cap \bigcap_{i=1}^{d} \{x \in \mathbb{R}^{n+1};\ (x, v_i) > c_i\}$$

is a non-empty open set in $\mathcal{S}^n$. Since x_0 belongs to an open set

$$\bigcap_{A\in\mathcal{E}\backslash\mathcal{C}} A^I \cap \bigcap_{A\in\mathcal{A}\backslash\mathcal{E}} A^c,$$

$\bigcap_{A\in\mathcal{E}} A^I \cap \bigcap_{A\in\mathcal{A}\backslash\mathcal{E}} A$ is a non-empty open set contained in C, and x_0 is in its closure, which completes the proof. □

LEMMA 6. *For $\mathcal{E} \subset \overline{\mathcal{D}}_n$ with $n \geq 1$ and $\#\mathcal{E} = k \geq 1$, there exists a general position $\mathcal{A}$ such that $\#\mathcal{A} = \#\mathcal{E}$ and $\#\mathbb{P}(\mathcal{A}) \geq \#\mathbb{P}(\mathcal{E})$.*

Proof. Denote each degenerate disk, say $\{x\} \in \mathcal{E}$, by $D(x, 1)$. Let $\mathcal{E} = \{D_n(v_i, c_i),\ i = 1, 2, \ldots, k\}$. For each $C \in \mathbb{P}(\mathcal{E})$, choose $x_C \in C$. There exists $\epsilon > 0$ such that if $C = \bigcap_{i\in\mathcal{C}} D_n(v_i, c_i) \cap \bigcap_{i\in\{1,2,\ldots,k\}\backslash\mathcal{C}} D_n(v_i, c_i)^c$, then

$$x_C \in \bigcap_{i\in\mathcal{C}} D_n(v_i, c_i - \epsilon)^I \cap \bigcap_{i\in\{1,2,\ldots,k\}\backslash\mathcal{C}} D_n(v_i, c_i - \epsilon)^c.$$

Hence, each x_C is an interior point of the corresponding element of the partition $\mathbb{P}(\{D_n(v_i, c_i - \epsilon);\ i = 1, 2, \ldots, k\})$.

We can also move v_i slightly to u_i, keeping x_C separated so that

$$\mathcal{A} := \{D_n(u_i, c_i - \epsilon),\ i = 1, 2, \ldots, k\}$$

is a general position. Then, $\#\mathcal{A} = k$ and $\#\mathbb{P}(\mathcal{A}) \geq \#\mathbb{P}(\mathcal{E})$, completing the proof. □

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LEMMA 7. *Let $\mathcal{A} \subset \mathcal{D}_n$ be a general position. Then $\#\mathbb{P}(\mathcal{A}) \leq \#\mathbb{Q}(\mathcal{A})$.*

Proof. If C and C' are distinct elements in $\mathbb{P}(\mathcal{A})$, then by Lemma 5, $C^I \neq \emptyset$ and $C'^I \neq \emptyset$. Take $x \in C^I$ and $x' \in C'^I$. There exists $A = D_n(v, c) \in \mathcal{A}$ such that either $C \subset A$ and $C' \cap A = \emptyset$ or $C' \subset A$ and $C \cap A = \emptyset$. Assume the former case without loss of generality. Then $(x, v) > c$ and $(x', v) < c$. Thus, x and x' belong to different elements in $\mathbb{Q}(\mathcal{A})$. This implies that $\#\mathbb{P}(\mathcal{A}) \leq \#\mathbb{Q}(\mathcal{A})$. □

Definition 8. Let $A = D_n(v, c)$ with $n \geq 2$. Let $\mathcal{D}_n^A$ be the set of $B \in \mathcal{D}_n$ such that $\partial A \cap \partial B$ is an $(n-2)$-dimensional sphere. Let $E_{v,c} := \{x \in \mathbb{R}^{n+1};\ (x, v) = c\}$; the space of elements $x - cv$ with $x \in E_{v,c}$ is identified with $\mathbb{R}^n$. Let $\phi_A : E_{v,c} \to \mathbb{R}^n$ be such that $\phi_A(x) = (x - cv)/\sqrt{1 - c^2}$.

LEMMA 9. *Let $A = D_n(v, c)$.*

(1) *The map $\phi_A : E_{v,c} \to \mathbb{R}^n$ is an affine map such that for $D_n(w, d) \in \mathcal{D}_n^A$, $\phi_A(D_n(w, d) \cap E_{v,c}) = D_{n-1}(w', d')$ with*

$$\begin{cases} w' = \dfrac{w - (v, w)v}{\sqrt{1 - (v, w)^2}} \\ d' = \dfrac{d - c(v, w)}{\sqrt{1 - c^2}\sqrt{1 - (v, w)^2}}. \end{cases}$$

(2) *Let $\mathcal{A} \subset \mathcal{D}_n$ be a general position with $\#\mathcal{A} \geq 2$ and $A \in \mathcal{A}$. Then $\{\phi_A(B \cap E_{v,c});\ B \in \mathcal{A} \cap \mathcal{D}_n^A\}$ is a general position in $\mathcal{D}^{n-1}$.*

Proof. (1) Let $B = D_n(w, d) \in \mathcal{D}_n^A$. It is clear that $w' \in \mathbb{R}^n$ and $\|w'\| = 1$. Since $\partial A \cap \partial B$ is an $(n-2)$-dimensional sphere, v and w are linearly independent and $\mathbf{c}\mathbf{A}^{-1}\mathbf{c}' < 1$ by Lemma 2, where $\mathbf{c} = (c, d)$ and

$$\mathbf{A} = \begin{pmatrix} 1 & (v, w) \\ (v, w) & 1 \end{pmatrix}.$$

Therefore,

$$\mathbf{c}\mathbf{A}^{-1}\mathbf{c}' = \frac{c^2 + d^2 - 2cd(v, w)}{1 - (v, w)^2} < 1,$$

and hence,

$$c^2 + d^2 - 2cd(v, w) < 1 - (v, w)^2.$$

This implies that

$$0 \leq c^2(v, w)^2 + d^2 - 2cd(v, w) < (1 - c^2)(1 - (v, w)^2),$$

and hence,

$$d'^2 = \frac{c^2(v, w)^2 + d^2 - 2cd(v, w)}{(1 - c^2)(1 - (v, w)^2)} < 1.$$

Thus, we have $-1 < d' < 1$.

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Let $y \in D_n(w, d) \cap E_{v,c}$. Then, we have $(y, v) = c$ and $(y, w) \geq d$. For $y' = \phi_A(y)$, we have

$$(y', w') = \left(\frac{y - cv}{\sqrt{1 - c^2}}, \frac{w - (v, w)v}{\sqrt{1 - (v, w)^2}} \right) \geq \frac{d - c(v, w)}{\sqrt{1 - c^2}\sqrt{1 - (v, w)^2}} = d'.$$

Therefore, $\phi_A(D_n(v, c) \cap E_{v,c}) \subset D_{n-1}(w', d')$.

Conversely, let $y' \in \mathbb{R}^n$ satisfy $\|y'\| = 1$ and $(y', w') \geq d'$. Note that the space $\mathbb{R}^n$ is identified with the space in $\mathbb{R}^{n+1}$ orthogonal to v. Hence, $(y', v) = 0$ holds. Let $y = \sqrt{1 - c^2}y' + cv \in E_{v,c}$. Then $\phi_A(y) = y'$. Moreover,

$$\begin{aligned}
(y, v) &= \sqrt{1 - c^2}(y', v) + c(v, v) = c, \\
(y, w) &= \sqrt{1 - c^2}(y', w) + c(v, w) \\
&= \sqrt{1 - c^2}(y', \sqrt{1 - (v, w)^2}w' + (v, w)v) + c(v, w) \\
&\geq \sqrt{1 - c^2}\sqrt{1 - (v, w)^2}d' + c(v, w) \\
&= \sqrt{1 - c^2}\sqrt{1 - (v, w)^2}\frac{d - c(v, w)}{\sqrt{1 - c^2}\sqrt{1 - (v, w)^2}} + c(v, w) \\
&= d - c(v, w) + c(v, w) = d.
\end{aligned}$$

Therefore, $y' \in \phi_A(D_n(v, c) \cap E_{v,c})$, and hence,

$$\phi_A(D_n(v, c) \cap E_{v,c}) \supset D_{n-1}(v', d'),$$

which completes the proof of (1).

(2) follows from (1) and the definition of a general position. □

LEMMA 10. *Let $\mathcal{A} \subset \mathcal{D}_n$ be a general position with $n \geq 1$ and $\#\mathcal{A} = k \geq 1$. Then*

$$\#\mathbb{Q}(\mathcal{A}) \leq 2\sum_{i=0}^{n} \binom{k-1}{i},$$

where we define $\binom{k}{i} = 0$ if $k < i$. Moreover, the equality holds if $\mathcal{A}$ is intersecting.

Proof. Let $\mathcal{A} \subset \mathcal{D}_n$ be a general position with $n \geq 1$ and $\#\mathcal{A} = k \geq 1$. If $k = 1$, then $\#\mathbb{Q}(\mathcal{A}) = 2$ and our lemma holds. Moreover, if $n = 1$, then ∂A is a two-point set for any $A \in \mathcal{A}$, and $\partial A \cap \partial B = \emptyset$ for any distinct elements A, $B \in \mathcal{A}$ since $\mathcal{A}$ is a general position. Therefore, $S^1 \setminus \bigcup_{A \in \mathcal{A}} \partial A$ has $2k$ connected components. Thus, our lemma holds for $n = 1$, since

$$\#\mathbb{P}(\mathcal{A}) = 2k = 2\sum_{i=0}^{1} \binom{k-1}{i} \quad (k = 1, 2, \ldots).$$

We use induction on k. Let $k \geq 2$ and assume that our lemma holds with $1, 2, \ldots, k-1$ in place of k. Assume also that $n \geq 2$. Take $A \in \mathcal{A}$. Then, $\{\phi_A(B \cap E_{v,c});\ B \in \mathcal{A} \cap \mathcal{D}_n^A\}$ is a general position by Lemma 9, which is intersecting if $\mathcal{A}$ is intersecting. Let $l = \#(\mathcal{A} \cap \mathcal{D}_n^A)$. Then, $l \leq k - 1$ and the equality holds if $\mathcal{A}$ is intersecting.

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Let $C \in \mathbb{Q}(\mathcal{A}\backslash\{A\})$. We count the number of elements in $\mathbb{Q}(\mathcal{A})$, $\mathbb{Q}(\mathcal{A}\backslash\{A\})$ and $\mathbb{Q}(\{\phi_A(B \cap E_{v,c});\ B \in \mathcal{A} \cap \mathcal{D}_n^A\})$ coming from C, say p, q and r. There are two cases.

Case 1. If $C \cap \partial A = \emptyset$, then $p = 1$, $q = 1$ and $r = 0$.

Case 2. If $C \cap \partial A \neq \emptyset$, then $C \cap \partial A$ is divided into connected components, which are elements in $\mathbb{Q}(\{\phi_A(B \cap E_{v,c});\ B \in \mathcal{A} \cap \mathcal{D}_n^A\})$. Let them be $\ell_1, \ell_2, \ldots, \ell_t$. Then ℓ_1 divides C into two connected components, say C_1 and C_2, and ℓ_2 divides one of C_1 and C_2 into two connected components, and hence, C is divided into three connected components by $\ell_1 \cup \ell_2$. In the same way, C is divided into $t+1$ connected components by $\ell_1 \cup \ell_2 \cup \cdots \cup \ell_t$. These connected components are elements of $\mathbb{Q}(\mathcal{A})$. Thus, $p = t+1$, $q = 1$ and $r = t$.

Hence, using the induction hypothesis,

$$\begin{aligned}
\#\mathbb{Q}(\mathcal{A}) &= \sum_C p = \sum_C (q+r) \\
&= \#\mathbb{Q}(\mathcal{A}\backslash\{A\}) + \#\mathbb{Q}(\{\phi_A(B \cap E_{v,c});\ B \in \mathcal{A} \cap \mathcal{D}_n^A\}) \\
&\le 2\sum_{i=0}^{n} \binom{k-2}{i} + 2\sum_{i=0}^{n-1} \binom{l-1}{i} \qquad (2) \\
&\le 2\sum_{i=0}^{n} \binom{k-2}{i} + 2\sum_{i=0}^{n-1} \binom{k-2}{i} \qquad (3) \\
&= 2\sum_{i=0}^{n} \binom{k-1}{i}.
\end{aligned}$$

In the above, if $\mathcal{A}$ is intersecting, then $\{\phi_A(B \cap E_{v,c});\ B \in \mathcal{A} \cap \mathcal{D}_n^A\}$ is also intersecting, and hence, the equality in (2) holds by the induction hypothesis. Moreover, since $l = k-1$ holds in this case, the equality in (3) holds. Thus, the equality holds if $\mathcal{A}$ is intersecting. □

Definition 11. Let $\mathcal{A} \subset \mathcal{S}^n$ be a general position with $n \ge 1$ and $\#\mathcal{A} = k \ge 1$. It is called *connected* if any C in $\mathbb{P}(\mathcal{A})$ is connected.

LEMMA 12. *For any $\mathcal{A} \subset \mathcal{D}^n$ with $n \ge 1$ and $\#\mathcal{A} \ge 1$, we have*

$$\#\mathbb{P}(\mathcal{A}) \le 2\sum_{i=0}^{n} \binom{k-1}{i}.$$

Moreover, the equality holds if $\mathcal{A}$ is a connected, intersecting general position.

Proof. The inequality follows from Lemmas 6, 7 and 10. The equality in the case where $\mathcal{A}$ is a connected, intersecting general position follows from Lemma 10, since $\#\mathbb{P}(\mathcal{A}) = \#\mathbb{Q}(\mathcal{A})$. □

3. *Existence of an optimal position*

Let $B_n(v, r) = \{x \in \mathbb{R}^n;\ \|x - v\| \le r\}$ be the n-dimensional ball with center $v \in \mathbb{R}^n$ and radius $r > 0$. Let $\mathcal{B}_n$ be the set of all such balls.

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Definition 13. A non-empty finite subset $\mathcal{A}$ of $\mathcal{B}_n$ is called a *general position* if for any subset $\{B_n(v_i, r_i);\ i = 1, 2, \ldots, l\}$ of $\mathcal{A}$ with cardinality l such that $1 \le l \le n+1$, we have that $\bigcap_{i=1}^{l} \partial B_n(v_i, r_i)$ is either an $(n-l)$-dimensional sphere or the empty set, where a 0-dimensional sphere is a two-point set and a (-1)-dimensional sphere is defined to be the empty set. It is called *intersecting* if the former case holds always. Moreover, a general position $\mathcal{A}$ is called *connected* if any C in $\mathbb{P}(\mathcal{A})$ is connected.

A bounded region of $\mathbb{R}^n$ can be embedded into a small region of $\mathcal{S}^n$ by a map which is close enough to a homothetic map. By this map, unit balls in $\mathbb{R}^n$ are mapped to small, slightly perturbed disks with almost uniform sizes in $\mathcal{S}^n$. Moreover, a general position $\mathcal{A} \subset \mathcal{B}_n$ corresponds to a position $\mathcal{A}' \subset \mathcal{D}_n$ having the same configuration as $\mathcal{A}$.

This correspondence gives the following theorem.

THEOREM 14. *For any $\mathcal{A} \subset \mathcal{B}_n$ with $n \ge 1$ and $\#\mathcal{A} = k \ge 1$, we have*

$$\#\mathbb{P}(\mathcal{A}) \le 2 \sum_{i=0}^{n} \binom{k-1}{i}.$$

Moreover, the equality holds if $\mathcal{A}$ is a connected and intersecting general position.

THEOREM 15. *Take $n \ge 2$ and any δ with $0 < \delta < 1$. Let*

$$\mathcal{A} = \{B_n(v_1, 1), B_n(v_2, 1), \ldots, B_n(v_k, 1)\}$$

with $\|v_i\| = \delta$ for any $i = 1, 2, \ldots, k$. Assume that if $k \le n$, then $v_1, v_2, \ldots, v_k$ are not on a plane of dimension $k-2$, and if $k \ge n+1$, then any subset of $v_1, v_2, \ldots, v_k$ with cardinality $n+1$ is not on a hyperplane. Then, $\mathcal{A}$ is a connected and intersecting general position.

Proof. Take any subset of $\mathcal{A}$ with cardinality l such that $2 \le l \le n$. Without loss of generality, let it be $\{B_n(v_1, 1), B_n(v_2, 1), \ldots, B_n(v_l, 1)\}$. By our assumption, $v_1, v_2, \ldots, v_l$ are not on a plane of dimension $l-2$, but exactly on an $(l-1)$-dimensional plane. Hence, they are on an $(l-2)$-dimensional sphere, say with center $u \in \mathbb{R}^n$ and radius ρ, such that $\|u\| < \delta$ and $\rho = \sqrt{\delta^2 - \|u\|^2}$. Hence, $\bigcap_{i=1}^{l} \partial B_n(v_i, 1)$ is an $(n-l)$-dimensional sphere

$$\{x \in \mathbb{R}^n;\ \|x - u\| = \sqrt{1 - \delta^2 + \|u\|^2},\ (x - u, v_i - u) = 0\ (i = 1, \ldots, l-1)\}.$$

If $k \ge n+1$, then take any subset of $\mathcal{A}$ with cardinality $n+1$. Let it be

$$\{B_n(v_1, 1), B_n(v_2, 1), \ldots, B_n(v_{n+1}, 1)\}$$

without loss of generality. Suppose that $\bigcap_{i=1}^{n+1} \partial B_n(v_i, 1)$ is non-empty. Let w be an element of it. Then

$$\begin{aligned}\{v_1, v_2, \ldots, v_{n+1}\} &\subset \{x \in \mathbb{R}^n;\ \|x\| = \delta\} \cap \{x \in \mathbb{R}^n;\ \|x - w\| = 1\} \\ &\subset \{x \in \mathbb{R}^n;\ (x, w) = (\|w\|^2 + \delta^2 - 1)/2\},\end{aligned}$$

which contradicts our assumption. Thus, $\mathcal{A}$ is an intersecting general position.

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To prove that $\mathcal{A}$ is connected, take any $C \in \mathbb{P}(\mathcal{A})$. Let

$$C = \bigcap_{A \in \mathcal{E}} A \cap \bigcap_{A \in \mathcal{A} \setminus \mathcal{E}} A^c$$

with some $\mathcal{E} \subset \mathcal{A}$. Without loss of generality, assume that

$$\mathcal{E} = \{B(v_i, 1);\ i = 1, 2, \ldots, d\}$$

with $0 \le d \le k$. For $i,\ j = 1, 2, \ldots, k$ with $i \ne j$, let

$$R_{i,j} = \{x \in \mathbb{R}^n;\ \|x - v_i\| < \|x - v_j\|\},$$

which is an open half-space bounded by the hyperplane $\partial R_{i,j}$ containing the origin O. Denote

$$\mathcal{R} = \bigcap_{i \le d, j > d} R_{i,j}.$$

Then $\mathcal{R}$ is an open polygonal cone with focus O containing C^I.

Let $\mathcal{L}$ be the set of unit vectors u in $\mathbb{R}^n$ such that $\{tu;\ t > 0\} \subset \mathcal{R}$. Define functions F and f from $\mathcal{L}$ to $\mathbb{R}_+$ such that

$$F(u) = \inf\{t > 0;\ \|tu - v_i\| > 1 \text{ for some } i = 1, \ldots, d\}$$
$$f(u) = \sup\{t > 0;\ \|tu - v_i\| \le 1 \text{ for some } i = d+1, \ldots, k\}.$$

Then it is clear that $f(u) < F(u)$ for any $u \in \mathcal{L}$ and

$$C^I = \{x \in \mathbb{R}^n;\ x = tu \text{ for some } u \in \mathcal{L} \text{ and } t \text{ with } f(u) < t < F(u)\}.$$

Moreover, since F and f are continuous functions, C^I is connected, and hence, C is connected since $C^I \subset C \subset \overline{C^I}$. Thus, $\mathcal{A}$ is connected. □

COROLLARY 16. *Let $X = G = \mathbb{R}^n$ $(n \ge 1)$; the action $g \in G$ to $x \in X$ is the translation $x - g$. If $D = B_n(0, 1)$ is the n-dimensional unit ball, then*

$$p^*_{X,G,D}(k) = 2 \sum_{i=0}^{n} \binom{k-1}{i}.$$

Moreover, if $n \ge 2$ and Σ is an infinite subset of G such that for some δ with $0 < \delta < 1$, $\Sigma \subset \{x \in \mathbb{R}^n;\ \|x\| = \delta\}$, and that any subset of Σ with cardinality $n+1$ is not on a hyperplane, then Σ is an optimal position of the triple (X, G, D). Also, if $n = 1$ and $\Sigma \subset (-1, 1)$, then Σ is an optimal position.

4. *Non-existence of an optimal position*

Let $X = G = \mathbb{R}^2$; the action $g \in G$ to $x \in X$ is the translation $x - g$. If $D = [0, 1] \times [0, 1]$ is the unit square, then the same result as for the unit disk holds for the maximal pattern complexity, that is, $p^*_{X,G,D}(k) = k^2 - k + 2$. But in this case, an optimal position does not exist.

To get $p^*_{X,G,D}(k) = k^2 - k + 2$, we put translations of the unit square, say $D_1,\ D_2, \ldots,$ one by one in the following way, where we let V_n denote the union of the vertices of $D_1,\ D_2, \ldots,\ D_n$.

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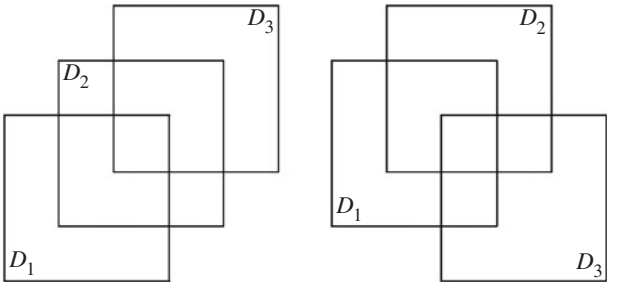


FIGURE 1. We have the configuration on the right, not the left.

(i) D_1 is put anywhere. D_2 is put so that one of its vertices is in the interior of D_1. Hence, $D_1 \cap D_2$ is a rectangle such that just two of its vertices situated diagonally are elements in V_2.

(ii) Assume that D_i $(i = 1, 2, \ldots, 2n)$ have been already put so that $\bigcap_{i=1}^{2n} D_i$ is a rectangle such that just two of its vertices situated diagonally, say u and v, are elements in V_{2n}. Put D_{2n+1} so that one of its vertices is in the interior of $\bigcap_{i=1}^{2n} D_i$ and it does not contain any of u and v. Then $\bigcap_{i=1}^{2n+1} D_i$ is a rectangle such that just one of its vertices, say w, belongs to V_{2n+1}. Put D_{2n+2} so that one of its vertices is in the interior of $\bigcap_{i=1}^{2n+1} D_i$ and it contains w. Then $\bigcap_{i=1}^{2n+2} D_i$ is a rectangle such that just two of its vertices situated diagonally belong to V_{2n+2}.

(iii) Repeat (ii) with $n + 1$ instead of n.

This choice of the translations of the unit square, called $\mathcal{A}$, realizes the maximal pattern complexity $p^*_{X,G,D}(k) = k^2 - k + 2$. On the other hand, the triple (X, G, D) does not admit an optimal position. We prove these results in the following lemmas.

LEMMA 17. *For the above triple* (X, G, D), $p^*_{X,G,D}(k) = k^2 - k + 2$ *and the above* $\mathcal{A} := \{D_i;\ i = 1, 2, \ldots, k\}$ *satisfies*

$$\#\mathbb{P}(\mathcal{A}) = \#\mathbb{Q}(\mathcal{A}) = k^2 - k + 2 \quad (k = 1, 2, \ldots).$$

Proof. Let $\mathcal{B}$ consist of k translations of D such that any two of their boundaries intersect exactly at two points. Then $\#\mathbb{Q}(\mathcal{B}) = k^2 - k + 2$. For any $\mathcal{E} \subset \{D + g;\ g \in \mathbb{R}^2\}$ with $\#\mathcal{E} = k$, by the same arguments as in the proofs of Lemmas 6 and 7, there exists $\mathcal{B}$ as above such that $\#\mathbb{P}(\mathcal{E}) \le \#\mathbb{P}(\mathcal{B}) \le \#\mathbb{Q}(\mathcal{B}) = k^2 - k + 2$. Hence, $p^*_{X,G,D}(k) \le k^2 - k + 2$.

By the above argument, $\#\mathbb{P}(\mathcal{A}) \le \#\mathbb{Q}(\mathcal{A}) = k^2 - k + 2$. To prove that $\#\mathbb{P}(\mathcal{A}) = k^2 - k + 2$, it is sufficient to prove that $\#\mathbb{P}(\mathcal{A}) = \#\mathbb{Q}(\mathcal{A})$. Hence, it is sufficient to prove that any element in $\mathbb{P}(\mathcal{A})$ is connected. We prove this by induction on k. This is clear for $k = 1, 2$. For $k = 3$, it holds since by our construction we have the configuration on the right in Figure 1, but not the one on the left.

Assume that this is true for $1, 2, \ldots, k - 1$ for some $k \ge 4$. Since one of the vertices, say left and down, of D_k is in the interior of $\bigcap_{i=1}^{k-1} D_i$, only two neighboring edges of D_k intersect with $\bigcup_{i=1}^{k-1} D_i$. Let $C \in \mathbb{P}(\{D_i;\ i = 1, 2, \ldots, k - 1\})$. By the induction hypothesis, C is connected. If $C \cap \partial D_k$ is connected, then ∂D_k divides C into two

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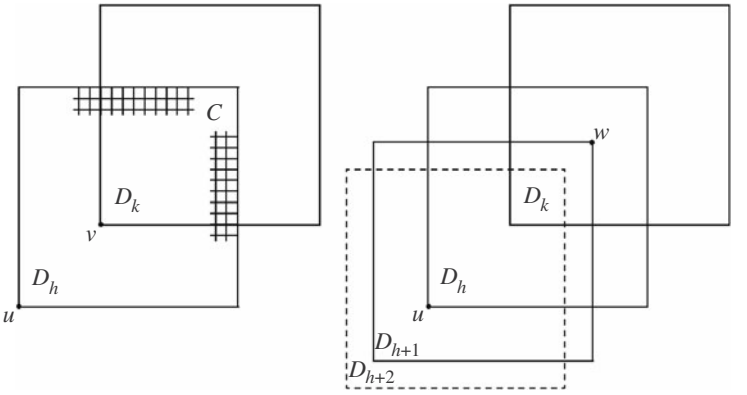


FIGURE 2. The region C and the configurations.

connected regions belonging to distinct elements in $\mathbb{P}(\{D_i;\ i = 1, 2, \ldots, k\})$. Therefore, it is sufficient to prove that this is the only possible case.

Suppose that $C \cap \partial D_k$ is not connected. Then, C should be a region extending vertically and horizontally at the same time and intersecting with two edges of D_k. Then there exists D_h in $\{D_i;\ i = 1, 2, \ldots, k-1\}$ which is located rightmost and uppermost at the same time, since otherwise there is no C as above. Hence, we have the situation on the left of Figure 2.

If $k = h + 1$, then u is a vertex of $\bigcap_{i=1}^{k-1} D_i$. In this case, we do not put D_k as on the left of Figure 2 in our construction, and we have a contradiction. If $k \geq h + 2$, then for any l with $h < l < k$, D_l should contain v by our construction. On the other hand, since D_h is further right and further up than D_l by the assumption on D_h, the left-down vertex of D_l should be further left and further down than u, which implies that the right-up vertex of D_l is in the interior of $D_h \cap D_k$. But in this case, we have the configuration on the right of Figure 2, which contradicts our choice of D_{h+2} in our construction. Hence, we have $k = h + 2$ and the configuration should be as on the right of Figure 2. In this case, $\bigcap_{i=1}^{k-1} D_i$ has two diagonally situated vertices in V_{k-1}, u and w, while we should not put D_k like this in our construction.

In any case, we have a contradiction coming from the supposition that $C \cap \partial D_k$ is not connected. Hence, it is connected. Thus, each $C \in \mathbb{P}(\mathcal{A})$ is connected, which completes the proof. □

LEMMA 18. *The above triple* (X, G, D) *does not admit an optimal position.*

Proof. Take any infinite subset Σ of G. If there exist three distinct elements (u_1, v_1), (u_2, v_2), (u_3, v_3) in Σ such that:

(1) either $u_1 \leq u_2 \leq u_3$ or $u_1 \geq u_2 \geq u_3$; and

(2) either $v_1 \leq v_2 \leq v_3$ or $v_1 \geq v_2 \geq v_3$,

then we have $A_1 \cap A_2^c \cap A_3 = \emptyset$ with $A_i = D + (u_i, v_i)$ $(i = 1, 2, 3)$. Hence, $\#\mathbb{P}(A_1, A_2, A_3) \leq 7$, and $\{A_1, A_2, A_3\}$ dose not attain $p^*_{X,G,D}(3) = 8$.

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We prove that Σ contains three distinct elements as above. Take any five distinct elements in Σ, say (u_i, v_i) $(i = 1, 2, 3, 4, 5)$ with $u_1 \le u_2 \le u_3 \le u_4 \le u_5$. Let $v_h = \min_i v_i$ and $v_k = \max_i v_i$ for some $h, k = 1, 2, 3, 4, 5$. If $h + 1 < k$ or $k + 1 < h$, then (u_h, v_h), (u_l, v_l), (u_k, v_k) satisfy the above condition with $l = h + 1 < k$ or $l = k + 1 < h$. Otherwise either $\{1, 2\} \cap \{h, k\} = \emptyset$ or $\{4, 5\} \cap \{h, k\} = \emptyset$ holds. Assume the former without loss of generality. If $v_1 \le v_2$, then (u_1, v_1), (u_2, v_2), (u_k, v_k) satisfy the above condition. If $v_1 \ge v_2$, then (u_1, v_1), (u_2, v_2), (u_h, v_h) satisfy the above condition. Thus, there exist three distinct elements as above, which completes the proof. □

5. *Primitive factor*

Let (X, G, D) be the triple in Corollary 16 with $n \ge 1$ and let Σ satisfy the conditions in Corollary 16. Let $\Omega \subset \{0, 1\}^\Sigma$ be the name set with respect to the optimal position Σ of the triple (X, G, D). That is, Ω is the closure of $\{\omega_q;\ q \in \mathbb{R}^n\}$, where $\omega_q \in \{0, 1\}^\Sigma$ is such that

$$\omega_q(v) = \begin{cases} 1 & (q \in B_n(v, 1)) \\ 0 & (q \notin B_n(v, 1)) \end{cases} \quad (v \in \Sigma). \tag{4}$$

THEOREM 19. *The above uniform set Ω has the unique primitive factor* [Ξ]*, where* $\Xi = \mathcal{P}((01)^l 0) \cup \mathcal{P}((10)^l 1)$ *if* $n = 2l$ *is even, and* $\Xi = \mathcal{P}((01)^{l+1}) \cup \mathcal{P}((10)^{l+1})$ *if* $n = 2l + 1$ *is odd.*

Proof. We sometimes identify $x \in \mathbb{R}^n$ with the vector $\vec{Ox}$, where O is the origin. Let $\mathcal{S}_1 := \{x \in \mathbb{R}^n;\ \|x\| = 1\}$.

Our theorem was essentially proved in [**4**] for the case $n = 1$.

Assume that $n \ge 2$. It is sufficient to prove that for any infinite subset Σ' of Σ, there exists an injection $\psi : \mathbb{N} \to \Sigma'$ satisfying $\Omega' \circ \psi = \Xi$, where Ω' is the name set with respect to Σ'. For simplicity, we denote this Σ' by Σ and Ω' by Ω.

There exist accumulation points of Σ. Take one of them and call it r_0. For $v \in \mathcal{S}_1$ and ϵ with $0 < \epsilon < 1/4$, let

$$\Theta(v, \epsilon) = \{w \in \mathbb{R}^n;\ (w, v) > (1 - \epsilon)\|w\|\}.$$

Take $\epsilon_0,\ \epsilon_1, \ldots, \epsilon_{n-1}$ with $0 < \epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1} < 1/4$ arbitrary.

Since r_0 is an accumulation point of Σ, $(\Sigma - r_0) \cap \{x \in \mathbb{R}^n;\ \|x\| < \epsilon\}$ is an infinite set for any $\epsilon > 0$. Let $\Sigma_0(\epsilon_0) = (\Sigma - r_0) \cap \{x \in \mathbb{R}^n;\ \|x\| < \epsilon_0\}$.

There exists $v_1 \in \mathcal{S}_1$ such that $\Sigma_0(\epsilon) \cap \Theta(v_1, \eta)$ is an infinite set for any $\epsilon > 0$ and $\eta > 0$. Let $\Sigma_1(\epsilon_0, \epsilon_1) = \pi_1(\Sigma_0(\epsilon_0) \cap \Theta(v_1, \epsilon_1))$, where $\pi_1(w) = w - (w, v_1)v_1$ is the projection defined on $\Sigma_0(\epsilon_0) \cap \Theta(v_1, \epsilon_1)$.

Since there is no subset of Σ of cardinality $n + 1$ which is on a hyperplane, $\pi_1(w) = 0$ holds at most for n elements in $\Sigma_0(\epsilon_0) \cap \Theta(v_1, \epsilon_1)$. On the other hand, since $w \in \Sigma_0(\epsilon_0) \cap \Theta(v_1, \epsilon)$ implies that

$$\|w - (w, v_1)v_1\|^2 < 2\epsilon\|w\|^2,$$

there exist infinitely many $w \in \Sigma_0(\epsilon_0) \cap \Theta(v_1, \epsilon)$ such that

$$0 < \|\pi_1(w)\|^2 < 2\epsilon\|w\|^2$$

for any ϵ with $0 < \epsilon \le \epsilon_1$. This implies that there exist infinitely many distinct elements in $\Sigma_1(\epsilon_0, \epsilon_1)$.

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Now, there exists $v_2 \in \mathcal{S}_1$ with $(v_1, v_2) = 0$ such that $\Sigma_1(\epsilon, \eta) \cap \Theta(v_2, \zeta)$ is an infinite set for any $\epsilon > 0$, $\eta > 0$ and $\zeta > 0$. Let

$$\Sigma_2(\epsilon_0, \epsilon_1, \epsilon_2) = \pi_2(\Sigma_1(\epsilon_0, \epsilon_1) \cap \Theta(v_2, \epsilon_2)),$$

where $\pi_2(w) = w - (w, v_2)v_2$ is the projection defined on $\Sigma_1(\epsilon_0, \epsilon_1) \cap \Theta(v_2, \epsilon_2)$.

If $n = 2$, then $\Sigma_2 = \{O\}$. If $n \geq 3$, then $\pi_2(w) = 0$ holds at most for n elements in $\Sigma_1(\epsilon_0, \epsilon_1) \cap \Theta(v_2, \epsilon_2)$. Then, by the same argument as above, there exist infinitely many distinct elements in $\Sigma_2(\epsilon_0, \epsilon_1, \epsilon_2)$.

In this way, we continue until we get $\Sigma_{n-1} := \Sigma_{n-1}(\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1})$ such that $\Sigma_{n-1}(\eta_0, \eta_1, \ldots, \eta_{n-1}) \cap \Theta(v_n, \epsilon)$ is an infinite set for any $\eta_0, \eta_1, \ldots, \eta_{n-1}$ and $\epsilon > 0$, but $\pi_n(\Sigma_{n-1} \cap \Theta(v_n, \epsilon)) = \{O\}$. Let

$$\Theta := \Theta(\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}) = \pi_1^{-1} \circ \pi_2^{-1} \circ \cdots \circ \pi_{n-1}^{-1} \Sigma_{n-1} \subset \Sigma - r_0.$$

Note that $v_1, v_2, \ldots, v_n$ are chosen independently of the choice of $\epsilon_0, \epsilon_1, \ldots, \epsilon_{n-1}$.

By the construction, Θ is an infinite set, and for any $w \in \Theta$ we have

$$w = (w, v_1)v_1 + (w, v_2)v_2 + \cdots + (w, v_n)v_n, \quad \|w\| < \epsilon_0,$$

and $v_1, v_2, \ldots, v_n$ are an orthonormal basis of $\mathbb{R}^n$. Moreover, we have

$$(w, v_j) > (1 - \epsilon_j)\sqrt{(w, v_j)^2 + (w, v_{j+1})^2 + \cdots + (w, v_n)^2}$$

for any $w \in \Theta$ and $j = 1, 2, \ldots, n$, since $(w, v_j)v_j + \cdots + (w, v_n)v_n \in \Theta(v_j, \epsilon_j)$. It follows that

$$\begin{aligned} &0 < (w, v_1) < \epsilon_0 \\ &0 < (w, v_{j+1}) < 2\sqrt{\epsilon_j}\,(w, v_j) \quad (j = 1, 2, \ldots, n-1). \end{aligned} \tag{5}$$

We need the following lemma to complete the proof.

LEMMA 20. *Let $A = (a_{ij})_{i,j=0,1,2,\ldots,n}$ be a matrix with $n \geq 2$ and*

$$\begin{aligned} a_{ij} &> 0 \quad (i, j = 0, 1, 2, \ldots, n), \\ a_{i0} &= 1 \quad (i = 0, 1, 2, \ldots, n), \quad \textit{and} \\ \frac{a_{i'j'}}{a_{i'j}} &< \frac{1}{2(n+1)!}\frac{a_{ij'}}{a_{ij}} \quad (0 \leq i < i' \leq n,\ 0 \leq j < j' \leq n). \end{aligned} \tag{6}$$

Then

$$\det A = (-1)^{n(n+1)/2} a_{0n} a_{1,n-1} \cdots a_{n0}(1 + \xi)$$

with $|\xi| < 1/2$. Hence, det *A has the same sign as $(-1)^{n(n+1)/2}$.*

Proof. Assume (6). Let τ be a permutation on $\{0, 1, \ldots, n\}$. Assume that there exist u, v with $u < v$ and $\tau(u) < \tau(v)$. Let $\tau' = \tau(u, v)$, where (u, v) is the transposition. Then

$$\frac{\prod_{i=0}^n a_{i\tau(i)}}{\prod_{i=0}^n a_{i\tau'(i)}} = \frac{a_{u\tau(u)}a_{v\tau(v)}}{a_{u\tau'(u)}a_{v\tau'(v)}} = \frac{a_{v\tau(v)}/a_{v\tau(u)}}{a_{u\tau(v)}/a_{u\tau(u)}} < \frac{1}{2(n+1)!}.$$

This implies that

$$0 < \prod_{i=0}^n a_{i\tau(i)} < \frac{1}{2(n+1)!} a_{0n} a_{1,n-1} \cdots a_{n0}$$

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for any permutation τ other than $(n, n-1, \dots, 1, 0)$. Hence,

$$\det A = (-1)^{n(n+1)/2} a_{0n} a_{1,n-1} \cdots a_{n0}(1+\xi)$$

with $|\xi| < 1/2$. □

Let us continue the proof of Theorem 19. Choose $\epsilon_0^0,\ \epsilon_1^0, \dots, \epsilon_{n-1}^0$ with $0 < \epsilon_0^0, \epsilon_1^0, \dots, \epsilon_{n-1}^0 < 1/4$ arbitrary, and take $w_0 \in \Theta(\epsilon_0^0, \epsilon_1^0, \dots, \epsilon_{n-1}^0)$. Let $a_{0j} = (w_0, v_j)$ $(j = 1, 2, \dots, n)$ and $a_{00} = 1$. Then by (5), $a_{0j} > 0$ $(j = 1, 2, \dots, n)$.

Choose $\epsilon_0^1,\ \epsilon_1^1, \dots, \epsilon_{n-1}^1$ with $0 < \epsilon_0^1, \epsilon_1^1, \dots, \epsilon_{n-1}^1 < 1/4$ such that

$$\epsilon_0^1 < \frac{a_{01}}{2(n+1)!}, \quad 2\sqrt{\epsilon_j^1} < \frac{1}{2(n+1)!}\frac{a_{0,j+1}}{a_{0j}} \quad (j = 1, 2, \dots, n-1)$$

and take $w_1 \in \Theta(\epsilon_0^1, \epsilon_1^1, \dots, \epsilon_{n-1}^1)$. Let $a_{1j} = (w_1, v_j)$ $(j = 1, 2, \dots, n)$ and $a_{10} = 1$. Then by (5), $a_{1j} > 0$ $(j = 1, 2, \dots, n)$ and

$$\frac{a_{1,j+1}/a_{1j}}{a_{0,j+1}/a_{0j}} < \frac{1}{2(n+1)!} \quad (j = 0, 1, \dots, n-1).$$

In this way, we can choose $w_0, w_1, w_2, \dots$ in $\Sigma - r_0$. Let $a_{0i} = 1,\ a_{ij} = (w_i, v_j)$ for $i = 0, 1, 2, \dots$ and $j = 1, 2, \dots, n$. Then $a_{ij} > 0$ $(j = 1, 2, \dots, n)$ and

$$\frac{a_{i,j+1}/a_{ij}}{a_{i-1,j+1}/a_{i-1,j}} < \frac{1}{2(n+1)!} \quad (j = 0, 1, \dots, n-1)$$

for any $i = 1, 2, \dots$.

Therefore, for any $l_0, l_1, \dots, l_n$ with $l_0 < l_1 < \cdots < l_n$, the matrix $A = (a_{l_i j})_{i,j=0,1,\dots,n}$ satisfies $(-1)^{n(n+1)/2} \det A > 0$ by Lemma 20.

Define an injection $\psi : \mathbb{N} \to \Sigma$ by $\psi(i) = r_0 + w_i$ $(i \in \mathbb{N})$. To complete the proof, we have to prove that $\Omega \circ \psi = \Xi$, where Ω is the name set with respect to Σ.

For $q \in \mathbb{R}^n$, let ω_q be as in (4). Then $\omega_q \circ \psi(i) = \omega_q(r_0 + w_i) = 1$ if and only if $r_0 + w_i \in B_n(q, 1)$. To prove that $\omega_q \circ \psi \in \Xi$, it is sufficient to prove that $\omega_q \circ \psi$ has i at most n in number such that

$$\omega_q \circ \psi(i) \neq \omega_q \circ \psi(i+1). \tag{7}$$

If $\|q\| \leq 1 - \delta$, then $\omega_q \circ \psi(i)$ can take value 0 for at most one i, while if $\|q\| \geq 1 + \delta$, then $\omega_q \circ \psi(i)$ can take value 1 for at most one i. Therefore, $\omega_q \circ \psi$ has at most two i as in (7).

Assume that $1 - \delta < \|q\| < 1 + \delta$. Let

$$T_q = \left\{ x \in \mathbb{R}^n;\ (x, q) = \frac{\|q\|^2 + \delta^2 - 1}{2} \right\}$$

be the hyperplane in $\mathbb{R}^n$. Then $\omega_q \circ \psi(i) = 1$ if and only if q and $r_0 + w_i$ are not separated by T_q since $\|r_0 + w_i\| = \delta$, where, by definition, $x, y \in \mathbb{R}^n$ are separated by T_q if x and y are not on T_q, but on the opposite sides of $\mathbb{R}^n$ separated by T_q.

Let T be a hyperplane in $\mathbb{R}^n$ and

$$M(T) := \{i \in \mathbb{N};\ q \text{ and } r_0 + w_i \text{ are not separated by } T\}.$$

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We can find a continuously moving family of hyperplanes $T(t)$ with $t \in [0, 1]$ such that $T(0) = T_q$, $T(1)$ contains n elements in $\{r_0 + w_i;\ i \in \mathbb{N}\}$ and $M(T(t)) = M(T_q)$ for any $t \in [0, 1)$. Let

$$T(1) \cap \{r_0 + w_i;\ \ i \in \mathbb{N}\} = \{r_0 + w_{l_1}, r_0 + w_{l_2}, \ldots, r_0 + w_{l_n}\}$$

with $l_1 < l_2 < \cdots < l_n$. Then, $T(1)$ is the plane defined by the equation

$$F(x_1, x_2, \ldots, x_n) = \begin{vmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ 1 & a_{l_1 1} & a_{l_1 2} & \cdots & a_{l_1 n} \\ 1 & a_{l_2 1} & a_{l_2 2} & \cdots & a_{l_2 n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & a_{l_n 1} & a_{l_n 2} & \cdots & a_{l_n n} \end{vmatrix} = 0,$$

where the coordinate system $(x_1, x_2, \ldots, x_n)$ is defined with respect to the orthonormal basis $v_1, v_2, \ldots, v_n$ and r_0 as its origin.

Then, for any $x, y \in \mathbb{R}^n$, x and y are separated by $T(1)$ if $F(x)$ and $F(y)$ have different signs, while x and y are not separated by $T(1)$ if $F(x)$ and $F(y)$ have the same sign. Hence, $\omega_q \circ \psi(i) \neq \omega_q \circ \psi(j)$ if $F(w_i)$ and $F(w_j)$ have different signs, while $\omega_q \circ \psi(i) = \omega_q \circ \psi(j)$ if $F(w_i)$ and $F(w_j)$ have the same sign.

Therefore, if $l_k < i < l_{k+1}$, then by Lemma 20, $F(w_i)$ has the same sign as $(-1)^k(-1)^{n(n+1)/2}$, and hence $\omega_q \circ \psi(i)$ is constant for $l_k < i < l_{k+1}$. Since $F(\omega_i)$ and $F(\omega_j)$ with $l_{k-1} < i < l_k$ and $l_{h-1} < j < l_h$ have the same sign if $h - k$ is even and have different signs if $h - k$ is odd, the 0–1-sequence $\omega_q \circ \psi(i), \omega_q \circ \psi(i+1), \ldots, \omega_q \circ \psi(j)$ with $l_{k-1} < i < l_k$, $l_{h-1} < j < l_h$ and $i + 1 = l_k, i + 2 = l_{k+1}, \ldots, j - 1 = l_{k+j-i-2}$ changed values at most $j - i - 1$ times, where $j - i - 1$ is the number of l_m in $\{i, i + 1, \ldots, j\}$. Therefore, the 0–1-sequence $\omega_q \circ \psi(0), \omega_q \circ \psi(1), \omega_q \circ \psi(2), \ldots$ changed values at most n times. Hence, $\omega_q \circ \psi \in \Xi$.

Conversely, for any choice of $\{l_1 < l_2 < \cdots < l_n\} \subset \mathbb{N}$, the hyperplane T containing $\{r_0 + w_{l_1}, r_0 + w_{l_2}, \ldots, r_0 + w_{l_n}\}$ is determined. By the assumption for Σ, no other point of $\{r_0 + w_i;\ \ i \in \mathbb{N}\}$ is on T.

Then an $(n - 2)$-dimensional plane $T \cap \{x \in \mathbb{R}^n;\ \ \|x\| = \delta\}$ is determined with center p and radius ρ, say. Let

$$q = \left(1 \pm \frac{\sqrt{1 - \rho^2}}{\sqrt{\delta^2 - \rho^2}}\right) p.$$

Then $\omega_q \circ \psi \in \Xi$. By moving q slightly, we can make the values at $\omega_q \circ \psi(l_j)$ $(j = 1, 2, \ldots, n)$ (where $F(w_{l_j}) = 0$) arbitrary. Hence, any element in $\{0, 1\}^{\mathbb{N}}$ which changes value at most n times can be realized by an element in $\Omega \circ \psi$, since by taking $l_{j+1} = l_j + 1$ and choosing the values at $\omega_q \circ \psi(l_j)$ and $\omega_q \circ \psi(l_{j+1})$, we can cancel some changes of value. Hence, $\Xi \subset \Omega \circ \psi$.

Thus, $\Omega \circ \psi = \Xi$, which completes the proof. □

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