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KMS states and branched points

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Abstract. We completely classify the Kubo–Martin–Schwinger (KMS) states for the gauge action on a C^*-algebra associated with a rational function R introduced in our previous work. The gauge action has a phase transition at $\beta = \log \deg R$. We can recover the degree of R, the number of branched points, the number of exceptional points and the orbits of exceptional points from the structure of the KMS states. We also classify the KMS states for C^*-algebras associated with some self-similar sets, including the full tent map and the Sierpinski gasket by a similar method.

1. *Introduction*

It is of fundamental interest in operator algebras to analyze the interplay between a geometric (or dynamical) object and a C^*-algebra associated with it. For example, if a topological dynamical system is free and minimal, then the associated crossed product C^*-algebra is simple. The invariant measures of the dynamical system yield the traces on the C^*-algebra. Topological Markov shifts are studied in terms of the associated Cuntz–Krieger algebras [**9**]. Their Ext-groups are identified with Bowen–Franks groups of the Markov shifts. The fixed point algebras by the gauge action are AF-algebras [**3**] and their K-groups are the dimension groups of the Markov shifts. The decidability of the isomorphism problem was studied in [**6**].

More detailed information on the dynamical system is often obtained by studying representations or states of the C^*-algebra. For example, Bratteli and Jorgensen [**4, 5**] studied a connection between filter functions corresponding to a multiresolution analysis in wavelet theory and the representations of the Cuntz algebras [**8**].

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In this paper we study relations between complex dynamical systems of rational functions and associated C^*-algebras with the gauge action. In particular, we investigate the structure of Kubo–Martin–Schwinger (KMS) states and factor type of the Gelfand–Naimark–Segal (GNS) representations of the KMS states. These are closely related with the singularity structure of complex dynamical systems.

For a branched covering, Deaconu and Muhly [**10**] introduced a C^*-algebra associated with it using an r-discrete groupoid. Dutkay and Jorgensen study a spectral theory on Hilbert spaces built on general onto and finite-to-one maps [**13**]. A typical example of a branched covering is a rational function regarded as a self-map of the Riemann sphere $\hat{\mathbb{C}}$. In order to capture information of the branched points for the complex dynamical system arising from a rational function R, the second- and third-named authors [**26**] introduced a slightly different construction of a C^*-algebra $\mathcal{O}_R(\hat{\mathbb{C}})$ (respectively $\mathcal{O}_R(J_R)$ and $\mathcal{O}_R(F_R)$) associated with R on $\hat{\mathbb{C}}$ (respectively the Julia set J_R and the Fatou set F_R of R). The C^*-algebra $\mathcal{O}_R(\hat{\mathbb{C}})$ is the Cuntz–Pimsner algebra of a Hilbert bimodule over the C^*-algebra $C(\hat{\mathbb{C}})$ of the set of continuous functions, and the other two are defined in a similar way.

We study the gauge action α on the C^*-algebra. If rational functions P and Q are topologically conjugate, then the associated bimodules are isomorphic. Therefore the C^*-dynamical systems $(\mathcal{O}_P(\hat{\mathbb{C}}), \alpha)$ and $(\mathcal{O}_Q(\hat{\mathbb{C}}), \alpha)$ are conjugate in the sense that there exists an isomorphism $\varphi : \mathcal{O}_P(\hat{\mathbb{C}}) \to \mathcal{O}_Q(\hat{\mathbb{C}})$ such that $\alpha_t\varphi = \varphi\alpha_t$ for any t. Hence the structure of the KMS states for the gauge action is an invariant for complex dynamical systems of rational functions up to topological conjugacy. The structure of the KMS states reflects that of the singular points of R as we expected. We completely classify the KMS states for the gauge action of $\mathcal{O}_R(\hat{\mathbb{C}})$. If R has no exceptional points, then the gauge action has a phase transition at $\beta = \log \deg R$ in the following sense: In the region $0 \leq \beta < \log \deg R$, no KMS state exists. A unique KMS state exists at $\beta = \log \deg R$, which is of type $\mathrm{III}_{1/\deg R}$ and corresponds to the Lyubich measure. The extreme β-KMS states at $\beta > \log \deg R$ are parameterized by the branched points of R and are factor states of type I. If R has exceptional points, then there appear additional β-KMS states for $0 < \beta \leq \log \deg R$ parameterized by exceptional points. We can recover the degree of R, the number of branched points and the number of exceptional points from the structure of the KMS states. The orbits of exceptional points are distinguished by 0-KMS states. Here we define a 0-KMS state to be an α-invariant tracial state.

For example, let $P(z) = z^3$, $Q(z) = z^3 + 1$ and $R(z) = (1/18)z^3 - (1/2)z^2 + 6$. We denote by E_P (respectively $B(P)$) the set of exceptional points (respectively branched points) of P. Then $E_P = B(P) = \{0, \infty\}$, $E_Q = \{\infty\}$, $B(Q) = \{0, \infty\}$, $E_R = \{\infty\}$ and $B(R) = \{0, 6, \infty\}$. Our results show that C^*-dynamical systems $(\mathcal{O}_P(\hat{\mathbb{C}}), \alpha)$, $(\mathcal{O}_Q(\hat{\mathbb{C}}), \alpha)$ and $(\mathcal{O}_R(\hat{\mathbb{C}}), \alpha)$ are not conjugate to each other.

We also classify the KMS states for the C^*-algebras associated with some self-similar sets for contractions by a similar method. In fact, if the inverse of the rational function R has continuous branches $\gamma_1, \ldots, \gamma_N$ on J_R, then our construction is arranged so that the C^*-algebra associated with the self-similar set is isomorphic to the C^*-algebra $\mathcal{O}_R(J_R)$. For example, the Sierpinski gasket is homeomorphic to the Julia set J_R of the rational function $R(z) = (z^3 - \frac{16}{27})/z$.

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In the final section, we show that any symmetry of the dynamical system gives rise to an automorphism of the algebra. We also give a criterion for outerness of such automorphisms, including the group of symmetries of a regular icosahedron.

We recall that Olsen and Pedersen [**39**] showed that a β-KMS state for the gauge action of the Cuntz algebra $\mathcal{O}_n$ exists if and only if $\beta = \log n$ and that the $\log n$-KMS state is unique. Since then, several authors have discussed KMS states for the gauge action (and its generalization) of the Cuntz–Pimsner algebra (and its generalization). Here we content ourselves with only giving an incomplete list of such works: [**11, 14, 15, 16, 17, 18, 23, 30, 31, 32, 34, 35, 38, 41, 42**]. Among the others, in this paper we follow Laca and Neshveyev's approach [**32**] where the structure of the KMS states is described in terms of a certain Perron–Frobenius type operator. Many works have been done on Perron–Frobenius type operators (see e.g. [**1, 24, 43**]). We give an explicit description of the Perron–Frobenius type operator in our cases, which allows us to perform a detailed analysis of the KMS states.

We shall now state our main theorems of the paper.

THEOREM A. *Let R be a rational function with $N = \deg R \geq 2$. Consider the gauge action α on the associated C^*-algebra $\mathcal{O}_R(\hat{\mathbb{C}})$. Then the set $\mathrm{ex}(K_\beta(\alpha))$ of extreme β-KMS states of α is completely classified as follows.*

Case 1. Suppose that R has two exceptional points $E_R = \{w_1, w_2\}$ such that $R(w_1) = w_1$ and $R(w_2) = w_2$. Then the branched points $B(R) = \{w_1, w_2\}$.

(1) *If $\beta = 0$, then the set of extreme α-invariant traces consists of two points φ_{β,w_1} and φ_{β,w_2} such that the restriction $\varphi_{\beta,w_i}|_{C(\hat{\mathbb{C}})}$ is the Dirac measure δ_{w_i} for $i = 1, 2$.*

(2) *If $0 < \beta < \log N$, then $\mathrm{ex}(K_\beta(\alpha))$ consists of two points φ_{β,w_1} and φ_{β,w_2} such that the restriction $\varphi_{\beta,w_i}|_{C(\hat{\mathbb{C}})}$ is the Dirac measure δ_{w_i} for $i = 1, 2$.*

(3) *If $\beta = \log N$, then $\mathrm{ex}(K_\beta(\alpha))$ consists of three points φ^L, φ_{w_1} and φ_{w_2}, where the restriction $\varphi^L|_{C(\hat{\mathbb{C}})}$ is the Lyubich measure and the restriction $\varphi_{w_i}|_{C(\hat{\mathbb{C}})}$ is the Dirac measure δ_{w_i} for $i = 1, 2$.*

(4) *If $\log N < \beta$, then $\mathrm{ex}(K_\beta(\alpha))$ consists of two points φ_{β,w_1} and φ_{β,w_2} such that the restriction $\varphi_{\beta,w_i}|_{C(\hat{\mathbb{C}})}$ is the Dirac measure δ_{w_i} for $i = 1, 2$.*

Case 2. Suppose that R has two exceptional points $E_R = \{w_1, w_2\}$ such that $R(w_1) = w_2$ and $R(w_2) = w_1$. Then $B(R) = \{w_1, w_2\}$.

(1) *If $\beta = 0$, then the set of extreme α-invariant traces consists of one point φ_{w_1,w_2} such that the restriction $\varphi_{w_1,w_2}|_{C(\hat{\mathbb{C}})}$ corresponds to $\frac{1}{2}\delta_{w_1} + \frac{1}{2}\delta_{w_2}$.*

(2) *If $0 < \beta < \log N$, then $\mathrm{ex}(K_\beta(\alpha))$ consists of two points $\varphi^{\beta,1}$ and $\varphi^{\beta,2}$ such that the restriction to $C(\hat{\mathbb{C}})$ corresponds to the measures*

$$\mu^{\beta,1} = \frac{1}{e^\beta + 1}\delta_{w_1} + \frac{e^\beta}{e^\beta + 1}\delta_{w_2} \quad \text{and} \quad \mu^{\beta,2} = \frac{e^\beta}{e^\beta + 1}\delta_{w_1} + \frac{1}{e^\beta + 1}\delta_{w_2}.$$

(3) *If $\beta = \log N$, then $\mathrm{ex}(K_\beta(\alpha))$ consists of three points φ^L, $\varphi^{\beta,1}$ and $\varphi^{\beta,2}$, where the restriction to $C(\hat{\mathbb{C}})$ corresponds to the Lyubich measure μ^L, $\mu^{\beta,1}$ and $\mu^{\beta,2}$.*

(4) *If $\log N < \beta$, then $\mathrm{ex}(K_\beta(\alpha))$ consists of two points $\varphi^{\beta,1}$ and $\varphi^{\beta,2}$, where the restriction to $C(\hat{\mathbb{C}})$ corresponds to $\mu^{\beta,1}$ and $\mu^{\beta,2}$.*

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Case 3. Suppose that R has one exceptional point, i.e. $E(R) = \{w\}$.

(1) *If $\beta = 0$, then the set of extreme α-invariant traces consists of one point $\varphi_{\beta,w}$ such that the restriction $\varphi_{\beta,w}|_{C(\hat{\mathbb{C}})}$ is the Dirac measure δ_w.*

(2) *If $0 < \beta < \log N$, then there exists a unique β-KMS state $\varphi_{\beta,w}$ and its restriction $\varphi_{\beta,w}|_{C(\hat{\mathbb{C}})}$ corresponds to the Dirac measure δ_w.*

(3) *If $\beta = \log N$, then $\mathrm{ex}(K_\beta(\alpha))$ consists of two points φ^L and φ_w, where the restriction to $C(\hat{\mathbb{C}})$ corresponds to the Lyubich measure μ^L and the Dirac measure δ_w.*

(4) *If $\log N < \beta$, then $\mathrm{ex}(K_\beta(\alpha))\varphi$ is parameterized by the branched points $B(R)$. For $w \in B(R)$, the corresponding extreme β-KMS state $\varphi_{\beta,w}$ is determined by its restriction $\varphi_{\beta,w}|_{C(\hat{\mathbb{C}})}$. The corresponding measure $\mu_{\beta,w}$ on $\hat{\mathbb{C}}$ is given by*

$$\mu_{\beta,w} = m_{\mu_{\beta,w}} \sum_{k=0}^{\infty} \frac{1}{e^{k\beta}} \sum_{z \in R^{-k}(w)} \delta_z,$$

where $m_{\mu_{\beta,w}}$ is the normalized constant.

Case 4. Suppose that R has no exceptional points.

(1) *If $\beta = 0$, then there exist no α-invariant traces.*

(2) *If $0 < \beta < \log N$, then there exist no β-KMS states.*

(3) *If $\beta = \log N$, then there exists a unique β-KMS state φ, and its restriction $\varphi|_{C(\hat{\mathbb{C}})}$ is the Lyubich measure.*

(4) *If $\log N < \beta$, then $\mathrm{ex}(K_\beta(\alpha))$ is parameterized by the branched points $B(R)$ as in case 3.*

In any case the extreme β-KMS state φ^L corresponding to the Lyubich measure μ^L is a factor state of type $III_{1/N}$. All the other extreme β-KMS states are factor states of type I.

Since Theorem A is the complete classification, we can capture some information about the singularity of the original complex dynamical system from the C^*-dynamical system $(\mathcal{O}_R(\hat{\mathbb{C}}), \alpha)$ as follows.

THEOREM B. *Let P and Q be rational functions of degree at least two. Consider the gauge action α. If C^*-dynamical systems $(\mathcal{O}_P(\hat{\mathbb{C}}), \alpha)$ and $(\mathcal{O}_Q(\hat{\mathbb{C}}), \alpha)$ are conjugate, then the covering degree $\deg P = \deg Q$, the number of branched points ${}^{\#}B(P) = {}^{\#}B(Q)$, the number of exceptional points ${}^{\#}E_P = {}^{\#}E_Q$ and the orbit structures of the exceptional points under P and Q are the same.*

This paper is an extended version of the preprint [**28**].

2. *Dynamical systems and Hilbert bimodules*

In this section, we recall our construction of the C^*-algebras associated with rational functions in [**26**] and self-similar sets in [**27**], which are constructed as the Cuntz–Pimsner algebras [**19, 40**].

2.1. *The Cuntz–Pimsner algebra.*

Let A be a C^*-algebra and X be a Hilbert right A-module. We denote by $L(X)$ the algebra of the adjointable bounded operators on X. For $\xi, \eta \in X$, the 'rank-one' operator $\theta_{\xi,\eta}$ is defined by $\theta_{\xi,\eta}(\zeta) = \xi(\eta|\zeta)_A$ for $\zeta \in X$. The closure of the linear span of the rank-one operators is denoted by $K(X)$.

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A sequence $(u_n)_n$ in X is said to be a countable basis of X over A if, for any $x \in X$, the series $\sum_{n=1}^{\infty} u_n(u_n|x)_A$ converges to x in norm. We note that $(u_n)_n$ converges unconditionally in the sense that the net $\left(\sum_{n\in F} u_n(u_n|x)_A\right)_F$, where F runs over all finite subsets of $\mathbb{N}$, converges to x in norm (see [**25**] on basis). We have $\|u_n\| \le 1$ and the sequence $\left(\sum_{k=1}^{n} \theta_{u_k,u_k}\right)_n$ is an approximate unit for $K(X)$.

We say that X is a Hilbert bimodule over A if X is a Hilbert right A-module with a *-homomorphism $\phi : A \to L(X)$. We always assume that X is full and ϕ is injective. Let $F(X) = \bigoplus_{n=0}^{\infty} X^{\otimes n}$ be the full Fock module of X with a convention $X^{\otimes 0} = A$. For $\xi \in X$, the creation operator $T_\xi \in L(F(X))$ is defined by

$$T_\xi(a) = \xi a \quad \text{and} \quad T_\xi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

We define $i_{F(X)} : A \to L(F(X))$ by

$$i_{F(X)}(a)(b) = ab \quad \text{and} \quad i_{F(X)}(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \phi(a)\xi_1 \otimes \cdots \otimes \xi_n$$

for $a, b \in A$. The Cuntz–Toeplitz algebra $\mathcal{T}_X$ is the C^*-algebra acting on $F(X)$ generated by $i_{F(X)}(a)$ with $a \in A$ and T_ξ with $\xi \in X$.

Let $j_K|K(X) \to \mathcal{T}_X$ be the homomorphism defined by $j_K(\theta_{\xi,\eta}) = T_\xi T_\eta^*$. We consider the ideal $I_X := \phi^{-1}(K(X))$ of A. Let $\mathcal{J}_X$ be the ideal of $\mathcal{T}_X$ generated by $\{i_{F(X)}(a) - (j_K \circ \phi)(a) : a \in I_X\}$. Then the Cuntz–Pimsner algebra $\mathcal{O}_X$ is defined as the quotient $\mathcal{T}_X/\mathcal{J}_X$. Let $\pi : \mathcal{T}_X \to \mathcal{O}_X$ be the quotient map. We set $S_\xi = \pi(T_\xi)$ and $i(a) = \pi(i_{F(X)}(a))$. Let $i_K : K(X) \to \mathcal{O}_X$ be the homomorphism defined by $i_K(\theta_{\xi,\eta}) = S_\xi S_\eta^*$. Then $\pi((j_K \circ \phi)(a)) = (i_K \circ \phi)(a)$ for $a \in I_X$.

The Cuntz–Pimsner algebra $\mathcal{O}_X$ is the universal C^*-algebra generated by $i(a)$ with $a \in A$ and S_ξ with $\xi \in X$ satisfying the conditions that $i(a)S_\xi = S_{\phi(a)\xi}$, $S_\xi i(a) = S_{\xi a}$, and $S_\xi^* S_\eta = i((\xi|\eta)_A)$ for $a \in A$, $\xi, \eta \in X$ and $i(a) = (i_K \circ \phi)(a)$ for $a \in I_X$. We usually identify $i(a)$ with a in A. We also identify S_ξ with $\xi \in X$ and simply write ξ instead of S_ξ.

There exists an action $\alpha : \mathbb{R} \to \operatorname{Aut} \mathcal{O}_X$ defined by $\alpha_t(\xi) = e^{it}\xi$ for $\xi \in X$ and $\alpha_t(a) = a$ for $a \in A$, which is called the *gauge action*.

2.2. *The case of rational functions.* Let R be a rational function with $N = \deg R$, that is, if $R(z) = P(z)/Q(z)$ with relatively prime polynomials $P(z)$ and $Q(z)$, the degree $\deg R$ is the maximum of those of $P(z)$ and $Q(z)$. We regard R as an N-fold branched covering map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The sequence $(R^n)_n$ of iterations of R gives a complex dynamical system on $\hat{\mathbb{C}}$. The Fatou set F_R of R is the maximal open subset of $\hat{\mathbb{C}}$ on which $(R^n)_n$ is equicontinuous (or a normal family), and the Julia set J_R of R is the complement of the Fatou set in $\hat{\mathbb{C}}$. We always assume that R is of degree at least two.

Recall that a *branched point* of R is a point z_0 around which R is not locally one-to-one. It is a zero of the derivative R' or a pole of R of order two or higher. The image $w_0 = R(z_0)$ is called a *branch value* of R. Using appropriate local charts, if $R(z) = w_0 + c(z - z_0)^n +$ (higher terms) with $n \ge 1$ and $c \ne 0$ on some neighborhood of z_0, then the integer $n = e(z_0)$ is called the *branch index* of R at z_0. Thus $e(z_0) \ge 2$ if z_0 is a branched point and $e(z_0) = 1$ otherwise. Therefore R is an $e(z_0) : 1$ map in a punctured neighborhood of z_0. Let $\mathcal{B}(R)$ be the set of branched points of R and $\mathcal{C}(R)$ be the set of the

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branch values of R. Then the restriction $R : \hat{\mathbb{C}} \setminus R^{-1}(\mathcal{C}(R)) \to \hat{\mathbb{C}} \setminus \mathcal{C}(R)$ is an N-to-one regular covering.

Let $A = C(\hat{\mathbb{C}})$ and $X = C(\text{graph } R)$ be the set of continuous functions on $\hat{\mathbb{C}}$ and graph R respectively, where graph $R = \{(x, y) \in \hat{\mathbb{C}}^2 : y = R(x)\}$ is the graph of R. Then X is an A–A bimodule by

$$(a \cdot \xi \cdot b)(x, y) = a(x)\xi(x, y)b(y), \quad a, b \in A, \ \xi \in X.$$

We define an A-valued inner product $(\ |\ )_A$ on X by

$$(\xi|\eta)_A(y) = \sum_{x \in R^{-1}(y)} e(x)\overline{\xi(x, y)}\eta(x, y), \quad \xi, \eta \in X, \ y \in \hat{\mathbb{C}}.$$

Thanks to the branch index $e(x)$, the inner product above gives a continuous function and X is a full Hilbert bimodule over A without completion. The left action of A is unital and faithful.

Since the Julia set J_R is completely invariant under R, i.e. $R(J_R) = J_R = R^{-1}(J_R)$, we can consider the restriction $R|_{J_R} : J_R \to J_R$, which will often be denoted by the same letter R. Let graph $R|_{J_R} = \{(x, y) \in J_R \times J_R : y = R(x)\}$ be the graph of the restriction map $R|_{J_R}$ and $X(J_R) = C(\text{graph } R|_{J_R})$. In the same way as above, $X(J_R)$ is a full Hilbert bimodule over $C(J_R)$. Since the Fatou set F_R is also completely invariant, $X(F_R) := C_0(\text{graph } R|_{F_R})$ is a full Hilbert bimodule over $C_0(F_R)$.

Definition 2.1. [**26**] The C^*-algebra $\mathcal{O}_R(\hat{\mathbb{C}})$ is defined as the Cuntz–Pimsner algebra of the Hilbert bimodule $X = C(\text{graph } R)$ over $A = C(\hat{\mathbb{C}})$. When the Julia set J_R is not empty (for example, $\deg R \geq 2$), we define the C^*-algebra $\mathcal{O}_R(J_R)$ as the Cuntz–Pimsner algebra of the Hilbert bimodule $X = C(\text{graph } R|_{J_R})$ over $A = C(J_R)$. When the Fatou set F_R is not empty, the C^*-algebra $\mathcal{O}_R(F_R)$ is defined similarly.

2.3. *The case of self-similar sets.* Let (Ω, d) be a separable, complete metric space Ω with a metric d. Let $\gamma = (\gamma_1, \ldots, \gamma_N)$ be a system of continuous maps from Ω to itself. We say that γ_i is a proper contraction if there exist positive constants c_1 and c_2 with $0 < c_1 \leq c_2 < 1$ satisfying the condition

$$c_1 d(x, y) \leq d(\gamma_i(x), \gamma_i(y)) \leq c_2 d(x, y), \quad i = 1, 2, \ldots, N, \quad \text{for all } x, y \in X.$$

We say that a non-empty compact set $K \subset \Omega$ is *self-similar* (in a weak sense) with respect to the system $\gamma = (\gamma_1, \ldots, \gamma_N)$ if

$$K = \bigcup_{i=1}^{N} \gamma_i(K).$$

If the contractions are proper, then there exists a unique self-similar set $K \subset X$. In this paper we usually ignore the ambient space Ω and assume that $\gamma = (\gamma_1, \ldots, \gamma_N)$ is a system of continuous functions on a self-similar set K.

We use the following notation:

$$\mathcal{B}(\gamma) = \{x \in K \mid x = \gamma_j(y) = \gamma_{j'}(y) \text{ for some } y \in K \text{ and } j \neq j'\},$$
$$\mathcal{C}(\gamma) = \{y \in K \mid \gamma_j(y) = \gamma_{j'}(y) \text{ for some } j \neq j'\}.$$

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We call a point in $\mathcal{B}(\gamma)$ a branched point, and a point in $\mathcal{C}(\gamma)$ a branch value. If $\gamma = (\gamma_1, \ldots, \gamma_N)$ is a system of branches of R^{-1} for a certain map R, the terms are compatible with those for R.

We set $\mathcal{G}$ to be the union of the cographs of γ_i for $i = 1, 2, \ldots, N$, that is,

$$\mathcal{G} = \mathcal{G}(\{\gamma_j : j = 1, 2, \ldots, N\}) := \bigcup_{i=1}^{N} \{(x, y) \in K^2 : x = \gamma_i(y)\}.$$

Let $A = C(K)$ and let $X = C(\mathcal{G})$. Then X is an A–A bimodule by

$$(a \cdot f \cdot b)(x, y) = a(x) f(x, y) b(y), \quad a, b \in A, \ f \in X.$$

We introduce an A-valued inner product $(\ |\ )_A$ on X by

$$(\xi|\eta)_A(y) = \sum_{i=1}^{N} \overline{\xi(\gamma_i(y), y)} \eta(\gamma_i(y), y), \quad \xi, g \in X, \ y \in K.$$

Definition 2.2. [**27**] Let (K, d) be a compact metric space and $\gamma = (\gamma_1, \ldots, \gamma_N)$ be a system of proper contractions on K. Assume that K is self-similar. The C^*-algebra $\mathcal{O}_\gamma(K)$ is defined as the Cuntz–Pimsner algebra $\mathcal{O}_X$ of the Hilbert bimodule $X = C(\mathcal{G})$ over $A = C(K)$.

3. *KMS states*

First we recall some facts on KMS states on general Cuntz–Pimsner algebras from Laca and Neshveyev [**32**]. Let A be a C^*-algebra and X be a full Hilbert A–A bimodule with a non-degenerate left A-action. Let α be the gauge action. For $\beta > 0$, we denote by $K_\beta(\alpha)$ the set of β-KMS states for $(\mathcal{O}_X, \alpha)$.

In the following we assume that there exists a countable basis $\{u_i\}_{i=1}^{\infty}$ for X as a right Hilbert A-module and assume that $N := \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} (u_i|u_i)_A \right\|$ is finite.

Much work has been done on Perron–Frobenius type operators (see e.g. [**1**], [**24**] and [**43**]). But for our purpose, we only need the following special version.

Definition 3.1. (Perron–Frobenius type operators) Let the notation be as above. We introduce a Perron–Frobenius type operator $F : A^* \to A^*$, which is a bounded positive map, by the following formula:

$$F(\omega)(a) = \sum_{i=1}^{\infty} \omega((u_i|a \cdot u_i)_A), \quad \omega \in A^*, a \in A.$$

For $\beta \geq 0$, we set $F_\beta = e^{-\beta} F$.

The operator F was discussed in [**41**] in the case where X is finitely generated and in [**32**] in the general case. When τ is a finite positive trace on A, it is known that $F(\tau)$ is again a finite positive trace, which does not depend on the choice of $\{u_i\}_{i=1}^{\infty}$. Therefore for positive $a \in A^+$ and a finite positive trace τ, we have

$$F(\tau)(a) = \sup_{v_i} \sum_{i=0}^{n} \tau((v_i|a \cdot v_i)),$$

where the supremum is taken over all $v_0, v_1, \ldots, v_n \in X$ satisfying $\sum_{i=1}^{n} \theta_{v_i, v_i} \leq I$ (see [**32**, Theorem 1.1]).

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THEOREM 3.2. (Laca–Neshveyev [**32**]) *Let A and X be as above and let α be the gauge action on $\mathcal{O}_X$. Then there is a bijective affine isomorphism between $K_\beta(\alpha)$ and the set $T(A)_\beta$ of tracial states τ on A satisfying the following conditions:*

$$F_\beta(\tau)(a) = \tau(a) \quad \text{for all } a \in I_X, \tag{K1}$$

$$F_\beta(\tau)(a) \le \tau(a) \quad \text{for all } a \in A^+. \tag{K2}$$

The correspondence is given by restriction.

Definition 3.3. Let A and X be as above. Following Exel and Laca [**18**], we say that a finite positive trace τ_1 on A is of *finite type* if there exists a finite positive trace τ_0 on A such that $\tau_1 = \sum_{n=0}^{\infty} F_\beta^n(\tau_0)$ in the weak* topology. We denote by $T_f(A, F_\beta)$ the set of finite positive traces on A of finite type. We say that a finite positive trace τ_2 on A is of *infinite type* if $F_\beta(\tau_2) = \tau_2$. We denote by $T_i(A, F_\beta)$ the set of finite positive traces on A of infinite type.

A β-KMS state φ for the gauge action α on $\mathcal{O}_X$ is said to be of *finite type* (respectively *infinite type*) if the restriction $\tau = \varphi|_A$ is of finite type (respectively infinite type). We denote by $K_\beta(\alpha)_f$ (respectively $K_\beta(\alpha)_i$) the set of β-KMS states of finite (respectively infinite) type.

Laca and Neshveyev [**32**, Proposition 2.4] showed that any tracial state $\tau \in T(A)_\beta$ is uniquely decomposed as

$$\tau = \tau_1 + \tau_2 \tag{3.1}$$

with $\tau_1 \in T_f(A, F_\beta)$ and $\tau_2 \in T_i(A, F_\beta)$. Moreover τ_1 is given by

$$\tau_1 = \sum_{n=0}^{\infty} F_\beta^n(\tau_0) \tag{3.2}$$

with

$$\tau_0 = \tau - F_\beta(\tau) \tag{3.3}$$

and τ_2 is given by

$$\tau_2 = \lim_n F_\beta^n(\tau) \quad \text{in the weak* topology.} \tag{3.4}$$

The above classification of the KMS states *a priori* depends on the construction of $\mathcal{O}_X$ from A and X, and we do not know whether there is an intrinsic characterization of it in terms of the system $(\mathcal{O}_X, \alpha)$.

Since the decomposition (3.1) is unique, we have

$$\mathrm{ex}(K_\beta(\alpha)) = \mathrm{ex}(K_\beta(\alpha)_f) \cup \mathrm{ex}(K_\beta(\alpha)_i), \tag{3.5}$$

where, for a convex set C, we denote by $\mathrm{ex}(C)$ the set of extreme points of C.

Equation (3.3) shows that τ_0 vanishes on I_X thanks to the condition (K1). Let $T(A/I_X)$ be the set of tracial states on A that vanish on I_X, which may be regarded as tracial states on A/I_X. We denote by $T(A/I_X)_\beta$ the set of $\tau \in T(A/I_X)$ such that $\sum_{n=0}^{\infty} F_\beta^n(\tau)$ converges in the weak* topology. When $\beta > \log N$, we have $\|F_\beta\| < 1$ and so $T(A/I_X) = T(A/I_X)_\beta$. For $\tau \in T(A/I_X)_\beta$, we set $\varphi_{\tau,\beta}$ to be the β-KMS state corresponding to the

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tracial state $\psi_{\tau,\beta} := m_{\tau,\beta} \sum_{n=0}^{\infty} F_\beta^n(\tau) \in T(A)_\beta$, where $m_{\tau,\beta}$ is a normalizing constant. That is, $m_{\tau,\beta} = \left(\sum_{n=0}^{\infty} F_\beta^n(\tau)(1)\right)^{-1}$ and

$$\varphi_{\tau,\beta}|_A = m_{\tau,\beta} \sum_{n=0}^{\infty} F_\beta^n(\tau) = \psi_{\tau,\beta}.$$

LEMMA 3.4. *The map* $T(A/I_X)_\beta \ni \tau \mapsto \varphi_{\tau,\beta} \in K_\beta(\alpha)_f$ *is a bijection, which is not necessarily affine, but sends* $\mathrm{ex}(T(A/I_X)_\beta) = T(A/I_X)_\beta \cap \mathrm{ex}(T(A/I_X))$ *onto* $\mathrm{ex}(K_\beta(\alpha)_f)$.

Proof. If a positive functional ω is dominated by a positive functional τ such that $\sum_{n=0}^{\infty} F_\beta^n(\tau)$ converges, then $\sum_{n=0}^{\infty} F_\beta^n(\omega)$ also converges. Hence $\mathrm{ex}(T(A/I_X)_\beta) = T(A/I_X)_\beta \cap \mathrm{ex}(T(A/I_X))$.

The fact that the above map is bijective follows from the fact that τ is uniquely given by

$$\tau = ((\varphi_{\tau,\beta}|_A - F_\beta(\varphi_{\tau,\beta}|_A))(1))^{-1}(\varphi_{\tau,\beta}|_A - F_\beta(\varphi_{\tau,\beta}|_A)).$$

Let $\tau \in T(A/I_X)_\beta$ and $\tau', \tau'' \in T(A/I_X)$ satisfy $\tau = c_1\tau' + c_2\tau''$ for some non-negative constants c_1 and c_2 with $c_1 + c_2 = 1$. Then τ' and τ'' belong to $T(A/I_X)_\beta$, and we get

$$\varphi_{\tau,\beta} = \frac{m_{\tau,\beta}c_1}{m_{\tau',\beta}}\varphi_{\tau',\beta} + \frac{m_{\tau,\beta}c_2}{m_{\tau'',\beta}}\varphi_{\tau'',\beta}.$$

Therefore if $\varphi_{\tau,\beta}$ is extreme, then so is τ. Conversely assume that $\tau \in \mathrm{ex}(T(A/I_X)_\beta)$ and that there exist $\varphi', \varphi'' \in K_\beta(\alpha)_f$ such that

$$\varphi_{\tau,\beta} = t_1\varphi' + t_2\varphi''$$

for some non-negative constants t_1, t_2 with $t_1 + t_2 = 1$. Then there exist τ' and $\tau'' \in T(A/I_X)_\beta$ such that $\varphi_{\tau',\beta} = \varphi'$ and $\varphi_{\tau'',\beta} = \varphi''$. Put

$$c_1 = \frac{m_{\tau',\beta}t_1}{m_{\tau,\beta}} \quad \text{and} \quad c_2 = \frac{m_{\tau'',\beta}t_2}{m_{\tau,\beta}}.$$

Then

$$\begin{aligned} m_{\tau,\beta} \sum_n F_\beta^n(\tau) = \varphi_{\tau,\beta}|_A &= t_1\varphi'|_A + t_2\varphi''|_A \\ &= m_{\tau,\beta}c_1 \sum_n F_\beta^n(\tau') + m_{\tau,\beta}c_2 \sum_n F_\beta^n(\tau'') \\ &= m_{\tau,\beta} \sum_n F_\beta^n(c_1\tau' + c_2\tau''), \end{aligned}$$

which shows that $\tau = c_1\tau' + c_2\tau''$. Since τ is extreme, we get $\tau' = \tau'' = \tau$ and so $\varphi' = \varphi'' = \varphi_{\tau,\beta}$, which finishes the proof. □

COROLLARY 3.5. *If* $\beta > \log N$, *then* $\mathrm{ex}(K_\beta(\alpha)) = \{\varphi_{\tau,\beta}\}_{\tau \in \mathrm{ex}(T(A/I_X))}$.

Proof. If $\beta > \log N$, then $\|F_\beta\| < 1$ holds, and so $K_\beta(\alpha)_i = \emptyset$. □

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In the rest of this section, we investigate the type of the GNS representation of a finite type β-KMS state φ on $\mathcal{O}_X$ for the gauge action.

Let $(\pi_\varphi, H_\varphi, \Omega_\varphi)$ be the GNS representation associated with φ. Since $\{u_i\}_{i=1}^\infty$ is a countable basis for X, the sequence $(a_n)_n := \left(\sum_{k=1}^n \theta_{u_k,u_k}\right)_n$ is an approximate unit for $K(X) \subset \mathcal{O}_X$. Therefore $(\pi_\varphi(a_n))_n$ converges to a projection $p \in \pi_\varphi(\mathcal{O}_X)''$. For $x \in X$, we have

$$p\pi_\varphi(S_x) = \sum_{i=1}^\infty \pi_\varphi(S_{u_i(u_i|x)_A}) = \pi_\varphi(S_x) \quad \text{in the strong topology.}$$

Thus for any $a \in A$

$$(1-p)\pi_\varphi(a)p = \sum_{i=1}^\infty (1-p)\pi_\varphi(S_{au_i}S_{u_i}^*) = 0 \quad \text{in the strong topology.}$$

In particular, p commutes with $\pi_\varphi(A)$. We denote by $\hat{\varphi}$ the weakly continuous extension of the state φ to $\pi_\varphi(\mathcal{O}_X)''$ given by $\hat{\varphi}(T) = (T\Omega_\varphi|\Omega_\varphi)$ for $T \in \pi_\varphi(\mathcal{O}_X)''$.

LEMMA 3.6. *Let φ be a β-KMS state for α and let the notation be as above. Then the following are equivalent:*

(1) *φ is of infinite type;*

(2) $\hat{\varphi}(p) = 1$.

Proof. Since φ is a β-KMS state, for any $a \in A$,

$$\begin{aligned}(F_\beta(\varphi|_A))(a) &= \sum_{n=1}^\infty \frac{1}{e^\beta}\varphi|_A((u_n|au_n)_A) = \sum_{n=1}^\infty \frac{1}{e^\beta}\varphi(S_{u_n}^* a S_{u_n}) \\ &= \sum_{n=1}^\infty \varphi(S_{u_n}S_{u_n}^* a) = \sum_{n=1}^\infty (\pi_\varphi(S_{u_n}S_{u_n}^* a)\Omega_\varphi|\Omega_\varphi) \\ &= (p\pi_\varphi(a)\Omega_\varphi|\Omega_\varphi) = \hat{\varphi}(p\pi_\varphi(a)).\end{aligned}$$

Putting $a = I$, we get $(F_\beta(\varphi|_A))(1) = \hat{\varphi}(p)$. Since φ is a β-KMS state, the condition (K2) in Theorem 3.2 implies that $F_\beta(\varphi|_A) \le \varphi|_A$. Therefore $1 - \hat{\varphi}(p) = 0$ if and only if $\varphi|_A(1) - (F_\beta(\varphi|_A))(1) = 0$, thus if and only if $\varphi|_A = F_\beta(\varphi|_A)$, i.e. φ is of infinite type. □

THEOREM 3.7. *In the above setting, let $\tau \in \mathrm{ex}(T(A/I_X)_\beta)$ and $\varphi := \varphi_{\tau,\beta}$ be the corresponding extreme β-KMS state of finite type. Let τ' be a state on A/I_X defined by $\tau'(a + I_X) = \tau(a)$. Then $\pi_\varphi(\mathcal{O}_X)''$ is of type I (respectively type II) if and only if $\pi'_\tau(A/I_X)''$ is of type I (respectively type II). In particular, if A is abelian, then $\varphi_{\tau,\beta}$ is a type I state.*

Proof. Since the β-KMS state φ is of finite type, $\hat{\varphi}(p) \neq 1$ by Lemma 3.6. Hence $p \neq I$. Let $H_0 = (I-p)H_\varphi$. Since p commutes with $\pi_\varphi(A)$, we can define a representation $\rho : A \to B(H_0)$ by $\rho(a) = (I-p)\pi_\varphi(a)|_{H_0}$. For any $x \in X^{\otimes n}$ and $y \in X^{\otimes m}$, we have $(I-p)\pi_\varphi(S_x S_y^*) = 0$. Therefore

$$(I-p)\pi_\varphi(\mathcal{O}_X)''(I-p) = (I-p)\pi_\varphi(A)''(I-p) = (I-p)\pi_\varphi(A)'' \cong \rho(A)''.$$

Put

$$\Omega_0 := \frac{1}{\sqrt{1-\hat{\varphi}(p)}}(I-p)\Omega_\varphi.$$

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Then $\|\Omega_0\| = 1$. For $a \in A$,

$$(\rho(a)\Omega_0|\Omega_0) = \frac{1}{1-\hat{\varphi}(p)}((I-p)\pi_\varphi(a)\Omega_\varphi|\Omega_\varphi) = \frac{1}{1-\hat{\varphi}(p)}\hat{\varphi}((I-p)\pi_\varphi(a))$$
$$= \frac{1}{1-\hat{\varphi}(p)}(\varphi|_A(a) - (F_\beta(\varphi|_A))(a)) = \tau(a).$$

Since $\rho(A)\Omega_0 = (I-p)\pi_\varphi(\mathcal{O}_X)\Omega_\varphi$, the vector Ω_0 is cyclic for ρ. Therefore ρ is unitarily equivalent to the GNS representation π_τ of A for τ. Hence $(I-p)\pi_\varphi(\mathcal{O}_X)''(I-p)$ is isomorphic to $\pi_\tau(A)''$. Since $\pi_\varphi(\mathcal{O}_X)''$ is a factor and $\pi_\tau(A)''$ is isomorphic to $\pi_{\tau'}(A/I_X)''$, we get the statement. □

4. *Branched points and Perron–Frobenius type operators*

In this section, we give a detailed description of the Perron–Frobenius type operators F introduced in the previous section for concrete examples. We treat the Hilbert bimodules arising from rational functions and self-similar sets in a unified setting, that is, a topological relation in the sense of Brenken [**7**] and a topological quiver with a weighted counting measure in the sense of Muhly and Solel [**36**] and Muhly and Tomforde [**37**].

Firstly we unify the notation of bimodules. Let K be a compact metric space and $\mathcal{G}$ be a closed subset of $K \times K$. For $(x, y) \in \mathcal{G}$, we set $p_1(x, y) = x$ and $p_2(x, y) = y$, and $\mathcal{G}_y = p_1 p_2^{-1}(y)$. We assume that $p_1(\mathcal{G}) = p_2(\mathcal{G}) = K$ and $N := \sup_{y\in K} \#p_2^{-1}(y) < \infty$. Let $A = C(K)$, $X = C(\mathcal{G})$. We introduce an A–A bimodule structure into X by $(a \cdot \xi \cdot b)(x, y) = a(x)\xi(x, y)b(y)$. We assume that there exists a positive function e on $\mathcal{G}$ such that $e(x, y) \geq 1$ for all $(x, y) \in \mathcal{G}$ and

$$K \ni y \mapsto (\xi|\eta)_A(y) := \sum_{x\in\mathcal{G}_y} e(x, y)\overline{\xi(x, y)}\eta(x, y)$$

is continuous for all $\xi, \eta \in X$.

LEMMA 4.1. *For every $(x_0, y_0) \in \mathcal{G}$, every open neighborhood W of (x_0, y_0), and every $\varepsilon > 0$, there exist open neighborhoods U of x_0 and V of y_0 such that $U \times V \subset W$, $p_2((U \times V) \cap \mathcal{G}) = V$, and*

$$\left| e(x_0, y_0) - \sum_{x\in U\cap\mathcal{G}_y} e(x, y) \right| < \varepsilon \quad \textit{for all } y \in V.$$

In particular, e is upper semicontinuous.

Proof. Let $\mathcal{G}_{y_0} = \{x_0, x_1, \ldots, x_n\}$. We choose open neighborhoods U_a of x_a for $0 \leq a \leq n$ and open neighborhood V_0 of y_0 such that $U_0 \times V_0 \subset W$ and $\overline{U_a} \cap \overline{U_b} = \emptyset$ for $a \neq b$. Then there exists an open neighborhood V_1 of y_0 with $V_1 \subset V_0$ such that, for all $y \in V_1$, $\mathcal{G}_y \subset \bigcup_{a=0}^n U_a$. Indeed, if there were no such V_1, there would exist a sequence $\{(x_n, y_n)\}_{n=1}^\infty$ in $\mathcal{G}$ such that $\{y_n\}_{n=1}^\infty$ converges to y_0 and $x_n \notin U_a$ for all a. An accumulation point of this sequence is of the form $(x', y_0) \in \mathcal{G}$ such that $x' \notin U_a$ for all a, which is a contradiction. Therefore V_1 as above exists. Let $\xi \in X$ such that $\xi(x, y) = 1$ for $(x, y) \in U_0 \times V_1$ and $\xi(x, y) = 0$, for all $x \in \bigcup_{a=1}^n U_a$. Then for $y \in V_1$, we have

$$(\xi|\xi)_A(y) = \sum_{x\in U_0\cap\mathcal{G}_y} e(x, y),$$

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which is continuous and $(\xi|\xi)_A(y_0) = e(x_0, y_0) \geq 1$. Note that by assumption $\mathcal{G}_y \neq \emptyset$ for each $y \in V_1$, and so $U_0 \cap \mathcal{G}_y \neq \emptyset$ for y sufficiently close to y_0. Replacing U_0 and V_1 with smaller neighborhoods U of x_0 and V of y_0 respectively, we get the first half of the statement. Since e is a positive function, for all $(x, y) \in (U \times V) \cap \mathcal{G}$ we have $e(x_0, y_0) + \varepsilon \geq e(x, y)$, which means that e is upper semicontinuous. □

Since e is an upper semicontinuous function on a compact set $\mathcal{G}$, e is bounded above. Therefore $(\xi|\eta)_A$ gives an A-valued inner product of X with the corresponding norm equivalent to the uniform norm of $C(\mathcal{G})$, and so X is a Hilbert bimodule over A.

Following [**28**], for $f \in C(K)$ we set $\tilde{f}(y) = \sum_{x \in \mathcal{G}_y} f(x)$, which is not necessarily continuous.

Let $u_0, u_1, \ldots, u_n \in X$. Then for $\xi \in X$ we have

$$\sum_{a=0}^{n} (\xi|\theta_{u_a,u_a}\xi)_A(y) = \sum_{a=0}^{n} |(u_a|\xi)(y)|^2$$
$$= \sum_{x,x' \in \mathcal{G}_y} \sum_{a=0}^{n} e(x', y)e(x, y)u_a(x', y)\overline{u_a(x, y)}\xi(x, y)\overline{\xi(x', y)}.$$

Let

$$A(y)_{x',x} = \sum_{a=0}^{n} \sqrt{e(x', y)e(x, y)}u_a(x', y)\overline{u_a(x, y)}.$$

Then

$$\sum_{a=0}^{n} \theta_{u_a,u_a} \leq I \tag{4.1}$$

is equivalent to the matrix $(A(y)_{x',x})_{x',x}$ being dominated by 1 for any y. In particular, (4.1) implies that

$$A(y)_{x,x} = \sum_{a=0}^{n} e(x, y)|u_a(x, y)|^2 \leq 1 \quad \text{for all } x \in \mathcal{G}_y,$$

and so

$$\sum_{a=0}^{n} (u_a|f \cdot u_a)(y) = \sum_{a=0}^{n} \sum_{x \in \mathcal{G}_y} f(x)e(x, y)|u_a(x, y)|^2 \leq \tilde{f}(y) \tag{4.2}$$

for all positive $f \in C(K)$.

In what follows, we often identify an element in $C(K)_+^*$ with the corresponding measure on K. In our setting, the Perron–Frobenius type operator F is a positive map $F : C(K)^* \to C(K)^*$ given by

$$F(\mu)(f) = \sup_{u_a} \sum_{a=0}^{n} \mu((u_a|f \cdot u_a)), \quad \mu \in C(K)_+^*, f \in C(K)_+, \tag{4.3}$$

where the supremum is taken over all $u_0, u_1, \ldots, u_n \in X$ satisfying (4.1). We have $\|F\| = N$.

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Since K is separable, X is countably generated and has a countable basis $\{v_i\}_{i=0}^{\infty}$. We introduce an increasing sequence $\{\varphi_n\}_{n=0}^{\infty}$ of positive maps of $C(K)$ by

$$\varphi_n(f) = \sum_{i=0}^{n} (v_i | f \cdot v_i)_A, \quad f \in C(K).$$

Then F is given by the limit

$$F(\mu)(f) = \lim_{n\to\infty} \mu(\varphi_n(f)). \tag{4.4}$$

THEOREM 4.2. *Let F be as above. Then*

$$F(\mu)(f) = \int_K \tilde{f}(y)\, d\mu(y), \quad f \in C(K)_+, \mu \in C(K)_+^*.$$

Proof. First, we show the statement for any Dirac measure $\mu = \delta_y$ on $y \in K$. Note that the above argument implies that

$$F(\mu)(f) \le \int_K \tilde{f}(y)\, d\mu(y), \quad f \in C(K)_+, \mu \in C(K)_+^*.$$

We fix $y_0 \in K$ and $\varepsilon > 0$. Let $\mathcal{G}_{y_0} = \{x_0, x_1, \ldots, x_n\}$. Applying Lemma 4.1, we choose open neighborhoods U_a of x_a and V of y_0 such that $U_a \cap U_b = \emptyset$ for $a \neq b$ and

$$\bigcup_{a=1}^{n} U_a \supset \mathcal{G}_y \quad \text{for all } y \in V,$$

$$\left| e(x_a, y_0) - \sum_{x \in U_a \cap V} e(x, y) \right| < \varepsilon \quad \text{for all } y \in V.$$

Let $u_a \in X$ be a continuous non-negative function such that $\mathrm{supp}(u_a) \subset U_a \times V$ and

$$u_a(x_a, y_0) = \frac{1}{\sqrt{e(x_a, y_0) + \varepsilon}},$$

$$0 \le u_a(x, y) \le \frac{1}{\sqrt{e(x_a, y_0) + \varepsilon}} \quad \text{for all } (x.y) \in (U_a \times V) \cap \mathcal{G}.$$

For $y \in V$, we define the matrix $A(y) = (A(y)_{x',x})_{x',x\in\mathcal{G}_y}$ as before. We set $I_a(y) = U_a \cap \mathcal{G}_y$ for $y \in V$ and $A_a(y) = (A(y)_{x',x})_{x',x\in I(a)}$. Then $A(y)$ is a direct sum of the $A_a(y)$. Let $c \in \ell^2(I(a))$. Then

$$\begin{aligned}
\langle A_a(y)c, c\rangle &= \sum_{x',x\in I_a(y)} \sum_{b=0}^{n} \sqrt{e(x', y)e(x, y)}\, u_b(x', y)\overline{c(x')u_b(x, y)}c(x) \\
&= \left| \sum_{x\in I_a(y)} \sqrt{e(x, y)}\, \overline{u_a(x, y)}c(x) \right|^2 \\
&\le \|c\|^2 \sum_{x\in I_a(y)} e(x, y)|u_a(x, y)|^2 \\
&\le \|c\|^2 \sum_{x\in I_a(y)} \frac{e(x, y)}{e(x_0, y_0) + \varepsilon} \le \|c\|^2,
\end{aligned}$$

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where we use Lemma 4.1. Thus $A_a(y) \leq 1$ and (4.1) is satisfied. Since

$$\sum_{a=0}^{n}(u_a|f\cdot u_a)_A(y_0) = \sum_{x\in\mathcal{G}_{y_0}}\sum_{a=0}^{n} f(x)e(x,\,y_0)|u_a(x,\,y_0)|^2$$
$$= \sum_{x\in\mathcal{G}_{y_0}} f(x)\frac{e(x,\,y_0)}{e(x,\,y_0)+\varepsilon} \geq \tilde{f}(y)\frac{1}{1+\varepsilon},$$

we get $F(\delta_{y_0})(f) = \tilde{f}(y_0)$.

The above computation implies that, for every $f \in C(K)_+$ and every $y \in K$, $\{\phi_n(f)(y)\}_{n=0}^{\infty}$ increasingly converges to $\tilde{f}(y)$. Thus thanks to the bounded (or monotone) convergence theorem, we get

$$F(\mu)(f) = \lim_{n\to\infty}\int_K \phi_n(f)(y)\,d\mu(y) = \int_K \tilde{f}(y)\,d\mu(y) \quad \text{for all } \mu \in C(K)_+^*,$$

which finishes the proof in the general case. □

We denote by $D(e)$ the set of discontinuous points of e. For example, if R is a rational function, $\mathcal{G} = \text{graph } R$ and $e(z,\,R(z)) = e(z)$ is the branch index, then $D(e)$ is the set $\{(z,\,R(z)) \in \text{graph } R \mid z \in \mathcal{B}(R)\}$ and $p_1(D(e)) = \mathcal{B}(R)$, where $\mathcal{B}(R)$ is the set of branched points of R. If $\gamma = (\gamma_1, \ldots, \gamma_N)$ is a system of proper contractions on K, $\mathcal{G} = \bigcup_{i=1}^{N}\{(x,\,y) \in K^2 : x = \gamma_i(y)\}$, and $e(x,\,y) = \#\{i \mid x = \gamma_i(y)\}$, then $D(e) = \{(x,\,y) \in K^2 \mid x = \gamma_j(y) = \gamma_{j'}(y) \text{ for some } j \neq j'\}$ and $p_1(D(e)) = \mathcal{B}(\gamma)$.

LEMMA 4.3. *Let $D(e)$ be the set of discontinuous points of e.*

(1) *For every $(x_0,\,y_0) \in \mathcal{G}\setminus D(e)$, there exist closed neighborhoods F of x_0 and G of y_0, and a continuous map $\gamma : F \to G$, such that*

$$(F\times G)\cap\mathcal{G} = \{(\gamma(y),\,y) \in K^2 : y \in G\}.$$

(2) *The set $D(e)$ is a closed set.*

Proof. (1) Let $0 < \varepsilon < 1/2$ and take a neighborhood W of $(x_0,\,y_0)$ such that

$$|e(x_0,\,y_0) - e(x,\,y)| < \varepsilon \quad \text{for all } (x,\,y) \in W.$$

By Lemma 4.1, there exist closed neighborhoods F of x_0 and G of y_0 such that $F\times G \subset W$, $p_2((F\times G)\cap\mathcal{G}) = G$, and

$$\left|e(x_0,\,y_0) - \sum_{x\in\mathcal{G}_y\cap F} e(x,\,y)\right| < \varepsilon \quad \text{for all } y \in G.$$

Since $e(x,\,y) \geq 1$ for all $(x,\,y) \in \mathcal{G}$, the set $\mathcal{G}_y \cap F$ is a singleton for all $y \in G$. Thus there exists a function $\gamma : G \to F$ such that

$$(F\times G)\cap\mathcal{G} = \{(\gamma(y),\,y) : y \in G\}.$$

Since $(F\times G)\cap\mathcal{G}$ is closed, γ is continuous.

(2) We choose $\xi \in X$ such that $\text{supp}(\xi) \subset F\times G$, $0 \leq \xi \leq 1$, and $\xi(x,\,y) = 1$ in a neighborhood W_1 of $(x_0,\,y_0)$. Then for $(x,\,y) \in W_1$, we have

$$(\xi|\xi)_A(y) = \sum_{y\in F\cap\mathcal{G}_y} e(x,\,y)|\xi(x,\,y)|^2 = e(\gamma(y),\,y),$$

which shows that e is continuous on $W_1 \cap \mathcal{G}$. Thus $D(e)$ is a closed set. □

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The following proposition shows that the set of singularities corresponds to the ideal I_X of A. We can prove it using the preceding lemmas as in [**26**] and in [**27**]. It is also shown in a general situation by Muhly and Tomforde [**37**, Corollary 3.12].

PROPOSITION 4.4. *In the above situation*

$$I_X = \{f \in C(K) : f|_{p_1(D(e))} = 0\} \cong C_0(K \setminus p_1(D(e))).$$

COROLLARY 4.5. *Let the notation be as above. Then the following hold.*

(1) *If $\beta > \log N$, then there is a one-to-one correspondence between $\mathrm{ex}(K_\beta(\alpha))_f = \mathrm{ex}(K_\beta(\alpha))$ and $p_1(D(e))$.*

(2) *If $\beta = \log N$, then there is a one-to-one correspondence between $K_\beta(\alpha)_i$ and the set of Borel probability measures μ on K satisfying*

$$\int_K f(x)\,d\mu(x) = \frac{1}{N}\int_K \tilde{f}(x)\,d\mu(x) \quad \textit{for all } f \in C(K).$$

In particular, such a measure μ satisfies $\mu\{y \in K : \#\mathcal{G}_y < N\} = 0$.

Proof. Part (1) follows from Proposition 4.4 and Lemma 3.4. The first statement of part (2) follows from Theorem 3.2. The case with $f = 1$ implies the second statement. □

5. *Classification of KMS states in the case of rational functions*

Throughout this section, we assume that R is a rational function of degree at least two. We shall completely classify the KMS states for the gauge actions on the C^*-algebras $\mathcal{O}_R(\hat{\mathbb{C}})$ and $\mathcal{O}_R(J_R)$. We can recover the degree of R, the number of branched points, the number of exceptional points and the orbits of exceptional points from information about the structure of the KMS states.

Recall that, for $f \in C(\hat{\mathbb{C}})$ and $y \in \hat{\mathbb{C}}$, we have

$$\tilde{f}(y) = \sum_{x \in R^{-1}(y)} f(x),$$

which is not necessarily continuous. For a bounded regular Borel measure μ on $\hat{\mathbb{C}}$, we denote by $\tau_\mu \in C(\hat{\mathbb{C}})^*$ the corresponding positive linear functional (though we often identify μ with τ_μ). Then the Perron–Frobenius type operator $F : C(\hat{\mathbb{C}})^* \to C(\hat{\mathbb{C}})^*$ with norm $N = \deg R$ is given by

$$F(\tau_\mu)(f) = \int_{\hat{\mathbb{C}}} \tilde{f}(z)\,d\mu(z).$$

In particular, for a Dirac measure δ_y on y we get

$$F(\delta_y) = \sum_{x \in R^{-1}(y)} \delta_x.$$

In our particular situation, the conditions (K1) and (K2) in Theorem 3.2 take the following forms:

$$\frac{1}{e^\beta}\int_{\hat{\mathbb{C}}} \tilde{a}(z)\,d\mu(z) = \int_{\hat{\mathbb{C}}} a(z)\,d\mu(z) \quad \text{for all } a \in C_0(\hat{\mathbb{C}} \setminus B(R)), \tag{K1}$$

$$\frac{1}{e^\beta}\int_{\hat{\mathbb{C}}} \tilde{a}(z)\,d\mu(z) \le \int_{\hat{\mathbb{C}}} a(z)\,d\mu(z) \quad \text{for all } a \in A_+, \tag{K2}$$

where we identify $C_0(\hat{\mathbb{C}} \setminus \mathcal{B}(R))$ with $I_X = \{f \in A \mid f(z) = 0 \text{ for } z \in \mathcal{B}(R)\}$.

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LEMMA 5.1. *If a measure μ satisfies the conditions (K1) and (K2), then the point mass $\mu(\{w\})$ for $w \in \hat{\mathbb{C}}$ must satisfy the following:*

$$e^{-\beta}\mu(\{R(w)\}) = \mu(\{w\}), \quad w \notin \mathcal{B}(R), \tag{K1$'$}$$

$$e^{-\beta}\mu(\{R(w)\}) \le \mu(\{w\}), \quad w \in \hat{\mathbb{C}}. \tag{K2$'$}$$

Proof. Let $w \notin \mathcal{B}(R)$. Then we can approximate the characteristic function $\chi_{\{w\}}$ by a monotone decreasing sequence $(a_n)_n$ of non-negative functions in I_X. Thus

$$\mu(\{w\}) = \lim_n \int_{\hat{\mathbb{C}}} a_n(z)\, d\mu(z) = \lim_n e^{-\beta} \int_{\hat{\mathbb{C}}} \sum_{x \in R^{-1}(z)} a_n(x)\, d\mu(z)$$

$$= e^{-\beta} \int_{\hat{\mathbb{C}}} \sum_{x \in R^{-1}(z)} \chi_{\{w\}}(x)\, d\mu(z) = e^{-\beta}\mu(\{R(w)\})$$

by the Lebesgue convergence theorem. The other one is similarly obtained. □

Let us recall some facts on exceptional points for R. For z and w in $\hat{\mathbb{C}}$, we define $z \sim_R w$ if there exist non-negative integers n and m with $R^n(z) = R^m(w)$. Then $\sim_R$ is an equivalence relation on $\hat{\mathbb{C}}$. We denote the equivalence class containing z by $[z]_R$. We also define the *backward orbit* $O^-(z)$ of z by

$$O^-(z) := \{w \in \hat{\mathbb{C}} \mid R^n(w) = z \text{ for some non-negative } n \in \mathbb{Z}\}.$$

Then we have $O^-(z) \subset [z]_R$.

Definition 5.2. A point z in $\hat{\mathbb{C}}$ is an *exceptional point* for R if the backward orbit $O^-(z)$ of z is finite. We denote by E_R the set of exceptional points.

It is known that E_R is a subset of $F_R \cap \mathcal{B}(R)$ and that z is an exceptional point if and only if $[z]_R$ is finite. A rational function R of degree at least two has at most two exceptional points by the Riemann–Hurwitz formula. The reader is referred to [**2**, §4.1] for basic properties of exceptional points, including the following result.

LEMMA 5.3. *Let R be a rational function of degree at least two. Then one of the following holds.*

(1) *The set $E_R = \emptyset$.*
(2) *The set E_R consists of one point z_0 and $\{z_0\} = [z_0]_R$. After a suitable conjugation by a Möbius transformation, we may assume that $z_0 = \infty$ and R is a polynomial.*
(3) *The set E_R consists of two points $z_0 \neq z_1$ and $[z_0]_R = \{z_0\}$ and $[z_1]_R = \{z_1\}$. After a suitable conjugation by a Möbius transformation, we may assume that $z_0 = 0$, $z_1 = \infty$, and $R(z) = z^d$ for some positive integer d.*
(4) *The set E_R consists of two points $z_0 \neq z_1$ and $[z_0]_R = [z_1]_R = \{z_0, z_1\}$. After a suitable conjugation by a Möbius transformation, we may assume that $z_0 = 0$, $z_1 = \infty$, and $R(z) = z^d$ for some negative integer d.*

The following proposition is crucial for the complete classification of the KMS states.

PROPOSITION 5.4. *Assume that a measure μ satisfies the conditions (K1) and (K2). If μ has a point mass at z for some $z \notin E_R$, then $\beta > \log N$.*

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Proof. Thanks to Lemma 5.1, we have $\mu\{w\} > 0$ for any $w \in O^-(z)$. To prove the statement, it suffices to show that there exists $w \in O^-(z)$ satisfying the following two conditions: (1) $O^-(w) \cap \mathcal{C}(R) = \emptyset$; and (2) $R^{-m}(w) \cap R^{-n}(w) = \emptyset$ for all distinct non-negative integers m, n. Indeed, Lemma 5.1 with such w implies that

$$\sum_{n=0}^{\infty} \left(\frac{N}{e^{\beta}}\right)^n \mu(\{w\}) \leq \mu(O^-(w)) \leq 1.$$

Assume that condition (1) does not hold for any $w \in O^-(z)$. Then since $\mathcal{C}(R)$ is a finite set, there exist two positive integers $m < n$ such that $R^{-m}(z) \cap R^{-n}(z)$ is not empty, and so $z = R^{n-m}(z)$. Let p be the minimal positive integer such that $R^p(z) = z$ and set $L = \{R^k(z)\}_{k=0}^{p-1}$. For any $w \in O^-(z)$, the same argument as above shows that there exists a positive integer q such that $w = R^q(w)$. Thus for a sufficiently large integer k, we get $w = R^{qk}(w) \in L$. Since $O^-(z)$ is an infinite set, this is a contradiction. Therefore condition (1) holds for some $z_1 \in O^-(z)$.

Assuming that condition (2) does not hold for any $w \in O^-(z_1)$, we get a contradiction in the same way, and so there exists $w \in O^-(z_1)$ satisfying conditions (1) and (2). □

We now recall basic properties of the Lyubich measure.

Definition 5.5. (Lyubich measure [**20, 33**]) Let $N = \deg R$ and δ_x be the Dirac measure on x for $x \in \hat{\mathbb{C}}$. For any $y \in \hat{\mathbb{C}} \setminus E_R$ and each $n \in \mathbb{N}$, we define a probability measure μ_n^y on the Riemann sphere $\hat{\mathbb{C}}$ by

$$\mu_n^y = \sum_{x \in R^{-n}(y)} N^{-n} e(x) e(R(x)) \cdots e(R^{n-1}(x)) \delta_x.$$

The sequence $(\mu_n^y)_n$ converges weakly to a measure μ^L, which is called the *Lyubich measure*. The measure μ^L is independent of the choice of $y \in \hat{\mathbb{C}} \setminus E_R$.

The measure μ^L has been studied by several authors, e.g. Freire, Lopes and Mäné [**20**] and Lyubich [**33**]. The support of μ^L is the Julia set J_R, and μ^L is an invariant measure in the sense that $\mu^L(E) = \mu^L(R^{-1}(E))$ for any Borel set E. Hence $\int f(R(x))\, d\mu^L(x) = \int f(x)\, d\mu^L(x)$. Moreover $\mu^L(R(E)) = N\mu(E)$ for any Borel set E on which R is injective. The measure μ^L is the unique measure of maximal entropy.

We introduce an operator $G : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ by setting

$$G(f)(w) = \sum_{z \in R^{-1}(w)} e(z) f(z)$$

and the contraction $\overline{G} = N^{-1}G$. The conjugate operator G^* is slightly different from F. However, if μ has no point mass on $\mathcal{C}(R)$, then G^* satisfies $G^*\tau_\mu = F\tau_\mu$.

PROPOSITION 5.6. (Lyubich [**33**]) *For any $a \in C(\hat{\mathbb{C}})$ and compact subset $K \subset \hat{\mathbb{C}}$ with $K \cap E_R = \emptyset$, we have*

$$\lim_{m \to \infty} \left\| \overline{G}^m a - \int_{\hat{\mathbb{C}}} a(z)\, d\mu^L(z) \right\|_K = 0,$$

where $\| \cdot \|_K$ is the uniform norm on K.

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We use the symbol τ^L for τ_{μ^L} for simplicity.

PROPOSITION 5.7. *Let μ be a bounded regular Borel measure on $\hat{\mathbb{C}}$. Suppose that μ satisfies one of the following conditions:*

(1) *μ satisfies conditions (K1) and (K2) and μ has no point mass on $\mathcal{B}(R)$;*

(2) *μ satisfies condition (K1) and μ has no point mass on $\mathcal{B}(R) \cup \mathcal{C}(R)$.*

Then we have the following. If $\beta \neq \log N$, then $\mu = 0$. If $\beta = \log N$, then μ coincides with the Lyubich measure up to a constant.

Proof. Suppose that μ is non-zero. We may and do assume that μ is a probability measure. First we consider the case (1). Since $e^{-\beta}\mu(\{R(w)\}) \leq \mu(\{w\})$ for $w \in \hat{\mathbb{C}}$ by Lemma 5.1, μ has no point mass on $\mathcal{C}(R)$. Therefore (K1) holds for all $a \in A$ and we may consider the case (2). For $a = 1$, we have $\tilde{a}(x) = N$ for almost every x with respect to μ and $\beta = \log N$. For any $a \in A$, we have

$$\tau_\mu(a) = F_\beta(\tau_\mu)(a) = \tau_\mu(\overline{G}(a)).$$

Note that $\mu(E_R) = 0$ holds as $E_R \subset \mathcal{B}(R)$. For any $\varepsilon > 0$, we can choose a compact subset $K \subset \hat{\mathbb{C}}$ satisfying $K \cap E_R = \emptyset$ and $\mu(K^c)\|a\| < \varepsilon$ by the regularity of μ. Thus we have

$$\begin{aligned}
|\tau_\mu(a) - \tau^L(a)| &= |\tau_\mu(\overline{G}^m(a)) - \tau^L(a)| \\
&= \left| \int_K \overline{G}^m(a)\, d\mu + \int_{K^c} \overline{G}^m(a)\, d\mu - (\tau^L(a)\mu(K) + \tau^L(a)\mu(K^c)) \right| \\
&\leq \left| \int_K (\overline{G}^m(a) - \tau^L(a))\, d\mu \right| + \left| \int_{K^c} \overline{G}^m(a)\, d\mu \right| + \tau^L(a)\mu(K^c) < 3\varepsilon
\end{aligned}$$

for sufficiently large m thanks to Proposition 5.6, which shows that $\mu = \mu^L$. □

PROPOSITION 5.8. *The Lyubich measure μ^L is extended to an infinite type $\log N$-KMS state φ^L on $\mathcal{O}_R(\hat{\mathbb{C}})$ for the gauge action.*

Proof. Since the Lyubich measure has no point mass, it satisfies $F(\tau^L) = G^*(\tau^L)$. On the other hand, for any $x \in \hat{\mathbb{C}}$ we have

$$G^*(\mu_n^x) = N \sum_{y \in R^{-1}(x)} \mu_{n+1}^y,$$

which implies that $G^*(\tau^L) = N\tau^L$. Thus by Corollary 4.5, the Lyubich measure μ^L corresponds to a $\log N$-KMS state φ^L. □

Now we are ready to state our classification results. Since we already know the result for $\beta > \log N$ thanks to Corollary 4.5, we assume that $0 < \beta \leq \log N$. Proposition 5.4 implies that every finite type β-KMS state, if it exists, arises from a point in E_R. Let μ be a probability measure corresponding to an extreme infinite type β-KMS state φ, which satisfies $F_\beta(\mu) = \mu$. Let $\mu = \mu_a + \mu_d$ be the decomposition of μ into the atomic part μ_a and the diffuse part μ_d. Then a similar argument as in the proof of Lemma 5.1 shows that $F_\beta(\mu_a)$ is atomic and $F_\beta(\mu_d)$ is diffuse again, and so we get $F_\beta(\mu_a) = \mu_a$ and $F_\beta(\mu_d) = \mu_d$. Therefore either $\mu_a = 0$ or $\mu_d = 0$ holds. If $\mu_d = 0$, Proposition 5.4 and Lemma 5.3 imply that μ would be supported by E_R. Since $R^{-1}(x)$ is a singleton for

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$x \in E_R$, this does not occur. Therefore $\mu_a = 0$ and Proposition 5.7 implies that $\beta = \log N$ and μ is the Lyubich measure.

The above observation with an easy case-by-case analysis using Lemma 5.3 shows the following theorems.

THEOREM 5.9. *For $\beta > 0$, there exists an infinite type β-KMS state for the gauge action α on $\mathcal{O}_R(\hat{\mathbb{C}})$ if and only if $\beta = \log N$. The set $K_{\log N}(\alpha)_i$ is a singleton consisting of φ_L given by the Lyubich measure.*

THEOREM 5.10. *For the finite type β-KMS states for the gauge action α on $\mathcal{O}_R(\hat{\mathbb{C}})$, the following hold.*

(1) *For $\beta > \log N$, the set $\mathrm{ex}(K_\beta(\alpha)_f)$ of the extreme finite type β-KMS states φ is parameterized by the branched points $\mathcal{B}(R)$. For $w \in \mathcal{B}(R)$, the corresponding extreme β-KMS state $\varphi_{\beta,w}$ is determined by its restriction $\varphi_{\beta,w}|_{C(\hat{\mathbb{C}})}$, i.e. the regular Borel measure $\mu_{\beta,w}$ on $\hat{\mathbb{C}}$ given by*

$$\mu_{\beta,w} = m_{\mu_{\beta,w}} \sum_{k=0}^{\infty} \frac{1}{e^{k\beta}} \sum_{z \in R^{-k}(w)} \delta_z,$$

where $m_{\mu_{\beta,w}}$ is the normalized constant.

(2) *For $0 < \beta \leq \log N$, the set $\mathrm{ex}(K_\beta(\alpha)_f)$ of the extreme finite type β-KMS states coincides with $\{\varphi_{\beta,w}\}_{w \in E_R}$.*

Remark 5.11. Corollary 4.5 shows that the GNS representation of every extreme finite type KMS state with $\beta > 0$ is a type I factor representation. In §7, we show that the GNS representation of φ^L is a factor representation of type $\mathrm{III}_{1/N}$.

Remark 5.12. Theorem 3.2 still holds for $\beta = 0$ if we define a 0-KMS state to be an α-invariant tracial state. Such a state exists if and only if E_R is not empty and we have the following.

(1) When E_R consists of one point w, then there exists a unique α-invariant trace state φ_w. The restriction of φ_w to $C(\hat{\mathbb{C}})$ is given by the Dirac measure δ_w.
(2) When E_R consists of two points w_1 and w_2 with $R(w_1) = w_1$ and $R(w_2) = w_2$, the set of α-invariant trace states has exactly two extreme points $\{\varphi_{w_i}\}_{i=1,2}$. The restriction of φ_{w_i} to $C(\hat{\mathbb{C}})$ is given by δ_{w_i} for $i = 1, 2$.
(3) When E_R consists of two points w_1 and w_2 with $R(w_1) = w_2$ and $R(w_2) = w_1$, then there exists a unique α-invariant trace φ. The restriction of φ to $C(\hat{\mathbb{C}})$ is given by $\frac{1}{2}(\delta_{w_1} + \delta_{w_2})$.

Note that the GNS representations of these states are not factor representations. It is routine work to show that they give finite type I von Neumann algebras.

Proofs of Theorems A and B in the introduction. The set of all extreme β-KMS states depends only on the system $(\mathcal{O}_R(\hat{\mathbb{C}}), \alpha)$. Since the set is obtained as $\mathrm{ex}(K_\beta(\alpha)) = \mathrm{ex}(K_\beta(\alpha)_f) \cup \mathrm{ex}(K_\beta(\alpha)_i)$, Theorem A follows from Theorems 5.9 and 5.10 and Remark 5.12. Theorem B immediately follows from Theorem A, because Theorem A is a complete classification. □

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Example 5.13. Let $R(z) = z^N$ with $N \geq 2$. Then $E_R = \{0, \infty\} = \mathcal{B}(R)$ with $R(0) = 0$ and $R(\infty) = \infty$. For every $\beta > 0$ and $w = 0, \infty$, we have $\mu_{\beta,w} = \delta_w$.

Example 5.14. Let $R(z) = z^{-N}$ with $N \geq 2$. Then we have $E_R = \mathcal{B}(R) = \{0, \infty\}$ with $R(0) = \infty$ and $R(\infty) = 0$. For every $\beta > 0$ we have

$$\mu_{\beta,0} = \frac{e^\beta}{e^\beta + 1}\delta_0 + \frac{1}{e^\beta + 1}\delta_\infty,$$
$$\mu_{\beta,\infty} = \frac{1}{e^\beta + 1}\delta_0 + \frac{e^\beta}{e^\beta + 1}\delta_\infty.$$

Example 5.15. Let $R(z) = z^N + 1$ with $N \geq 2$. Then $E_R = \{\infty\}$ and $\mathcal{B}(R) = \{0, \infty\}$. For any $\beta \geq 0$ the Dirac measure δ_∞ extends to a β-KMS state $\varphi_{\beta,\infty}$. The measure

$$\mu_{\beta,0} = \left(1 - \frac{N}{e^\beta}\right) \sum_{k=0}^{\infty} \left(\frac{1}{e^{k\beta}} \sum_{z \in R^{-k}(0)} \delta_z \right)$$

corresponds to a β-KMS state $\varphi_{\beta,0}$ for $\beta > \log N$. For $0 \leq \beta < \log N$, the set of β-KMS states consists of one point $\varphi_{\beta,\infty}$. For $\beta = \log N$, the set of extreme β-KMS states consists of two points φ^L and $\varphi_{\beta,\infty}$. For $\log N < \beta$, the set of extreme β-KMS states consists of two points $\varphi_{\beta,\infty}$ and $\varphi_{\beta,0}$.

Finally, we consider the C^*-algebras $\mathcal{O}_R = \mathcal{O}_R(J_R)$ associated with a rational function R on the Julia set J_R, which is purely infinite and simple [**26**].

THEOREM 5.16. *For the gauge action α on the C^*-algebras $\mathcal{O}_R(J_R)$, the following hold.*

(1) *For $0 < \beta < \log N$, there exist no β-KMS states.*
(2) *For $\beta = \log N$, there exists a unique β-KMS state φ^L, and its restriction $\varphi^L|_{C(J_R)}$ is the Lyubich measure. The GNS representation of φ^L is a factor representation of type $III_{1/N}$.*
(3) *For $\log N < \beta$, the set $\mathrm{ex}(K_\beta(\alpha))$ of extreme β-KMS states is parameterized by the branched points $\mathcal{B}(R) \cap J_R$. For $w \in \mathcal{B}(R) \cap J_R$, the corresponding extreme β-KMS state $\varphi_{\beta,w}$ is determined by its restriction $\varphi_{\beta,w}|_{C(J_R)}$, which is given by the same formula as in Theorem 5.10. The GNS representation of $\varphi_{\beta,w}$ is a factor representation of type I.*

Proof. The proof follows from a similar argument as in the case of $\mathcal{O}_R(\hat{\mathbb{C}})$ with the fact that the Julia set J_R contains no exceptional point by [**2**]. □

6. *Classification of KMS states in the case of self-similar sets*

Let (K, d) be a compact metric space and let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N)$ be a system of proper contractions satisfying $K = \bigcup_{i=1}^N \gamma_i(K)$. In this section, we consider the structure of the KMS states for the gauge action on $\mathcal{O}_X = \mathcal{O}_\gamma(K)$. Recall that $A = C(K)$ and $I_X = \{f \in C(K) : f|_{\mathcal{B}(\gamma)} = 0\} = C_0(K \setminus \mathcal{B}(\gamma))$ in this case. The set $\mathcal{B}(\gamma)$ is finite if and only if $\mathcal{C}(\gamma)$ is finite.

For $y \in K$, we set $\gamma(y) = \bigcup_{i=1}^N \{\gamma_i(y)\}$. The Borel function $\tilde{a}$ for $a \in A$ is

$$\tilde{a}(y) = \sum_{x \in \gamma(y)} a(x) = \sum_{j=1}^N \frac{1}{e(\gamma_j(y), y)} a(\gamma_j(y)). \tag{6.1}$$

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Note that, if $\mathcal{C}(\gamma)$ is not empty, $\tilde{a}$ is not necessarily continuous. Theorem 4.2 shows that

$$F_\beta(\delta_y) = e^{-\beta} \sum_{j=1}^{N} \frac{1}{e(\gamma_j(y),\, y)} \delta_{\gamma_j(y)} = e^{-\beta} \sum_{x \in \gamma(y)} \delta_x. \tag{6.2}$$

If a probability measure μ on K satisfies (K2), we have $\mu \geq F_\beta(\mu) \geq \mu(\{x\}) F_\beta(\delta_x)$ and so the following holds:

$$\mu(\{\gamma_i(x)\}) \geq e^{-\beta} \mu(\{x\}). \tag{6.3}$$

LEMMA 6.1. *Let μ be a probability regular Borel measure on K satisfying conditions (K1) and (K2). If $\mu(\mathcal{C}(\gamma)) = 0$, then $\log N \leq \beta$.*

Proof. Suppose that $\mu(\mathcal{C}(\gamma)) = 0$. Then

$$\tilde{1}(y) = \sum_{j=1}^{N} \frac{1}{e(\gamma_j(y),\, y)} 1(\gamma_j(y)) = N, \quad \text{a.e. } y \text{ with respect to } \mu.$$

Condition (K2) implies that

$$\frac{N}{e^\beta} = \frac{1}{e^\beta} \int_K \tilde{1}(z)\, d\mu(z) \leq \int_K 1\, d\mu(z) = 1.$$

Hence $\log N \leq \beta$. □

For $x \in K$ and $n \geq 0$, we set the nth orbit of x by

$$O_n(x) = \{\gamma_{i_1} \cdots \gamma_{i_2} \cdots \gamma_{i_n}(x) \in K : 1 \leq i_1, i_2, \ldots, i_n \leq N\}$$

and set $O(x) = \bigcup_{n=0}^{\infty} O_n(x)$. Note that if a measure μ satisfying (K2) has a point mass at x, it has a point mass at y for all $y \in O(x)$ thanks to (6.3).

LEMMA 6.2. *Let $0 < \beta \leq \log N$ and let μ be a probability regular Borel measure on K satisfying conditions (K1) and (K2). Assume that either $0 < \beta < \log N$ or μ is of finite type. Let $y \in K$. If there exists $x \in O(y)$ such that $O(x) \cap \mathcal{C}(\gamma) = \emptyset$, then μ has no point mass at y.*

Proof. First we assume that $0 < \beta < \log N$. If μ had a point mass at y, then μ would have a point mass at x as well and we would have $\mu \geq \mu(\{x\}) F_\beta^n(\delta_x)$. However, by assumption we get

$$1 \geq \mu(\{x\}) F_\beta^n(\delta_x)(K) = \mu(\{x\})(N e^{-\beta})^n \to \infty \quad (\text{as } n \to \infty),$$

which is a contradiction.

Now we assume that $\beta = \log N$ and the corresponding KMS state is of finite type. Then there is a measure ν supported by $\mathcal{B}(\gamma)$ such that

$$\mu = \sum_{n=0}^{\infty} F_\beta^n(\nu).$$

Suppose that μ has a point mass at y. Then there would exist n and $c > 0$ such that $F_\beta^n(\nu) \geq c\delta_x$, and so $F_\beta^{n+m}(\nu) \geq c F_\beta^m(\delta_x)$. As we have $F_\beta^m(\delta_x)(1) = 1$, we get

$$1 = \mu(1) = \sum_{k=0}^{\infty} F_\beta^k(\nu)(1) \geq \sum_{k=n}^{\infty} c = \infty,$$

which is a contradiction. □

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To describe infinite type KMS states, we recall the notion of the Hutchinson measure for a self-similar set (see [**22**] for details).

Definition 6.3. Let $\overline{G} : C(K) :\to G(K)$ be the unital positive map defined by

$$\overline{G}(a) = \frac{1}{N}\sum_{i=1}^{N} a \cdot \gamma_i, \quad a \in C(K).$$

The *Hutchinson measure* is a unique $\overline{G}^*$-invariant probability measure.

LEMMA 6.4. *Unless K consists of one point, the Hutchinson measure μ^H has no point mass.*

Proof. Let $\mu^H = \mu_a + \mu_d$ be the decomposition of μ^H into the atomic part μ_a and the diffuse part μ_d. Then it is easy to show that $\overline{G}^*(\mu_a)$ is atomic and $\overline{G}^*(\mu_d)$ is diffuse. Thus either $\mu_a = 0$ or $\mu_d = 0$ holds thanks to the uniqueness of a $\overline{G}$-invariant measure.

Suppose that $\mu_d = 0$. We set $m = \max\{\mu\{x\} : x \in K\}$ and $L = \{x \in K : \mu\{x\} = m\}$, which is a finite set. Since

$$\overline{G}^*(\delta_x) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\gamma_i(x)}$$

and $\overline{G}^*(\mu^H) = \mu^H$, we have

$$m(\#L) = \mu^H(L) = \overline{G}^*(\mu^H)(L) = \frac{1}{N}\sum_{i=1}^{N}\sum_{y\in K} \mu\{y\}\delta_{\gamma_i(y)}(L)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\sum_{x\in L, y\in K} \mu\{y\}\delta_{x,\gamma_i(y)}.$$

This shows that $\gamma_i(L) = L$ for all i and $L = K$. Since γ_i is a proper contraction and L is a finite set, we conclude that K consists of one point. □

Note that when a measure μ satisfies $\mu(\mathcal{C}(\gamma)) = 0$, we have $F(\mu) = N\overline{G}(\mu)$. Thus when the Hutchinson measure μ^H satisfies $\mu^H(\mathcal{C}(\gamma)) = 0$, it gives rise to an infinite type $\log N$-KMS state, which we denote by φ^H. Thanks to Lemma 6.4, the condition $\mu^H(\mathcal{C}(\gamma)) = 0$ holds whenever $\mathcal{C}(\gamma)$ is a countable set. Conversely, we have the following result.

LEMMA 6.5. *Let φ be a β-KMS state of infinite type for $\beta = \log N$ and let μ be the corresponding measure on K. Then $\varphi = \varphi^H$ and $\mu^H(\mathcal{C}(\gamma)) = 0$.*

Proof. The condition $F_\beta(\mu) = \mu$ implies that

$$0 = \int_K (N - \tilde{1}(x))\, d\mu(x) = \int_K \sum_{j=1}^{N}\left(1 - \frac{1}{e(\gamma_j(x), x)}\right) d\mu(x)$$

$$= \int_{\mathcal{C}(\gamma)} \sum_{j=1}^{N}\left(1 - \frac{1}{e(\gamma_j(x), x)}\right) d\mu(x),$$

which shows that $\mu(\mathcal{C}(\gamma)) = 0$. Thus $\mu = F_\beta(\mu) = \overline{G}(\mu)$ and so $\mu = \mu^H$. □

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THEOREM 6.6. *Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N)$ be a system of proper contractions on a compact metric space K such that $K = \bigcup_{i=1}^{N} \gamma_i(K)$. Suppose either that $\mathcal{B}(\gamma)$ is empty or that $\mathcal{B}(\gamma)$ is a finite set and, for every $y \in \mathcal{C}(\gamma)$, there exists $x \in O(y)$ such that $O(x) \cap \mathcal{C}(\gamma) = \emptyset$. For the gauge action α on the C^*-algebras $\mathcal{O}_\gamma(K)$, the following hold.*

(1) *For $0 < \beta < \log N$, there exist no β-KMS states.*
(2) *For $\beta = \log N$, there exists a unique β-KMS state φ^H, and its restriction $\varphi^H|_{C(K)}$ is the Hutchinson measure μ^H.*
(3) *For $\log N < \beta$, the set $\mathrm{ex}(K_\beta(\alpha))$ of extreme β-KMS states φ is parameterized by the branched points $\mathcal{B}(\gamma)$.*

Proof. For $0 < \beta < \log N$, Lemmas 6.1 and 6.2 show that there is no β-KMS state. Assume that μ is a measure corresponding to a finite type β-KMS state for $\beta = \log N$. Then Lemma 6.2 implies that $\mu(\mathcal{C}(\gamma)) = 0$ and so $\mu(K) = F_\beta(\mu)(K)$ would hold. However, this means that μ is of infinite type, which is a contradiction. The rest follows from Lemma 6.5 and Corollary 4.5. □

Remark 6.7. In §7, we show that the GNS representation of φ^H is a factor representation of type $\mathrm{III}_{1/N}$.

Example 6.8. (Inverse branches of the tent map) Let $K = [0, 1]$, $\gamma_1(y) = \frac{1}{2}y$, and $\gamma_2(y) = 1 - \frac{1}{2}y$. Then $\mathcal{B}(\gamma) = \{\frac{1}{2}\}$ and $\mathcal{C}(\gamma) = \{1\}$. Since $O(1/2) \cap \mathcal{C}(\gamma) = \emptyset$, the assumption of Theorem 6.6 is satisfied. The Hutchinson measure μ^H is the normalized Lebesgue measure on $[0, 1]$. Hence for $0 < \beta < \log 2$, there exist no β-KMS states. For $\beta = \log 2$, there exists a unique β-KMS state φ^H, and its restriction $\varphi^H|_{C([0,1])}$ is the normalized Lebesgue measure. For $\log 2 < \beta$, there exists a unique β-KMS state $\varphi_{\beta,1/2}$, and its restriction $\varphi|_{C([0,1])}$ is

$$\mu_{\beta,1/2} = \left(1 - \frac{2}{e^\beta}\right) \sum_{n=0}^{\infty} \frac{1}{e^{n\beta}} \sum_{(j_1,\ldots,j_n)\in\{1,2\}^n} \delta_{\gamma_{j_1}\cdots\gamma_{j_n}(1/2)}.$$

Example 6.9. Let $K = [0, 1]$, $\gamma_1(y) = \frac{1}{2}y$, and $\gamma_2(y) = \frac{1}{2}(y + 1)$. Then $\mathcal{B}(\gamma) = \phi$ and $\mathcal{C}(\gamma) = \phi$ and the Hutchinson measure μ^H is the normalized Lebesgue measure on $[0, 1]$. The C^*-algebra $\mathcal{O}_\gamma(K)$ with α is isomorphic to the Cuntz algebra $\mathcal{O}_2$ with the usual gauge action. Thus a β-KMS state on $\mathcal{O}_\gamma(K)$ exists if and only if $\lambda = \log 2$, and its restriction to $C([0, 1])$ is the normalized Lebesgue measure.

Example 6.10. (Sierpinski gasket) Let Ω be a regular triangle in $\mathbb{R}^2$ with three vertices $c_1 = (1/2, \sqrt{3}/2)$, $c_2 = (0, 0)$ and $c_3 = (1, 0)$. The middle point of c_1c_2 is denoted by b_1, the middle point of c_1c_3 is denoted by b_2 and the middle point of c_2c_3 is denoted by b_3. We define proper contractions γ_i for $i = 1, 2, 3$ by

$$\gamma_1(x, y) = \left(\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{\sqrt{3}}{4}\right), \quad \gamma_2(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right), \quad \gamma_3(x, y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right).$$

Then the self-similar set K associated with γ is the Sierpinski gasket. It is known that K is homeomorphic to the Julia set J_R of the rational function

$$R(z) = \frac{z^3 - \frac{16}{27}}{z}.$$

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However, these three contractions cannot be identified with the inverse branches of R because $\gamma_1(c_2) = \gamma_2(c_1)$. Let $\tilde{\gamma}_1 = \gamma_1$, $\tilde{\gamma}_2 = \rho_{-2\pi/3} \circ \gamma_2$, and $\tilde{\gamma}_3 = \rho_{2\pi/3} \circ \gamma_3$, where ρ_θ is a rotation by the angle θ. The self-similar set for $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3)$ is K as well and they are inverse branches of a map $h : K \to K$, which is conjugate to $R : J_R \to J_R$. Therefore $\mathcal{O}_R(J_R)$ is isomorphic to $\mathcal{O}_{\tilde{\gamma}}(K)$, which is known to be a purely infinite and simple C^*-algebra. The C^*-algebras $\mathcal{O}_\gamma(K)$ and $\mathcal{O}_{\tilde{\gamma}}(K)$ have non-isomorphic K-groups and hence they are not isomorphic (see [**26**] for details).

Since $\mathcal{B}(\gamma) = \emptyset$ and $\mathcal{C}(\gamma) = \emptyset$, there exists a β-KMS state for the gauge automorphism for $\mathcal{O}_\gamma(K)$ if and only if $\lambda = \log 3$.

For $\tilde{\gamma}$, we have $\mathcal{B}(\tilde{\gamma}) = \{b_1, b_2, b_3\}$, $\mathcal{C}(\tilde{\gamma}) = \{c_1, c_2, c_3\}$, and the assumption of Theorem 6.6 is satisfied. Thus for $0 < \beta < \log 3$, there exist no β-KMS states. For $\beta = \log 3$, there exists a unique β-KMS state whose restriction to $C(K)$ is given by the Hutchinson measure μ^H. For $\log 3 < \beta$, the set $\mathrm{ex}(K_\beta(\alpha))$ of extreme β-KMS states is parameterized by the branched points $\mathcal{B}(\gamma) = \{b_1, b_2, b_3\}$. For $b_i \in \mathcal{B}(\gamma)$, the corresponding extreme β-KMS state is given by the measure

$$\mu_{\beta,b_i} = \left(1 - \frac{3}{e^\beta}\right) \sum_{n=0}^{\infty} \frac{1}{e^{n\beta}} \sum_{(j_1,\ldots,j_n)\in\{1,2,3\}^n} \delta_{\gamma_{j_1}\cdots\gamma_{j_n}(b_i)}.$$

7. *Type III representations by Lyubich and Hutchinson measures*

In this section, we show that the KMS states arising from the Lyubich measures and the Hutchinson measures give type $\mathrm{III}_{1/N}$ representations. Our proof is based on the facts that the Cuntz algebra $\mathcal{O}_N$ arises from the Bernoulli N-shift and that the unique KMS state of the gauge action of $\mathcal{O}_N$ gives a type $\mathrm{III}_{1/N}$ representation.

Let N be an integer greater than one and $K_N = \{1, 2, \ldots, N\}^{\mathbb{N}}$. We denote by ν_N the Bernoulli measure with weight $(1/N, \ldots, 1/N)$, i.e. the infinite product measure of the uniform measure of $\{1, 2, \ldots, N\}$. Let σ be the Bernoulli shift

$$\sigma(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

For $j = 1, 2, \ldots, N$, we set ρ_j to be the inverse branches of σ, that is,

$$\rho_j(x_1, x_2, \ldots) = (j, x_1, x_2, \ldots).$$

Note that ν_N is nothing but the Hutchinson measure of $\rho = (\rho_1, \rho_2, \ldots, \rho_N)$.

Let Y be the $C(K_N)$–$C(K_N)$ bimodule giving $\mathcal{O}_\rho(K_N)$. Since

$$\bigcup_{j=1}^{N} \{(\rho_j(y), y) \in K_N^2 : x \in K_N\} = \{(x, \sigma(x)) \in K_N^2 : x \in K_N\},$$

in what follows, we identify Y with $C(K_N)$ whose Hilbert C^*-bimodule structure is given by

$$(a \cdot f \cdot b)(x) = a(x) f(x) b(\sigma(x)),$$
$$(f|g)_{C(K_N)}(y) = \sum_{j=1}^{N} \overline{f(\rho_j(y))} g(\rho_j(y)).$$

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We use the symbol 1_E for the characteristic function of a subset $E \subset K_N$. For $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \{1, 2, \ldots, N\}^n$, we denote by E_ξ the cylinder set

$$E_\xi = \{x \in K_N : x_k = \xi_k, 1 \le k \le n\}.$$

It is known that the C^*-algebra $\mathcal{O}_\rho(K_N)$ is isomorphic to the Cuntz algebra $\mathcal{O}_N$. Indeed, $\mathcal{O}_\rho(K_N)$ is generated by $S_{1_{E_1}}, S_{1_{E_2}}, \ldots, S_{1_{E_N}}$, which satisfy the Cuntz algebra relation (see [**27**, Proposition 4.1] and [**41**, §4]). Note that the isomorphism $\mathcal{O}_N \ni S_j \mapsto S_{1_{E_j}} \in \mathcal{O}_\rho(K_N)$ intertwines the gauge actions, and in consequence the KMS states. We give a W^*-version of this argument first.

7.1. *The case of Hutchinson measures.* Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N)$ be an N-tuple of proper contractions of a compact metric space K satisfying the self-similar condition $K = \bigcup_{j=1}^N \gamma_j(K)$. We assume that the Hutchinson measure μ^H satisfies $\mu_H(\mathcal{C}(\gamma)) = 0$ and denote by φ^H the corresponding KMS state of the gauge action α on $\mathcal{O}_\gamma(K)$ as before.

For $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \{1, 2, \ldots, N\}^n$, we use the notation

$$\gamma_\xi = \gamma_{\xi_1} \cdot \gamma_{\xi_2} \cdots \gamma_{\xi_n}.$$

Since all the maps $\gamma_1, \gamma_2, \ldots, \gamma_N$ are proper contractions, there exists a surjective continuous map $q : K_N \to K$ such that

$$\{q(x)\} = \bigcap_{n=0}^{\infty} \gamma_{x_1} \cdot \gamma_{x_2} \cdots \gamma_{x_n}(K).$$

We have $q \circ \rho_i = \gamma_i \circ q$ for $i = 1, \ldots, N$. Since

$$\sum_{j=1}^N \gamma_{j*} q_* \nu_N = \sum_{j=1}^N q_* \rho_{j*} \nu_N = N q_* \nu_N,$$

the uniqueness of the Hutchinson measure μ^H implies that $\mu_H = q_* \nu_N$.

LEMMA 7.1. *For all $f, g \in L^2(K, \mu_N)$, the following holds:*

$$(f|g)_\mu = \lim_{n\to\infty} \frac{1}{N^n} \sum_{\xi \in \{1,2,\ldots,N\}^n} (f \cdot \gamma_\xi | 1)_\mu (1 | g \cdot \gamma_\xi)_\mu,$$

where $(\cdot|\cdot)_\mu$ is the inner product of $L^2(K, \mu)$.

Proof. Since $q^* : L^2(K, \mu_H) \ni f \mapsto f \cdot q \in L^2(K_N, \nu_N)$ is an isometry satisfying $\rho_j^* q^* = q^* \gamma_j^*$, it suffices to show the statement for (K_N, ρ, ν_N) instead of (K, γ, μ).

Let $\mathcal{F}_n$ be the σ-field generated by the first n coordinate functions of K_N, and $E(f|\mathcal{F}_n)$ be the conditional expectation of f given $\mathcal{F}_n$. Then for $f \in L^1(K_N, \nu_N)$ and $\xi \in \{1, 2, \ldots, N\}^n$, we have

$$E(f|\mathcal{F}_n)(\xi) = \int_{K_N} f \cdot \rho_\xi(x)\, d\nu_N(x).$$

Thus for $f, g \in L^2(K_N, \nu_N)$, we get

$$\frac{1}{N^n} \sum_{\xi \in \{1,2,\ldots,N\}^n} (f \cdot \rho_\xi | 1)_{\nu_N} (1 | g \cdot \gamma_\xi)_{\nu_N} = (E(f|\mathcal{F}_n) | E(g|\mathcal{F}_n))_{\nu_N},$$

which tends to $(f|g)_{\nu_N}$ as n goes to infinity. □

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Let $A = C(K)$ and X be the A–A bimodule giving $\mathcal{O}_\gamma(K)$. We denote by (π, H, Ω) the GNS triple for φ^H. We set $M = \pi(\mathcal{O}_\gamma(K))''$. Let $\hat{\varphi}^H$ be the normal extension of φ^H to M given by $\hat{\varphi}^H(m) = (m\Omega|\Omega)$ for $m \in M$. Since φ^H is a KMS state, $\hat{\varphi}^H$ is faithful on M.

Let τ^H be the trace on $A = C(K_N)$ given by μ^H and let (π_0, H_0, Ω_0) be the GNS triple for τ^H. We can identify H_0 with $L^2(K, \mu^H)$ and $\pi_0(a)$ with the multiplication operator by $a \in A$. Thus π_0 extends to a normal representation of $L^\infty(K, \mu^H)$. On the other hand, since τ^H is the restriction of φ^H to A, the Hilbert space H_0 is naturally identified with the closure of $\pi(A)\Omega$ and the restriction $\pi|_A$ of π to A is quasi-equivalent to π_0 [**23**, Lemma 4.1]. Therefore $\pi|_A$ extends to a normal representation of $B := L^\infty(K, \mu^H)$, which is denoted by $\hat{\pi}$.

For a measurable function f on K, we denote by $\|f\|_p$ the L^p-norm of f in $L^p(K, \mu^H)$.

For $m \in M$, we set $\|m\|_2 = \|m\Omega_L\|$. It is well known that the strong operator topology on the unit ball M^1 of M coincides with the topology given by $\|\cdot\|_2$, and M^1 is complete with respect to $\|\cdot\|_2$.

Let $\{h_n\}_{n=1}^\infty$ be an increasing sequence of non-negative functions in A such that $h_n(x) = 0$ for $x \in \mathcal{C}(\gamma)$ and $\{h_n(y)\}_{n=1}^\infty$ converges to 1 for all $y \in K \setminus \mathcal{C}(\gamma)$. For $1 \le j \le N$, we introduce $u_{j,n} \in X$ by setting $u_{j,n}(\gamma_j(x), x) = h_n(x)$ and $u_{j,n}(\gamma_k(x), x) = 0$ for $k \ne j$.

LEMMA 7.2. *Let the notation be as above. Then the sequence $\{\pi(S_{u_{j,n}})\}_{n=1}^\infty$ converges to an isometry, say $\tilde{S}_j$, in the $*$ strong operator topology. The isometries $\{\tilde{S}_j\}_{j=1}^N$ satisfy the Cuntz algebra relation. For the modular automorphism group $\{\sigma_t^{\hat{\varphi}^H}\}_{t\in\mathbb{R}}$, the following holds: $\sigma_t^{\hat{\varphi}^H}(\tilde{S}_j) = e^{-t\log N\sqrt{-1}}\tilde{S}_j$, for all $t \in \mathbb{R}$.*

Proof. By straightforward computation, we have $\|S_{u_{j,m}}\| \le 1$ and

$$\|\pi(S_{u_{j,m}} - S_{u_{j,n}})\|_2^2 = \|h_m - h_n\|_2^2,$$
$$\|\pi(S_{u_{j,m}}^* - S_{u_{j,n}}^*)\|_2^2 = \frac{1}{N}\|h_m - h_n\|_2^2.$$

Since we assume that $\mu(C(\gamma)) = 0$, these imply that $\{\pi(S_{u_{j,n}})\}_{n=1}^\infty$ and $\{\pi(S_{u_{j,n}})^*\}_{n=1}^\infty$ are Cauchy sequences with respect to $\|\cdot\|_2$, and so $\{\pi(S_{u_{j,n}})\}_{n=1}^\infty$ converges in the $*$ strong operator topology. As we have $S_{u_{j,n}}^* S_{u_{k,n}} = \delta_{j,k}\pi(|h_n|^2)$, which converges to $\delta_{j,k}1$ in the strong operator topology, the operators $\{\tilde{S}_j\}_{j=1}^N$ are isometries with mutual orthogonal ranges. Since $\sigma_t^{\hat{\varphi}^H}$ is a normal extension of $\alpha_{-t\log N}$, we have $\sigma_t^{\hat{\varphi}^H}(\tilde{S}_j) = e^{-t\log N\sqrt{-1}}\tilde{S}_j$, which implies that

$$\sum_{j=1}^N \hat{\varphi}^H(\tilde{S}_j\tilde{S}_j^*) = \sum_{j=1}^N \frac{1}{N}\hat{\varphi}^H(\tilde{S}_j^*\tilde{S}_j) = 1.$$

Since $\hat{\varphi}^H$ is faithful, this implies $\sum_{j=1}^N \tilde{S}_j\tilde{S}_j^* = 1$. □

THEOREM 7.3. *Let the notation be as above. Then M is the approximately finite dimensional (AFD) type $III_{1/N}$ factor.*

Proof. Thanks to the above lemma, it suffices to show that M is generated by $\{\tilde{S}_j\}_{j=1}^N$.

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Since $aS_{u_{j,n}} = S_{u_{j,n}} a \cdot \gamma_j$ holds for all $a \in A$, we get $\pi(a)\tilde{S}_j = \tilde{S}_j \pi(a \cdot \gamma_j)$. For $f \in X$, we define $f_j \in A$ by $f_j(x) = f(\gamma_j(x), x)$. Since $\mu(C(\gamma)) = 0$, we have $\tilde{S}_j^* \pi(S_f) = \pi(f_j)$, and so

$$\pi(S_f) = \sum_{j=1}^{N} \tilde{S}_j \pi(f_j).$$

Therefore the linear span of the elements of the form $\tilde{S}_\eta \pi(a) \tilde{S}_\zeta^*$ with $a \in C(K)$, $\eta \in \{1, 2, \ldots, N\}^m$, and $\zeta \in \{1, 2, \ldots, N\}^n$ is a dense $*$-algebra of M, where $\tilde{S}_\eta = \tilde{S}_{\eta_1} \tilde{S}_{\eta_2} \cdots \tilde{S}_{\eta_m}$. We understand $\tilde{S}_\eta f \tilde{S}_\zeta^* = f$ for $\eta = \zeta = \emptyset$. This observation shows that to finish the proof it suffices to prove that $\pi(A) \subset \{\tilde{S}_j\}_{j=1}^N{}''$.

For $a \in A$, we set

$$a_n = \sum_{\eta \in \{1,2,\ldots,N\}^n} \hat{\varphi}^H(\tilde{S}_\eta^* \pi(a) \tilde{S}_\eta) \tilde{S}_\eta \tilde{S}_\eta^* = \sum_{\eta \in \{1,2,\ldots,N\}^n} \hat{\varphi}^H(\pi(a \cdot \gamma_\eta)) \tilde{S}_\eta \tilde{S}_\eta^*.$$

Then $\{a_n\}_{n=1}^\infty$ is a bounded sequence in $\{\tilde{S}_j\}_{j=1}^N{}''$. We show that $\{a_n\}_{n=1}^\infty$ converges to $\pi(a)$ in the strong operator topology. Indeed, we have

$$\begin{aligned} \|\pi(a) - a_n\|_2^2 &= \hat{\varphi}^H(\pi(|a|^2) - \pi(a^*)a_n - a_n^* \pi(a) + |a_n|^2) \\ &= \|a\|_2^2 - \frac{1}{N^n} \sum_{\eta \in \{1,2,\ldots,N\}^n} |(f \cdot \gamma_\eta | 1)_{\nu_N}|^2. \end{aligned}$$

Lemma 7.2 shows that the right-hand side tends to 0 as n goes to infinity. Thus we have $\pi(A) \subset \{\tilde{S}_j\}_{j=1}^N{}''$. □

7.2. *The case of Lyubich measures.* Let R be a rational function with $N = \deg R \geq 2$, $A = C(\hat{\mathbb{C}})$, and $X = C(\text{graph } R)$ be as before. Since $\hat{\mathbb{C}} \ni z \mapsto (z, R(z)) \in \text{graph } R$ is a homeomorphism, we identify X with $C(\hat{\mathbb{C}})$, whose A–A bimodule structure is given by

$$\begin{aligned} (a \cdot f \cdot b)(z) &= a(z) f(z) b(R(z)), \\ (f|g)_A(z) &= \sum_{w \in R^{-1}(z)} e(w) \overline{f(w)} g(w). \end{aligned}$$

The following theorem relies on Heicklen and Hoffman's remarkable result [**21**] saying that $(\hat{\mathbb{C}}, R, \mu^H)$ is conjugate to (K_N, σ, ν_N) as measurable dynamical systems.

THEOREM 7.4. *Let R be a rational function with $N = \deg R \geq 2$. Let μ^L be the Lyubich measure and φ^L be the corresponding* log N*-KMS state on $\mathcal{O}_R(\hat{\mathbb{C}})$ for the gauge action. Then the GNS representation of φ^L is a factor representation of type $III_{1/N}$.*

Proof. Let (π, H, Ω) be the GNS triple for φ^L. We set $M = \pi(\mathcal{O}_R(\hat{\mathbb{C}}))''$. Let $\hat{\varphi}^L$ be the normal extension of φ^L to M. We denote by $\hat{\pi}$ the normal extension of the restriction $\pi|_A$ to $B := L^\infty(\hat{\mathbb{C}}, \mu^L)$. For $m \in M$, we set $\|m\|_2 = \|m\Omega_L\|$. We denote by $\|\cdot\|_p$ the L^p-norm of $L^p(\hat{\mathbb{C}}, \mu^L) = L^p(J_R, \mu^L)$.

For $f \in X$, the condition $\mu^H(\mathcal{C}(R)) = 0$ implies that

$$\|f\|_\infty \leq \|\pi_L(S_f)\| = \|\pi_L((f|f)_A)\|^{1/2} \leq \sqrt{N} \|f\|_\infty,$$

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$$\|\pi_L(S_f)\|_2^2 = \int_{\hat{\mathbb{C}}} \sum_{w\in R^{-1}(z)} e(w)|f(w)|^2\, d\mu^L(z)$$
$$= \int_{\hat{C}} \sum_{w\in R^{-1}(z)} |f(z)|^2\, d\mu^L(z) = N\|f\|_2^2.$$

The KMS condition implies that $\|\pi(S_f^*)\|_2 = \|f\|_2/\sqrt{N}$.

For $f \in L^\infty(\hat{\mathbb{C}}, \mu^L)$, we choose a sequence $\{f_n\}_{n=1}^\infty$ in $C(\hat{\mathbb{C}})$ satisfying the following three conditions: (1) $\{\|f_n\|_\infty\}_{n=1}^\infty$ is dominated by $\|f\|_\infty$; (2) $\{\|f_n - f\|_2\}_{n=1}^\infty$ converges to 0; and (3) $\{f_n(z)\}_{n=1}^\infty$ converges to $f(z)$ for almost every z with respect to μ^L. Then $\{\pi_L(S_{f_n})\}_{n=1}^\infty$ converges in the $*$ strong operator topology, whose limit is denoted by $\hat{S}_f$. Note that $\hat{S}_f$ does not depend on the choice of the sequence $\{f_n\}_{n=1}^\infty$. Let $g \in L^\infty(\hat{\mathbb{C}}, \mu^L)$ and $\{g_n\}_{n=1}^\infty$ be a sequence in $C(\hat{\mathbb{C}})$ satisfying the same properties as above for g instead of f. Then

$$\hat{S}_f^* \hat{S}_g = \lim_{n\to\infty} \pi_L((f_n|g_n)_A),$$

where the limit is taken in the $*$ strong operator topology. For $f, g \in B$, we define $(f|g)_B \in B$ by the same formula as the A-valued inner product of X. Then we get

$$\hat{S}_f^* \hat{S}_g = \hat{\pi}((f|g)_B), \quad f, g \in L^\infty(\hat{\mathbb{C}}, \mu^L). \tag{7.1}$$

In a similar way, we have

$$\hat{\pi}(a)\hat{S}_f\hat{\pi}(b) = \hat{S}_{afb\cdot R}, \quad f, a, b \in L^\infty(\hat{\mathbb{C}}, \mu^L). \tag{7.2}$$

Since $\sigma_t^{\hat{\varphi}^L}$ is a normal extension of $\alpha_{-t\log N}$, we have

$$\sigma_t^{\hat{\varphi}^L}(\hat{\pi}(a)) = \hat{\pi}(a), \quad a \in L^\infty(\hat{\mathbb{C}}, \mu^L), \tag{7.3}$$
$$\sigma_t^{\hat{\varphi}^L}(\hat{S}_f) = e^{-t\log N\sqrt{-1}}\hat{S}_f, \quad f \in L^\infty(\hat{\mathbb{C}}, \mu^L). \tag{7.4}$$

Thanks to Heicklen and Hoffman [**21**], there exists a von Neumann algebra isomorphism $\phi : L^\infty(K_N, \nu_N) \to L^\infty(\hat{\mathbb{C}}, \mu^L)$ satisfying $\phi(f)\cdot R = \phi(f\cdot\sigma)$ for all $f \in L^\infty(K_N, \nu_N)$. We claim that there exists a representation $(\tilde{\pi}, H)$ of $\mathcal{O}_\rho(K_N)$ satisfying

$$\tilde{\pi}(a) = \hat{\pi}(\phi(a)), \quad a \in C(K_N),$$
$$\tilde{\pi}(S_f) = \hat{S}_{\phi(f)}, \quad f \in Y = C(K_N).$$

Thanks to (7.1) and (7.2), such a representation would exist for the Cuntz–Toeplitz algebra $\mathcal{T}_Y$. Since Y has a finite basis $\{1_{E_j}\}_{j=1}^N$, to prove the claim it suffices to show that

$$\sum_{j=1}^N \hat{S}_{\phi(1_{E_j})}\hat{S}_{\phi(1_{E_j})}^* = I.$$

Indeed, thanks to the KMS condition we have

$$\hat{\varphi}^H\left(I - \sum_{j=1}^N \hat{S}_{\phi(1_{E_j})}\hat{S}_{\phi(1_{E_j})}^*\right) = 1 - \frac{1}{N}\sum_{j=1}^N \hat{\varphi}^H(\hat{S}_{\phi(1_{E_j})}^*\hat{S}_{\phi(1_{E_j})}) = 0.$$

Since $\hat{\varphi}^H$ is faithful, we get the claim.

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It is routine work to show that $\tilde{\pi}(\mathcal{O}_\rho(K_N))'' = M$, and so $(\tilde{\pi}, H, \Omega)$ is a cyclic representation. Equations (7.3) and (7.4) imply that $(\tilde{\pi}, H, \Omega)$ is unitarily equivalent to the GNS representation of the unique $\log N$-KMS state of the gauge action on $\mathcal{O}_\rho(K_N)$. Thus, Theorem 7.3 shows that M is the AFD type $\mathrm{III}_{1/N}$ factor. □

8. *Symmetry*

Since our construction of the Cuntz–Pimsner algebra from a dynamical system is natural, a symmetry of the dynamical system gives rise to an automorphism of the algebra. In this section, we give a criterion for outerness of such automorphisms. Quasi-free automorphisms on the Cuntz–Pimsner algebras for bimodules with finite bases are studied by Katayama and Takehana [**29**] and Zacharias [**45**], while the bimodules we treat in this paper do not necessarily have finite bases.

Let M be the AFD type $\mathrm{III}_{1/N}$ factor acting on a Hilbert space. Since M arises from the GNS representation of the unique KMS state of the gauge action on the Cuntz algebra $\mathcal{O}_N$, we may find a generating set $\{S_j\}_{j=1}^N$ of M consisting of isometries satisfying the following conditions: (1) they satisfy the Cuntz algebra relation; and (2) there exists a cyclic and separating vector $\Omega \in H$ for M such that, if we denote the vector state for Ω by φ, then $\sigma_t^\varphi(S_j) = N^{-t\sqrt{-1}} S_j$ holds for $t \in \mathbb{R}$ and $j = 1, 2, \ldots, N$. The canonical shift endomorphism Φ of M is defined by

$$\Phi(x) = \sum_{j=1}^{N} S_j x S_j^*, \quad x \in M.$$

Let B be the maximal abelian subalgebra of M generated by $\bigcup_{n=1}^\infty \{S_\xi S_\xi^*\}_{\xi \in \{1,2,\ldots,N\}^n}$.

LEMMA 8.1. *Let the notation be as above. Assume that θ is an automorphism of M commuting with the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ such that $\theta(B) = B$ and $\Phi \cdot \theta(x) = \theta \cdot \Phi(x)$ for all $x \in B$. If there exists $b \in B$ satisfying $\theta(b) \neq b$, then θ is outer.*

Proof. We first fix the notation. Let $\mathcal{F}_n = \Phi^n(M)' \cap M$, which is isomorphic to $M_{N^n}(\mathbb{C})$. The centralizer M_φ is a type II_1 factor generated by $\bigcup_{n=1}^\infty \mathcal{F}_n$ and M_φ satisfies $M \cap M_\varphi' = \mathbb{C}$. In addition, B is a maximal abelian subalgebra of M_φ as well.

Suppose that there exists a unitary $u \in M$ satisfying $\theta(x) = uxu^*$ for all $x \in M$. We claim that $u \in M_\varphi$. For $n \in \mathbb{Z}$, we set

$$u_n = \frac{1}{T} \int_0^T N^{tn\sqrt{-1}} \sigma_t^\varphi(x)\, dt,$$

where $T = 2\pi/\log N$. Then $u_m x = \theta(x) u_m$ holds for all $x \in M_\varphi$, which shows that $u_m^* u_m$, $u_m u_m^* \in M \cap M_\varphi' = \mathbb{C}$. Thus u_m is a multiple of a unitary. If $u_m \neq 0$ for some $m \neq 0$, the KMS condition would imply that $\varphi(u_m^* u_m) \neq \varphi(u_m u_m^*)$, which is a contradiction. Thus we get $u = u_0 \in M_\varphi$.

Let E_n be the trace-preserving conditional expectation from M_φ onto $\mathcal{F}_n$. Then $\{\|u - E_n(u)\|_2\}_{n=1}^\infty$ converges to 0. For every $b \in B$, we have

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$$\begin{aligned}\|b-\theta(b)\|_2 &= \|\Phi^n(b)-\Phi^n(\theta(b))\|_2 = \|\Phi^n(b)-\theta(\Phi^n(b))\|_2\\ &= \|\Phi^n(b)u-u\Phi^n(b)\|_2 = \|\Phi^n(b)(u-E_n(u))-(u-E_n(u))\Phi^n(b)\|_2\\ &\le 2\|u-E_n(u)\|_2\|b\|,\end{aligned}$$

which shows that $b=\theta(b)$. This is a contradiction and θ is outer. □

Let R be a rational function with $\deg R \ge 2$ and let θ be a homeomorphism of $\hat{\mathbb{C}}$ commuting with R. Then, by naturality, we get an automorphism of $\mathcal{O}_R(\hat{\mathbb{C}})$ and $\mathcal{O}_R(J_R)$, which we denote by $\theta_{\hat{\mathbb{C}}}$ and θ_{J_R}, respectively. Namely, we define the actions of θ on $a \in C(\hat{\mathbb{C}})$ and $f \in X = C(\text{graph } R)$ by $\theta \cdot a(z) = a(\theta^{-1}(z))$ and by $\theta \cdot f(z, R(z)) = f(\theta^{-1}(z), R(\theta^{-1}(z)))$, respectively.

THEOREM 8.2. *Let the notation be as above and let μ^L be the Lyubich measure. If the automorphism of $L^\infty(\hat{\mathbb{C}}, \mu^L) = L^\infty(J_R, \mu^L)$ induced by θ is non-trivial, then $\theta_{\hat{\mathbb{C}}}$ and θ_{J_R} are outer.*

Proof. We use the notation in the proof of Theorem 7.4. Since $\theta_{\hat{\mathbb{C}}}$ preserves φ^L, there exists an automorphism θ_M of M satisfying $\theta_M \cdot \pi = \pi \cdot \theta_{\hat{\mathbb{C}}}$. Since π factors through $\mathcal{O}_R(J_R)$, to prove the theorem it suffices to show that θ_M is outer.

Let $S_j = \hat{S}_{\phi(1_{E_j})}$ and

$$\Phi(x) = \sum_{j=1}^{N} S_j x S_j^*, \quad x \in M.$$

We claim that, for any countable basis $\{u_n\}_{n=1}^\infty$ of X and $f \in L^\infty(\hat{\mathbb{C}}, \mu^L)$, the following holds:

$$\Phi(\hat{\pi}(f)) = \sum_{j=1}^{\infty} \pi(S_{u_j})\hat{\pi}(f)\pi(S_{u_j}^*).$$

Indeed, thanks to Lemma 3.6, the right-hand side converges in the strong operator topology. Since $S_k^*\pi(S_{u_j})$ commutes with $\hat{\pi}(A)''$, we have

$$\begin{aligned}\pi(S_{u_j})\hat{\pi}(f)(S_{u_j}^*) &= \sum_{k=1}^{N} S_k S_k^*\pi(S_{u_j})\hat{\pi}(f)\pi(S_{u_j}^*) = \sum_{k=1}^{N} S_k\hat{\pi}(f)S_k^*\pi(S_{u_j})\pi(S_{u_j}^*)\\ &= \Phi(\hat{\pi}(f))\pi(S_{u_j})\pi(S_{u_j}^*),\end{aligned}$$

which shows the claim.

Since $\{\theta_{\hat{\mathbb{C}}}(u_j)\}_{j=1}^\infty$ is a basis of X too, we get $\theta_M(\Phi(\hat{\pi}(f))) = \Phi(\theta_M(\hat{\pi}(f)))$ for every $f \in L^\infty(\hat{\mathbb{C}}, \mu^L) = \phi(L^\infty(K_N, \nu_N))$. Now the statement follows from Lemma 8.1, as the proof of Theorem 7.3 shows that $\pi(A)''$ coincides with B in Lemma 8.1. □

Example 8.3. We denote by G_R the group of Möbius transformations commuting with R. Then G_R is a finite subgroup of $PSL(2, \mathbb{C})$ (see [**2**, p. 104]) and G_R is either a cyclic group, a dihedral group or the group of symmetries of a regular tetrahedron, octahedron or icosahedron. It is known that there exist exactly three rational functions with $2 \le \deg R \le 30$ such that G_R is the group of symmetries of a regular icosahedron, which is isomorphic to A_5 (see [**12**] and [**44**]).

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Let $R(z) = z^n$ for $n \geq 2$ and let ω be the primitive $(n-1)$th root of unity. Then

$$G_R = \left\{ z, \omega z, \ldots, \omega^{n-2} z, \frac{1}{z}, \frac{\omega}{z}, \ldots, \frac{\omega^{n-2}}{z} \right\}.$$

Thus G_R is isomorphic to the dihedral group D_{n-1} for $n \geq 3$ and is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for $n = 2$. Therefore the G_R-action on $\mathcal{O}_R(\hat{\mathbb{C}})$ transitively acts on the set of extreme β-KMS states for $\beta > \log n$.

Example 8.4. If R has only real coefficients, then $\theta(z) = \overline{z}$ commutes with R.

Remark 8.5. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N)$ be a system of proper contractions on a compact metric space K satisfying the self-similarity condition. Then for a homeomorphism θ of K satisfying $\theta \cdot \gamma_j \cdot \theta^{-1} = \gamma_{p(j)}$, where p is a permutation of $\{1, 2, \ldots, N\}$, a result similar to Theorem 8.2 follows from Lemma 8.1.

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