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The entropy of a special overlapping dynamical system

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Abstract. The term special overlapping refers to a certain simple type of piecewise continuous function from the unit interval to itself and also to a simple type of iterated function system (IFS) on the unit interval. A correspondence between these two classes of objects is used (1) to find a necessary and sufficient condition for a fractal transformation from the attractor of one special overlapping IFS to the attractor of another special overlapping IFS to be a homeomorphism and (2) to find a formula for the topological entropy of the dynamical system associated with a special overlapping function.

1. *Introduction*

Iterated maps on an interval provide the simplest examples of dynamical systems. Parameterized families of geometrically simple continuous dynamical systems on an interval have a rich history because of their intricate behaviour, the insights they provide into higher dimensional systems, and diverse applications. Numerous papers have been written concerning their invariant measures, entropies, and behaviours; we note in particular the works of Collet and Eckmann [**5**], and Milnor and Thurston [**11**]. Many results in the literature concern the topological entropy of continuous systems [**1, 12**], but piecewise continuous maps have also received attention, [**6, 13, 19**].

One of the main results of this paper is a formula for the topological entropy of a dynamical system $([0, 1], T)$, where T is a piecewise continuous function from the interval $[0, 1]$ onto itself consisting of two continuous pieces, as shown on the left in Figure 1. More precisely, we are interested in functions $T : [0, 1] \to [0, 1]$ of the form

$$T(x) = \begin{cases} g_0(x) & \text{if } 0 \le x < q, \\ g_1(x) & \text{if } q \le x \le 1, \end{cases} \tag{1}$$

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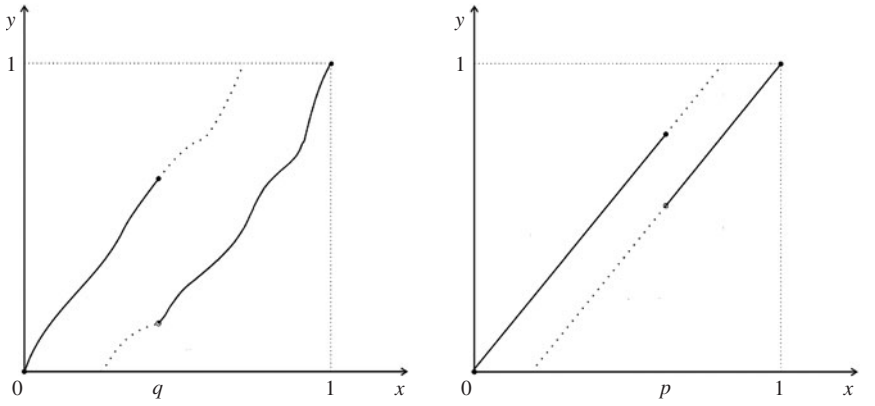


FIGURE 1. A special overlapping dynamical system (left) and the uniform case (right).

or

$$T(x) = \begin{cases} g_0(x) & \text{if } 0 \le x \le q, \\ g_1(x) & \text{if } q < x \le 1, \end{cases} \tag{2}$$

where $g_0 : [0, 1] \to [0, 1]$ and $g_1 : [0, 1] \to [0, 1]$ are continuous increasing functions such that:

(1) $g_0(0) = 0,\ g_1(1) = 1$;
(2) $0 < g_1(q) < g_0(q) < 1$ for some $q \in (0, 1)$; and
(3) there is an $s > 1$ such that $|g_i(x) - g_i(y)| \ge s|x - y|$ for $i = 0, 1$ and for all $x \in [0, 1]$.

Note that condition (3) holds, for example, if g_0, g_1 are differentiable and there is an $s > 1$ such that $g_0'(x) \ge s$ and $g_1'(x) \ge s$ for all $x \in [0, 1]$. Call such a dynamical system a *special overlapping* dynamical system. It is 'overlapping' in the sense that $g_0([0, q)) \cap g_1((q, 1]) \ne \emptyset$ and in the sense that, in the associated iterated function system (IFS) as described in §2, the images of the unit interval under the two functions of the IFS are overlapping.

Although many important IFS properties, like the Hausdorff dimension of the attractor, can be obtained under the assumption of the open set condition (OSC), there is a developing literature on overlapping IFSs. The papers [**15, 20**] contain results on the absolute continuity of the invariant measure of certain IFSs with probability weights; in particular this solved a conjecture of Erdös on Bernoulli convolutions. The papers [**14, 16**] further explore this direction. The papers [**8, 9, 17**] concern digit expansions (β-expansions) related to overlapping IFSs on the reals. Interesting results can be obtained for overlapping IFSs that satisfy a weak separation condition of Lau and Ngai [**10, 21**] (weaker than the OSC); IFSs that satisfy the weak separation condition include those related to Bernoulli convolutions associated with PV numbers and also include contractive IFSs consisting of inverses of integer matrices. The latter example applies to properties of wavelets and to properties of self-replicating tilings.

Associated with the dynamical system T with point of discontinuity q, there are two special itineraries, called *critical itineraries* $\alpha := \alpha_0, \alpha_1, \alpha_2 \ldots$ and $\beta := \beta_0, \beta_1, \beta_2, \ldots,$

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where $\alpha_n, \beta_n \in \{0, 1\}$ for all $n \geq 0$ (see Definition 12). Our main theorem (Theorem 1.1 given below) states that the topological entropy of T is $-\ln r(q)$ where $r(q)$ is the smallest solution $x \in (0, 1)$ to the equation

$$\sum_{n=0}^{\infty} \alpha_n x^n = \sum_{n=0}^{\infty} \beta_n x^n. \tag{3}$$

The proof of this theorem relies on finding a 'uniform' dynamical system that is topologically conjugate to the dynamical system $([0, 1], T)$. By uniform we mean a function $U := U_{r,p}$ of the form shown on the right in Figure 1, where the two branches are lines of equal slope r. For such a dynamical system it is well-known that the entropy is $\ln r$. That there exists such a topologically conjugate uniform dynamical system is also proved in [**13**], but only under an additional transitivity assumption. Additionally, what was not known prior to this work, is the explicit relationship between T, on the left in Figure 1, and the parameters p and r that uniquely determine U, on the right in Figure 1. In this paper we construct such a topologically conjugate U by determining the parameters p and r in terms of just the two critical itineraries α and β of T. Specifically, our main theorem is the following.

THEOREM 1.1. *Let $([0, 1], T)$ be any special overlapping dynamical system with point of discontinuity q, critical itineraries α and β, and $r(q)$ as defined in equation (**??**).*

(1) *The dynamical system $([0, 1], T)$ is topologically conjugate to the uniform dynamical system $([0, 1), U_{r,p})$, where $r = r(q)$ and $p = (1 - r) \sum_{n=0}^{\infty} \alpha_n r^n$.*
(2) *The entropy of dynamical system $([0, 1], T)$ is $-\ln r(q)$.*

Our approach is constructive in character. We make use of an analogue of the kneading determinant of [**11**], appropriate for discontinuous interval maps, and thereby avoid a measure-theoretic existential proof such as those in [**6, 13**].

Our proof of Theorem 1.1 depends on a correspondence between the dynamics of a single function, on the one hand, and IFSs on the other. See [**18**] and the references therein for similar connections between the dynamics related to β-expansions and Bernoulli convolutions and properties of self-similar sets obtained from an IFS. The correspondence in this paper is such that two dynamical systems are topologically conjugate if and only if the attractors of the two corresponding IFSs are related by a fractal homeomorphism. Indeed, one motivation for undertaking this research was our desire to establish, and to be able to compute, fractal homeomorphisms between attractors of IFSs—for applications such as those in [**2**].

An IFS is a standard method for constructing a self-referential fractal, the attractor of the IFS usually being a fractal. Given two IFSs with the same number of functions, a method for transforming the attractor of one to the attractor of the other has been laid out in [**3**]. Figure 2 shows such a fractal transformation. Even if the attractors themselves are mundane, the fractal transformations between them may be interesting. In Figure 3, for example, the attractors are simply the unit square $\square$. To visualize the fractal transformation we can observe its effect on a 'picture'. By *picture* we mean a function $c : \square \to \mathcal{C}$, where $\mathcal{C}$ denotes the color palate, for example $\mathcal{C} = \{0, 1, 2, \ldots, 255\}^3$. A fractal transformation $h : \square \to \square$ induces a map from a picture on one attractor to a picture on the other attractor

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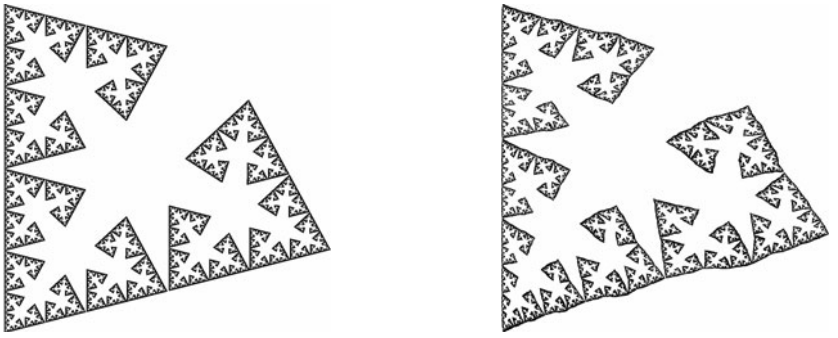


FIGURE 2. The attractor of an IFS (left) and its image under a fractal homeomorphism (right).

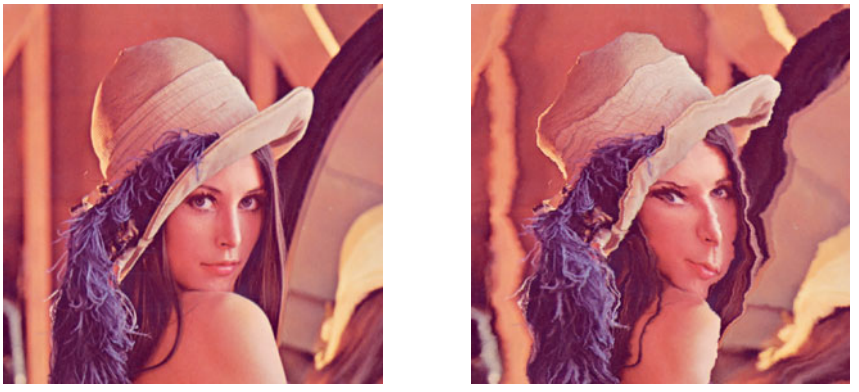


FIGURE 3. A fractal homeomorphism applied to the original picture.

given by

$$h(c) := c \circ h.$$

The question of when a fractal transformation is a homeomorphism is difficult, an answer previously known only for a few special situations. For example, the fractal transformation depicted in Figure 3 is a homeomorphism because it is a case of the particular type of IFS studied in [**4**]. In this paper a complete solution to when a fractal transformation is a homeomorphism is provided by Theorem 5.2 for the case of a *special overlapping* IFS on the unit interval, i.e. for an IFS $([0, 1];\ f_0,\ f_1)$ consisting of two contractions f_0, f_1 defined on the unit interval $[0, 1]$ such that $[0, 1] = f_0([0, 1]) \cup f_1([0, 1])$ and $f_0([0, 1]) \cap f_1([0, 1]) \neq \emptyset$. One necessary and sufficient condition proved as part of Theorem 5.2 is as follows: a fractal transformation is a homeomorphism if and only if the critical itineraries α and β associated with one IFS equal the critical itineraries associated with the other. That the two-dimensional fractal transformation depicted in Figure 2 is a homeomorphism follows from our one-dimensional result because the two-dimensional IFS is the product of two one-dimensional IFSs of the type considered in this paper.

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The organization of the paper is as follows. Basic definitions and facts about IFSs and their attractors are reviewed in §2. The dynamical system associated with an IFS is also defined in that section. Fractal transformations and how they are constructed using masks (Theorems 2.1 and 3.1) are the subjects of §3. The particular type of IFS that is central to this paper, a special overlapping IFS, is defined in §4. A uniform IFS, a particular case of a special overlapping IFS, is also discussed in that section. Each point of the attractor of an IFS can be assigned an address. The address space of the attractor of a special overlapping IFS is the topic of §5. The two critical itineraries are defined in this section, and two important results are stated. Theorem 5.1 characterizes the address space of a special overlapping IFS in terms of the critical itineraries. Theorem 5.2 states that the following four conditions are equivalent: (1) the address spaces of two special overlapping IFSs are equal; (2) the corresponding critical itineraries are equal; (3) the two IFSs are related by a fractal homeomorphism; and (4) the two associated dynamical systems are topologically conjugate. Theorems 5.1 and 5.2 lead to the main result on topological entropy, Theorem 6.1, stated and proved, with the aid of several lemmas, in §6.

2. *An IFS and its associated dynamical system*

Basic results on IFSs and their associated dynamical systems are contained in this section. We begin in the setting of a complete metric space and specialize to the unit interval on the real line in §4.

Let $\mathbb{X}$ be a complete metric space. If $f_m : \mathbb{X} \to \mathbb{X}$, $m = 1, 2, \ldots, M$, are continuous maps, then $\mathcal{F} = (\mathbb{X};\ f_1,\ f_2, \ldots,\ f_M)$ is called an *iterated function system* (IFS). To define the attractor of an IFS, first define

$$\mathcal{F}(B) = \bigcup_{f \in \mathcal{F}} f(B)$$

for any $B \subset \mathbb{X}$. By slight abuse of terminology we use the same symbol $\mathcal{F}$ for the IFS, the set of functions in the IFS, and for the above map. For $B \subset \mathbb{X}$, let $\mathcal{F}^k(B)$ denote the k-fold composition of $\mathcal{F}$, the union of $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}(B)$ over all finite words $i_1 i_2 \cdots i_k$ of length k. Define $\mathcal{F}^0(B) = B$. A non-empty compact set $A \subset \mathbb{X}$ is said to be an *attractor* of the IFS $\mathcal{F}$ if:

(1) $\mathcal{F}(A) = A$; and
(2) $\lim_{k\to\infty} \mathcal{F}^k(B) = A$, for all compact sets $B \subset \mathbb{X}$, where the limit is with respect to the Hausdorff metric.

A function $f : \mathbb{X} \to \mathbb{X}$ is called a *contraction* with respect to a metric d if there is a $0 \le s < 1$ such that $d(f(x),\ f(y)) \le s\, d(x,\ y)$ for all $x,\ y \in \mathbb{R}^n$. An IFS with the property that each function is a contraction will be called a *contractive* IFS. In his seminal paper Hutchinson [**7**] proved that a contractive IFS on a complete metric space has a unique attractor.

For a contractive IFS, it is possible to assign to each point of the attractor an 'address' as follows. Let $\Omega = \{1, 2, \ldots, N\}^\infty$ denote the set of infinite strings using symbols $1, 2, \ldots, N$. For a string $\omega \in \Omega$, denote the nth element, $n \ge 0$, in the string by ω_n, and denote by $\omega|_n$ the string consisting of the first $n + 1$ symbols in ω, i.e., $\omega|_n = \omega_0\omega_1 \cdots \omega_n$.

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Moreover, we use the notation

$$f_{\omega|_n} := f_{\omega_0} \circ f_{\omega_1} \circ \cdots \circ f_{\omega_n}.$$

The set Ω can be given the product topology induced from the discrete topology on $\{1, 2, \ldots, N\}$. The product topology is the same as the topology induced by the metric $d(\omega, \sigma) = 2^{-k}$ where k is the least index such that $\omega_k \neq \sigma_k$. The space (Ω, d) is a compact metric space.

Definition 1. Let $\mathcal{F} = (\mathbb{X}; f_1, f_2, \ldots, f_N)$ be a contractive IFS on a complete metric space $\mathbb{X}$ with attractor A. The map $\pi : \Omega \to A$ defined by

$$\pi(\sigma) := \lim_{k\to\infty} f_{\sigma|_k}(x)$$

is called the *coding map* of $\mathcal{F}$.

For a contractive IFS it is well-known [**7**] that the limit exists and is independent of $x \in \mathbb{X}$. Moreover π is continuous, onto, and satisfies the following commuting diagram for each $n = 1, 2, \ldots, N$.

$$\begin{array}{ccc} \Omega & \xrightarrow{S_n} & \Omega \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{X} & \xrightarrow[f_n]{} & \mathbb{X} \end{array}$$

The symbol $s_n : \Omega \to \Omega$ denotes the inverse shift map defined by $s_n(\sigma) = n\sigma$.

Definition 2. A *section* of the coding map π is a function $\tau : A \to \Omega$ such that $\pi \circ \tau$ is the identity. For $x \in A$, the string $\tau(x)$ is referred to as the *address* of x. Call the set $\Omega_\tau := \tau(A)$ the *address space* of the section τ.

Definition 3. Let S denote the shift operator on Ω, i.e., $S(n\sigma) = \sigma$ for any $n \in \{1, 2, \ldots, N\}$ and any $\sigma \in \Omega$. A subset $W \subseteq \Omega$ will be called *shift invariant* if $S(W) \subseteq W$. If Ω_τ is shift invariant, then τ is called a *shift invariant section*.

The following example demonstrates the naturalness of shift invariance.

Example 1. Consider the IFS $\mathcal{F} = (\mathbb{R}\,;\ f_0, f_1)$ where $f_0(x) = \frac{1}{2}\,x$ and $f_1(x) = \frac{1}{2}\,x + \frac{1}{2}$. The attractor is the interval $[0, 1]$. The coding map is

$$\pi(\omega_0\omega_1\omega_2\cdots) = \frac{1}{2}\sum_{k=0}^{\infty}\omega_k\left(\frac{1}{2}\right)^k = \sum_{k=0}^{\infty}\omega_k\left(\frac{1}{2}\right)^{k+1}.$$

A section τ, evaluated at a point x, is therefore a binary representation of x. If the section is shift invariant, then $\tau(x)$ is, for all x, either a binary representation that does not end with $111\cdots$ or, for all x, a binary representation that does not end with $000\cdots$. For example, if $\tau(\frac{1}{4}) = .00111\cdots$, then $\tau(\frac{1}{2}) = .0111\cdots$, not $\tau(\frac{1}{2}) = .100\cdots$.

Call an IFS *injective* if each function in the IFS is injective. Theorem 2.1 below, which is proved in [**4**], states that every shift invariant section of an injective IFS can be obtained from a mask.

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Definition 4. For an IFS $\mathcal{F}$ with attractor A, a *mask* is a partition $M = \{M_i : 1 \le i \le N\}$ of A such that $M_i \subseteq f_i(A)$ for all $f_i \in \mathcal{F}$.

Definition 5. Given an injective IFS $\mathcal{F}$ with a mask $M = \{M_i : 1 \le i \le N\}$, the *section* τ_M *associated with mask* M is the function $\tau_M : A \to \Omega$ defined as follows. Let Ω_k denote the set of all finite strings of length k in the symbols $\{1, 2, \ldots, N\}$. For each $k \ge 0$ define a partition $M^k = \{M_\sigma : \sigma \in \Omega_k\}$ of A recursively by taking $M^1 = M$ and

$$M^{k+1} = \{M_{\sigma j} = M_\sigma \cap f_\sigma(M_j) : \sigma \in \{1, 2, \ldots, N\}^k, 1 \le j \le N\}.$$

A straightforward induction shows that M^k is indeed a partition of A for every $k \ge 0$, and that each such partition is a refinement of the previous partition, in particular $M_{\sigma j} \subseteq M_\sigma$ for all finite σ and all $j \in \{1, 2, \ldots, N\}$. (Note that, for some values of σ, the sets M_σ may be empty.) Moreover, since the functions in the IFS are contractions, the maximum diameter of the sets in M^k approaches 0 as $k \to \infty$. Since each $x \in \mathbb{X}$ lies in a unique nested sequence

$$M_{i_0} \supseteq M_{i_0 i_1} \supseteq M_{i_0 i_1 i_2} \supseteq \cdots,$$

we can define $\tau_M(x) = i_0 i_1 i_2 \cdots$. Note that this definition of τ_M is equivalent to saying that

$$x \in f_{i_0} \circ f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_{k-1}}(M_{i_k})$$

for all $k \ge 0$. That τ_M is indeed a section is part of Theorem 2.1 below, whose proof appears in [**4**].

LEMMA 2.1. *With notation as above, for any injective* IFS *and for any finite string* σ *and symbol* j*, we have* $M_{j\sigma} = M_j \cap f_j(M_\sigma)$.

Proof. The result will be proved by induction on the length of σ. Concerning length 1, it is easy to check, from the definition of the partition, that $M_{ji} = M_j \cap f_j(M_i)$. Now

$$\begin{aligned} M_{j\sigma i} &= M_{j\sigma} \cap f_{j\sigma}(M_i) = M_j \cap f_j(M_\sigma) \cap f_{j\sigma}(M_i) \\ &= M_j \cap f_j(M_\sigma) \cap f_j(f_\sigma(M_i)) = M_j \cap f_j(M_\sigma \cap f_\sigma(M_i)) = M_j \cap f_j(M_{\sigma i}), \end{aligned}$$

the second to last equality using that f_j is injective. □

THEOREM 2.1. *Let* $\mathcal{F}$ *be a contractive and injective* IFS.

(1) *If* M *is a mask, then* τ_M *is a shift invariant section of* π.

(2) *If* τ *is a shift invariant section of* π*, then* $\tau = \tau_M$ *for some mask* M.

Definition 6. Let $\mathcal{F}$ be an injective IFS with attractor A. Given a mask M for $\mathcal{F}$, define a function $T_{(\mathcal{F},M)} : A \to A$ by

$$T_{(\mathcal{F},M)}(x) := f_i^{-1}(x) \quad \text{when } x \in M_i.$$

The pair $(A, T_{(\mathcal{F},M)})$ will be called the *dynamical system associated with* $\mathcal{F}$ *and* M. The *itinerary* of a point $x \in A$ is the string $i_0 i_1 i_2 \cdots \in \Omega$, where i_k is the unique integer, $1 \le i_k \le N$, such that

$$T^k_{(\mathcal{F},M)}(x) \in M_{i_k}.$$

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PROPOSITION 2.1. *If $\mathcal{F}$ is an injective masked* IFS *with associated dynamical system $(A, T_{(\mathcal{F},M)})$, then, for all $x \in A$, the itinerary of x is its address $\tau_M(x)$.*

Proof. By its definition, $i_0\, i_1 \ldots$ is the itinerary of x if and only if $f_{i_{k-1}}^{-1} \circ \cdots \circ f_{i_1}^{-1} \circ f_{i_0}^{-1}(x) \in M_{i_k}$ for all $k \geq 0$. But this is equivalent to $x \in f_{i_0} \circ f_{i_1} \circ \cdots \circ f_{i_{k-1}}(M_{i_k})$ for all $k \geq 0$, which, as noted above, defines the sections. □

3. *Fractal transformation*

Consider two contractive IFSs $\mathcal{F} = (\mathbb{X};\, f_1,\, f_2, \ldots,\, f_N)$ and $\mathcal{G} = (\mathbb{Y};\, g_1, g_2, \ldots, g_N)$ with the same number N of functions on complete metric spaces $\mathbb{X}$ and $\mathbb{Y}$. Basically a fractal transformation from $\mathcal{F}$ to $\mathcal{G}$ is a map $h : A_F \to A_G$ that sends a point in the attractor A_F of $\mathcal{F}$ to the point in the attractor A_G of $\mathcal{G}$ with the same address. More specifically we have the following definition.

Definition 7. Let A_F and A_G be the attractors and π_F and π_G the coding maps of contractive IFSs $\mathcal{F}$ and $\mathcal{G}$, respectively. A map $h : A_F \to A_G$ is called a *fractal transformation* if there exist shift invariant sections τ_F and τ_G such that the following diagram commutes:

$$\begin{array}{ccc} A_F & \xrightarrow{\quad h \quad} & A_G \\ & {\scriptstyle \tau_F} \searrow \quad \swarrow {\scriptstyle \tau_G} & \\ & \Omega & \end{array} \tag{4}$$

i.e., the transformation h takes each point $x \in A_F$ with address $\sigma = \tau_F(x)$ to the point $y \in A_G$ with the same address $\sigma = \tau_G(y)$. A fractal transformation that is a homeomorphism is called a *fractal homeomorphism*.

Theorem 3.1, proved in [**4**], states that the fractal transformations between A_F and A_G are exactly maps of the form $\pi_G \circ \tau_F$ or $\pi_F \circ \tau_G$ for some shift invariant sections $\tau_F,\ \tau_G$.

THEOREM 3.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be contractive* IFS*s. With notation as above:*

(1) *if $h : A_F \to A_G$ is a fractal homeomorphism with respect to sections τ_F and τ_G, then $h = \pi_G \circ \tau_F$ and $h^{-1} = \pi_F \circ \tau_G$;*
(2) *if τ_F is a shift invariant section for $\mathcal{F}$ and $h := \pi_G \circ \tau_F$ is a homeomorphism, then h is a fractal homeomorphism.*

4. *Overlapping IFS*

The type of IFS that is the subject of this paper is what will be called a special overlapping IFS on the unit interval of the real line.

Definition 8. A *special overlapping* IFS is an IFS

$$\mathcal{F} = ([0, 1];\, f_0,\, f_1),$$

where the functions are continuous, increasing, contractions that satisfy

$$f_0(0) = 0, \quad f_1(1) = 1, \quad 0 < f_1(0) < f_0(1) < 1.$$

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The attractor of a special overlapping IFS is the unit interval [0, 1]. If it were the case that $f_1(0) = f_0(1)$, then the two sets $f_0([0, 1])$ and $f_1([0, 1])$ would be 'just touching'; since $f_1(0) < f_0(1)$, they are 'overlapping'.

Next we fix some notation used in the remainder of the paper. The coding map for $\mathcal{F}$ will be denoted by $\pi := \pi_{\mathcal{F}}$. We consider masks for a special overlapping IFS $\mathcal{F}$ of the form

$$M_q^+ = \{[0, q), [q, 1]\} \text{ or } M_q^- = \{[0, q], (q, 1]\} \quad \text{where } f_1(0) < q < f_0(1).$$

Definition 9. The point q will be called the *mask point*.

For a masked special overlapping IFS let τ_q^+ and τ_q^- denote the sections corresponding to M_q^+ and M_q^-, respectively. The two respective address spaces are denoted by

$$\Omega_q^- = \tau_q^-([0, 1]) \quad \text{and} \quad \Omega_q^+ = \tau_q^+([0, 1]).$$

For a masked special overlapping IFS, the associated dynamical systems, as defined in the previous section, are $([0, 1], T_q^+)$ and $([0, 1], T_q^-)$, where

$$T_q^+(x) = \begin{cases} f_0^{-1} & \text{if } x < q, \\ f_1^{-1} & \text{if } x \geq q, \end{cases} \quad \text{and} \quad T_q^-(x) = \begin{cases} f_0^{-1} & \text{if } x \leq q, \\ f_1^{-1} & \text{if } x > q. \end{cases}$$

Since f_0 and f_1 are contractions, the inverses $g_0 = f_0^{-1}$ and $g_1 = f_1^{-1}$ satisfy the expanding condition (3) in the introduction. Also since $f_1(0) < q < f_0(1)$, we have $0 < g_1(q) < g_0(q) < 1$. Therefore the dynamical system associated with a special overlapping IFS is a special overlapping dynamical system as defined and discussed in the introduction. We will refer to such a dynamical system as a *special overlapping dynamical system*. So there is a bijection between special overlapping dynamical systems and masked special overlapping IFSs.

For our purposes, the following can serve as a definition of the entropy of a dynamical system. Note that $|\Omega_{q,n}^+| = |\Omega_{q,n}^-|$, so this definition is consistent with the one in [**13**].

Definition 10. The *topological entropy* $h(T_q^{\pm})$ of a special overlapping dynamical system $([0, 1], T_q^{\pm})$ is

$$h(T_q) = \lim_{n\to\infty} \frac{1}{n} \log |\Omega_{q,n}^+| = \lim_{n\to\infty} \frac{1}{n} \log |\Omega_{q,n}^-|,$$

where $\Omega_{q,n}^{\pm} := \{\omega|_n : \omega \in \Omega_q^{\pm}\}$.

The following particular case of a special overlapping IFS plays an important role.

Definition 11. The IFS $\mathcal{U}_a = ([0, 1]; L_0, L_1)$, where

$$\begin{aligned} L_0(x) &= ax, \\ L_1(x) &= ax + 1 - a, \end{aligned} \tag{5}$$

will be called a *uniform* IFS. The graphs of the two functions L_0 and L_1 are parallel lines. When $\frac{1}{2} < a < 1$ the IFS $\mathcal{U}_a$ is special overlapping. For the IFS $\mathcal{U}_a$ the coding map will be denoted by π_a. For a uniform IFS with mask point p, the sections will be denoted by $\mu_{(a,p)}^+$ and $\mu_{(a,p)}^-$, and the associated dynamical systems by $([0, 1], U_{(a,p)}^+)$ and $([0, 1], U_{(a,p)}^-)$, where $a < p < 1 - a$.

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The following result concerning the uniform case follows readily from Parry [**13**].

THEOREM 4.1. *The topological entropy of the uniform dynamical systems* $([0, 1], U^{\pm}_{(a,q)})$ *is equal to* $-\ln(a)$.

LEMMA 4.1. *Let* $a \in (0, 1)$ *and* $\omega \in \{0, 1\}^{\infty}$. *For the* IFS $\mathcal{U}_a$ *we have*

$$\pi_a(\omega) = (1 - a) \sum_{k=0}^{\infty} \omega_k \, a^k.$$

In particular, $\pi_a(\omega)$ *is a continuous function of* a *in the interval* $[0, 1)$.

Proof. For the IFS $\mathcal{U}_a$ we have $L_i(x) = ax + i(1 - a)$ for $i = 0, 1$. Iterating

$$L_{\omega_0} \circ L_{\omega_1} \circ L_{\omega_2} \circ \cdots \circ L_{\omega_k}(x) = a^k x + (a^{k-1}\omega_{k-1} + \cdots + a\omega_1 + \omega_0)(1 - a).$$

Therefore

$$\pi_a(\omega) = \lim_{k\to\infty} L_{\omega|_k}(x) = (1 - a) \sum_{k=0}^{\infty} \omega_k \, a^k$$

for any $x \in [0, 1]$. Clearly the series converges for $0 \le a < 1$, and it is continuous inside the radius of convergence. □

5. *The address space*

The lexicographic order $\preceq$ on $\{0, 1\}^{\infty}$ is the total order defined by $\sigma \prec \omega$ if $\sigma \neq \omega$ and $\sigma_k < \omega_k$, where k is the least index such that $\sigma_k \neq \omega_k$. For $\sigma, \omega \in \{0, 1\}^{\infty}$ with $\sigma \preceq \omega$, define the interval

$$[\sigma, \omega] := \{\zeta \in \{0, 1\}^{\infty} : \sigma \preceq \zeta \preceq \omega\},$$

and similarly for (σ, ω), $(\sigma, \omega]$, and $[\sigma, \omega)$. We use the notation $\overline{0} = 000\cdots$ and $\overline{1} = 111\cdots$. Greek letters, other than coding map π and section τ, will denote strings; lower case Roman letters will denote real numbers. Two itineraries play a special role.

Definition 12. For a special overlapping masked IFS $\mathcal{F}$, the itineraries

$$\alpha_q := \tau_q^-(q) \quad \text{and} \quad \beta_q := \tau_q^+(q)$$

will be called the *critical itineraries*.

THEOREM 5.1. *For a special overlapping masked* IFS $\mathcal{F}$ *with mask point* q, *let* $\overline{\Omega}_q = \Omega_q^+ \cup \Omega_q^-$. *Then:*

(1) *if* $x, y \in [0, 1]$ *and* $x > y$, *then* $(\tau_q^-)(x) \succ (\tau_q^+)(y)$;
(2) *the sections* $\tau_q^+ : [0, 1] \to \Omega^+$ *and* $\tau_q^- : [0, 1] \to \Omega^-$ *are strictly increasing functions;*
(3) $\Omega_q^- = \{\omega \in \{0, 1\}^{\infty} : S^n(\omega) \in [\overline{0}, \alpha_q] \cup (\beta_q, \overline{1}]$ *for all* $n \ge 0\}$;
(4) $\Omega_q^+ = \{\omega \in \{0, 1\}^{\infty} : S^n(\omega) \in [\overline{0}, \alpha_q) \cup [\beta_q, \overline{1}]$ *for all* $n \ge 0\}$;
(5) $\overline{\Omega}_q = \{\omega \in \{0, 1\}^{\infty} : S^n(\omega) \in [\overline{0}, \alpha_q] \cup [\beta_q, \overline{1}]$ *for all* $n \ge 0\}$;
(6) $\overline{\Omega}_q$ *is the closure of* Ω_q^+ *and the closure of* Ω_q^- *in the metric space* $\{0, 1\}^{\infty}$.

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Proof. Since the mask is fixed, we suppress the index q throughout the proof. Also, when the superscript $+$ or $-$ is omitted, we mean either one.

Concerning statement 1, if $x > y$, then $(T^-)(x) > (T^+)(y)$ as long as $x, y \le q$ or $x, y \ge q$. Hence $x > y$ implies that $\tau^-(x) \succeq \tau^+(y)$. If $\tau^-(x) = \tau^+(y)$, then $x = \pi(\tau^-(x)) = \pi(\tau^+(y)) = y$, a contradiction.

Statement 2 follows directly from statement 1 since $\tau^+(x) \succeq \tau^-(x)$ for all $x \in [0, 1]$.

We next prove statement 3; the proof of statement 4 is omitted since it is done in essentially the same way. To show that Ω_q^- is contained in $\{\omega \in \{0, 1\}^\infty : S^n(\omega) \in [\overline{0}, \alpha] \cup (\beta, \overline{1}] \text{ for all } n \ge 0\}$, assume that $\omega \in \Omega^-$, and hence that $\omega = \tau^-(x)$ for some x. If ω begins with a 0, then $x \le q$, which by the monotonicity of τ^- implies that $\omega = \tau^-(x) \preceq \tau^-(q) = \alpha$. If ω begins with 1, then $x > q$, which implies, using statement 1, that $\omega = \tau^-(x) \succ \tau^+(q) = \beta$. By shift invariance of Ω^-, the shift $S\omega \in \Omega^-$ and the same argument shows that $S\omega$ lies in the set $\{\omega \in \{0, 1\}^\infty : S^n(\omega) \in [\overline{0}, \alpha] \cup (\beta, \overline{1}] \text{ for all } n \ge 0\}$.

To prove containment in the other direction in statement 3, assume that $\omega = \omega_0\omega_1\omega_2 \cdots \in \{\omega \in \{0, 1\}^\infty : S^n(\omega) \in [\overline{0}, \alpha] \cup (\beta, \overline{1}] \text{ for all } n \ge 0\}$. By definition $\omega \in \Omega^-$ if ω lies in the image of $[0, 1]$ under the section map. By the definition of the section map, it is then sufficient to show that $M_{\omega|_k} \ne \emptyset$ for all k. We will show more, namely that $M_{(S^n\omega)|_k)} \ne \emptyset$ for all k and all n. This will be done by induction on k. The statement is obviously true for $k = 0$. Assuming it true for k, we will prove it for $k + 1$. Fix n and let $j\sigma = (S^n\omega)|_{k+1}$. There are two cases, $j = 0$ and $j = 1$. We will let $j = 0$; the proof for $j = 1$ is essentially the same. By Lemma 2.1 it is sufficient to show that $M_0 \cap f_0(M_\sigma) = M_{j\sigma} \ne \emptyset$. Equivalently it must be shown that there is an $x \in M_\sigma$ such that $f_0(x) \le q$. By the induction hypothesis $M_\sigma \ne \emptyset$. Since $0\sigma \preceq \alpha|_k$ and $\alpha_0 = 0$, also $\sigma \preceq \widehat{\alpha} := \alpha_1\alpha_2 \cdots \alpha_k$. Since $\alpha \in \Omega^-$, we know that $S^n\alpha \in \Omega^-$, and hence $M_{\widehat{\alpha}} \ne \emptyset$, which implies that there is a $y \in M_{0\widehat{\alpha}}$ such that $f_0(y) \le q$. But it follows easily from the definition of the partition M^k that if $\sigma \preceq \widehat{\alpha}$, then the interval M_σ precedes (or is equal to) the interval $M_{\widehat{\alpha}}$. Therefore there is an $x \in M_\sigma$ such that $x \le y$. Since $f_0(y) \le q$ and f_0 is an increasing function, we arrive at the required $f_0(x) \le q$.

To prove statement 5, let $\Gamma = \{\omega \in \{0, 1\}^\infty : S^n(\omega) \in [\overline{0}, \alpha_q] \cup [\beta_q, \overline{1}] \text{ for all } n \ge 0\}$. Clearly $\Omega^+ \subseteq \Gamma$ and $\Omega^- \subseteq \Gamma$. Conversely $\Gamma \subseteq \Omega^+ \cup \Omega^-$ unless there is a $\sigma \in \Gamma$ and integers m and n such that $S^n(\alpha) = \beta$ and $S^m(\beta) = \alpha$. Depending on whether $n > m$ or $m > n$, this implies that there is an integer k such that $S^k(\alpha) = \beta$ or $S^k(\beta) = \alpha$. To show, by contradiction, that neither of these equalities is possible, assume that $S^k(\alpha) = \beta$. Since $\alpha \in \Omega^-$ and Ω^- is shift invariant, also $\beta = S^k(\alpha) \in \Omega^-$. But this contradicts the characterization of Ω^- given in statement 4. The equality $S^k(\beta) = \alpha$ is likewise contradicted.

Statement 6 follows from statements 3–5. □

LEMMA 5.1. *For a masked special overlapping* IFS, *the section* $\tau_q^+ : [0, 1] \to \Omega_q^+$ *is continuous at all points except those in the set* $X^+ := \{x : S^n(\tau_q^+(x)) = \beta_q \text{ for some } n\}$, *and is continuous from the right everywhere. Moreover, if* $x \in X^+$ *and* n *is the least integer such that* $S^n(\tau_q^+(x)) = \beta_q$, *then*

$$\lim_{y \to x^-} \tau_q^+(y) = \tau_q^+(x)|_n \, \alpha.$$

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Likewise, the section $\tau_q^- : [0, 1] \to \Omega_q^-$ *is continuous at all points except those in the set* $X^- := \{x : S^n(\tau_q^-(x)) = \alpha_q \text{ for some } n\}$, *and is continuous from the left everywhere. Moreover, if* $x \in X^-$ *and* n *is the least integer such that* $S^n(\tau_q^+(x)) = \alpha_q$, *then*

$$\lim_{y \to x^+} \tau_q^-(y) = \tau_q^-(x)|_n \, \beta.$$

Proof. To simplify notation, the subscript q is omitted. Consider the section τ^+; the statement for τ^- is proved similarly. The continuity at points not in X^+ follows directly from the continuity of f_0 and f_1 and the fact that τ^+ can be viewed as an itinerary as described in Proposition 2.1, likewise for the continuity from the right for points in X^+. From the definition of the dynamical system associated with the IFS, it is easy to verify that the following diagram commutes.

$$\begin{array}{ccc} [0,1] & \xrightarrow{T^{\pm}} & [0,1] \\ {\scriptstyle \tau^{\pm}}\downarrow & & \downarrow{\scriptstyle \tau^{\pm}} \\ \Omega_F & \xrightarrow[S]{} & \Omega_F \end{array} \tag{6}$$

By the commuting diagram above, $\tau^+((T^+)^n(x)) = S^n(\tau^+(x)) = \beta$, which implies that $(T^+)^n(x) = q$. Since n is the first such integer and if y is sufficiently close to x, then $\tau^+(y)|_n = \tau^+(x)|_n$. If $y < x$, then $(T^+)^n(y) < (T^+)^n(x) = q$. Now

$$\begin{aligned} \lim_{y \to x^-} \tau^+(y) = \tau^+(x)|_n \lim_{y \to x^-} \tau^+((T^+)^n y) = \tau^+(x)|_n \lim_{y \to ((T^+)^n x)^-} \tau^+(y) \\ = \tau^+(x)|_n \lim_{y \to q^-} \tau^+(y) = \tau^+(x)|_n \lim_{y \to q^-} \tau^-(y) = \tau^+(x)|_n \, \alpha, \end{aligned}$$

the second to last equality because, for any m the first m entries in the itineraries of $\tau^-(y)$ and $\tau^+(y)$ are equal if y is sufficiently close to (and to the left of) x. □

Two dynamical systems $(\mathbb{X}, T)$ and $(\mathbb{Y}, S)$ are *topologically conjugate* if there exists a homeomorphism $\phi : \mathbb{X} \to \mathbb{Y}$ such that $T = \phi^{-1} \circ S \circ \phi$. Note that conditions 1 and 3 of Theorem 5.2 below alone provide a necessary and sufficient condition for the fractal transformation from one overlapping IFS to another to be a homeomorphism. The condition is simply that the critical itineraries of the associated dynamical systems be equal.

THEOREM 5.2. *Given two special overlapping masked* IFS*s* $\mathcal{F}$ *and* $\mathcal{G}$ *with respective mask points* q *and* p, *sections* $\tau_F^{\pm}$ *and* $\tau_G^{\pm}$, *dynamical systems* $T_F^{\pm}$ *and* $T_G^{\pm}$, *and address spaces* $\Omega_F^{\pm}$ *and* $\Omega_G^{\pm}$, *the following statements are equivalent.*

(1) *The fractal transformations* $\pi_G \circ \tau_F^{\pm}$ *and* $\pi_F \circ \tau_G^{\pm}$ *are homeomorphisms.*
(2) *The address spaces are equal:* $\Omega_F^+ = \Omega_G^+$ *and* $\Omega_F^- = \Omega_G^-$.
(3) *The critical itineraries are equal:* $\tau_F^+(q) = \tau_G^+(p)$ *and* $\tau_F^-(q) = \tau_G^-(p)$.
(4) *The dynamical systems* T_F^+ *and* T_G^+ *are topologically conjugate, as are* T_F^- *and* T_G^-.

Proof. To simplify notation we omit the superscript $\pm$. We will show that $1 \Rightarrow 4 \Rightarrow 3 \Leftrightarrow 2 \Rightarrow 1$.

$(1 \Rightarrow 4)$ Assume that $h := \pi_G \circ \tau_F$ is a homeomorphism. Since h is bijective, $\Omega_F = \Omega_G$. From the commuting diagram (6) above and the fact that $\pi_G = \tau_G^{-1}$ on $\Omega_G = \Omega_F$, we

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have another commutative diagram for G.

$$\begin{array}{ccc} \Omega_F & \xrightarrow{S} & \Omega_F \\ \downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi_G} \\ [0,1] & \xrightarrow[T_g]{} & [0,1] \end{array}$$

Combining the two commutative diagrams we arrive at $T_G \circ h = h \circ T_F$ or $T_G = h_{FG} T_F h^{-1}$.

$(4 \Rightarrow 3)$ Let topologically conjugate dynamical systems T_F and T_G be related by $T_G \circ h = h \circ T_F$, where h is a homeomorphism. If q is the mask point of $\mathcal{F}$ and p is the mask point of $\mathcal{G}$, we claim that $p = h(q)$. Otherwise, $h \circ T_F$ is discontinuous in some neighborhood of q while $T_G \circ h$ is continuous in some neighborhood of q, a contradiction. Now $T_F^n(q) \geq q$ if and only if $T_G^n(p) = T_G^n(h(q)) = h(T_F^n(q)) \geq h(q) = p$. This implies statement (3).

$(3 \Leftrightarrow 2)$ That $(3 \Rightarrow 2)$ follows directly from statements 3 and 4 of Theorem 5.1. The same statements imply that the largest element of Ω^- that starts with 0 is α, and the smallest element of Ω^+ that starts with 1 is β. Therefore $(2 \Rightarrow 3)$.

$(2 \Rightarrow 1)$ To simplify notation we omit the subscript q. Assuming statement 2, we will show that $\pi_G \circ \tau_F^+$ is a homeomorphism. Essentially the same proof shows that $\pi_G \circ \tau_F^-$ is a homeomorphism. Since $(2 \Rightarrow 3)$ we know that the critical itineraries α and β of $\mathcal{F}$ are equal to the respective critical itineraries of $\mathcal{G}$, and moreover, for mask point p,

$$\pi_G(\alpha) = (\pi_G \circ \tau_G^-)(p) = p = (\pi_G \circ \tau_G^+)(p) = \pi_G(\beta).$$

Since it follows immediately from Definition 7 that $\pi_G \circ \tau_F^+$ is a bijection, it suffices to show that it is continuous. Because π_G is continuous, Lemma 5.1 implies $\pi_G \circ \tau_F^+$ is continuous at all points except perhaps those in the set $X := \{x : S^n(\tau^+(x)) = \beta \text{ for some } n\}$. Let $x \in X$. Again by Lemma 5.1, it suffices to prove that $\pi_G \circ \tau_F^+$ is continuous from the left. But

$$\begin{aligned} \lim_{y \to x^-} \pi_G(\tau_F^+(y)) &= \pi_G\left(\lim_{y \to x^-} \tau_F^+(y)\right) = \pi_G(\tau_F^+(x)|_n \alpha) = f_{\tau_F^+(x)|_n}(\pi_G \alpha) \\ &= f_{\tau_F^+(x)|_n}(\pi_G \beta) = \pi_G(\tau_F^+(x)|_n \beta) = \pi_G(\tau_F^+(x)). \end{aligned}$$ □

6. *Entropy of a special overlapping dynamical system*

Throughout this section, $\mathcal{F}$ is a special overlapping IFS with mask point q, critical itineraries α and β, and $\mathcal{U}_a$ is a uniform IFS with coding map π_a. In Lemmas 6.1 and 6.2 we assume that there exists an $a \in (0, 1)$ such that $\pi_a(\alpha) = \pi_a(\beta)$. In this case let

$$r(q) := \min\{a \in (0, 1) : \pi_a(\alpha) = \pi_a(\beta)\}. \tag{7}$$

According to Lemma 4.1

$$r(q) = \min\left\{x \in (0, 1) : \sum_{n=0}^{\infty} \alpha_n x^n = \sum_{n=0}^{\infty} \beta_n x^n\right\}. \tag{8}$$

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Note that the function $\sum_{n=0}^{\infty}(\beta_n - \alpha_n)z^n$ is analytic inside the unit disk in the complex plane, and hence can have at most finitely many zeros within any closed disk of radius less than 1. In particular,

$$\pi_r(\alpha) = \pi_r(\beta).$$

LEMMA 6.1. (1) *Assume that there exists an* $a \in (0, 1)$ *such that* $\pi_a(\alpha) = \pi_a(\beta)$ *and let* $r = r(q)$. *The map* $\pi_a : \overline{\Omega}_q \to [0, 1]$ *is increasing for* $0 < a \leq r$ *and strictly increasing for* $0 < a < r$.

(2) *If there is no* $a \in (0, 1)$ *such that* $\pi_a(\alpha) = \pi_a(\beta)$, *then the map* $\pi_a : \overline{\Omega}_q \to [0, 1]$ *is strictly increasing for all* $a \in (0, 1)$.

Proof. Since it is fixed throughout the proof, the subscript q is omitted. Note that, for any IFS $\mathcal{U}_a$ with $a < \frac{1}{2}$, it is easy to check, either because the attractor is totally disconnected or directly from the power series, that if $\sigma \prec \omega$, then $\pi_a(\sigma) < \pi_a(\omega)$.

Concerning statement 1, let

$$s = \inf\{a \in (0, 1) : \pi_a(\sigma) = \pi_a(\omega) \text{ for some } \sigma, \omega \in \overline{\Omega},\ \sigma_0 \neq \omega_0\}. \tag{9}$$

Note that $s \leq r$ because $\alpha_0 = 0$, $\beta_0 = 1$ and $\alpha, \beta \in \overline{\Omega}$. Using the continuity of $\pi_a(\sigma)$ in a (see Lemma 4.1) and σ (see the comments following Definition 2), and the compactness of $\overline{\Omega}$, it follows that there exist $\sigma, \omega \in \overline{\Omega}$ such that $\pi_s(\sigma) = \pi_s(\omega)$. We claim that $r = s$. Assume, by way of contradiction, that $s < r$. If we assume, without loss of generality, that $\sigma_0 = 0$ and $\omega_0 = 1$, then

$$\pi_{\frac{1}{3}}(\sigma) \leq \pi_{\frac{1}{3}}(\alpha) < \pi_{\frac{1}{3}}(\beta) \preceq \pi_{\frac{1}{3}}(\omega) \tag{10}$$

because by Theorem 5.1 we have $\sigma \preceq \alpha$ and $\beta \preceq \omega$ and, as mentioned above, $\pi_{\frac{1}{3}}$ is order preserving. Consider $\pi_a(\sigma), \pi_a(\alpha), \pi_a(\beta), \pi_a(\omega)$ as functions of $a \in [1/3, r]$. (It is helpful to visualize the graphs of these four functions.) Since $s < r$, we have

$$\begin{aligned} &\pi_a(\alpha) < \pi_a(\beta) \quad \text{for } \tfrac{1}{3} \leq a \leq s, \\ &\pi_a(\sigma) < \pi_a(\omega) \quad \text{for } \tfrac{1}{3} \leq a < s, \\ &\pi_s(\sigma) = \pi_s(\omega). \end{aligned} \tag{11}$$

By the continuity of π_a with respect to a and the intermediate value theorem, the formulas (10) and (11) imply that either there is a $t \in (\frac{1}{3}, s)$ such that $\pi_t(\sigma) = \pi_t(\alpha)$ with $\sigma \neq \alpha$ or there is a $t \in (\frac{1}{3}, s)$ such that $\pi_t(\omega) = \pi_t(\beta)$ with $\omega \neq \beta$. Since the proof is essentially the same in either case, assume that $\pi_t(\sigma) = \pi_t(\alpha)$ with $\sigma \neq \alpha$. Since $t < s$, this would contradict the minimality of s (in equation (9)) if $\sigma_0 = 0$ and $\alpha_0 = 1$. This is not the case, however, because $\alpha_0 = 0$. In order to get the contradiction, we define two related strings σ' and α' such that $\pi_t(\sigma') = \pi_t(\alpha')$ and $\sigma'_0 = 0$ and $\alpha'_0 = 1$. To do this, let k be the least integer such that $(S^k\sigma)_0 \neq (S^k\alpha)_0$ and let $\sigma' = S^k\sigma$ and $\alpha' = S^k\alpha$, which forces $\sigma_0 \neq \omega'_0$. We are now done because $L_{\sigma|_k}(\pi_t(\sigma')) = \pi_t(\sigma) = \pi_t(\omega) = L_{\omega|_k}(\pi_t(\omega')) = L_{\sigma|_k}(\pi_t(\omega'))$ implies, because $L_{\sigma|_k}$ is invertible, that $\pi_t(\sigma') = \pi_t(\omega')$. The shift invariance of $\overline{\Omega}$ guarantees that $\sigma', \omega' \in \overline{\Omega}$. Therefore $s = r$.

To conclude the proof of statement 1 of the lemma, assume that $a \prec r$, $\sigma, \omega \in \overline{\Omega}$, and $\sigma \prec \omega$. If $\sigma_0 = 0$ and $\omega_0 = 1$, then $\pi_a(\sigma) \neq \pi_a(\omega)$ by what was proved in the paragraph

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above. Since $\pi_{\frac{1}{3}}(\sigma) < \pi_{\frac{1}{3}}(\omega)$, it would follow that $\pi_a(\sigma) < \pi_a(\omega)$; otherwise the crossing graphs would contradict $s = r$. Even if $\sigma_0 = \omega_0$, we claim that $\pi_a(\sigma) \neq \pi_a(\omega)$. Assume otherwise, that $\pi_a(\sigma) = \pi_a(\omega)$, then by letting σ' and ω' be shifts of σ and ω, respectively, exactly as was done in the paragraph above, we get $\pi_a(\sigma') = \pi_a(\omega')$ with $\sigma'_0 \neq \omega'_0$, which contradicts $s = r$.

In the case $a = r$ and $\sigma \prec \omega$, clearly $\pi_a(\sigma) > \pi_a(\omega)$ could contradict the continuity of π_a at $a = r$; therefore $\pi_a(\sigma) \leq \pi_a(\omega)$.

Lastly consider statement 2, i.e. the case $\pi_a(\alpha) \neq \pi_a(\beta)$ for all $a \in (0, 1)$. Essentially the same proof as above shows that $s = 1$ and consequently that if $\sigma \prec \omega$ then $\pi_a(\sigma) < \pi_a(\omega)$ for all $a < s = 1$. □

LEMMA 6.2. *Assume that there exists an $a \in (0, 1)$ such that $\pi_a(\alpha) = \pi_a(\beta)$ and let $r = r(q)$. For any integer $n > 0$, if $S^n(\beta) \prec \alpha$, then $\pi_r(S^n(\beta)) < \pi_r(\alpha)$. Similarly if $S^n(\alpha) \succ \beta$, then $\pi_r(S^n(\alpha)) > \pi_r(\beta)$.*

Proof. The following are readily verifiable facts about the partitions of [0, 1] that are part of Definition 5 of the sections associated with the masks M_q^+ and M_q^-. Denote the kth partitions by $(M^k)^+$ and $(M^k)^-$.

(1) The sets in partitions $(M^k)^+$ (except the last) and $(M^k)^-$ (except the first) are half open intervals of the form $[\cdot, \cdot)$ and $(\cdot, \cdot]$, respectively.
(2) The intervals in $(M^k)^+$ have the same endpoints as the intervals in $(M^k)^-$. Denote the set of open intervals by M^k.
(3) Given any interval I in M^k, the first k elements in the address (either + or − address) of any two points in I are equal.
(4) If (x, y) is an interval in M^k whose elements have address beginning with $\theta\theta_1\theta_2$, where θ has length $k - 2$ and $\theta_1, \theta_2 \in \{0, 1\}$, then the address $\tau^+(x)$ of x is

$$\begin{cases} \theta 0\beta & \text{if } \theta_1\theta_2 = 01, \\ \theta\beta & \text{if } \theta_1\theta_2 = 10, \end{cases}$$

and the address $\tau^-(y)$ of y is

$$\begin{cases} \theta\alpha & \text{if } \theta_1\theta_2 = 01, \\ \theta 1\alpha & \text{if } \theta_1\theta_2 = 10. \end{cases}$$

We will prove that $S^n(\beta) \prec \alpha$ implies $\pi_r(S^n(\beta)) < \pi_r(\alpha)$. That $S^n(\alpha) \succ \beta$ implies $\pi_r(S^n(\alpha)) > \pi_r(\beta)$ has essentially the same proof. Assume that $S^n(\beta) \prec \alpha$. There exists a k (sufficiently large) and three open intervals $I_1 = (x_1, y_1)$, $I_2 = (x_2, y_2)$, $I_3 = (x_3, y_3) \in M^k$ with the following properties:

(5) $y_1 \leq x_2 < y_2 \leq x_3$;
(6) $\tau(z)|_k = \alpha|_k$ for all $z \in I_3$;
(7) $y_3 = q$;
(8) $\tau(z)|_k = S^n(\beta)|_k$ for all $z \in I_1$;
(9) either the last two elements $\tau(z)|_k$ are 01 for all $z \in I_2$, or the last two elements $\tau(z)|_k$ are 10 for all $z \in I_2$; and
(10) $S^n(\beta) = \tau^+(z_0)$ for some $z_0 \in [x_1, y_1)$.

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The existence of the intermediate interval I_2 follows from the facts that the right endpoint y_3 of I_3 is fixed at q (statement 7) and that the lengths of the intervals of M^k tend to 0 as $k \to \infty$. If statement 9 were false, then there would exist a k, an interval $I \in M^k$, and a finite string θ such that $\tau(z) = \theta\overline{0}$ or $\tau(z) = \theta\overline{1}$ for all $z \in I$, which is impossible (again because the lengths of the intervals of M^k tend to 0 as $k \to \infty$).

By statement 2 of Theorem 5.1 and by property 4 above, if the last two elements of the finite I_2-address are 01, then

$$S^n(\beta) = \tau^+(z_0) \preceq \tau^+(x_2) = \theta 0\beta \quad \text{and} \quad \theta\alpha = \tau^-(y_2) \preceq \tau^-(y_3) = \alpha. \tag{12}$$

If the last two elements of the finite I_2-address are 10, then

$$S^n(\beta) = \tau^+(z_0) \preceq \tau^+(x_2) = \theta\beta \quad \text{and} \quad \theta 1\alpha = \tau^-(y) \preceq \tau^-(y_3) = \alpha. \tag{13}$$

Consider the first case above; the proof for the second case is essentially the same. From the inequalities above and by Lemma 6.1 (since $\theta 0\beta$ and $\theta\alpha$ lie in $\overline{\Omega}$), we have $\pi_r(S^n(\beta)) \le \pi_r(\theta 0\beta)$ and $\pi_r(\theta\alpha) \le \pi_r(\alpha)$. The proof is complete if $\pi_r(\theta 0\beta) < \pi_r(\theta\alpha)$. But using Lemma 4.1,

$$\begin{aligned}\pi_r(\theta\alpha) - \pi_r(\theta 0\beta) &= r^k(\pi_r(\alpha) - \pi_r(0\beta)) = r^k(\pi_r(\alpha) - r\pi_r(\beta)) \\ &= r^k(\pi_r(\alpha) - r\pi_r(\alpha)) = r^k(1-r)\pi_r(\alpha) > 0.\end{aligned}$$

□

LEMMA 6.3. *There exists an $a \in (0, 1)$ such that $\pi_a(\alpha) = \pi_a(\beta)$.*

Proof. Assume, by way of contradiction, that $\pi_a(\alpha) < \pi_a(\beta)$ for all $a \in (0, 1)$. Let a be arbitrary in the interval $(0, 1)$. By statement 2 of Lemma 6.1, the map π_a is strictly increasing on $\overline{\Omega}_q$. Let $p = \pi_a(\beta)$. For $\omega \in \Omega_q^+$ we claim that

$$U^+_{(a,p)}(\pi_a\omega) = \pi_a(S\omega),$$

where $U^+_{(a,p)}$ is the uniform dynamical system. This would imply that the address space Ω_q^+ is an invariant subset of the dynamical system $U^+_{(a,p)}$. This, in turn, would imply that the entropy of the special overlapping dynamical system T_q^+ is less than or equal to the entropy of the uniform dynamical system $U^+_{(a,p)}$, which, according to Theorem 4.1, equals $-\ln a$. Since this is true for all $a \in (0, 1)$, the entropy of T_q^+ must be 0, which is not possible for a dynamical system where the two continuous branches are expansive.

To prove the claim, let $U = U^+_{(a,p)}$. First note, from Lemma 6.1, that if $\pi_a(\omega) < p = \pi_a(\beta)$ then $\omega \prec \beta$, and hence $\omega_0 = 0$. Likewise, if $\pi_a(\omega) \ge p = \pi_a(\beta)$ then $\omega \succeq \beta$, and hence $\omega_0 = 1$. Therefore, if $\pi_a(\omega) < p$ then

$$\begin{aligned}U(\pi_a(\omega)) &= U\left((1-a)\sum_{n=0}^{\infty}\omega_n a^n\right) = (1-a)\left(\sum_{n=0}^{\infty}\omega_{n+1}a^n + \frac{\omega_0}{a}\right) \\ &= (1-a)\sum_{n=0}^{\infty}\omega_{n+1}a^n = \pi_a(S\omega),\end{aligned}$$

and if $\pi_a(\omega) \ge p$, then

$$\begin{aligned}U(\pi_a(\omega)) &= U\left((1-a)\sum_{n=0}^{\infty}\omega_n a^n\right) = (1-a)\left(\sum_{n=0}^{\infty}\omega_{n+1}a^n + \frac{\omega_0}{a} - \frac{1}{a}\right) \\ &= (1-a)\sum_{n=0}^{\infty}\omega_{n+1}a^n = \pi_a(S\omega).\end{aligned}$$

□

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LEMMA 6.4. *Let $r = r(q)$ and $p = \pi_r(\alpha) = \pi_r(\beta)$. If the uniform* IFS $\mathcal{U}_r$ *with coding map π_r has mask point p and sections $\mu^+_{(r,p)}$ and $\mu^-_{(r,p)}$, then $\mu^-_{r,p}(p) = \alpha$ and $\mu^+_{r,p}(p) = \beta$.*

Proof. We will prove that $\mu^+_{(r,p)}(p) = \beta$; the proof that $\mu^-_{(r,p)}(p) = \alpha$ is essentially the same. Let $U := U^+_{(r,p)}$ be the dynamical system associated with the uniform IFS and let $\omega := \mu^+_{(r,p)}(p)$. For all $n \geq 0$, we will prove the following by induction on n:

(1) $\omega_n = \beta_n$; and

(2) $U^n(p) = \pi_r(S^n\beta)$.

Since both β and ω begin with a 1, both statements are true for $n = 0$. Assuming the two statements true for $n-1$, we will prove that they are true for n.

Starting with statement 2:

$$U^n(p) = U(U^{n-1}(p)) = U(\pi_r(S^{n-1}\beta)) = U\left((1-r)\sum_{k=0}^{\infty}\beta_{n-1+k}r^k\right)$$

$$= (1-r)\sum_{k=0}^{\infty}\beta_{n+k}r^k = \pi_r(S^n\beta).$$

The second to last equality above comes from the following direct calculation: if $\omega_{n-1} = 0$, then by the induction hypothesis $\beta_{n-1} = 0$ and

$$U\left((1-r)\sum_{k=0}^{\infty}\beta_{n-1+k}r^k\right) = (1-r)\sum_{k=0}^{\infty}\beta_{n+k}r^k + \frac{\beta_{n-1}}{r}$$

$$= (1-r)\sum_{k=0}^{\infty}\beta_{n+k}r^k = \pi_r(S^n\beta),$$

and if $\omega_{n-1} = 1$, then $\beta_{n-1} = 1$ and

$$U\left((1-r)\sum_{k=0}^{\infty}\beta_{n-1+k}r^k\right) = (1-r)\sum_{k=0}^{\infty}\beta_{n+k}r^k + \frac{\beta_{n-1}}{r} - \frac{1}{r}$$

$$= (1-r)\sum_{k=0}^{\infty}\beta_{n+k}r^k = \pi_r(S^n\beta).$$

Concerning statement 1, if $\beta_n = 0$, then by statement 4 of Theorem 5.1 we have $S^n\beta \prec \alpha$. Therefore, by Lemma 6.2 and statement 2 which we have just proved, we have $U^n(p) = \pi_r(S^n\beta) < \pi_r(\alpha) = p$. By the definition of the itinerary of p this implies that $\omega_n = 0$, and hence $\omega_n = \beta_n$. If, on the other hand, $\beta_n = 1$, then by statement 3 of Theorem 5.1 we have $S^n\beta \succeq \beta$, and therefore $U^n(p) = \pi_r(S^n\beta) \geq \pi_r(\beta) = p$. Again by definition of the itinerary of p, we have $\omega_n = 1$ and hence $\omega_n = \beta_n$. □

THEOREM 6.1. *Let $([0,1], T)$ be any special overlapping dynamical system with mask point q, critical itineraries α and β, and $r(q)$ as defined in equations (7) or (8).*

(1) *The dynamical system $([0,1], T)$ is topologically conjugate to the uniform dynamical system $([0,1), U_{r,p})$, where $r = r(q)$ and $p = (1-r)\sum_{n=0}^{\infty}\alpha_n r^n$.*

(2) *The entropy of dynamical system $([0,1], T)$ is $-\ln r$, where r the smallest solution $x \in [0,1]$ to the equation*

$$\sum_{n=0}^{\infty}\alpha_n x^n = \sum_{n=0}^{\infty}\beta_n x^n.$$

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Proof. Statement 1 follows immediately from Lemma 6.4 and from the equivalence of statements 3 and 4 of Theorem 5.2. Statement 2 then follows immediately from Theorem 4.1 and the fact that two topologically conjugate dynamical systems have the same entropy. □

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