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# RESTRICTED PARTITION FUNCTIONS AND ADDITIVE COMPLEMENTS

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**Abstract**

Let $\mathbb{N}$ be the set of positive integers. For subsets $\mathcal{A}, \mathcal{M} \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $p(n, \mathcal{A}, \mathcal{M})$ denote the number of representations of $n$ in the form $n = \sum_{a\in\mathcal{A}} m_a a$, where $m_a \in \mathcal{M} \cup \{0\}$ for all $a \in \mathcal{A}$ and only finitely many $m_a$ are nonzero. We prove that there exist two infinite sets $\mathcal{A} = \{a_n\}_{n=1}^{\infty}$ and $\mathcal{M}$ of positive integers such that $\lim_{n\to\infty}((\log a_{n+1} - \log a_n)/\log n) = +\infty$, $p(n, \mathcal{A}, \mathcal{M}) > 0$ for every $n \in \mathbb{N}$ and $p$ has polynomial growth. More generally, we provide a construction that associates restricted partition functions of polynomial growth with additive complements satisfying a simple counting condition. This answers a question of Dai and Chen ['On two problems of Ljujić and Nathanson', *C. R. Math. Acad. Sci. Paris* **354** (2016), 235–238] in the affirmative.

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Let $\mathbb{N}$ be the set of positive integers and put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$ and given subsets $\mathcal{A}, \mathcal{M}$ of $\mathbb{N}$, Canfield and Wilf [2] introduced the restricted integer partition function $p(n, \mathcal{A}, \mathcal{M})$, defined as the number of representations of $n$ in the form $n = \sum_{a\in\mathcal{A}} m_a a$, where $m_a \in \mathcal{M} \cup \{0\}$ for all $a \in \mathcal{A}$ and only finitely many $m_a$ are nonzero. The function $p = p(n, \mathcal{A}, \mathcal{M})$ is said to have polynomial growth if there exist a constant $c > 0$ and an integer $n_0$ such that $p(n, \mathcal{A}, \mathcal{M}) \le n^c$ for all $n \ge n_0$.

Answering a question of Canfield and Wilf [2] in a stronger form, Alon [1] proved that there are two explicit infinite sets $\mathcal{A}$ and $\mathcal{M}$ of positive integers such that $p(n, \mathcal{A}, \mathcal{M}) = 1$ for all $n \ge 1$. For $\mathcal{A} = \{n!\}_{n\ge1}$ or for $\mathcal{A} = \{n^n\}_{n\ge1}$, Alon further proved that there exist $n_0$ and an infinite set $\mathcal{M}$ of positive integers such that

$$0 < p(n, \mathcal{A}, \mathcal{M}) < n^{8+o(1)}$$

for all $n > n_0$, thereby answering two further problems of Ljujić and Nathanson [4] in the affirmative. In 2016, via an elementary inductive argument, Dai and Chen [3]

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improved Alon's exponent from $8 + o(1)$ to $2 + o(1)$. More generally, Dai and Chen obtained a unified result for all subsets $\mathcal{A} = \{1 = a_1 < a_2 < \cdots\}$ satisfying

$$c_1(n+1)^{\theta_1} a_n \le a_{n+1} \le c_2(n+1)^{\theta_2} a_n, \quad n = 1, 2, 3, \ldots,$$

where $c_2 > c_1 > 0$ and $\theta_2 \ge \theta_1 > 0$ are constants. This led them to the following interesting question.

QUESTION 1 (Dai and Chen, [3, Question 3.5]). Do there exist two infinite sets $\mathcal{A} = \{a_n\}_{n=1}^{\infty}$ and $\mathcal{M}$ of positive integers such that

$$\lim_{n\to\infty} \frac{\log a_{n+1} - \log a_n}{\log n} = +\infty,$$

$p(n, \mathcal{A}, \mathcal{M}) > 0$ for all sufficiently large $n$ and $p$ has polynomial growth?

In this note, we show that the answer to Question 1 is affirmative. Before proving this corollary, we establish a more general result that relates additive complements to Question 1. Let $\mathcal{B}$ and $\mathcal{S}$ be subsets of $\mathbb{N}_0$. We call $\mathcal{B}$ and $\mathcal{S}$ *complete additive complements* if every nonnegative integer can be represented as the sum of an element of $\mathcal{B}$ and an element of $\mathcal{S}$, that is, $\mathcal{B} + \mathcal{S} = \mathbb{N}_0$. For a set $\mathcal{Y} \subseteq \mathbb{N}_0$, let $\mathcal{Y}(x)$ denote the number of elements of $\mathcal{Y}$ not exceeding $x$. Our main result is stated as the following theorem.

THEOREM 2. *Let $\mathcal{B}$ and $\mathcal{S}$ be complete additive complements satisfying*

$$\mathcal{B}(x)\mathcal{S}(x) \ll x.$$

*Suppose that*

$$\mathcal{A} = \{2^b : b \in \mathcal{B}\} \quad \text{and} \quad \mathcal{M} = \left\{ \sum_{d\in\mathcal{D}} 2^d : \mathcal{D} \subset \mathcal{S},\ 0 < |\mathcal{D}| < \infty \right\}.$$

*Then, $p(n, \mathcal{A}, \mathcal{M}) > 0$ for every $n \in \mathbb{N}$ and $p$ has polynomial growth.*

PROOF. By the hypothesis of the theorem, we have

$$\mathcal{B} + \mathcal{S} = \mathbb{N}_0. \tag{1}$$

We first show that $p(n, \mathcal{A}, \mathcal{M}) > 0$ for every $n \in \mathbb{N}$. By the binary expansion of $n$, there is a finite nonempty set $\mathcal{J} \subset \mathbb{N}_0$ such that $n = \sum_{j\in\mathcal{J}} 2^j$. By (1), each $j \in \mathcal{J}$ can be written as $j = s_j + b_j$, where $s_j \in \mathcal{S}$ and $b_j \in \mathcal{B}$. Thus,

$$n = \sum_{j\in\mathcal{J}} 2^{s_j} 2^{b_j}. \tag{2}$$

Combining the terms in (2) with the same value of $b_j$,

$$n = \sum_{b\in\mathcal{B}} \Bigg( \sum_{j\in\mathcal{J}:\ b_j = b} 2^{s_j} \Bigg) 2^b = \sum_{a\in\mathcal{A}} m_a a.$$

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For each fixed $b$, the corresponding $s_j$ are distinct, because the $j$ are distinct. Hence, each nonzero inner sum belongs to $\mathcal{M}$. Therefore, $m_a \in \mathcal{M} \cup \{0\}$ for every $a \in \mathcal{A}$ and only finitely many $m_a$ are nonzero.

It remains to show that $p$ has polynomial growth. Let $n$ be sufficiently large and put $N = \log n/\log 2$. Take an arbitrary representation counted by $p(n, \mathcal{A}, \mathcal{M})$ as

$$n = \sum_{b\in\mathcal{B}} m_{2^b} 2^b. \tag{3}$$

For each $b$ with $m_{2^b} \neq 0$, the uniqueness of the binary expansion gives a unique finite nonempty set $\mathcal{D}_b \subset \mathcal{S}$ such that

$$m_{2^b} = \sum_{s\in\mathcal{D}_b} 2^s.$$

Thus, the representation (3) determines a finite set

$$\mathcal{P} = \{(s, b) : b \in \mathcal{B},\ s \in \mathcal{D}_b\} \subseteq \mathcal{S} \times \mathcal{B}$$

such that

$$n = \sum_{(s,b)\in\mathcal{P}} 2^{s+b}.$$

Since all summands are positive, each pair in $\mathcal{P}$ satisfies $s + b \leq N$. Note that the map from representations to such sets $\mathcal{P}$ is injective and so

$$p(n, \mathcal{A}, \mathcal{M}) \leq 2^{f(N)},$$

where

$$f(N) = \#\{(s, b) \in \mathcal{S} \times \mathcal{B} : s + b \leq N\}.$$

By the hypothesis of our theorem,

$$f(N) \leq \mathcal{S}(N)\mathcal{B}(N) \leq c_0 N = \frac{c_0}{\log 2} \log n, \tag{4}$$

where $c_0 > 0$ is a constant. It follows from (4) that

$$p(n, \mathcal{A}, \mathcal{M}) \leq 2^{c_0 \log n/\log 2} \leq n^{c_0},$$

completing the proof that $p$ has polynomial growth. □

Now, we turn back to Question 1. We shall use the following result of Ruzsa on additive complements of powers of two.

PROPOSITION 3 (Ruzsa, [5]). *There is a subset $\mathcal{S} \subset \mathbb{N}$ with $\mathcal{S}(x) \ll x/\log x$ such that every sufficiently large integer is of the form $2^k + s$, where $k \in \mathbb{N}$ and $s \in \mathcal{S}$.*

COROLLARY 4. *The answer to Question 1 is affirmative.*

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PROOF. Let

$$\mathcal{B} = \{0\} \cup \{2^k : k = 0, 1, 2, \ldots\}.$$

By Proposition 3, after adding finitely many nonnegative integers to $\mathcal{S}$ if necessary, there exists some $\mathcal{S} \subseteq \mathbb{N}_0$ such that $\mathcal{S}(x) \ll x/\log x$ and

$$\mathcal{B} + \mathcal{S} = \mathbb{N}_0.$$

Indeed, finitely many missing values may be put into $\mathcal{S}$, since $0 \in \mathcal{B}$, and this does not affect the estimate $\mathcal{S}(x) \ll x/\log x$. Since $\mathcal{B}(x) \ll \log x$ for $x \geq 2$,

$$\mathcal{B}(x)\mathcal{S}(x) \ll x.$$

Let

$$\mathcal{A} = \{2^b : b \in \mathcal{B}\} := \{a_n\}_{n=1}^{\infty},$$

where the elements of $\mathcal{A}$ are listed in increasing order. Then, $a_1 = 1$ and $a_n = 2^{2^{n-2}}$ for every $n \geq 2$. Hence,

$$\lim_{n\to\infty} \frac{\log a_{n+1} - \log a_n}{\log n} = \lim_{n\to\infty} \frac{(2^{n-1} - 2^{n-2})\log 2}{\log n} = +\infty.$$

Put

$$\mathcal{M} = \left\{ \sum_{d\in\mathcal{D}} 2^d : \mathcal{D} \subset \mathcal{S},\ 0 < |\mathcal{D}| < \infty \right\}.$$

The set $\mathcal{S}$ is infinite, otherwise $\mathcal{B} + \mathcal{S}$ would have counting function $O(\log x)$ and could not be all of $\mathbb{N}_0$. Hence, $\mathcal{M}$ is infinite. By Theorem 2, $p(n, \mathcal{A}, \mathcal{M}) > 0$ for every $n \in \mathbb{N}$ and $p$ has polynomial growth, completing the proof of Corollary 4. □

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During the preparation of this work, the author used ChatGPT (OpenAI, GPT-5.5 Thinking) to explore possible approaches to Question 1. In particular, ChatGPT suggested considering sets of the form $\mathcal{A} = 2^{\mathcal{D}}$ and coefficients represented by binary subset sums, and pointed to a theorem of Ruzsa on additive complements of lacunary sequences. The author rewrote the proof and obtained a more general theorem.

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