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3-Primary exponents

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0. *Introduction*

The purpose of this paper is to show that 3^n annihilates the 3-primary component of the homotopy groups of the $2n+1$-dimensional sphere. In the terminology of (2) and (3), S^{2n+1} has exponent 3^n at 3.

In fact, a stronger result is proved. Localize at 3 and let $\Omega^{2n}S^{2n+1}\langle 2n+1\rangle$ denote the $2n$-fold loop space of the $2n+1$-connected cover of S^{2n+1}. Then $\Omega^{2n}S^{2n+1}\langle 2n+1\rangle$ has a null homotopic 3^n-th power map.

It is also shown that a simply connected mod 3^r Moore space has 3-torsion of order 3^{r+1} in its homotopy groups.

The main point of this paper is that, after suitable modifications, the methods and results of (2) and (3) which were restricted to primes greater than 3 actually extend to the prime 3. It is necessary to work harder at the prime 3 because of the fact that Samelson products in mod 3 homotopy do not satisfy the Jacobi identity. At primes greater than 3, the Jacobi identity is used in the proofs of certain Bockstein formulas. (See Lemma 6·4.) These formulas are important in the proofs of the main results.

The main technical idea in this paper is to analyse the deviation from the Jacobi identity in mod 3^r homotopy. It is shown that the Bockstein formulas hold up to an error term which does not influence the final results.

This paper is organized as follows. In Section 1, properties of Samelson products are reviewed. In Section 2, it is shown that higher order torsion exists in the homotopy groups of Moore spaces. In Section 3, this higher order torsion is used to prove product decompositions of certain loop spaces. In Section 4, the decomposition theorems are applied to exponent results. Section 5 contains the proofs of useful algebraic identities. And, in Section 6, modifications of Bockstein formulas (upon which all else depends) are proved.

Most results are stated so that they are valid for any odd prime. The notation, conventions, terminology, and style are fairly consistent with (2) and (3) so that the reader is encouraged to refer to them if clarification is ever needed.

I have chosen to repeat some of the arguments in (2) and (3). The reason for this is twofold. First, many ideas from (2) and (3) must be recalled. To do this without putting these ideas in their appropriate context would be ugly and incomprehensible. Second, improvements can be made in some of the arguments and results.

Throughout this paper, p denotes an odd prime and all spaces are localized at p. All Lie algebras are graded and connected.

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1. *Samelson products*

The proofs of the theorems of this paper depend on Samelson products in mod p^r homotopy groups. Bocksteins and Hurewicz maps also play a role. First, we recall their properties and the notations for any odd prime. Second, we discuss differences between the prime 3 and greater primes. Complete details for this section can be found in (9).

The Moore spaces $S^{m-1} \cup_{p^r} e^m$, which are denoted $P^m(p^r)$, define the mod p^r homotopy groups of a space X by $\pi_m(X; Z/p^rZ) = [P^m(p^r); X]$ for m $\geqslant 2$. When $\pi_1(X)$ is abelian, extend this definition by $\pi_1(X; Z/p^rZ) = \pi_1(X) \otimes Z/p^rZ$.

One should think of $\pi_m(X; Z/p^rZ)$ as standing in the same relationship to $\pi_m(X)$ as $H_m(X; Z/p^rZ)$ stands to $H_m(X)$. There are Bocksteins β: $\pi_m(X; Z/p^rZ) \to \pi_{m-1}(X)$ and reduction maps ρ: $\pi_m(X) \to \pi_m(X; Z/p^rZ)$. There is a homotopy Bockstein spectral sequence $E^r_\pi(X)$ with Bockstein differentials β^r and with $E^1_\pi(X) = \pi_m(X; Z/pZ)$ and $\beta^1 = \rho\beta$.

Furthermore, the Hurewicz maps

$$\phi\colon \pi_m(X) \to H_m(X) \quad \text{and} \quad \phi\colon \pi_m(X; Z/p^rZ) \to H_m(X; Z/p^rZ)$$

satisfy $\phi\beta = \beta\phi$ and $\phi\rho = \rho\phi$. These Hurewicz maps induce morphisms

$$\phi\colon E^r_\pi(X) \to E^r_H(X)$$

of homotopy Bockstein spectral sequences to homology Bockstein spectral sequences.

Now, let p be an odd prime and let G be a group-like space. The Samelson product is a bilinear pairing $[\,,]\colon \pi_m(G; Z/p^rZ) \times \pi_n(G; Z/p^rZ) \to \pi_{m+n}(G; Z/p^rZ)$. It satisfies:

LEMMA 1·1. (i) The Samelson product is bilinear.

(ii) It is anti-symmetric, i.e. $[x, y] = -(-1)^{\deg x \deg y}[y, x]$.

(iii) The Hurewicz map preserves brackets, i.e.

$$\phi[x, y] = [\phi x, \phi y] = \phi x \phi y - (-1)^{\deg x \deg y} \phi y \phi x.$$

(iv) $\rho[x, y] = [\rho x, \rho y]$ and $\rho\beta[x, y] = [\rho\beta x, y] + (-1)^{\deg x}[x, \rho\beta y]$.

(v) The Samelson product induces a pairing of $E^r_\pi(F)$ in which the $\beta^{(r)}$ are derivations and the Hurewicz maps ϕ: $E^r_\pi(G) \to E^r_H(G)$ are morphisms in the sense of (iii).

The notation $[x, y]$ for the Samelson product suggests that it should make $\pi_*(G; Z/p^rZ)$ and $E^r_\pi(G)$ into Lie algebras. This is the case if p is greater than 3.

LEMMA 1·2. If p is greater than 3, then the Jacobi identity holds, i.e.

$$[x, [y, z]] = [[x, y], z] + (-1)^{\deg x \deg y}[y, [x, z]].$$

Note that the Jacobi identity implies that $3[x, [x, x]] = 0$ if $\deg x$ is odd. Hence, if p is greater than 3, then $[x, [x, x]] = 0$ if $\deg x$ is odd.

By definition, a Lie algebra has a bilinear anti-symmetric pairing which satisfies the Jacobi identity and the 'triple product identity' $[x, [x, x]] = 0$ if $\deg x$ is odd. Even

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though the Jacobi identity is valid for classical integral Samelson products, $\pi_*(G)$ is not a Lie algebra since the triple product identity fails. For example, if ι generates $\pi_{2n-1}(\Omega S^{2n})$ and $n > 1$, then $[\iota,[\iota,\iota]] \neq 0$ [6, 12].

We do without the triple product identity at the prime 3, but we need to know to what extent the Jacobi identity fails. We describe this now.

Let α_1 be a class in $\pi_{m+2}(S^{m-1})$ which is detected by the Steenrod operation $\mathscr{P}^1$, i.e. $\mathscr{P}^1$ is nonzero in the mod 3 cohomology of the complex $S^{m-1} \cup_{\alpha_1} e^{m+3}$. Let α be the image of α_1 under the inclusion $S^{m-1} \to P^m(3^r)$. The deviation from the Jacobi identity is:

LEMMA 1·3. If $p = 3$, then the Jacobi identity holds modulo linear combinations of $[[\rho\beta x, \rho\beta y], \rho\beta z](\rho\alpha)$ and $[\rho\beta y, [\rho\beta x, \rho\beta z]](\rho\alpha)$ where ρ denotes the mod 3^r reduction.

The class α has order 3 and hence, if $r = 1$, then $\rho\alpha$ is in the image of the first Bockstein differential $\beta^{(1)}$. Thus, the deviation from the Jacobi identity is in the image of $\beta^{(1)}$.

Lemma 1·3 yields:

COROLLARY 1·4. *If $p = 3$, then the Jacobi identity is valid if any one of $\rho\beta x$, $\rho\beta y$, $\rho\beta z$ is 0. In particular, if $r > 1$, then the Jacobi identity is valid in $E_\pi^r(G)$.*

The deviation in Lemma 1·3 introduced compositions into the work, making it useful to have formulas which relate compositions to Samelson products. To this end, define $\{x, y\} = [\rho\beta x, y] - (-1)^{\deg x}[x, \rho\beta y]$.

LEMMA 1·5. If p is an odd prime, then there are linear operators
$D\colon \pi_k(P^m(p^r);\ Z/p^rZ) \to \pi_{k+2}(P^{m+1}(p^r);\ Z/p^rZ)$ such that,
for x in $\pi_m(G;\ Z/p^rZ)$, y in $\pi_n(G; Z/p^rZ)$, and h in $\pi_k(P^m(p^r);\ Z/p^rZ)$, one has

$$[xh, y] = [x, y]\,(\Sigma^n h) + \{x, y\}\,(\Sigma^{n-2} D(h)).$$

The operator D satisfies $D(\Sigma h) = \Sigma D(h)$. Recall that $(a+b)\,k = ak + bk$ if k is a suspension. If h is a suspension or if $n > 2$, then 1·5 yields

$$[xh, y] = [x, y]\,(\Sigma^n h) + [\rho\beta x, y]\,(\Sigma^{n-2} D(h)) - (-1)^{\deg x}[x, \rho\beta y]\,(\Sigma^{n-2} D(h)).$$

Finally, we remark that the results in this section extend to relative Samelson products. Let $F \to E \to B$ be a fibration sequence. Then $\Omega F \to \Omega E \to \Omega B$ is also a fibration sequence. If p is an odd prime, then there are relative Samelson products

$$[\,,\,]\colon \pi_m(\Omega E;\ Z/p^rZ) \times \pi_n(\Omega F;\ Z/p^rZ) \to \pi_{m+n}(\Omega F;\ Z/p^rZ)$$

and

$$[\,,\,]\colon \pi_n(\Omega F;\ Z/p^rZ) \times \pi_m(\Omega E;\ Z/p^rZ) \to \pi_{m+n}(\Omega F;\ Z/p^rZ)$$

which satisfy the transcriptions of the formulas of this section.

2. *Higher-order torsion*

In this section, it is shown that the Moore space $P^{2n+1}(p^r)$ contains torsion of order p^{r+1} in its homotopy groups. Since Samelson products are the main tool, it is convenient to work with the loop space $\Omega P^{2n+1}(p^r)$.

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Auxiliary spaces are needed. They are defined by the following homotopy commutative diagram in which the rows and columns are fibration sequences up to homotopy. In this diagram, * denotes a point, p^r: $S^{2n+1} \to S^{2n+1}$ denotes the degree p^r map, and q: $P^{2n+1}(p^r) \to S^{2n+1}$ denotes the map which pinches the bottom cell of $P^{2n+1}(p^r)$ to a point.

$$\begin{array}{ccccccc}
 & & E^{2n+1}\{p^r\} & \xrightarrow{r'} & P^{2n+1}(p^r) & \xrightarrow{q'} & S^{2n+1}\{p^r\} \\
 & & \downarrow{\scriptstyle t'} & & \downarrow{\scriptstyle 1} & & \downarrow \\
\Omega S^{2n+1} & \xrightarrow{\partial} & F^{2n+1}\{p^r\} & \xrightarrow{r} & P^{2n-1}(p^r) & \xrightarrow{q} & S^{2n+1} \\
\downarrow{\scriptstyle \Omega p^r} & & \downarrow{\scriptstyle t} & & \downarrow & & \downarrow{\scriptstyle p^r} \\
\Omega S^{2n+1} & \xrightarrow{=} & \Omega S^{2n+1} & \longrightarrow & * & \longrightarrow & S^{2n+1}.
\end{array}$$

The loop functor Ω is applied to the above diagram to give the diagram which is called the main diagram. The main diagram is a homotopy commutative diagram of loop spaces and loop maps in which the rows and columns are fibration sequences up to homotopy.

The next two paragraphs recall notation and terminology for the graded objects that are used to describe and mod p homology of the spaces $\Omega F^{2n+1}\{p^r\}$, $\Omega P^{2n+1}(p^r)$, and ΩS^{2n+1} in the main diagram.

If $\{x_\alpha\}$ is a graded set, then $L(x_\alpha)$ denotes the free Lie algebra generated by $\{x_\alpha\}$ and $\langle x_\alpha \rangle$ denotes the abelian Lie algebra generated by $\{x_\alpha\}$. If K is a Lie algebra, then UK denotes its universal enveloping algebra. Recall that $UL(x_\alpha)$ is the free associative algebra $T(x_\alpha)$ generated by $\{x_\alpha\}$ and $U\langle x_\alpha \rangle$ is the free commutative algebra $S(x_\alpha)$ generated by $\{x_\alpha\}$.

Let u have degree $2n-1$ and let v and z both have degree $2n$. Denote by L the free Lie algebra $L(u, v)$ over the field Z/pZ and denote by $L^{(0)}$ the kernel of the map $L \to \langle \iota \rangle$ which sends v to ι and u to 0. Any subalgebra of a free Lie algebra is also a free Lie algebra ((2), Proposition 3·9). Hence, $L^{(0)}$ is a free Lie algebra. In fact, ((2), Proposition 9·2) asserts that:

LEMMA 2·1. $L^{(0)}$ is the free Lie algebra generated by the elements $x_k = ad^{k-1}(v)(u)$ with $k \geqslant 1$.

Lemma 2·2 below is Proposition 9·3 of (2). Lemmas 2·3, 2·4, and 2·5 are also in [2], but they are easy exercises for the reader.

LEMMA 2·2. Applying the mod p homology functor to the fibration sequence

$$\Omega F^{2n+1}\{p^r\} \to \Omega P^{2n+1}(p^r) \to \Omega S^{2n+1}$$

yields the short exact sequence of Hopf algebras $UL^{(0)} \to UL \to U\langle \iota \rangle$.

LEMMA 2·3. In the mod p homology Bockstein spectral sequence $E_H^s(\Omega S^{2n+1})$, $\beta^s = 0$ for all s. Hence, $U\langle \iota \rangle = E_H^s(\Omega S^{2n+1})$ for all s.

LEMMA 2·4. In $E_H^s(\Omega P^{2n+1}(p^r))$, $\beta^s = 0$ for all $s \leqslant r$. The elements u and v may be chosen so that $\beta^r v = u$ and $\beta^r u = 0$. Hence, $UL = E_H^s(\Omega P^{2n+1}(p^r))$ for $s \leqslant r$ and $Z/pZ = E_H^s(\Omega P^{2n+1}(p^r))$ for $s > r$.

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Since $UL^{(0)}$ injects into UL, 2·4 implies:

LEMMA 2·5. In $E_H^s(\Omega F^{2n+1}\{p^r\})$, $\beta^s = 0$ for $s \leqslant r$ and β^r is the restriction of the β^r in 2·4. Hence, $UL^{(0)} = E_H^s(\Omega F^{2n+1}\{p^r\})$ for $s \leqslant r$.

Setting $\beta^r v = u$, $\beta^r u = 0$, $\beta^r \iota = 0$ determines differential Lie algebra structures on $L^{(0)}$, L, and $\langle \iota \rangle$. Then $L^{(0)} \to L \to \langle \iota \rangle$ is a short exact sequence of differential Lie algebras. The resulting sequence of differential Hopf algebras $UL^{(0)} \to UL \to U\langle \iota \rangle$ is the sequence $E_H^r(\Omega F^{2n+1}\{p^r\}) \to E_H^r(\Omega P^{2n+1}(p^r)) \to E_H^r(\Omega S^{2n+1})$.

In order to describe the terms $E_H^s(\Omega F^{2n+1}\{p^r\})$ with $s > r$, the following definition is necessary.

Definition 2·6. For $k \geqslant 0$, define $\tau_k(v)$ to be $x_{p^k} = ad^{p^k-1}(v)(u)$. Thus, $\tau_0(v) = u$. For $l \geqslant 1$, define $\sigma_l(v)$ to be

$$\frac{1}{2}\sum_{j=1}^{p^l-1} p^{-1}(j, p^l - j)\,[x_j, x_{p^l-j}].$$

Corollary 4·6 of (2) asserts that $\beta^r \tau_k(v) = \beta^r \sigma_l(v) = 0$ and Theorem 10·3 of (2) asserts that:

LEMMA 2·7. $H_*(UL^{(0)}, \beta^r) = E_H^{r+1}(\Omega F^{2n+1}\{p^r\}) = S(\tau_k(v), \sigma_l(v))_{\substack{k \geqslant 0 \\ l \geqslant 1}}$. If $k \geqslant 1$, then $\beta^{r+1}\tau_k(v) = \lambda_k \sigma_k(v)$ where $\lambda_k \not\equiv 0 \bmod p$. Hence, $S(\tau_0(v)) = E_H^s(\Omega F^{2n+1}\{p^r\})$ for $s > r+1$.

The elements $\tau_k(v)$ and $\sigma_l(v)$ are used in the construction of a filtration of the differential Lie algebra $L^{(0)}$. Recall Lemma 12·3 of (2):

LEMMA 2.8. There exists a decreasing sequence $L^{(k)}$ of subdifferential Lie algebras of $L^{(0)}$ such that:

(i) If $k \geqslant 0$, then $L^{(k+1)}$ is an ideal in $L^{(k)}$.

(ii) There is a short exact sequence $L^{(1)} \to L^{(0)} \xrightarrow{\pi^{(1)}} \langle u \rangle$ with $\pi^{(1)}(\tau_0(v)) = u$.

(iii) If $k \geqslant 1$, then there is a short exact sequence $L^{(k+1)} \to L^{(k)} \xrightarrow{\pi^{(k+1)}} \langle \tau_k, \sigma_k \rangle$ with $\pi^{(k+1)}(\tau_k(v)) = \tau_k$ and $\pi^{(k+1)}(\sigma_k(v)) = \sigma_k$.

(iv) If $k \geqslant 1$, then $H_*(UL^{(k)}, \beta^r) = S(\tau_j(v), \sigma_j(v))_{j \geqslant k}$.

(v) Let $L^{(\infty)} = \cap_{k=1}^{\infty} L^{(k)}$. Then there is a graded set $\{x_\alpha, y_\alpha\}$ such that $L^{(\infty)} = L(x_\alpha, y_\alpha)$ and $\beta^r x_\alpha = y_\alpha$.

Remark. Introduce a trigrading on L by giving v tridegree $(2n-1, 1, 0)$ and u tridegree $(2n-2, 0, 1)$. The differential β^r has tridegree $(-1, -1, 1)$. This trigrading is compatible with the proof of [2, Lemma 12·3]. That is, all the maps in 2·8 could be chosen to respect this trigrading. We shall suppose this done. In particular, $L^{(\infty)}$ contains all elements of third degree $\geqslant 2$.

The results of this paper are consequences of Lemma 2·8 and of Lemma 2·9 below. The proof of 2·9 is given in Section 6.

LEMMA 2·9. For $k \geqslant 1$, there exist mod p^{r+1} homotopy classes

$$\bar{\delta}_k \colon P^{2p^k n-1}(p^{r+1}) \to \Omega E^{2n+1}\{p^r\}$$

with the following properties:

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(i) Let $\delta_k = (\Omega t')\,\bar{\delta}_k\colon P^{2p^kn-1}(p^{r+1}) \to \Omega E^{2n+1}\{p^r\} \to \Omega F^{2n+1}\{p^r\}$ and let $\tau'_k = \rho\delta_k$, the mod p reduction of δ_k. Then the Hurewicz image $\phi\tau'_k$ is in $L^{(k)}$ and $\pi^{(k+1)}(\phi\tau'_k) = \tau_k$; that is, $\phi\tau'_k - \tau_k(v)$ is in $L^{(k+1)}$.

(ii) Let $\sigma'_k = \beta^{r+1}\tau'_k$. Then $\phi\sigma'_k$ is in $L^{(k)}$ and $\pi^{(k+1)}(\phi\sigma'_k) = \lambda_k\sigma_k$ where $\lambda_k \not\equiv 0 \bmod p$; that is, $\phi\sigma_k - \lambda_k\sigma_k(v)$ is in $L^{(k+1)}$.

The next result is the goal of this section.

THEOREM 2·10. *Let p be an odd prime and $n \geqslant 1$. For all $k \geqslant 1$, the homotopy group $\pi_{2p^kn-2}(\Omega P^{2n+1}(p^r))$ contains a summand isomorphic to $Z/p^{r+1}Z$.*

Proof. Let $E^s_\pi = E^s_\pi(\Omega P^{2n+1}(p^r))$ and let $E^s_H = E^s_H(\Omega P^{2n+1}(p^r))$. To prove the theorem, it is sufficient to exhibit nonzero classes in the image of β^{r+1} in dimensions $2p^kn-2$ of E^{r+1}_π.

Let σ''_k be the composition $(\Omega r)\,\sigma'_k\colon P^{2p^kn-2}(p) \to \Omega F^{2n+1}\{p^r\} \to \Omega P^{2n+1}(p^r)$. Since $\beta^s\sigma'_k = 0$ for $s \leqslant r$, the same is true for σ''_k. Hence, σ''_k represents a class in E^{r+1}_π. Since σ'_k is in the image of β^{r+1}, so is σ''_k. It is only necessary to show that σ''_k does not represent zero in E^{r+1}_π.

Let PE^r_H be the module of primitives in E^r_H. The Hurewicz map $\phi\colon E^r_\pi \to E^r_H$ factors through PE^r_H. Thus, there is an induced map $H\phi\colon E^{r+1}_\pi \to H_*(PE^r_H, \beta^r)$. It is sufficient to show that $(H\phi)\,\sigma''_k \neq 0$.

In E^s_H with $s \leqslant r$, 2·9 implies that $\phi\sigma''_k = \lambda_k\sigma_k(v) + \alpha$ where α is in $L^{(k+1)}$. It follows from 2·8 (iv) and from ((2), lemma 3·14) that $L^{(k)} \cong \langle\tau_k(v), \sigma_k(v)\rangle \oplus L^{(k+1)}$ as chain complexes. Hence, $(H\phi)\,\sigma''_k$ is not zero in

$$H_{2p^kn-2}(L^{(k)}, \beta^r) = H_{2p^kn-2}(L, \beta^r) = H_{2p^kn-2}(PE^r_H, \beta^r).$$

The equalities hold since $L^{(k)}$ is L in dimensions greater than $2p^{k-1}n-1$ and L is PE^r_H in dimensions not of the form $2p^l$, $l \geqslant 1$. The latter follows from the fact that PE^r_H is generated by L and p^l-th powers of even dimensional classes in L ((2), proposition 2·8). ∎

Remark. If p is an odd prime and $n \geqslant 1$, then $\Omega P^{2n+2}(p^r)$ has the same homotopy type as $S^{2n+1}\{p^r\} \times \Omega \bigvee_{k=0}^{\infty} P^{4n+2kn+3}(p^r)$ by ((2), theorem 11·1). Thus, 2·10 implies that $\pi_*(P^m(p^r))$ contains summands isomorphic to $Z/p^{r+1}Z$ if $m \geqslant 3$.

3. *Decompositions of loop spaces*

In this section, the loop spaces $\Omega F^{2n+1}\{p^r\}$ and $\Omega E^{2n+1}\{p^r\}$ are shown to decompose up to homotopy into infinite products. The symbol Π is used to denote the weak infinite product, i.e. the direct limit of the finite products.

THEOREM 3·1. *Let p be an odd prime and $n \geqslant 1$. There exists a homotopy equivalence*

$$\psi\colon \Omega \bigvee_\alpha P^{n_\alpha}(p^r) \times \prod_{k=1}^{\infty} S^{2p^kn-1}\{p^{r+1}\} \times S^{2n-1} \to \Omega F^{2n+1}\{p^r\}$$

where n_α runs over an index sets of integers greater than $4n-1$ and there are only finitely many n_α with a given value.

Proof. The proof is based on the following lemma ((2), remark 3·5, lemma 3·6, and proposition 3·7).

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LEMMA 3·2. Let $K' \xrightarrow{i} K \xrightarrow{j} K''$ be a short exact sequence of finite type Lie algebras. Then $UK' \xrightarrow{U_i} UK \xrightarrow{U_j} UK''$ is a short exact sequence of Hopf algebras. In particular, if $\gamma\colon UK'' \to UK$ is any linear map such that the composition $(Uj)\gamma\colon UK'' \to UK \to UK''$ is an isomorphism, then the map $UK' \otimes UK'' \xrightarrow{Ui\otimes\gamma} UK \otimes UK \xrightarrow{\text{mult}} UK$ is an isomorphism of left UK' modules. In fact, γ can be chosen to be a map of coalgebras and, in this case, the map mult$\cdot(Ui\otimes\gamma)$ is an isomorphism of left UK' module coalgebras.

Apply 3·2 to the filtration $L(^k)$ of $L(^0)$ which is described in 2·8. Thus, $UL^{(0)}$ is isomorphic to $UL^{(i)} \otimes U\langle u\rangle$ and $UL^{(k)}$ is isomorphic to $UL^{(k+1)} \otimes U\langle \tau_k, \sigma_k\rangle$ as left differential coalgebras. Iteration gives:

LEMMA 3·3. $UL^{(0)}$ is isomorphic to $UL^{(\infty)} \otimes \bigotimes_{k=1}^{\infty} U\langle \tau_k, \sigma_k\rangle \otimes U\langle u\rangle$ as left $UL^{(\infty)}$ differential module coalgebras.

Recall that $H_*(\Omega F^{2n+1}\{p^r\}; Z/pZ)$ is isomorphic to $UL^{(0)}$. The tensor product decomposition of $UL^{(0)}$ in 3·3 is the decomposition of $H_*(\Omega F^{2n+1}\{p^r\}; Z/pZ)$ which would follow from 3·1 and the Kunneth theorem. The proof consists of realizing this algebraic decomposition by maps of spaces. It is necessary to produce maps $f\colon \Omega \bigvee_\alpha P^{n_\alpha}(p^r) \to \Omega F^{2n+1}\{p^r\}$, $g_k\colon S^{2p^k n-1}\{p^{r+1}\} \to \Omega F^{2n+1}\{p^r\}$, and $\bar{\mu}\colon S^{2n-1} \to \Omega F^{2n+1}\{p\}^r$ with the following homological properties: f_* should map $H_*(\Omega \bigvee_\alpha P^{n_\alpha}(p^r); Z/pZ)$ isomorphically onto $UL^{(\infty)}$, g_{k*} should map $H_*(S^{2p^k n-1^a}\{p^{r+1}\}; Z/pZ)$ isomorphically onto $U\langle \tau_k, \sigma_k\rangle$, and $\bar{\mu}_*$ should map $H_*(S^{2n-1}; Z/pZ)$ isomorphically onto $U\langle u\rangle$. Given these maps, use the loop multiplication of $\Omega F^{2n+1}\{p^r\}$ to multiply them. This constructs the map ψ of 3·1. Lemma 3·2 implies that ψ is a mod p homology isomorphism. Hence, ψ is a homology isomorphism of spaces localized at p. Therefore, ψ is a homotopy equivalence. If $n = 1$, the spaces are not simply connected. However, they are H-spaces and, in this situation, a homology isomorphism still implies a homotopy equivalence.

The proof is completed by producing the maps f, g_k, and $\bar{\mu}$. Let

$$\nu\colon P^{2n}(p^r) \to \Omega P^{2n+1}(p^r) \quad \text{and} \quad \mu\colon P^{2n-1}(p^r) \to \Omega F^{2n+1}\{p^r\}$$

represent mod p^r homotopy classes such that the Hurewicz images of their mod p reductions are given by $\phi\rho\nu = v$ and $\phi\rho\mu = u$. With these homotopy classes, one can form relative Samelson products which lie in $\pi_*(\Omega F^{2n+1}\{p^r\}; Z/p^rZ)$. See Section 1.

Recall from 2·8 that $L^{(\infty)} = L(x_\alpha, y_\alpha)$ with $\beta^r x_\alpha = y_\alpha$. Let $n_\alpha - 1 =$ degree x_α. Since x_α is a linear combination of Lie brackets in v and u, there is a linear combination of relative Samelson products χ_α in ν and μ such that $\chi_\alpha\colon P^{n_\alpha-1}(p^r) \to \Omega F^{2n+1}\{p^r\}$ and $\phi\rho\chi_\alpha = x_\alpha$. Therefore, $\chi_{\alpha*}$ maps $\bar{H}_*(P^{n_\alpha-1}(p^r); Z/pZ)$ isomorphically onto the vector space generated by x_α and y_α.

Add the χ_α to get a map $\chi\colon \bigvee_\alpha P^{n_\alpha-1}(p^r) \to \Omega F^{2n+1}\{p^r\}$. Extend χ to the multiplicative extension $f\colon \Omega\Sigma(\bigvee_\alpha P^{n_\alpha-1}(p^r)) \to \Omega F^{2n+1}\{p^r\}$. Then f_* is an isomorphism onto $UL^{(\infty)}$.

In the cofibration sequence $S^{2m-1} \xrightarrow{p^r} S^{2m-1} \xrightarrow{j} P^{2m}(p^r)$, the composition jp^r is null homotopic. Therefore, there is a map of fibration sequences up to homotopy:

$$\begin{array}{ccccccc}
\Omega S^{2m-1} & \longrightarrow & S^{2m-1}\{p^r\} & \longrightarrow & S^{2m-1} & \xrightarrow{p^r} & S^{2m-1} \\
\downarrow{\scriptstyle \Omega j} & & \downarrow{\scriptstyle h} & & \downarrow & & \downarrow{\scriptstyle j} \\
\Omega P^{2m}(p^r) & \longrightarrow & \Omega P^{2m}(p^r) & \longrightarrow & * & \longrightarrow & P^{2m}(p^r).
\end{array}$$

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The mod p homology of $S^{2m-1}\{p^r\}$ can be computed by using the Serre spectral sequence of the principal fibration sequence $\Omega S^{2m-1} \to S^{2m-1}\{p^r\} \to S^{2m-1}$. The spectral sequence collapses at E^2. The induced map h_* in mod p homology can be computed by using the fact that h is equivariant with respect to the action of ΩS^{2m-1} and the map Ωj. Verification of this paragraph is left as an exercise.

Let δ_k: $P^{2p^kn-1}(p^{r+1}) \to \Omega F^{2n+1}\{p^r\}$ be the map in 2·9. Extend δ_k to its unique multiplicative extension h_k: $\Omega P^{2p^kn}(p^{r+1}) \to \Omega F^{2n+1}\{p^r\}$. Form the composition $g_k = h_k h$ with the map h in the preceding diagram. Then g_{k*} gives an isomorphism onto $U\langle \tau_k, \sigma_k \rangle$.

Finally, let $\bar{\mu}$: $S^{2n-1} \to \Omega F^{2n+1}\{p^r\}$ represent a generator of $\pi_{2n-1}(\Omega F^{2n+1}\{p^r\})$. |

In order to give a decomposition of $\Omega E^{2n+1}\{p^r\}$, let $C(n)$ denote the fibre of the double suspension Σ^2: $S^{2n-1} \to \Omega^2 S^{2n+1}$. Refer to the main diagram in Section 2 and see that the map $\bar{\mu}$ in the proof of 3·1 fits into a map of fibration sequences:

$$\begin{array}{ccc} C(n) & \xrightarrow{\tilde{\mu}} & \Omega E^{2n+1}\{p^r\} \\ \downarrow & & \downarrow \\ S^{2n-1} & \xrightarrow{\bar{\mu}} & \Omega F^{2n+1}p^r \\ \downarrow & & \downarrow \\ \Omega^2 S^{2n+1} & \xrightarrow{=} & \Omega^2 S^{2n+1}. \end{array}$$

Relative Samelson products are zero in the fibration sequence

$$\Omega^2 S^{2n+1} \to * \to \Omega S^{2n+1}.$$

Hence, the main diagram implies that the relative Samelson products χ_α factor through the space $\Omega E^{2n+1}\{p^r\}$. It follows that the multiplicative map

$$f\colon \Omega \bigvee_\alpha P^{n_\alpha}(p^r) \to \Omega F^{2n+1}\{p^r\}$$

factors through a multiplicative map f: $\Omega \bigvee_\alpha P^{n_\alpha}(p^r) \to \Omega E^{2n+1}\{p^r\}$.

In 2·9, the map δ_k factors through $\bar{\delta}_k$. Hence, the maps g_k factor through maps $\tilde{g}_k$: $S^{2p^kn-1}\{p^{r+1}\} \to \Omega E^{2n+1}\{p^r\}$.

Consider the fibration sequence

$$\Omega \bigvee_\alpha P^{n_\alpha}(p^r) \times \prod_{k=1}^{\infty} S^{2p^kn-1}\{p^{r+1}\} C(n) \to \Omega \bigvee_\alpha P^{n_\alpha}(p^n) \times \prod_{k=1}^{\infty} S^{2p^kn-1}\{p^{r+1}\} \times S^{2n-1} \to \Omega^2 S^{2n+1}$$

in which the map to $\Omega^2 S^{2n+1}$ is the composition $\Sigma^2 \pi_3$ where π_3 is the projection of the product on the factor S^{2n+1}. Let ψ' be the result of multiplying the maps $\tilde{f}$, $\tilde{g}_k$, and $\tilde{\mu}$. There is a map of fibration sequences up to homotopy:

$$\begin{array}{ccc} \Omega \bigvee_\alpha P^{n_\alpha}(p^n) \times \prod_{k=1}^{\infty} S^{2p^kn-1}\{p^{n+1}\} \times C(n) & \xrightarrow{\psi'} & \Omega E^{2n+1}\{p^r\} \\ \downarrow & & \downarrow \\ \Omega \bigvee_\alpha P^{n_\alpha}(p^n) \times \prod_{k=1}^{\infty} S^{2p^kn-1}\{p^{r+1}\} \times S^{2n-1} & \xrightarrow{\psi} & \Omega F^{2n+1}(p^r)\} \\ \downarrow & & \downarrow \\ \Omega^2 S^{2n+1} & \xrightarrow{=} & \Omega^2 S^{2n+1}. \end{array}$$

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THEOREM 3·4. *The map ψ' is a homotopy equivalence.*

Proof. Since $1_{\Omega^3 S^{2n+1}}$ and ψ are homotopy equivalences, the five lemmas imply that ψ' is a homotopy equivalence. ∎

4. *Application to exponents*

In this section, the decomposition theorems in Section 3 are applied to determine the odd primary exponents of the homotopy groups of the sphere S^{2n+1} and of the homotopy groups of the fibre $C(n)$ of the double suspension $\Sigma^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$.

To begin, set $r = 1$ in Theorem 3·1 and let ψ^{-1} be a homotopy inverse for the map ψ. Define $\pi\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ to be the composition

$$\pi_3 \psi^{-1}(\Omega\partial)\colon \Omega^2 S^{2n+1} \to \Omega F^{2n+1}\{p^r\} \to \Omega \bigvee_\alpha P^{n_\alpha}(p) \times \prod_{k=1}^{\infty} S^{2p^k n-1}\{p^2\} \times S^{2n-1} \to S^{2n-1}$$

where π_3 is the projection on the last factor and $\Omega\partial$ is the map from the main diagram in Section 2.

THEOREM 4·1. *The composition $\pi\Sigma^2$ is homotopic to the degree p map $p\colon S^{2n-1} \to S^{2n-1}$. The composition $\Sigma^2\pi$ is homotopic to $\Omega^2 p\colon \Omega^2 S^{2n+1} \to \Omega^2 S^{2n+1}$.*

Proof. The first statement follows from the second statement. The latter is a consequence of the main diagram and of the diagram which precedes Theorem 3.4. ∎

If G is an H-space, then a k-th power map, denoted by $k\colon G \to G$, induces multiplication by k on $\pi_*(G)$. Hence, if $k\colon G \to G$ is null homotopic, then k annihilates $\pi_*(G)$. The loop space ΩG has two natural multiplications, but they are homotopic. Therefore, $\Omega k \simeq k$. If G has a null homotopic k-th power, so does ΩG.

Localized at an odd prime, the sphere S^{2n-1} is an H-space (1). Choose the multiplication which makes the double suspension $\Sigma^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$ into an H-map. With this choice, the fibration sequence $C(n) \to S^{2n-1} \to \Omega^2 S^{2n+1}$ is a sequence of H-spaces and H-maps. The degree p map on S^{2n-1} is the p-th power map. Note also that $\Omega^2 p \simeq p$ on $\Omega^2 S^{2n+1}$.

It follows that 4·1 implies:

COROLLARY 4·2. *The kernel and cokernel of $\Sigma^2_*\colon \pi_*(S^{2n-1}) \to \pi_*(\Omega^2 S^{2n+1})$ are annihilated by p.*

Apply induction and 4·2 to get:

COROLLARY 4·3. *If p is an odd prime, then the p-primary component of $\pi_*(S^{2n+1})$ is annihilated by p^n.*

The case $n = 1$ of 4·3 was first proved by Selick (10).

Let $S^{2n+1}\langle 2n+1\rangle$ denote the $2n+1$-connected cover of S^{2n+1}.

THEOREM 4·4. *Localized at an odd prime p, the H-space $\Omega^{2n} S^{2n+1}\langle 2n+1\rangle$ has a null homotopic p^n-th power map. Localized at an odd prime p, $C(n)$ has a null homotopic p-th power map.*

Proof. The result is proved in the following manner. First, show that $\Omega^2 S^2\langle 3\rangle$ has a null homotopic p-th power. Then show that $C(n)$ has a null homotopic p-th power.

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This is all that must be proved. If $n > 1$, then the result for $\Omega^{2n}S^{2n+1}\langle 2n+1\rangle$ follows immediately by induction using the fibration sequence

$$\Omega^{2n-2}S^{2n-1}\langle 2n-1\rangle \to \Omega^{2n}S^{2n+1}\langle 2n+1\rangle \to \Omega^{2n-3}C(n).$$

Note that $\Sigma^2\colon S^1 \to \Omega^2S^3$ splits up to homotopy. Therefore, $\Omega^2S^3 \simeq S^1 \times \Omega^2S^3\langle 3\rangle$ and $\Omega^2S^3\langle 3\rangle$ is a retract of Ω^2S^3.

Consider the following homotopy commutative diagram in which the columns are fibration sequences up to homotopy and the rows are maps of fibration sequences.

$$\begin{array}{ccccc}
\Omega^2S^3\langle 3\rangle & \longrightarrow & * & \longrightarrow & \Omega^2S^3\langle 3\rangle \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^2S^3 & \xrightarrow{\pi} & S^1 & \xrightarrow{\Sigma^2} & \Omega^2S^3 \\
\downarrow & & \downarrow & & \downarrow \\
K(Z,1) & \xrightarrow{p} & K(Z,1) & \xrightarrow{=} & K(Z,1).
\end{array}$$

The composition in the middle row is Ω^2p which is the p-th power in Ω^2S^3. Since $\Omega^2S^3\langle 3\rangle$ is a retract of Ω^2S^3, $p\colon \Omega^2S^3\langle 3\rangle \to \Omega^2S^3\langle 3\rangle$ is up to homotopy the only map which makes the following diagram homotopy commutative.

$$\begin{array}{ccc}
\Omega^2S^3\langle 3\rangle & \xrightarrow{p} & \Omega^2S^3\langle 3\rangle \\
\downarrow & & \downarrow \\
\Omega^2S^3 & \xrightarrow{\Omega^2p} & \Omega^2S^3.
\end{array}$$

Therefore, $p\colon \Omega^2S^3\langle 3\rangle \to \Omega^2S^3\langle 3\rangle$ is null homotopic.

Now consider the homotopy commutative diagram below in which the columns are fibration sequences.

$$\begin{array}{ccccc}
C(n) & \longrightarrow & * & \longrightarrow & C(n) \\
\downarrow & & \downarrow & & \downarrow \\
S^{2n-1} & \xrightarrow{p} & S^{2n-1} & \xrightarrow{=} & S^{2n-1} \\
\downarrow{\scriptstyle \Sigma^2} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \Sigma^2} \\
\Omega^2S^{2n+1} & \xrightarrow{\pi} & S^{2n-1} & \xrightarrow{\Sigma^2} & \Omega^2S^{2n+1}.
\end{array}$$

In fact, the above diagram is fibre homotopy commutative. The only obstruction lies in $H^{2n-1}(S^{2n-1}; \pi_{2n-1}(C(n))) = 0$. It follows that the top row is $p\colon C(n) \to C(n)$ up to homotopy. Hence, $p\colon C(n) \to C(n)$ is null homotopic. |

5. *Quasi-Lie algebras*

In this section, we introduce two convenient definitions, namely, that of an anti-symmetric algebra and that of a quasi-Lie algebra. Inside such objects various identities hold. These identities are applied in Section 6.

Let R be a commutative ring in which 2 is a unit. The ring R is the ground ring for the graded objects defined below.

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We adopt the convention that the degree of an element x is denoted by $|x|$.

Definition 5·1. Let L be a graded R-module together with a bilinear pairing

$$[\,,\,]\colon L_n \times L_m \to L_{n+m}.$$

(i) L is an antisymmetric algebra if $[x,y] = -(-1)^{|x|\,|y|}[y,x]$ for all x and y in L.

(ii) L is a quasi-Lie algebra if L is an antisymmetric algebra and if the Jacobi identity is valid, that is, if $[x,[y,z]] = [[x,y],z] + (-1)^{|x|\,|y|}[y,[x,2]]$ for all x, y and z in L.

(iii) L is a Lie algebra if L is a quasi-Lie algebra and if the triple product identity is valid, that is, if $[x,[x,x]] = 0$ for all x of odd degree in L.

Definition 5·2. If L is an algebra of one of the three types defined in 5·1, then L is a differential antisymmetric algebra, a differential quasi-Lie algebra, or a differential Lie algebra if L is endowed with a degree-1 linear map $d\colon L_n \to L_{n-1}$ which satisfies:

(i) d is a differential, that is, $d^2 = 0$.

(ii) d is a derivation, that is, $d[x,y] = [dx,y] + (-1)^{|x|}[x,dy]$ for all x and y in L.

Examples of the notions defined in 5·1 and 5·2 have already been provided in Section 1. To be precise, let X be a space. Then Samelson products in $\pi_*(\Omega X; Z/pZ)$ provide bilinear pairings so that:

(i) if $p > 2$ and $r \geqslant 1$, then $E_\pi^r(\Omega X)$ together with Bockstein differential β^r is a differential antisymmetric algebra;

(ii) if $p > 2$ and $r \geqslant 2$, then it is a differential quasi-Lie algebra, and

(iii) if $p > 3$ and $r \geqslant 1$, then it is a differential Lie algebra.

Lemma 5·3. Let v denote an element in an antisymmetric algebra, and let x, y, z and w denote elements in a quasi-Lie algebra. Then

(i) If v has even degree, then $[v,v] = 0$. In particular, $[v,[v,v]] = 0$ for even degree elements.

(ii) If x has odd degree, then $3[x,[x,x]] = 0$.

(iii) $[[x,y],z] = (-1)^{|y|\,|z|}[[x,z],y] + [x,[y,z]]$.

(iv) If x has odd degree, then $[w,[x,x]] = 2[[w,x],x] = -2[x,[x,w]] = -[[x,x],w]$.

(v) If x and y have the same odd degree, then

$$[x+y,[x+y,x+y]] = [x,[x,x]] + [y,[y,y]].$$

(vi) If x has odd degree, then $[w,[x,[x,x]]] = 0$.

Proof. The first four identities are well known, and their proofs are left as exercises. For this proof only, use the notation $[x,y] = xy$.

To prove (v), note that

$$\begin{aligned}(x+y)\,((x+y)\,(x+y)) - x(xx) - y(yy) &= x(xy) + x(yx) + x(yy) + y(xx) + y(xy) + y(yx)\\ &= x(xy) + x(xy) - 2y(xy) - 2x(xy) + y(xy) + y(yx) = 0.\end{aligned}$$

To prove (vi), note that

$$\begin{aligned}w(x(xx)) &= (wx)\,(xx) + (-1)^{|w|}x(w(xx))\\ &= 2((wx)\,x)\,x - (w(xx))\,x = 2((wx)\,x)\,x - 2((wx)\,x)\,x = 0. \;\blacksquare\end{aligned}$$

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The identity in 5·3 (vi) is interesting since it asserts that the triple product identity, $[x,[x,x]] = 0$ if x has odd degree, is valid in a quasi-Lie algebra if it is used in the interior of a bracketed word in a quasi-Lie algebra. This fact is used in the proof of the next technical lemma, which makes more precise the difference between a Lie algebra and a quasi-Lie algebra.

Let L' be a quasi-Lie algebra. Suppose that α_i is a set of generators for L'; that is, suppose that each element in L' is a linear combination of bracketed words in the α_i's. Assume that there is a homomorphism $f\colon L' \to L$ where L is the free Lie algebra generated by the images $f\alpha_i$. Choose a basis $\overline{\omega}_i$ for L. Thus, each $\overline{\omega}_i$ is a linear combination of bracketed words in the $f\alpha_i$s. Replacing each occurrence of $f\alpha_i$ by α_i gives a bracketed word ω_i such that $f\omega_i = \overline{\omega}_i$.

LEMMA 5·4. If γ is any element in L', then γ is a linear combination of the ω_i's and of the $[\omega_j,[\omega_j,\omega_j]]$'s where ω_j has odd degree.

Proof. There are scalars a_i such that $f\gamma = \Sigma_i a_i \overline{\omega}_i$. That is, antisymmetry, the Jacobi identity, and the triple product identity allow $f\gamma$ to be transformed into $\Sigma_i a_i \overline{\omega}_i$. Performing the same operations on γ and noting 5·6 (vi) gives that $\gamma \equiv \Sigma a_i \omega_i$ modulo linear combinations of elements of the form $[\beta,[\beta,\beta]]$ with β of odd degree. Applying the same reasoning to β and noting 5·6 (v) gives that

$$[\beta,[\beta,\beta]] = [\Sigma_j c_j\omega_j, [\Sigma_j c_j\omega_j, \Sigma_j c_j\omega_j]] = \Sigma c_j^3 [\omega_j,[\omega_j,\omega_j]]$$

for scalars c_j. ∎

Finally, let x be an element of even degree in a differential quasi-Lie algebra, and let $x_1 = dx$. For $k > 1$, let $x_k = [x, x_{k-1}] = ad(x)(x_{k-1})$. Then:

LEMMA 5·5.

$$dx_k = \tfrac{1}{2}\textstyle\sum_{j=1}^{k-1}(j,k-j)\,[x_j, x_{k-j}].$$

Proof. Since $dx_1 = 0$, the formula is valid for $k = 1$. Suppose that $k > 1$ and that the formula has been proved for $k-1$. Then

$$\begin{aligned} dx_k &= d[x, x_{k-1}] \\ &= [dx, x_{k-1}] + [x, dx_{k-1}] \\ &= [x_1, x_{k-1}] + \tfrac{1}{2}\textstyle\sum_{j=1}^{k-2}(j,k-j-1)\,[x,[x_j, x_{k-j-1}]] \\ &= [x_1, x_{k-1}] + \tfrac{1}{2}\textstyle\sum_{j=1}^{k-2}(j,k-j-1)\,\{[x_{j+1}, x_{k-j-1}] + [x_j, x_{k-j}]\} \\ &= \tfrac{1}{2}\textstyle\sum_{j=1}^{k-1}(j,k-j)\,[x_j, x_{k-j}]. \end{aligned}$$ ∎

Remark 5·6. In the next section, the Lemmas 5·3, 5·4 and 5·5 are interpreted in a combinatorial fashion, as in the proof of 5·4. That is, they are interpreted to assert that antisymmetry and the Jacobi identity may be used to transform one bracketed word into another bracketed word. With this interpretation, the lemmas make sense and are valid for the relative Samelson products discussed in Section 1, at least when $p \neq 3$.

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6. *Bockstein formulas*

This section is the technical heart of this paper. It contains the proof of the Bockstein formulas which imply the existence of higher order torsion in the homotopy groups of the Moore spaces $P^m(p^r)$ with $m \geqslant 3$ and p an odd prime. The case $p = 3$ and $r = 1$ is by far the most complicated. If it were not necessary to treat this case, this section would be much shorter.

Refer to Section 3 and recall the map $f\colon \Omega \bigvee_\alpha P^{n_\alpha}(p^r) \to \Omega F^{2n+1}\{p^r\}$. Since f is a loop map, there is a fibration sequence up to homotopy

$$\Omega \bigvee_\alpha P^{n_\alpha}(p^r) \to \Omega F^{2n+1}\{p^r\} \to U^{2n+1}\{p^r\} \to \bigvee_\alpha P^{n_\alpha}(p^r) \to F^{2n+1}\{p^r\}.$$

We begin by studying the principal fibration sequence

$$\Omega \bigvee_\alpha P^{n_\alpha}(p^r) \to \Omega F^{2n+1}\{p^r\} \to U^{2n+1}\{p^r\}.$$

Let $\bar{f}$ denote the composition

$$\Omega \bigvee_\alpha P^{n_\alpha}(p^r) \xrightarrow{f} \Omega F^{2n+1}\{p^r\} \xrightarrow{\Omega r} \Omega P^{2n+1}(p^r).$$

The following lemma is proved in (4):

LEMMA 6·1. There exists a map $\bar{j}\colon \Omega P^{2n+1}(p^r) \to \Omega \bigvee_\alpha P^{n_\alpha}(p^r)$ such that $\bar{j}\bar{f}$ is homotopic to the identity.

This yields

COROLLARY 6·2. *The principal fibration sequence*

$$\Omega \bigvee_\alpha P^{n_\alpha}(p^r) \xrightarrow{f} \Omega F^{2n+1}\{p^r\} \to U^{2n+1}\{p^r\}$$

has a section

$$s\colon U^{2n+1}\{p^r\} \to \Omega F^{2n+1}\{p^r\}.$$

Proof. Set $j = \bar{j}\Omega r\colon \Omega F^{2n+1}\{p^r\} \to \Omega \bigvee_\alpha P^{n_\alpha}(p^r)$. Then jf is homotopic to the identity. Let U be the homotopy theoretic fibre of j. The exact homotopy sequence of a fibration yields that the composition $U \to \Omega F^{2n+1}\{p^r\} \to U^{2n+1}\{p^r\}$ is a homotopy equivalence. Hence, $U \to \Omega F^{2n+1}\{p^r\}$ can be altered by a homotopy equivalence to give the section s. ∎

Multiplying the maps f and s in 6·2 gives a homotopy equivalence

$$\Omega \bigvee_\alpha P^{n_\alpha}(p^r) \times U^{2n+1}\{p^r\} \to \Omega F^{2n+1}\{p^r\}.$$

Hence, the homotopy groups of $\Omega \bigvee_\alpha P^{n_\alpha}(p^r)$ are a direct summand of those of $\Omega F^{2n+1}\{p^r\}$.

Recall from Section 2 that $H_*(\Omega F^{2n+1}\{p^r\}; Z/pZ)$ is isomorphic to $UL^{(0)}$, and recall from Section 3 that $H_*(\Omega \bigvee_\alpha P^{n_\alpha}(p^r); Z/pZ)$ is isomorphic to $UL^{(\infty)}$. It follows that there is a map of coalgebras $Z/pZ \otimes_{UL^{(\infty)}} UL^{(0)} \to H_*(U^{2n+1}\{p^r\}; Z/pZ)$. Lemma 3·3 implies that $Z/pZ \otimes_{UL^{(\infty)}} UL^{(0)}$ is isomorphic to $\otimes_{k=1}^{\infty} U\langle \tau_k, \sigma_k \rangle \otimes U\langle u \rangle$.

LEMMA 6·3. The map

$$Z/pZ \otimes_{UL^{(\infty)}} UL^{(0)} \to H_*(U^{2n+1}\{p^r\}; Z/pZ)$$

is an isomorphism.

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Proof. Let $G \to E \to B$ be any principal fibration sequence. There is a spectral sequence in (8) which abuts to $H_*(B; Z/pZ)$. Its E^2-term is

$$\mathrm{Tor}_{H_*(G;\, Z/pZ)}(Z/pZ, H_*(E; Z/pZ)).$$

If $H_*(E; Z/pZ)$ is a free $H_*(G; Z/pZ)$ module, then E^2 is isomorphic to

$$Z/pZ \otimes_{H_*(G;\, Z/pZ)} H_*(E; Z/pZ),$$

and the spectral sequence collapses at E^2. In this case, the natural map

$$Z/pZ \otimes_{H_*(G;\, Z/pZ)} H_*(E; Z/pZ) \to H_*(B; Z/pZ)$$

is an isomorphism.

Apply the above to the principal fibration sequence

$$\Omega \bigvee_\alpha P^{n_\alpha}(p^r) \to \Omega F^{2n+1}\{p^r\} \to U^{2n+1}\{p^r\}$$

to get the required answer. Note that $UL^{(\infty)}$ is a sub-Hopf algebra of $UL^{(0)}$. Hence, ((7), Theorem 4·4) asserts that $UL^{(0)}$ is a free $UL^{(\infty)}$ module. |

Remark. In the mod p homology Bockstein spectral sequence,

$$E^s_H(U^{2n+1}\{p^r\}) = E^{r+1}_H(U^{2n+1}\{p^r\}) \quad \text{for} \quad s \leqslant r+1.$$

The map

$$E^{r+1}_H(\Omega F^{2n+1}\{p^r\}) \to E^{r+1}_H(U^{2n+1}\{p^r\})$$

is an isomorphism. These statements follow from 2·5 and 3·3.

For the next proposition, introduce the following notation. Let ν_1 and μ_1 denote the respective mod p reductions of the two homotopy classes $\nu\colon P^{2n}(p^r) \to \Omega P^{2n+1}(p^r)$ and $\mu\colon P^{2n-1}(p^r) \to \Omega F^{2n+1}\{p^r\}$ which are defined in Section 3. For $k \geqslant 1$, extend this notation by letting μ_k denote the relative Samelson product $ad^{k-1}(\nu_1)(\mu_1)$ in

$$\pi_*(\Omega F^{2n+1}\{p^r\}; Z/pZ).$$

PROPOSITION 6·4. *In the mod p homotopy Bockstein spectral sequence $E^s_\pi(\Omega F^{2n+1}\{p^r\})$, the following formulas are valid.*

(i) If $s < r$, then $\beta^s\mu_k = 0$.

(ii) If $p > 3$ or if $p = 3$ and $r > 1$, then $\beta^r\mu_k = \frac{1}{2}\sum_{j=1}^{k-1}(j, k-j)[\mu_j, \mu_{k-j}]$.

(iii) If $p = 3$ and $r = 1$, then the formula in (ii) is valid modulo

$$\pi_*(\Omega \bigvee_\alpha P^{n_\alpha}(3); Z/3Z).$$

Grant Proposition 6·4 for the moment. Then we can prove the following weaker form of Lemma 2·9. (Refer to Section 2 for the notation.)

LEMMA 6·5. (i) For $k \geqslant 1$, there exist mod p^{r+1} homotopy classes

$$\delta_k\colon P^{2p^k n-1}(p^{r+1}) \to \Omega F^{2n+1}\{p^r\}$$

with the following properties. If $\tau'_k = \rho\delta_k$ (the mod p reduction of δ_k), then the Hurewicz image $\phi\tau'_k$ is in $L^{(k)}$ and $\pi^{(k+1)}(\phi\tau'_k) = \tau_k$.

(ii) Let $\sigma'_k = \beta^{r+1}\tau'_k$. Then $\phi\sigma'_k$ is in $L^{(k)}$ and $\pi^{(k+1)}(\phi\sigma'_k) = \lambda_k\sigma_k$ where $\lambda_k \not\equiv \bmod p$.

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Proof of 6·5. Suppose that $p > 3$ or that $p = 3$ and $r > 1$. Set

$$\tau_k(\nu) = \mu_{p^k} = ad^{p^k-1}(\nu_1)(\mu_1).$$

Then 6·4 implies that $\beta^r\tau_k(\nu) = 0$. Since $\phi\tau_k(\nu) = \tau_k$, 2·7 implies that $\beta^{r+1}\tau_k(\nu) = \lambda_k\sigma_k(\nu)$ where $\lambda_k \not\equiv 0 \bmod p$ and where $\sigma_k(\nu)$ is some nonzero element in $E_\pi^{r+1}(\Omega F^{2n+1}\{p^r\})$ such that $\phi\sigma_k(\nu) = \sigma_k(\nu)$. Let ϵ_k: $S^{2p^kn-2} \to \Omega F^{2n+1}\{p^r\}$ be a homotopy class whose mod p reduction $\rho\epsilon_k$ represents $\lambda_k\sigma_k(\nu)$. Then ϵ_k has order p^{r+1} and extends over the inclusion $S^{2p^kn-2} \to P^{2p^kn-1}(p^{r+1})$ to a map δ_k: $P^{2p^kn-1}(p^{r+1}) \to \Omega F^{2n+1}\{p^r\}$. With this choice of δ_k, the elements τ_k' and σ_k' are defined and σ_k' represents $\lambda_k\sigma_k(\nu)$. Since $L^{(k)}$ contains all primitives in the dimensions of τ_k' and σ_k', the Hurewicz images $\phi\tau_k'$ and $\phi\sigma_k'$ are in $L^{(k)}$. Clearly, $\pi^{(k+1)}(\phi\sigma_k') = \lambda_k\sigma_k$, and $\beta^{r+1}\tau_k' = \sigma_k'$ implies that $\pi^{(k+1)}(\phi\tau_k') = \tau_k$.

Now, suppose that $p = 3$ and $r = 1$. Corollary 6·2 gives a section s to the map which will be denoted by π: $\Omega F^{2n+1}\{p^r\} \to U^{2n+1}\{p^r\}$. Set

$$\tau_k(\nu) = s_*\pi_*\mu_{p^k} = s_*\pi_* ad^{p^k-1}(\nu_1)(\mu_1).$$

Then 6·4 implies that $\beta^1\tau_k(\nu) = 0$. The previous argument can be repeated with this choice of $\tau_k(\nu)$. ∎

Remark 6·6. Lemma 6·5 can be used in place of Lemma 2·9 to prove Theorem 3·1. The proof is identical to that given in Section 3.

Proof of 6·4. If $s < r$, then $\beta^s\mu_1 = \beta^s\nu_1 = 0$. Since β^s is a derivation on relative Samelson products, it follows that $\beta^s\mu_k = 0$, which proves 6·4 (i).

Suppose now that $p > 3$ or that $p = 3$ and $r > 1$. In the first case, the identities for a Lie algebra hold for all relative Samelson products. In the second case, 1·4 states that the identities for a quasi-Lie algebra hold in the mod 3 homotopy Bockstein spectral sequence E_π^r provided that $r \geqslant 2$. In either case, 6·4 (ii) follows from the proof of 5·5.

Assume for the remainder of this proof that $p = 3$ and $r = 1$. The argument in this case requires several technical definitions and lemmas.

Note that $\nu = \nu_1$ and $\mu = \mu_1$. Write β for β^1 so that $\beta\nu = \mu$.

Notation 6·7. Let h: $P^m(3) \to P^q(3)$ denote a mod 3 homotopy class which is a suspension. Let ω: $P^q(3) \to \Omega F^{2n+1}\{3\}$ denote a relative Samelson product of length $\geqslant 2$ formed from ν and μ with at least one occurrence of μ.

Definition 6·8. Define A to be the linear span of those homotopy classes in

$$\pi_*(\Omega F^{2n+1}\{3\}; Z/3Z)$$

which can be represented as a composition ωh.

Since h is assumed to be a suspension, the identity $(\omega_1 + \omega_2)h = \omega_1 h + \omega_2 h$ is valid. Hence, A is unchanged if ω is allowed to be a linear combination of relative Samelson products.

Definition 6·9. The relative Samelson product ω_1 is said to have Bockstein filtration $\geqslant q$ if there are at least q occurrences of μ in ω_1. This defines a decreasing filtration of A by setting F^qA equal to the linear span of those ωh where ω has Bockstein filtration $\geqslant q$.

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Note that $F^1A = A$. The Bockstein filtration is finite in each degree. Hence, $F^qA = 0$ for q sufficiently large in a fixed degree.

LEMMA 6·10. Let ω_1, ω_2, and ω_3 have Bockstein filtrations $\geqslant q_1, q_2$, and q_3, respectively.

(i) Then $[\omega_1, [\omega_2, \omega_3]] \equiv [[\omega_1, \omega_2], \omega_3] + (-1)^{\deg \omega_1 \deg \omega_2}[\omega_2, [\omega_1, \omega_2]]$ modulo

$$F^{q_1+q_2+q_3+3}A.$$

(ii) And $[\omega_1 h, \omega_2]$ is in $F^{q_1+q_2}A$.

Proof. Part (i) follows from 1·3 and the fact that $\beta\omega_i$ has Bockstein filtration $\geqslant q_i + 1$. Part (ii) follows from 1·5. ∎

Recall from Section 3 the linear combinations χ_α of relative Samelson products.

Definition 6·11. Denote by $A^{(\infty)}$ the subspace of A which is the linear span of those ωh with ω generated by the χ_α and the $\beta\chi_\alpha$.

It is immediate that $A^{(\infty)}$ is contained in $\pi_*(\Omega \bigvee_\alpha P^{n_\alpha}(3); Z/3Z)$. Hence, 6·4 (ii) follows from the following two lemmas.

LEMMA 6·12. Modulo F^5A, $\beta\mu_k \equiv \frac{1}{2}\sum_{j=1}^{k-1}(j, k-j)\,[\mu_j, \mu_{k-j}]$.

LEMMA 6·13. $A^{(\infty)}$ contains F^4A.

Proof of 6·12. The proof is by induction on k. It is based upon the equation

$$\beta\mu_k = \beta[\nu, \mu_{k-1}] = [\mu_1, \mu_{k-1}] + [\nu, \beta\mu_{k-1}],$$

which follows from the fact that β is a derivation.

Note that $\beta\mu_1 = 0$ and that $\beta\mu_2 = [\mu_1, \mu_1]$. And $\beta\mu_3 = [\mu_1, \mu_2] + [\nu, [\mu_1, \mu_1]]$. Since $\beta\mu_1 = 0$, 1·4 implies that the Jacobi identity can be applied to the second term. Thus, 5·3 (iv) yields that $\beta\mu_3 = 3[\mu_1, \mu_2] = 0$.

It is possible to continue in this way to prove that the formula for $\beta\mu_k$ holds modulo zero provided that $k \leqslant 8$. But an error term occurs in the formula for $\beta\mu_9$.

Anyway, it is sufficient to note that the formula is proved for $k \leqslant 3$. Make the inductive assumption that, modulo an error term γ which is in F^5A, the formula is proved for $k-1$.

The argument in 5·5 shows that the formula for $\beta\mu_k$ holds modulo two error terms.

The first error term is $[\nu, \gamma]$. Since γ is in F^5A, 6·10 (ii) implies that $[\nu, \gamma]$ is in F^5A.

The second error term is created by 1·3 and by the use of the Jacobi identity in the proof of 5·5. It is a linear combination of

$$[[\mu_1, \beta\mu_j], \beta\mu_{k-j-1}]\,(\rho\alpha) \quad \text{and} \quad [\beta\mu_j, [\mu_1, \beta\mu_{k-j-1}]]\,(\rho\alpha)$$

which are clearly in F^5A. This completes the proof of 6·12. ∎

Proof of 6·13. Let ω have Bockstein filtration $\geqslant q$ with $q \geqslant 4$.

Let ω_j be linear combinations of relative Samelson products such that the ω_j map to a basis for $L^{(0)}$ under the map which sends μ to u and ν to v. We may assume that there is a subset C of the ω_j such that the elements of C are generated by the χ_α and the $\beta\chi_\alpha$ and such that C maps to a basis of $L^{(\infty)}$.

The remark which follows 2·8 shows that we may assume that the ω_j respect a trigrading which ensures that each ω_j is homogeneous in the number of occurrences of μ and ν. In particular, C contains all ω_j with at least two occurrences of μ in each term.

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The proof of 5·4, together with 6·10, shows that, modulo $F^{q+3}A$, ω is a linear combination of the ω_j and of those $[\omega_j, [\omega_j, \omega_j]]$ with ω_j of odd degree. Since ω has Bockstein filtration $\geqslant 4$, all of these ω_j lie in C. Hence, ωh is contained in the sum of $A^{(\infty)}$ and $F^{q+3}A$.

Since the Bockstein filtration is finite in each degree, one can make the inductive assumption that $F^{q+3}A$ is contained in $A^{(\infty)}$. Hence, ωh is in $A^{(\infty)}$.

This completes the proof of 6·13 and of 6·4. |

The remainder of this paper is devoted to proving Lemma 2·9, which involves doing for $\Omega E^{2n+1}\{p^r\}$ something like what we have just done for $\Omega F^{2n+1}\{p^r\}$. Extra difficulties arise since neither of the mod p^r homotopy classes μ nor ν are in $\pi_*(\Omega E^{2n+1}\{p^r\}; Z/p^rZ)$. For example, neither $[\nu, \mu]$ nor $[\mu, \mu]$ may be regarded as a relative Samelson product lying in $\pi_*(\Omega E^{2n+1}\{p^r\}; Z/p^rZ)$.

In the fibration sequence $\Omega E^{2n+1}\{p^r\} \to \Omega P^{2n+1}(p^r) \to \Omega S^{2n+1}\{p^r\}$, the base space is the loops on an H-space. Hence, the base space is homotopy commutative via a canonical homotopy. This implies that all Samelson products vanish in

$$\pi_*(\Omega S^{2n+1}\{p^r\}; Z/p^rZ).$$

It follows that all Samelson products in $\pi_*(\Omega P^{2n+1}(p^r); Z/p^rZ)$ are the images of homotopy classes in $\pi_*(\Omega E^{2n+1}\{p^r\}; Z/p^rZ)$. However, it may not be possible to choose these homotopy classes so that antisymmetry and the Jacobi identities hold (at least up to the error term in 1·3 when $p = 3$). John Moore and I conjecture that this is in fact always possible.

Conjecture. Let $X \to Y \to Z$ be a fibration sequence with Z an H-space. Then the Samelson products of classes in the homotopy of ΩY pull back to the homotopy of ΩX in such a way that antisymmetry and the Jacobi identity (up to the error term when $p = 3$) are valid.

If the above is true, then 2·9 admits a proof which is almost exactly like that of 6·5. But it is not known. It is necessary to do the case-by-case analysis which follows.

LEMMA 6·14. There exist mod p^r homotopy classes $[\widetilde{\mu,\mu}]$, $[\widetilde{\nu,\mu}]$, $[\widetilde{\mu,\nu}]$ and $[\widetilde{\nu,\nu}]$ in $\pi_*(\Omega E^{2n+1}\{p^r\}; Z/p^rZ)$ such that their respective images in $\pi_*(\Omega F^{2n+1}\{p^r\}; Z/p^rZ)$ are $[\mu, \mu]$, $[\nu, \mu]$, $[\mu, \nu]$ and $[\nu, \nu]$. ($[\nu, \nu]$ is interpreted to be 0.) If $p > 3$ or if $n > 1$, then these classes are unique.

Proof. Since relative Samelson products are zero in the fibration sequence

$$\Omega^2 S^{2n+1} \to * \to \Omega S^{2n+1},$$

the exact homotopy sequence of $\Omega E^{2n+1}\{p^r\} \to \Omega F^{2n+1}\{p^r\} \to \Omega^2 S^{2n+1}$ shows that such classes exist.

A diagram in Section 3 extends to give the following homotopy commutative diagram in which the rows are fibration sequences up to homotopy.

$$\begin{array}{ccccccc}
\Omega^3 S^{2n+1} & \longrightarrow & C(n) & \longrightarrow & S^{2n-1} & \longrightarrow & \Omega^2 S^{2n+1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^3 S^{2n+1} & \longrightarrow & \Omega E^{2n+1}\{p^r\} & \longrightarrow & \Omega F^{2n+1}\{p^r\} & \longrightarrow & \Omega^2 S^{2n+1}.
\end{array}$$

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We now prove uniqueness when $p > 3$ or $n > 1$. Any two choices for such a class differ by an element which comes from the homotopy of $\Omega^3 S^{2n+1}$ and hence from the homotopy of $C(n)$. Since $C(n)$ is $2pn-4$ connected and the homotopy classes are in dimensions $4n-2$, $4n-1$, $4n-1$, and $4n$, respectively, uniqueness follows from the fact that $2pn-4 \geqslant 4n$. ∎

Remark 6·15. If $p > 3$ or if $n > 1$, then the uniqueness statement in 6·14 implies that $[\widetilde{\nu,\nu}] = 0$, $[\widetilde{\nu,\mu}] = -[\widetilde{\mu,\nu}]$, and $\rho\beta[\widetilde{\nu,\mu}] = [\widetilde{\mu,\nu}]$, where $\rho\beta$ is the mod p^r reduction of the Bockstein. If $p = 3$ and $n = 1$, first choose some $[\widetilde{\nu,\mu}]$ and then choose $[\widetilde{\mu,\nu}] = -[\widetilde{\nu,\mu}]$ and $[\widetilde{\mu,\mu}] = \rho\beta[\widetilde{\nu,\mu}]$. And, in this case, choose $[\widetilde{\nu,\nu}] = 0$. Finally, if $[\widetilde{x,y}]$ denotes any one of the elements $[\widetilde{\mu,\mu}]$, $[\widetilde{\nu,\mu}]$, $[\widetilde{\mu,\nu}]$, or $[\widetilde{\nu,\nu}]$ and if c and d are scalars, then set $[\widetilde{cx,dy}] = cd[\widetilde{x,y}]$.

For the fibration sequence $\Omega E^{2n+1}\{p^r\} \to \Omega P^{2n+1}(p^r) \to \Omega S^{2n+1}$, interpret a relative Samelson product of length 2 in μ and ν in the manner specified by 6·14 and 6·15. For example, $[\nu,\mu]$ is interpreted to be $[\widetilde{\nu,\mu}]$ in $\pi_*(\Omega E^{2n+2}\{p^r\}; Z/p^rZ)$. Having done this, there is a natural interpretation of the higher length relative Samelson products as classes in $\pi_*(\Omega E^{2n+1}\{p^r\}; Z/p^rZ)$. For example, $[\nu,[\nu,\mu]]$ is interpreted to be the relative Samelson product $[\nu,[\widetilde{\nu,\mu}]]$. There is no ambiguity in this interpretation. For example, $[[\nu,\mu],[\widetilde{\mu,\mu}]] = [[\widetilde{\nu,\mu}],[\mu,\mu]]$. This follows from:

LEMMA 6·16. Let $F \xrightarrow{i} E \xrightarrow{j} B$ be a fibration sequence of connected spaces. If x and y are classes in $\pi_*(\Omega F; Z/p^rZ)$, then $[x,y] = [i_*x,y] = [x,i_*y]$ in $\pi_*(\Omega F; Z/p^rZ)$.

Proof. This is implied by ((9), Proposition 11·7) and by the remark which follows it. ∎

Therefore, the relative mod p^r Samelson products

$$\chi_\alpha\colon P^{n_\alpha-1}(p^r) \to \Omega F^{2n+1}\{p^r\}$$

factor into compositions

$$P^{n_\alpha-1}(p^r) \xrightarrow{\tilde{\chi}_\alpha} \Omega E^{2n+1}\{p^r\} \to \Omega F^{2n+1}\{p^r\}$$

where the $\tilde{\chi}_\alpha\colon P^{n_\alpha-1}(p^r) \to \Omega E^{2n+1}\{p^r\}$ are relative Samelson products in the above interpretation. Add these $\tilde{\chi}_\alpha$ together to get a map

$$\tilde{\chi}\colon \bigvee_\alpha P^{n_\alpha-1}(p^r) \to \Omega E^{2n+1}\{p^r\}.$$

The unique multiplicative extension of $\tilde{\chi}$ is a map $\tilde{f}\colon \Omega \bigvee_\alpha P^{n_\alpha}(p^r) \to \Omega E^{2n+1}\{p^r\}$ such that the composition $\Omega \bigvee_\alpha P^{n_\alpha}(p^r) \to \Omega E^{2n+1}\{p^r\} \to \Omega F^{2n+1}\{p^r\}$ is the map f of Section 3.

There is a homotopy commutative diagram

$$\begin{array}{ccccccc}
\Omega \bigvee_\alpha P^{n_\alpha}(p^r) & \longrightarrow & \Omega E^{2n+1}\{p^r\} & \longrightarrow & V^{2n+1}\{p^r\} & \longrightarrow & \bigvee_\alpha P^{n_\alpha}(p^r) \longrightarrow E^{2n+1}\{p^r\} \\
\downarrow = & & \downarrow & & \downarrow & & \downarrow = \quad \downarrow \\
\Omega \bigvee_\alpha P^{n_\alpha}(p^r) & \longrightarrow & \Omega F^{2n+1}\{p^r\} & \longrightarrow & U^{2n+1}\{p^r\} & \longrightarrow & \bigvee_\alpha P^{n_\alpha}(p^r) \longrightarrow F^{2n+1}\{p^r\}
\end{array}$$

in which the rows are fibration sequences up to homotopy. Lemma 6·1 and the argument in 6·2 imply that the principal fibration sequence

$$\Omega \bigvee_\alpha P^{n_\alpha}(p^r) \to \Omega E^{2n+1}\{p^r\} \to V^{2n+1}\{p^r\}$$

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has a section s': $V^{2n+1}\{p^r\} \to \Omega E^{2n+1}\{p^r\}$. Hence, there is a homotopy equivalence

$$\Omega \bigvee_\alpha P^{n_\alpha}(p^r) \times V^{2n+1}\{p^r\} \to \Omega E^{2n+1}\{p^r\}.$$

We need to show that Bockstein formulas are valid.

Consider the mod p reductions ν_1 and μ_1 of ν and μ, respectively. Choose $[\widetilde{\mu_1, \mu_1}]$, $[\widetilde{\nu_1, \mu_1}]$, $[\widetilde{\mu_1, \nu_1}]$, and $[\widetilde{\nu_1, \nu_1}]$ to be the mod p reductions of $[\widetilde{\mu, \mu}]$, $[\widetilde{\nu, \mu}]$, $[\widetilde{\mu, \nu}]$ and $[\widetilde{\nu, \nu}]$, respectively. If $[\widetilde{x_1, y_1}]$ denotes any one of the elements $[\widetilde{\mu_1, \mu_1}]$, $[\widetilde{\nu_1, \mu_1}]$, $[\widetilde{\mu_1, \nu_1}]$, or $[\widetilde{\nu_1, \nu_1}]$ and if c and d are scalars, then set $[cx_1, dy_1] = cd[x_1, y_1]$. The following lemma is immediate.

LEMMA 6·17. The transcriptions of 6·14 and of 6·15 are valid with the above choices. Furthermore, if $s < r$, then β^s is zero on these choices and $\beta^r[\widetilde{\nu_1, \mu_1}] = [\widetilde{\mu_1, \mu_1}]$.

Interpret relative Samelson products formed from μ_1 and ν_1 to lie in

$$\pi_*(\Omega E^{2n+1}\{p^r\};\ Z/pZ)$$

in the same manner as the previous interpretation of relative Samelson products formed from μ and ν. Then the Bockstein differentials act as derivations on these products by 6·17.

Let $\mu_k = ad^{k-1}(\nu_1)(\mu_1) = ad^{k-2}(\nu_1)[\widetilde{\nu_1, \mu_1}]$ in $\pi_*(\Omega E^{2n+1}\{p^r\};\ Z/pZ)$.

Suppose for the moment that we can replace $\Omega F^{2n+1}\{p^r\}$ in 6·4 by $\Omega E^{2n+1}\{p^r\}$. To prove 2·9, we need to show the existence of the homotopy classes $\bar{\delta}_k$. But this can be done by replacing $\Omega F^{2n+1}\{p^r\}$ and $U^{2n+1}\{p^r\}$ by $\Omega E^{2n+1}\{p^r\}$ and $V^{2n+1}\{p^r\}$ in that part of the proof of 6·4 which constructs the homotopy classes δ_k. Detailed verification is left to the reader. Hence, to prove 2·9 and to complete this paper, it is only necessary to prove 6·4 with $\Omega F^{2n+1}\{p^r\}$ replaced by $\Omega E^{2n+1}\{p^r\}$.

We need to consider length 3 Jacobi identities in μ_1 and ν_1. All such are either trivially true or follow from the next lemma. For example,

$$[\nu_1, [\widetilde{\nu_1, \mu_1}] = 0 + [\nu_1, [\widetilde{\nu_1, \mu_1}]] = [[\widetilde{\nu_1, \nu_1}], \mu_1] + [\nu_1, [\widetilde{\nu_1, \mu_1}]]$$

is trivially true and

$$[\mu_1, [\widetilde{\nu_1, \mu_1}]] = [[\mu_1, \nu_1], \mu_1] + [\nu_1, [\mu_1, \mu_1]]$$

follows from 6·18(i).

LEMMA 6·18. In $\pi_*(\Omega E^{2n+1}\{p^r\};\ Z/pZ)$, the following identities are valid.

(i) $\Delta_1 = [\nu_1, [\mu_1, \mu_1]] - 2[[\nu_1, \mu_1], \mu_1] = 0$.

(ii) $\Delta_2 = 3[\mu_1, [\mu_1, \mu_1]] = 0$.

Proof. Suppose that $p > 3$. Since Δ_1 and Δ_2 map to zero in $\pi_*(\Omega F^{2n+1}\{p^r\};\ Z/pZ)$, the uniqueness argument in 6·14 (and $2pn - 4 \geqslant 6n - 2$) implies that $\Delta_1 = \Delta_2 = 0$.

Suppose that $p = 3$ for the remainder of this proof.

Then $\Delta_2 = 0$ since 3 annihilates $\pi_*(\Omega E^{2n+1}\{3^r\};\ Z/3Z)$.

Assume that $\Delta_1 \neq 0$. Since Δ_1 maps to zero in $\pi_*(\Omega F^{2n+1}\{3^r\};\ Z/3Z)$, there is an element Δ_1' in $\pi_{6n-2}(\Omega^3 S^{2n+1};\ Z/3Z)$ which maps to Δ_1. Let Δ_1'' be the image of Δ_1' in $\pi_{6n-2}(C(n);\ Z/3Z)$. The cases $n > 1$ and $n = 1$ are treated separately.

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Let $n > 1$. There is a map $C(n) \to K(Z/3Z, 6n-3)$ which is a mod 3 homology isomorphism in dimensions $\leqslant 6n$. Since Δ_1'' is nonzero, it follows that the induced map

$$\Delta_{1*}'': H_{6n-3}(P^{6n-2}(3); Z/3Z) \to H_{6n-3}(C(n); Z/3Z)$$

is nonzero. If γ is the composition $S^{6n-3} \xrightarrow{\text{incl}} P^{6n-2}(3) \xrightarrow{\Delta_1'} \Omega^3 S^{2n+1}$, then the composition $S^{6n-3} \xrightarrow{\gamma} \Omega^3 S^{2n+1} \to C(n)$ does not induce zero in mod 3 homology. Since $n > 1$, this contradicts the nonexistence of elements of mod p Hopf invariant one [5, 6, 11, 12]. Hence, $\Delta_1 = 0$.

Let $n = 1$. Theorem 3·1 (which is proved by 6.6) yields a decomposition of $\Omega F^3\{3^r\}$ into $\Omega \bigvee_\alpha P^{n_\alpha}(3^r) \times \prod_{k=1}^{\infty} S^{2\cdot 3^k-1}\{3^{r+1}\} \times S^1$. Since the first factor lifts to $\Omega E^3\{3^r\}$, it maps null homotopically into $\Omega^2 S^3$. The proof of 3·1 shows that $\pi_4(S^5\{3^{r+1}\}; Z/3Z)$ is generated by a class ϵ with $\epsilon \equiv [\mu_1, [\nu_1, \mu_1]]$ modulo $\pi_*(\Omega \bigvee_\alpha P^{n_\alpha}(3^r); Z/3Z)$. Therefore, the map $\pi_*(\Omega F^3\{3^r\}; Z/3Z \to \pi_*(\Omega^2 S^3; Z/3Z)$ is zero in dimensions 2, 3, and 4. By exactness of the homotopy sequence, $\pi_*(\Omega^3 S^3; Z/3Z) \to \pi_*(\Omega E^3\{3^r\}; Z/3Z)$ is an injection in dimensions 1, 2, and 3.

Localized at 3, $\pi_6(S^3) \approx Z/3Z$ and $\pi_5(S^3) = \pi_7(S^3) = 0$. Therefore,

$$\beta^1: \pi_4(\Omega^3 S^3; Z/3Z) \to \pi_3(\Omega^3 S^3; Z/3Z)$$

is an isomorphism. Hence, $\beta^1 \Delta_1' \neq 0$. Injectivity implies that $\beta^1 \Delta_1 \neq 0$.

But $\beta^1 \Delta_1 = 0$. If $r > 1$, then 6·17 implies this immediately. If $r = 1$, then 6·17 yields that $\beta^1 \Delta_1 = [\mu_1, [\widetilde{\mu_1, \mu_1}]] - 2[[\mu_1, \mu_1], \mu_1] = 3[\mu_1, [\widetilde{\mu_1, \mu_1}]] = 0$. The contradiction implies that $\Delta_1 = 0$. |

LEMMA 6·19. Let $[\omega_1, [\omega_2, \omega_3]]$ be an interpretation of a relative Samelson product in $E_\pi^r(\Omega E^{2n+1}\{p^r\})$ where each ω_i is formed from μ_1 and ν_1. Then the Jacobi identity, $[\omega_1, [\omega_2, \omega_3]] = [[\omega_1, \omega_2], \omega_3] + (-1)^{\deg \omega_1 \deg \omega_2}[\omega_2, [\omega_1, \omega_3]]$, is valid (modulo the error term given in 1·3 when $p = 3$ and $r = 1$).

Proof. If the ω_i all have length 1, this follows from 6·18. If any one of the ω_i has length $\geqslant 2$, then this ω_i is in $E_\pi^r(\Omega E^{2n+1}\{p^r\})$. Then $[\omega_1, [\omega_2, \omega_3]]$ is a relative Samelson product in the standard sense. |

In $E_\pi^r(\Omega E^{2n+1}\{p^r\})$, the interpretation of relative Samelson products formed from μ_1 and ν_1 satisfies: (i) It is antisymmetric, (ii) it has the Jacobi identity (up to the error term when $p = 3$ and $r = 1$), (iii) it is bilinear, and (iv) the Bockstein differentials are derivations. Therefore, the proof of 6·4 applies without change to prove 6·4 with $\Omega F^{2n+1}\{p^r\}$ replaced by $\Omega E^{2n+1}\{p^r\}$.

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REFERENCES

(1) ADAMS, J. F. The sphere, considered as an H-space mod p. *Quart. J. Math. Oxford Ser.* (2) **12** (1961), 52–60.

(2) COHEN, F. R., MOORE, J. C. and NEISENDORFER, J. A. Torsion in homotopy groups. *Ann. of Math.* **109** (1979), 121–168.

https://doi.org/10.1017/S0305004100058539 Published online by Cambridge University Press

(3) COHEN, F. R., MOORE, J. C. and NEISENDORFER, J. A. The double suspension and exponents of the homotopy groups of spheres. *Ann. of Math.* **110** (1979), 549–565.
(4) COHEN, F. R., MOORE, J. C. and NEISENDORFER, J. A. Moore spaces have exponents. (To appear.)
(5) HUSEMOLLER, D. *Fibre Bundles*, 2nd ed. (Springer-Verlag, 1966).
(6) LIULEVICIUS, A. The factorization of cyclic reduced powers by secondary cohomology operations. *Mem. Amer. Math. Soc.*, no. 42, 1962.
(7) MILNOR, J. W. and MOORE, J. C. On the structure of Hopf algebras. *Ann. of Math.* **81** (1965), 211–264.
(8) MOORE, J. C. Algèbre homologique et homologie des espaces classifiants, exposé 7. Séminaire H. Cartan–J. C. Moore, 1959/1960. Périodicité des groupes d'homotopie stable des groupes classiques, d'après Bott (Secrétariat Mathématique, Paris, 1961).
(9) NEISENDORFER, J. A. *Primary Homotopy Theory. Mem. Amer. Math. Soc.*, no. 232, 1980.
(10) SELICK, P. S. Odd primary torsion in $\pi_k(S^3)$. *Topology* **17** (1978), 407–412.
(11) SHIMADA, N. and YAMANOSHITA, T. On triviality of the mod p Hopf invariant. *Jap. J. Math.* **31** (1961), 1–25.
(12) TODA, H. p-primary components of homotopy groups II, mod p Hopf invariant. *Mem. Coll. Sci. Univ. Kyoto, Ser. A.* **31** (1958), 143–160.

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