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AN EXTENDED CLASS OF UNIVARIATE AND MULTIVARIATE GENERALIZED PÓLYA PROCESSES

JI HWAN CHA,* *Ewha Womans University*

Abstract

In this paper, we consider an extended class of univariate and multivariate generalized Pólya processes and study its properties. In the generalized Pólya process considered in [8], each occurrence of an event increases the stochastic intensity of the counting process. In the extended class studied in this paper, on the contrary, it decreases the stochastic intensity of the process, which induces a kind of negative dependence in the increments in the disjoint time intervals. First, we define the extended class of generalized Pólya processes and derive some preliminary results which will be used in the remaining part of the paper. It is seen that the extended class of generalized Pólya processes can be viewed as generalized pure death processes, where the death rate depends on both the state and the time. Based on the preliminary results, the main properties of the multivariate extended generalized Pólya process and meaningful characterizations are obtained. Finally, possible applications to reliability modeling are briefly discussed.

Keywords: Extended generalized Pólya process; pure death process; restarting property; mixture counting process; reliability application

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1. Introduction

The most basic and important counting process models for modeling random recurrent events are the renewal process and the nonhomogeneous Poisson process (NHPP); these models share the homogeneous Poisson process (HPP) as a special case. Up to the present, there have been great and continuous efforts to generalize such basic counting processes to more general ones to extend their applicability. Some examples of such generalizations include the compound Poisson process (see Chapter 2 of [23]), filtered Poisson process [18], two-dimensional Poisson process [22], and marked Poisson process (see Chapter 6 of [16]), which generalize the HPP or the NHPP (see also [15]). More recently, semi-Markov processes, which can also be regarded as a generalization of renewal processes, have been intensively studied and used in various applications (see [19, 7]).

Recently, [8] studied the generalized Pólya process (GPP), which can also be regarded as a generalization of the NHPP. In the current paper, we consider a further extended class of GPPs. In the GPP, each event occurrence increases the stochastic intensity of the counting process and, unlike the NHPP, it possesses a positively-dependent-increments property. By contrast, in the extended class studied in this paper, we assume that the occurrence of each

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* Postal address: Department of Statistics, Ewha Womans University, Seoul, 120-750, Korea. Email: jhcha@ewha.ac.kr
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event decreases the stochastic intensity of the process, and thus the corresponding counting process would exhibit a kind of negative dependence in the increments. In this way, by filling the gap which has existed since [8], this work completes the generalization of the NHPP to the class of GPPs, exploring undiscovered models and properties.

In the area of reliability, many different types of repairs have been suggested and applied based on various counting process models (see Section 5 of [8]). It is important to understand that *one counting process model corresponds to one repair type and vice versa* in reliability applications. For example, by a renewal process, the process of consecutive perfect repairs can be modeled. Thus, the new counting process model developed in this paper could be used for modeling a new repair process and could be applied to various maintenance optimization problems in the area of reliability. More detailed discussions of some reliability applications will be given in the later part of this paper.

Although the counting process model studied in this paper is an extension of the GPP, it possesses completely different stochastic properties and characteristics. One of the most significant differences is that the extended GPPs studied in this paper can be classified into complete and incomplete processes, depending on whether the potential events are fully observed or not. In an incomplete process, some potential events may not eventually occur. It is shown that an incomplete process can be represented as a mixture of complete processes. It is further shown that an incomplete process can be repeatedly separated into the baseline counting process and the corresponding mixing distribution under some conditions.

This paper is organized as follows. In Section 2, the extended class of GPPs is defined and some preliminary results which are needed in the remaining part of the paper are obtained. In Section 3, the main properties of the multivariate extended GPP, such as the joint distribution of the numbers of events and the characterization of the extended class of GPPs as the mixture of another group of processes, are derived. Relying on the latter property, the hierarchical structure of the extended GPP is investigated. In Section 4, further characterization and properties such as the superposition of the processes, the dependence property, the conditional joint distribution of arrival times, and the marginal processes in the multivariate process are studied. Finally, in Section 5, possible applications of the extended GPP to reliability maintenance modeling are discussed.

2. An extended class of GPPs and some preliminaries

The GPP was introduced as an extended model of the NHPP, and it is characterized in [8]. In this section, we consider an extended class of GPPs. We start our discussion with the definition of the GPP, which is based on the concept of the stochastic intensity. For a regular counting process $\{N(t),\ t \geq 0\}$, the stochastic intensity is defined by the following ([3, 4, 13]):

$$\lambda_t \equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}\left(N(t, t+\Delta t) = 1 | \mathcal{H}_{t-}\right)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbb{E}\left[N(t, t+\Delta t) | \mathcal{H}_{t-}\right]}{\Delta t}, \tag{1}$$

where $\mathcal{H}_{t-} \equiv \{N(u),\ 0 \leq u < t\}$ is the history of the process in $[0, t)$ and $N(t_1, t_2)$, $t_1 < t_2$, stands for the number of events in $[t_1, t_2)$. See p. 1150 of [8] for the interpretation of the stochastic intensity in (1) and several examples of λ_t. Note that, in a regular counting process, the joint distribution for the event occurrences in a time interval can be stated using the stochastic intensity in (1), which is essential in deriving various properties to be obtained in this paper (e.g., the distributions of number(s) of events, the characterization of the process as the mixture of

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other processes, the superposition of the processes, the characterization of the marginal processes in a multivariate process). Thus, the stochastic intensity plays a key role in this paper and is a measure of great importance.

Definition 1. (Generalized Pólya process, [8].) *A counting process* $\{N(t),\ t \geq 0\}$ *is called the generalized Pólya process (GPP) with the set of parameters* $(\lambda(t), \alpha, \beta)$*, where* $\lambda(t) \geq 0$*,* $\forall t \geq 0$*,* $\alpha \geq 0$*,* $\beta > 0$*, if*

(i) $N(0) = 0$*;*

(ii) $\lambda_t = (\alpha N(t-) + \beta)\lambda(t)$.

As mentioned on p. 1150 of [8], the GPP with $\alpha = 0$ and $\beta = 1$ reduces to the NHPP having intensity function $\lambda(t)$. In defining the GPP, except for this specific case with $\alpha = 0$, the parameter α is restricted to a positive real number, and because of this, the susceptibility to the event occurrence in the next time interval increases as the number of events occurring in the previous time interval increases. This is a kind of positive dependence property in the increments. In this paper, we consider an extended class of GPPs when the parameter α is a negative real value and, accordingly, the corresponding counting process model would exhibit a kind of negative dependence in the increments. However, as a counting process is not properly defined if the stochastic intensity can take negative value(s), we define the extended class of GPPs as follows, by setting the parameter $\beta = \alpha l_0$ with l_0 a fixed positive integer, not depending on t.

Definition 2. (Extended generalized Pólya process.) *A counting process* $\{N(t),\ t \geq 0\}$ *is called the extended generalized Pólya process (EGPP) with the set of parameters* $(\lambda(t), \alpha, l_0)$*, where* $\lambda(t) \geq 0$*,* $\forall t \geq 0$*,* $\alpha > 0$*, and* l_0 *is a positive integer, if*

(i) $N(0) = 0$*;*

(ii) $\lambda_t = (-\alpha N(t-) + \alpha l_0)\lambda(t)$.

For convenience, the EGPP with the set of parameters $(\lambda(t), \alpha, l_0)$ will be denoted by EGPP$(\lambda(t), \alpha, l_0)$, and the function $\lambda(t)$ will be called the 'base intensity function'. According to Definition 2, when $N(t-) < l_0$, the stochastic intensity of EGPP$(\lambda(t), \alpha, l_0)$ satisfies $\lambda_t > 0$ (if the base intensity function $\lambda(t)$ is positive at t), whereas, when $N(t-) = l_0$, it is given by $\lambda_t = 0$, which implies that additional events cannot occur anymore. Thus, the EGPP in Definition 2 has $\lambda_t \geq 0$, $t \geq 0$, and the corresponding counting process is well defined. Obviously, in EGPP$(\lambda(t), \alpha, l_0)$, $\lambda_t = 0,\ t > T_{l_0}$, where T_{l_0} is the occurrence time of the l_0th event, and no event occurs after the occurrence of the l_0th event.

Remark 1. In Definition 2, one could consider a further extended model by replacing (ii) with

$$\lambda_t = \max\{(-\alpha N(t-) + \beta)\lambda(t), 0\},$$

where β is an arbitrary positive real value, not necessarily a multiple of α. For this model, to derive stochastic properties, basically the same procedures as those which will be applied to the model in Definition 2 can be used. However, the results cannot be stated in explicit forms; for this reason, the discussion in this paper will just be focused on the model in Definition 2.

As the GPP has been extended to a multivariate version in [9], the EGPP in Definition 2 can be generalized to a multivariate process. For consistency and convenience, we employ similar

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notation as that of [9]. Let $\{\mathbf{N}(t),\ t \geq 0\}$, where $\mathbf{N}(t) = (N_1(t), N_2(t), \cdots, N_m(t))$, be a multivariate counting process, and let $\{M(t),\ t \geq 0\}$, where $M(t) = N_1(t) + N_2(t) + \cdots + N_m(t)$, be the corresponding pooled process. A multivariate process is said to be *regular* if the pooled process is regular, whereas it is called a marginally regular process if its marginal processes, considered as univariate point processes, are all regular (see also [9]). In this paper, the multivariate process $\{\mathbf{N}(t),\ t \geq 0\}$ of interest is assumed to be a regular process.

In the following discussion, for convenience, the marginal point processes are denoted by $\{N_i(t),\ t \geq 0\}$, and they will be called the type i point processes, $i = 1, 2, \cdots, m$. In addition, the events occurring in the type i point process $\{N_i(t),\ t \geq 0\}$ will be called type i events. Denote by $\mathcal{H}_{t-}^{(P)} \equiv \{M(u),\ 0 \leq u < t\}$ the history of the pooled process in $[0,t)$. Define $M(t-)$ as the total number of events in $[0,t)$ and $T_i^{(P)}$ as the time from 0 until the arrival of the ith event in the pooled process $\{M(t),\ t \geq 0\}$. Similarly, define the marginal histories of the marginal processes $\mathcal{H}_{t-}^{(i)} \equiv \{N_i(u),\ 0 \leq u < t\}$, $i = 1, 2, \cdots, m$, and the sequential arrival points of the events $0 \leq T_{i1} \leq T_{i2} \leq \cdots \leq T_{iN_i(t-)} < t$ in $[0,t)$, $i = 1, 2, \cdots, m$, where $N_i(t-)$ is the total number of events of the type i point process in $[0,t)$, $i = 1, 2, \cdots, m$.

A multivariate counting process can be efficiently defined through the stochastic intensity approach (see [9, 10]). As mentioned in [9], a regular multivariate process can be specified by the following set of intensity functions (see also [10]):

$$\lambda_t^{(i)} \equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}\left(N_i(t, t+\Delta t) = 1 | \mathcal{H}_{t-}^{(1)}; \mathcal{H}_{t-}^{(2)}; \cdots; \mathcal{H}_{t-}^{(m)}\right)}{\Delta t}, \ i = 1, 2, \cdots, m,$$

where $N_i(t_1, t_2)$, $t_1 < t_2$, represents the number of events of the type i point process in $[t_1, t_2)$, $i = 1, 2 \cdots, m$.

Now the multivariate extended generalized Pólya process is defined based on the intensity functions.

Definition 3. (Multivariate extended generalized Pólya process.) *A multivariate counting process* $\{\mathbf{N}(t),\ t \geq 0\}$ *is called the multivariate extended generalized Pólya process (MEGPP) with the set of parameters* $(\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, l_0)$*, where* $\lambda_i(t) \geq 0,\ t \geq 0,\ i = 1, 2, \cdots, m$*,* $\alpha > 0$*, and* l_0 *is a positive integer, if*

(i) $N_i(0) = 0$*, for* $i = 1, 2, \cdots, m$*;*

(ii) $\lambda_t^{(i)} = \left(-\alpha\left(\sum_{i=1}^m N_i(t-)\right) + \alpha l_0\right)\lambda_i(t)$*,* $i = 1, 2, \cdots, m$*.*

Remark 2.

(i) In EGPP$(\lambda(t), \alpha, l_0)$, if we start with $N(0) = l_0$ and decrease the counting variable by one unit on each occurrence of the event, the EGPP defined in Definition 2 can be regarded as a generalized pure death process, where the death rate depends on both the state and the time. (See, e.g., Section 6.2, pp. 345–349, of [25] for the ordinary pure death process.) Similarly, the MEGPP in Definition 3 can also be interpreted as a generalized multivariate pure death process, where each death is due to one of m types of causes.

(ii) The MEGPP in Definition 3 can be regarded as an extension of the MVGPP in [9] to the case when the parameter α in the definition of MVGPP is a negative real value.

The main properties of the MEGPP will be derived in the next section. For this, as preliminaries, we need some basic results on the univariate process EGPP, which will be discussed

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in the remaining part of this section. Before deriving the basic results on the EGPP, we will state an important property of the EGPP. To this end, we recall the definition of the 'restarting property' of a stochastic process.

Definition 4. (Restarting property, [8].) *Let $t > 0$ be an 'arbitrary' time point. If the conditional future stochastic process from t, given the history until time t, follows the same type of stochastic process with a possibly different set of process parameters, then the process is said to possess the restarting property. A stochastic process with the restarting property is called a restarting process.*

Several examples of counting process models which possess this restarting property are given on p. 1151 of [8]. To state our results, let us now define some necessary notation. Denote by $0 \le T_1 \le T_2 \le \cdots$ the sequential arrival points in $\{N(t),\ t \ge 0\}$. We also define $N^{(u)}(t) \equiv N(u+t) - N(u)$, $u, t > 0$, and $\Lambda(t) \equiv \int_0^t \lambda(s)ds$. From the definition of the EGPP in Definition 2, the following proposition clearly holds.

Proposition 1. *Let $\{N(t),\ t \ge 0\}$ be EGPP$(\lambda(t), \alpha, l_0)$. At an arbitrary time $u > 0$, given $\{N(u-) = n, T_1 = t_1, T_2 = t_2, \cdots, T_n = t_n\}$, where $n < l_0$, the conditional future process $\{N^{(u)}(t),\ t \ge 0\}$ is EGPP$(\lambda(u+t), \alpha, l_0 - n)$, $t \ge 0$.*

Now we will state the results on the distribution of the number(s) of events in the EGPP. The results in Theorem 1 and their proofs will form an important base for the derivations of the stochastic properties of the MEGPP. Throughout this paper, for convenience, we will use the following notation:

$$\binom{n}{n_1, n_2, \cdots, n_k} \equiv \frac{n!}{n_1! n_2! \cdots n_k!},$$

where $n = \sum_{i=1}^{k} n_i$, and

$$n^{\underline{i}} \equiv n(n-1)(n-2)\cdots(n-i+1).$$

Theorem 1. *Let $\{N(t),\ t \ge 0\}$ be EGPP$(\lambda(t), \alpha, l_0)$. For $t > 0$ and $0 \equiv u_0 < u_1 < u_2 < \cdots < u_r$, the following properties hold:*

$$(i)\ \mathbb{P}(N(t) = n) = \binom{l_0}{n} (1 - \exp\{-\alpha\Lambda(t)\})^n (\exp\{-\alpha\Lambda(t)\})^{l_0 - n},\ n = 0, 1, \cdots, l_0.$$

$$(ii)\ \mathbb{P}(N(u_2) - N(u_1) = n)$$
$$= \binom{l_0}{n} (\exp\{-\alpha\Lambda(u_1)\} - \exp\{-\alpha\Lambda(u_2)\})^n$$
$$\times (1 - \exp\{-\alpha\Lambda(u_1)\} + \exp\{-\alpha\Lambda(u_2)\})^{l_0 - n},\ n = 0, 1, \cdots, l_0.$$

$$(iii)\ \mathbb{P}(N(u_i) - N(u_{i-1}) = n_i,\ i = 1, 2, \cdots, r)$$
$$= \binom{l_0}{n_1, n_2, \cdots, n_r, l_0 - \sum_{k=1}^{r} n_k}$$
$$\times \left[\prod_{i=1}^{r} \left(\exp\{-\alpha\Lambda(u_{i-1})\} - \exp\{-\alpha\Lambda(u_i)\}\right)^{n_i}\right] \left(\exp\{-\alpha\Lambda(u_r)\}\right)^{l_0 - \sum_{k=1}^{r} n_k},$$
$$0 \le n_1 + n_2 + \cdots + n_r \le l_0.$$

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$$(iv)\ \mathbb{P}(N(u_2)-N(u_1)=n_2|N(u_1)=n_1)$$
$$=\binom{l_0-n_1}{n_2}(1-\exp\{-\alpha(\Lambda(u_2)-\Lambda(u_1))\})^{n_2}$$
$$\times(\exp\{-\alpha(\Lambda(u_2)-\Lambda(u_1))\})^{l_0-n_1-n_2},\ n_2=0,1,\cdots,l_0-n_1.$$

Proof. The proof is deferred to the appendix. □

A counting process is said to possess independent increments if the numbers of events that occur in disjoint time intervals are independent (see, e.g., Section 1.9, p. 42, of [23]). Clearly, it can be easily seen from Definition 2 that the EGPP does not have the independent increments property. Based on Theorem 1, the dependence property of the EGPP is now formally discussed. The following proposition states that the increments in any two disjoint time intervals have negative covariance and correlation coefficient.

Proposition 2. *For* $0\le u_1<u_2\le u_3<u_4$,

$$Cov(N(u_2)-N(u_1),N(u_4)-N(u_3))$$
$$=-l_0\prod_{i=2,4}\Big(\exp\{-\alpha\Lambda(u_{i-1})\}-\exp\{-\alpha\Lambda(u_i)\}\Big)<0,$$

and

$$Corr(N(u_2)-N(u_1),N(u_4)-N(u_3))$$
$$=-\sqrt{\frac{\prod_{i=2,4}\Big(\exp\{-\alpha\Lambda(u_{i-1})\}-\exp\{-\alpha\Lambda(u_i)\}\Big)}{\prod_{i=2,4}\Big(1-\exp\{-\alpha\Lambda(u_{i-1})\}+\exp\{-\alpha\Lambda(u_i)\}\Big)}}<0.$$

Proof. First assume that $u_1>0$. It can be shown from Theorem 1(iii) that

$$\mathbb{P}(N(u_i)-N(u_{i-1})=n_i,\ i=2,4)$$
$$=\frac{l_0!}{n_2!n_4!(l_0-n_2-n_4)!}\left[\prod_{i=2,4}\Big(\exp\{-\alpha\Lambda(u_{i-1})\}-\exp\{-\alpha\Lambda(u_i)\}\Big)^{n_i}\right]$$
$$\times\Big(1-\sum_{i=2,4}\Big(\exp\{-\alpha\Lambda(u_{i-1})\}-\exp\{-\alpha\Lambda(u_i)\}\Big)\Big)^{l_0-n_2-n_4},$$

where $0\le n_2+n_4\le l_0$. Then, by the properties of the multinomial distribution (see, e.g., [14], p. 34), the results can be directly obtained. In the case when $u_1=0$, the results can be shown in the same way. □

As mentioned in Proposition 1, for the EGPP, the restarting property holds conditionally as in the case of the GPP. On the other hand, for the GPP, the future process from an arbitrary time point u is 'unconditionally' the GPP as well (see Theorem 2 in [8], pp. 1152–1153). The importance of this unconditional restarting property is explained in [8], on pp. 1152–1153. Now, for the EGPP, we will check whether the future process from an arbitrary time point u is still unconditionally the EGPP or not. Employing similar notation as in [8] for consistency, for a fixed $u>0$, define $T_i^{(u)}$ as the time from 0 to the arrival of the event that occurred in (u,∞) at the ith time, $u\le T_1^{(u)}\le T_2^{(u)}\le\cdots$. Denote by $N^{(u)}(t_1,t_2)$, $t_1<t_2$, the number of events in

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$[u+t_1, u+t_2)$ and by $\mathcal{H}_{[u,u+t)}$ the history of the process in $[u, u+t)$. We now need to derive the stochastic intensity λ_t^u of the future process $\{N^{(u)}(t),\ t \geq 0\}$, which is defined by

$$\lambda_t^u \equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}\left(N^{(u)}(t, t+\Delta t) = 1 | \mathcal{H}_{[u,u+t)}\right)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbb{E}\left[N^{(u)}(t, t+\Delta t) | \mathcal{H}_{[u,u+t)}\right]}{\Delta t}.$$

Note that $\mathcal{H}_{[u,u+t)}$ can completely be defined by the number of events and the sequential arrival times during the interval $[u, u+t)$. The following theorem implies that, 'unconditionally', the future process of the EGPP is also the EGPP.

Theorem 2. *Let* $\{N(t),\ t \geq 0\}$ *be EGPP*$(\lambda(t), \alpha, l_0)$. *The stochastic intensity* λ_t^u *is given by*

$$\lambda_t^u = \Big(-\alpha[N((u+t)-) - N(u-)] + \alpha l_0\Big) \frac{\lambda(u+t)\exp\{-\alpha\Lambda(u+t)\}}{1-\exp\{-\alpha\Lambda(u)\}+\exp\{-\alpha\Lambda(u+t)\}},$$

and $\{N^{(u)}(t),\ t \geq 0\}$ *is unconditionally EGPP with the set of parameters*

$$\left(\frac{\lambda(u+t)\exp\{-\alpha\Lambda(u+t)\}}{1-\exp\{-\alpha\Lambda(u)\}+\exp\{-\alpha\Lambda(u+t)\}}, \alpha, l_0\right).$$

Proof. The proof is deferred to the appendix. □

Now the conditional distribution of the arrival times in the EGPP will be obtained. As a more general result, we consider that of the arrival times in an 'arbitrary time interval' $(u, v]$, $0 \leq u < v$:

$$\left(T_1^{(u)}, T_2^{(u)}, \cdots, T_{(N(v)-N(u))}^{(u)} | N(v) - N(u) = n\right).$$

Theorem 3. *Let* $\{N(t),\ t \geq 0\}$ *be EGPP*$(\lambda(t), \alpha, l_0)$. *The conditional joint probability density function of*

$$\left(T_1^{(u)}, T_2^{(u)}, \cdots, T_{(N(v)-N(u))}^{(u)} | N(v) - N(u) = n\right), \qquad 0 \leq u < v,$$

is given by

$$n! \prod_{i=1}^{n} \left(\frac{\alpha\lambda\left(t_i^{(u)}\right)\exp\left\{-\alpha\Lambda\left(t_i^{(u)}\right)\right\}}{\exp\{-\alpha\Lambda(u)\} - \exp\{-\alpha\Lambda(v)\}} \right),$$

where $u < t_1^{(u)} \leq t_2^{(u)} \leq \cdots \leq t_n^{(u)} \leq v$.

Proof. The proof is deferred to the appendix. □

3. Main properties of the MEGPP

The following proposition provides some basic properties of the MEGPP, to help with structural understanding of the process. The results in Proposition 3 play crucial roles in the analysis of the MEGPP. For convenience and consistency, we define the necessary notation as follows: $N_i^{(u)}(t) \equiv N_i(u+t) - N_i(u)$, $\Lambda_i(t) \equiv \int_0^t \lambda_i(s)ds$, $i = 1, 2, \cdots, m$, $\lambda(t) \equiv \lambda_1(t) + \lambda_2(t) + \cdots + \lambda_m(t)$, $\Lambda(t) = \int_0^t \lambda(s)ds = \Lambda_1(t) + \Lambda_2(t) + \cdots + \Lambda_m(t)$, and $p_i(t) \equiv \lambda_i(t)/\lambda(t)$, $i = 1, 2, \cdots, m$. We also use the concept of thinning of a univariate process, from Definition 4 of [9].

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Proposition 3. *Let* $\{\mathbf{N}(t),\ t \geq 0\}$ *be MEGPP*$(\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, l_0)$. *Then the following hold:*

(i) $\{M(t),\ t \geq 0\}$ *is EGPP*$(\lambda(t), \alpha, l_0)$.

(ii) *The process* $\{\mathbf{N}(t),\ t \geq 0\}$ *is constructed by thinning of* $\{M(t),\ t \geq 0\}$ *with thinning probabilities* $p_i(t)$, $i = 1, 2, \cdots, m$:

$$\left\{\left(M_{p_1(\cdot)}(t), M_{p_2(\cdot)}(t), \cdots, M_{p_m(\cdot)}(t)\right),\ t \geq 0\right\}.$$

(iii) *Given* $\left(\mathcal{H}_{u-}^{(1)}, \mathcal{H}_{u-}^{(2)}, \cdots, \mathcal{H}_{u-}^{(m)}\right)$, *we have that the process* $\{\mathbf{N}^{(u)}(t),\ t \geq 0\}$, *where*

$$\mathbf{N}^{(u)}(t) = \left(N_1^{(u)}(t), N_2^{(u)}(t), \cdots, N_m^{(u)}(t)\right),$$

is MEGPP$(\lambda_1(u+t), \lambda_2(u+t), \cdots, \lambda_m(u+t), \alpha, l_0 - (n_1 + n_2 + \cdots + n_m))$, *where* n_i *is the realization of* $N_i(u-)$, $i = 1, 2, \cdots, m$.

(iv) *For any fixed* $u \geq 0$, $\{\mathbf{N}^{(u)}(t),\ t \geq 0\}$ *is 'unconditionally' MEGPP with the set of parameters* $(\phi_1(t,u), \phi_2(t,u), \cdots, \phi_m(t,u), \alpha, l_0)$, *where*

$$\phi_i(t,u) \equiv \frac{\lambda_i(u+t)\exp\{-\alpha\Lambda(u+t)\}}{1-\exp\{-\alpha\Lambda(u)\}+\exp\{-\alpha\Lambda(u+t)\}}, \qquad i = 1, 2, \cdots, m.$$

Proof. The properties in Proposition 3 can be readily proved using Definition 2, Definition 3, Proposition 1, and Theorem 2, following a procedure similar to that in the proof of Proposition 1 in [9]. □

Now we will derive the main stochastic properties of the MEGPP. First of all, the joint distribution of the numbers of events is of prime interest to us.

Theorem 4. *Let* $\{\mathbf{N}(t),\ t \geq 0\}$ *be MEGPP*$(\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, l_0)$ *and* $t > 0$, $0 \equiv u_0 < u_1 < u_2 < \cdots < u_r$. *Then the following hold:*

(i)

$$\mathbb{P}(N_i(t) = n_i,\ i = 1, 2, \cdots, m) = \binom{l_0}{n_1, n_2, \cdots, n_m, l_0 - \sum_{i=1}^m n_i} \times \prod_{i=1}^m \left(\alpha \int_0^t \lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx\right)^{n_i} \left(\exp\{-\alpha\Lambda(t)\}\right)^{l_0 - \sum_{i=1}^m n_i},$$
$$0 \leq n_1 + n_2 + \cdots + n_m \leq l_0;$$

(ii)

$$\mathbb{P}(N_i(u_2) - N_i(u_1) = n_i,\ i = 1, 2, \cdots, m) = \binom{l_0}{n_1, n_2, \cdots, n_m, l_0 - \sum_{i=1}^m n_i} \times \prod_{i=1}^m \left(\alpha \int_{u_1}^{u_2} \lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx\right)^{n_i} \times \left(1 - \exp\{-\alpha\Lambda(u_1)\} + \exp\{-\alpha\Lambda(u_2)\}\right)^{l_0 - \sum_{i=1}^m n_i},$$
$$0 \leq n_1 + n_2 + \cdots + n_m \leq l_0;$$

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(iii)

$$\begin{aligned}&\mathbb{P}(N_i(u_j)-N_i(u_{j-1})=n_{ij},\ i=1,2,\cdots,m, j=1,2,\cdots,r)\\&\quad=\binom{l_0}{n_{11},n_{21},\cdots,n_{m1},n_{12},n_{22},\cdots,n_{m2},\cdots,n_{1r},n_{2r},\cdots,n_{mr},l_0-\sum_{i=1}^m\sum_{j=1}^r n_{ij}}\\&\qquad\times\prod_{i=1}^m\prod_{j=1}^r\left[\left(\alpha\int_{u_{j-1}}^{u_j}\lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx\right)^{n_{ij}}\right]\\&\qquad\times\left(\exp\{-\alpha\Lambda(u_r)\}\right)^{l_0-\sum_{i=1}^m\sum_{j=1}^r n_{ij}},\qquad 0\le\sum_{i=1}^m\sum_{j=1}^r n_{ij}\le l_0.\end{aligned}$$

Proof. First, let us prove Property (i). We have

$$\begin{aligned}&\mathbb{P}(N_i(t)=n_i,\ i=1,2,\cdots,m)\\&=\mathbb{P}\left(N_i(t)=n_i,\ i=1,2,\cdots,m|M(t)=\sum_{i=1}^m n_i\right)\mathbb{P}\left(M(t)=\sum_{i=1}^m n_i\right),\end{aligned}$$

where, by Proposition 3(i) and Theorem 1,

$$\mathbb{P}\left(M(t)=\sum_{i=1}^m n_i\right)=\binom{l_0}{\sum_{i=1}^m n_i}(1-\exp\{-\alpha\Lambda(t)\})^{\sum_{i=1}^m n_i}(\exp\{-\alpha\Lambda(t)\})^{l_0-\sum_{i=1}^m n_i}.$$

From Theorem 3, given $M(t)=\sum_{i=1}^m n_i$, the sequential arrival times

$$0\le T_1^{(P)}\le T_2^{(P)}\le\cdots\le T_{\sum_{i=1}^m n_i}^{(P)}\le t$$

have the following joint probability density function:

$$\left(\sum_{i=1}^m n_i\right)!\prod_{i=1}^{\sum_{i=1}^m n_i}\left(\frac{\alpha\lambda(t_i)\exp\{-\alpha\Lambda(t_i)\}}{1-\exp\{-\alpha\Lambda(t)\}}\right).$$

Thus they have the same distribution as the order statistics corresponding to $\sum_{i=1}^m n_i$ independent random variables identically distributed according to the probability density function

$$\left(\frac{\alpha\lambda(x)\exp\{-\alpha\Lambda(x)\}}{1-\exp\{-\alpha\Lambda(t)\}}\right),\qquad 0<x\le t.$$

Thus, if an event has occurred in (0,t], the probability that it will be a type i event is given by

$$\psi_i(t)\equiv\left(\frac{\alpha\int_0^t\lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx}{1-\exp\{-\alpha\Lambda(t)\}}\right),\qquad i=1,2,\cdots,m.$$

Thus,

$$\mathbb{P}\left(N_i(t)=n_i,\ i=1,2,\cdots,m|M(t)=\sum_{i=1}^m n_i\right)=\binom{\sum_{i=1}^m n_i}{n_1,n_2,\cdots,n_m}\prod_{i=1}^m\left(\psi_i(t)\right)^{n_i}.$$

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Therefore,

$$\mathbb{P}(N_i(t)=n_i,\ i=1,2,\cdots,m)=\binom{l_0}{n_1,n_2,\cdots,n_m,l_0-\sum_{i=1}^m n_i}\times\prod_{i=1}^m\left(\alpha\int_0^t\lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx\right)^{n_i}\left(\exp\{-\alpha\Lambda(t)\}\right)^{l_0-\sum_{i=1}^m n_i}.$$

By using Proposition 3(iv) and the above Property (i), Property (ii) can be directly obtained.

Now we show Property (iii). By Proposition 3(iii), given $N_i(u_1)-N_i(u_0)=n_{i1},\ i=1,2,\cdots,m$, we have that $\{\mathbf{N}^{(u_1)}(t),\ t\geq 0\}$ is MEGPP($\lambda_1(u_1+t),\lambda_2(u_1+t),\cdots,\lambda_m(u_1+t),\alpha,l_0-\sum_{i=1}^m n_{i1}$). Then, relying on the above Property (i),

$$\begin{aligned}&\mathbb{P}(N_i(u_2)-N_i(u_1)=n_{i2},\ i=1,2,\cdots,m|N_i(u_1)-N_i(u_0)=n_{i1},\ i=1,2,\cdots,m)\\&=\binom{l_0-\sum_{i=1}^m n_{i1}}{n_{12},n_{22},\cdots,n_{m2},l_0-\sum_{i=1}^m\sum_{j=1}^2 n_{ij}}\\&\quad\times\prod_{i=1}^m\left(\alpha\int_{u_1}^{u_2}\lambda_i(x)\exp\{-\alpha[\Lambda(x)-\Lambda(u_1)]\}dx\right)^{n_{i2}}\\&\qquad\times\left(\exp\{-\alpha[\Lambda(u_2)-\Lambda(u_1)]\}\right)^{l_0-\sum_{i=1}^m\sum_{j=1}^2 n_{ij}}.\end{aligned}$$

Thus,

$$\begin{aligned}&\mathbb{P}(N_i(u_j)-N_i(u_{j-1})=n_{ij},\ i=1,2,\cdots,m,j=1,2)\\&=\prod_{k=1}^2\left[\binom{l_0-\sum_{j=1}^{k-1}\left(\sum_{i=1}^m n_{ij}\right)}{n_{1k},n_{2k},\cdots,n_{mk},l_0-\sum_{j=1}^{k}\left(\sum_{i=1}^m n_{ij}\right)}\right.\\&\qquad\times\prod_{i=1}^m\left(\alpha\int_{u_{k-1}}^{u_k}\lambda_i(x)\exp\{-\alpha[\Lambda(x)-\Lambda(u_{k-1})]\}dx\right)^{n_{ik}}\\&\qquad\left.\times\left(\exp\{-\alpha[\Lambda(u_k)-\Lambda(u_{k-1})]\}\right)^{l_0-\sum_{i=1}^m\sum_{j=1}^k n_{ij}}\right],\end{aligned}$$

where $\sum_{j=1}^{k-1}(\cdot)\equiv 0$ when $k=1$. By applying a similar procedure recursively, we obtain the desired result. □

Note that the results in Theorems 1–4 hold whether $\Lambda(\infty)\equiv\int_0^\infty\lambda(s)ds=\infty$ or $\Lambda(\infty)<\infty$. In the EGPP, if $\Lambda(\infty)<\infty$, then $N(\infty)\equiv\lim_{t\to\infty}N(t)<l_0$ with positive probability. Thus, from the pure-death-process point of view stated in Remark 2, the process may *end incompletely* with positive probability. In this sense, if $\Lambda(\infty)<\infty$, then the corresponding EGPP (MEGPP) will be said to be *incomplete*, whereas an EGPP (MEGPP) with $\Lambda(\infty)=\infty$ will be called a *complete* EGPP (MEGPP). In the following theorem, it will be shown that an incomplete EGPP (MEGPP) can be characterized as a mixture of complete EGPPs (MEGPPs). Throughout this paper, $X\sim\text{Bin}(n,p)$ signifies that the random variable X follows the binomial distribution with parameters (n,p). In the following discussion, MEGPP($\lambda_1(t),\lambda_2(t),\cdots,\lambda_m(t),\alpha,l_0$) with $l_0=0$ represents a counting process whose counting random variables are given by 0 with probability 1 for all $t\geq 0$.

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Theorem 5. *Let* $\{\mathbf{N}(t),\ t \geq 0\}$ *be MEGPP*$(\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, l_0)$ *with* $\Lambda(\infty) < \infty$. *Let* L *be a binomial random variable:* $L \sim \text{Bin}(l_0, 1 - \exp\{-\alpha\Lambda(\infty)\})$. *For* $t > 0$, *let*

$$\phi_i(t) \equiv \frac{\lambda_i(t)\exp\{-\alpha\Lambda(t)\}}{\exp\{-\alpha\Lambda(t)\} - \exp\{-\alpha\Lambda(\infty)\}}, \qquad i = 1, 2, \cdots, m.$$

If $\{\mathbf{N}^*(t),\ t \geq 0\}$ *satisfies* $\{\mathbf{N}^*(t),\ t \geq 0\}|L \sim \text{MEGPP}(\phi_1(t), \phi_2(t), \cdots, \phi_m(t), \alpha, L)$, *then it is stochastically equivalent to* $\{\mathbf{N}(t),\ t \geq 0\}$.

Proof. The proof is deferred to the appendix. □

From Proposition 3(i), the properties of the EGPP can be directly obtained from those of the MEGPP, as the pooled process of MEGPP$(\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, l_0)$ is EGPP$(\lambda(t), \alpha, l_0)$. Thus, from Theorem 5, we have the following corollary.

Corollary 1. *Let* $\{N(t),\ t \geq 0\}$ *be EGPP*$(\lambda(t), \alpha, l_0)$ *with* $\Lambda(\infty) < \infty$. *Let* L *be a binomial random variable:* $L \sim \text{Bin}(l_0, 1 - \exp\{-\alpha\Lambda(\infty)\})$. *For* $t > 0$, *let*

$$\phi(t) \equiv \frac{\lambda(t)\exp\{-\alpha\Lambda(t)\}}{\exp\{-\alpha\Lambda(t)\} - \exp\{-\alpha\Lambda(\infty)\}}, \qquad t \geq 0.$$

If $\{N^*(t),\ t \geq 0\}$ *satisfies* $\{N^*(t),\ t \geq 0\}|L \sim \text{EGPP}(\phi(t), \alpha, L)$, *then it is stochastically equivalent to* $\{N(t),\ t \geq 0\}$.

Note that, in Corollary 1, if $\Lambda(\infty) < \infty$, then

$$\frac{\lambda(t)\exp\{-\alpha\Lambda(t)\}}{\exp\{-\alpha\Lambda(t)\} - \exp\{-\alpha\Lambda(\infty)\}} > \lambda(t), \qquad \forall t > 0.$$

Thus, roughly reinterpreting Corollary 1, an incomplete EGPP with the base intensity function $\lambda(t)$ can be characterized as a mixture of complete EGPPs with a larger base intensity function. A natural question in a more general sense could be the following: Is the mixture representation unique? Or can an incomplete EGPP be also characterized as the mixture of another group of EGPPs with a larger base intensity function? The following theorem provides the answer to this question.

Theorem 6. *Assume that* $\Lambda(\infty) < \infty$. *Let* $\{\mathbf{N}(t),\ t \geq 0\}$ *be a regular multivariate counting process and let* L *be a binomial random variable:* $L \sim \text{Bin}(l_0, p)$, *where* $l_0 \geq 1$ *and* $1 - \exp\{-\alpha\Lambda(\infty)\} < p < 1$. *For* $t > 0$, *let*

$$\phi_i(t) \equiv \frac{\lambda_i(t)\exp\{-\alpha\Lambda(t)\}}{\exp\{-\alpha\Lambda(t)\} - (1-p)}, \qquad i = 1, 2, \cdots, m.$$

If $\{\mathbf{N}(t),\ t \geq 0\}$ *satisfies* $\{\mathbf{N}(t),\ t \geq 0\}|L \sim \text{MEGPP}(\phi_1(t), \phi_2(t), \cdots, \phi_m(t), \alpha, L)$, *then it is MEGPP*$(\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, l_0)$.

Proof. Similarly to the proof of Theorem 5, it is sufficient to show the property for the univariate process. By the condition that $1 - \exp\{-\alpha\Lambda(\infty)\} < p < 1$, it holds that $\phi(t) \equiv \sum_{i=1}^{m} \phi_i(t) \geq 0$ for all $t \geq 0$. Then, following similar procedures to those described in the proof of Theorem 5, it can be shown that the stochastic intensity λ_t of the univariate process $\{N(t),\ t \geq 0\}$ is given by

$$\lambda_t = (-\alpha N(t-) + \alpha l_0)\lambda(t),$$

which implies that $\{N(t),\ t \geq 0\}$ is EGPP$(\lambda(t), \alpha, l_0)$. □

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From Theorem 6, we can obtain the following corollary. In the remaining part of this paper, other properties of the EGPP can also be derived from those of the MEGPP in similar ways, and we do not state them.

Corollary 2. *Assume that* $\Lambda(\infty) < \infty$. *Let* $\{N(t),\ t \geq 0\}$ *be a regular counting process and let* L *be a binomial random variable:* $L \sim \text{Bin}(l_0, p)$, *where* $l_0 \geq 1$ *and* $1 - \exp\{-\alpha\Lambda(\infty)\} < p < 1$. *For* $t > 0$, *let*

$$\phi(t) \equiv \frac{\lambda(t)\exp\{-\alpha\Lambda(t)\}}{\exp\{-\alpha\Lambda(t)\} - (1-p)}.$$

If $\{N(t),\ t \geq 0\}$ *satisfies* $\{N(t),\ t \geq 0\}|L \sim \text{EGPP}(\phi(t), \alpha, L)$, *then it is EGPP*$(\lambda(t), \alpha, l_0)$.

Note that, in Corollary 2, $\Phi(\infty) \equiv \int_0^\infty \phi(s)ds < \infty$. Thus, in Corollary 2, an incomplete EGPP has been characterized as the mixture of another group of incomplete EGPPs with a larger base intensity function. Corollary 2 (Theorem 6) improves understanding of the 'hierarchical structure' of an incomplete EGPP (MEGPP). By Corollary 2, if $\Lambda(\infty) < \infty$, then EGPP$(\lambda(t), \alpha, l_0)$ can be represented as the mixture of the baseline counting process

$$\text{EGPP}\left(\frac{\lambda(t)\exp\{-\alpha\Lambda(t)\}}{\exp\{-\alpha\Lambda(t)\} - (1-p_1)}, \alpha, l_1\right), \quad \text{where } p_1 > 1 - \exp\{-\alpha\Lambda(\infty)\}, \tag{2}$$

with the mixing distribution $L_1 \sim \text{Bin}(l_0, p_1)$. Again by Corollary 2, the baseline counting process in (2) for each $l_1 = 1, \cdots, l_0$ can also be represented as the mixture of the baseline counting process

$$\text{EGPP}\left(\frac{\lambda(t)\exp\{-\alpha\Lambda(t)\}}{\exp\{-\alpha\Lambda(t)\} - (1-p_1p_2)}, \alpha, l_2\right), \quad \text{where } p_1p_2 > 1 - \exp\{-\alpha\Lambda(\infty)\},$$

with the mixing distribution $L_2 \sim \text{Bin}(l_1, p_2)$. This separation procedure can be performed repeatedly. Note that, at each separation stage, the third parameter of the baseline EGPP can be $1, 2, \cdots, l_0$. Thus, if $p_1, p_2, \cdots$ are appropriately determined, it can be seen that any incomplete EGPP is separable into the corresponding baseline counting process and the mixing distribution for an infinite number of times. For example, choose p^* and n_0 so that $1 - \exp\{-\alpha\Lambda(\infty)\} < p^* < 1$ and $p^* + 1/(1+n_0) < 1$. Furthermore, determine p_i, $i = 1, 2, \cdots$, as follows:

$$p_1 = p^* + \frac{1}{1+n_0}, \qquad p_i = \frac{p^* + 1/(i+n_0)}{p^* + 1/((i-1)+n_0)}, \qquad i = 2, 3, \cdots.$$

Then $\prod_{i=1}^m p_i = p^* + 1/(m+n_0) > 1 - \exp\{-\alpha\Lambda(\infty)\}$, for $m = 1, 2, \cdots$, and the separation procedure can be performed for an infinite number of times. In this case, the baseline counting process at the mth stage is given by

$$\text{EGPP}\left(\frac{\lambda(t)\exp\{-\alpha\Lambda(t)\}}{\exp\{-\alpha\Lambda(t)\} - \left(1 - \prod_{i=1}^m p_i\right)}, \alpha, l_m\right),$$

and the base intensity function becomes larger and larger as the number of iterations increases; the 'upper bound function' for the base intensity function is given by

$$\phi^*(t) \equiv \frac{\lambda(t)\exp\{-\alpha\Lambda(t)\}}{\exp\{-\alpha\Lambda(t)\} - \exp\{-\alpha\Lambda(\infty)\}}, \qquad t \geq 0,$$

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which is the base intensity function in Corollary 1. In point process theory, a point process is said to be *infinitely divisible* if, for every k, it can be represented as the superposition of k independent, identically distributed point process components (see, e.g., [11], p. 255). In a similar sense, the property stated in Theorem 6 (Corollary 2) can be called the *infinitely separable* property of the incomplete MEGPP (EGPP).

4. Further characterization and properties of the MEGPP

In this section, we will discuss the superposition of MEGPPs, the dependence property, the conditional distribution of arrival times, and the marginal processes in the MEGPP. The following theorem characterizes the MEGPP as a superposition of another group of MEGPPs.

Theorem 7. *Let* $\{\tilde{\mathbf{N}}_i(t),\ t \geq 0\}$*, where* $\tilde{\mathbf{N}}_i(t) = (\tilde{N}_{i1}(t), \tilde{N}_{i2}(t), \cdots, \tilde{N}_{im}(t))$*, be MEGPP with the set of parameters* $(\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, l_i)$*,* $i = 1, 2, \cdots, r$*. Suppose that* $\{\tilde{\mathbf{N}}_i(t),\ t \geq 0\}$*,* $i = 1, 2, \cdots, r$*, are independent. Then* $\{\tilde{\mathbf{N}}(t),\ t \geq 0\}$*, where*

$$\tilde{\mathbf{N}}(t) = \left(\sum_{i=1}^{r} \tilde{N}_{i1}(t), \sum_{i=1}^{r} \tilde{N}_{i2}(t), \cdots, \sum_{i=1}^{r} \tilde{N}_{im}(t) \right),$$

is MEGPP$(\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, \sum_{i=1}^{r} l_i)$.

Proof. Due to Proposition 3(ii), a MEGPP can be constructed by the thinning of an EGPP. Thus, for the proof of Theorem 7, it is sufficient to show the result for the univariate process. Thus, letting $\{\tilde{N}_i(t),\ t \geq 0\}$ be independent EGPP$(\lambda(t), \alpha, l_i)$, $i = 1, 2, \cdots, r$, we will show that $\{\tilde{N}(t),\ t \geq 0\}$, where $\tilde{N}(t) \equiv \sum_{i=1}^{r} \tilde{N}_i(t)$, is EGPP$\left(\lambda(t), \alpha, \sum_{i=1}^{r} l_i\right)$. Denote by $\tilde{\lambda}_t$ and $\tilde{\lambda}_t^{(i)}$ the stochastic intensities of $\{\tilde{N}(t),\ t \geq 0\}$ and $\{\tilde{N}_i(t),\ t \geq 0\}$, $i = 1, 2, \cdots, r$, respectively, and by $\tilde{\mathcal{H}}_{t-}$ and $\tilde{\mathcal{H}}_{t-}^{(i)}$, $i = 1, 2, \cdots, r$, the corresponding histories. Then

$$\begin{aligned}
\sum_{i=1}^{r} \tilde{\lambda}_t^{(i)} &= \lim_{\Delta t \to 0} \sum_{i=1}^{r} \frac{\mathbb{P}\left(\tilde{N}_i(t, t+\Delta t) = 1 | \tilde{\mathcal{H}}_{t-}^{(j)},\ j = 1, 2, \cdots, r\right)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\mathbb{P}\left(\bigcup_{i=1}^{r} \{\tilde{N}_i(t, t+\Delta t) = 1\} | \tilde{\mathcal{H}}_{t-}^{(j)},\ j = 1, 2, \cdots, r\right)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\mathbb{P}\left(\sum_{i=1}^{r} \tilde{N}_i(t, t+\Delta t) = 1 | \tilde{\mathcal{H}}_{t-}^{(j)},\ j = 1, 2, \cdots, r\right)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\mathbb{P}\left(\tilde{N}(t, t+\Delta t) = 1 | \tilde{\mathcal{H}}_{t-}\right)}{\Delta t} = \tilde{\lambda}_t,
\end{aligned}$$

and $\tilde{\lambda}_t^{(i)} = (-\alpha \tilde{N}_i(t-) + \alpha l_i)\lambda(t)$, $i = 1, 2, \cdots, r$. Therefore,

$$\tilde{\lambda}_t = \left(-\alpha \sum_{i=1}^{r} \tilde{N}_i(t-) + \alpha \left(\sum_{i=1}^{r} l_i \right) \right) \lambda(t) = \left(-\alpha \tilde{N}(t-) + \alpha \left(\sum_{i=1}^{r} l_i \right) \right) \lambda(t). \qquad \square$$

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Now we will discuss the dependence property of the MEGPP. The following proposition states that there is a negative dependency between the increments in two different types of processes in any two (not necessarily disjoint) time intervals.

Proposition 4. *Let* $\{\mathbf{N}(t),\ t \geq 0\}$ *be MEGPP*$(\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, l_0)$. *For any* $t_{12} > t_{11}$ *and* $t_{22} > t_{21}$, *and for any* $1 \leq k_1 \neq k_2 \leq m$,

$$\begin{aligned} &Cov(N_{k_1}(t_{12}) - N_{k_1}(t_{11}), N_{k_2}(t_{22}) - N_{k_2}(t_{21})) \\ &= -l_0 \prod_{i=1,2} \left(\alpha \int_{t_{i1}}^{t_{i2}} \lambda_{k_i}(x) \exp\{-\alpha \Lambda(x)\} dx \right) < 0, \end{aligned}$$

and

$$\begin{aligned} &Corr(N_{k_1}(t_{12}) - N_{k_1}(t_{11}), N_{k_2}(t_{22}) - N_{k_2}(t_{21})) \\ &= -\sqrt{\frac{\prod_{i=1,2} \left(\alpha \int_{t_{i1}}^{t_{i2}} \lambda_{k_i}(x) \exp\{-\alpha \Lambda(x)\} dx \right)}{\prod_{i=1,2} \left(1 - \alpha \int_{t_{i1}}^{t_{i2}} \lambda_{k_i}(x) \exp\{-\alpha \Lambda(x)\} dx \right)}} < 0. \end{aligned}$$

Proof. For any sequence of the elements in $(t_{11}, t_{12}, t_{21}, t_{22})$ satisfying $t_{12} > t_{11}$ and $t_{22} > t_{21}$, from Theorem 4(iii), it can be shown that

$$\begin{aligned} &\mathbb{P}(N_{k_1}(t_{12}) - N_{k_1}(t_{11}) = n_1, N_{k_2}(t_{22}) - N_{k_2}(t_{21}) = n_2) \\ &= \binom{l_0}{n_1, n_2, l_0 - n_1 - n_2} \left[\prod_{i=1,2} \left(\alpha \int_{t_{i1}}^{t_{i2}} \lambda_{k_i}(x) \exp\{-\alpha \Lambda(x)\} dx \right)^{n_i} \right] \\ &\quad \times \left(1 - \sum_{i=1,2} \left(\alpha \int_{t_{i1}}^{t_{i2}} \lambda_{k_i}(x) \exp\{-\alpha \Lambda(x)\} dx \right) \right)^{l_0 - n_1 - n_2}, \end{aligned}$$

where $0 \leq n_1 + n_2 \leq l_0$. Then, according to the properties of the multinomial distribution (see also [14], p. 34), the results can be directly obtained. □

We will now derive the conditional joint distribution of arrival times of events given that $N_i(t) = n_i$, $i = 1, 2, \cdots, m$.

Theorem 8. *Let* $\{\mathbf{N}(t),\ t \geq 0\}$ *be MEGPP*$(\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, l_0)$. *The conditional joint arrival times distribution of*

$$(T_{i1}, T_{i2}, \cdots, T_{in_i}, i = 1, 2, \cdots, m | N_i(t) = n_i, i = 1, 2, \cdots, m)$$

is given by

$$\prod_{i=1}^{m} \left[n_i! \prod_{j=1}^{n_i} \left(\frac{\lambda_i(t_{ij}) \exp\{-\alpha \Lambda(t_{ij})\}}{\int_0^t \lambda_i(x) \exp\{-\alpha \Lambda(x)\} dx} \right) \right],$$

$0 < t_{i1} < \cdots < t_{in_i} \leq t$, $i = 1, 2, \cdots, m$.

Proof. Following a similar procedure to the proof of Theorem 1 and using Proposition 3(ii), it can be shown that the joint distribution of

$$(T_{i1}, T_{i2}, \cdots, T_{in_i}, N_i(t) = n_i, i = 1, 2, \cdots, m)$$

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is given by

$$(l_0)^{\overline{\sum_{i=1}^m n_i}} \cdot \prod_{i=1}^m \left(\prod_{j=1}^{n_i} \alpha p_i(t_{ij}) \lambda(t_{ij}) \exp\{-\alpha \Lambda(t_{ij})\} \right) \cdot \exp\left\{ -\left(l_0 - \sum_{i=1}^m n_i \right) \alpha \Lambda(t) \right\}$$
$$= (l_0)^{\overline{\sum_{i=1}^m n_i}} \cdot \prod_{i=1}^m \left(\prod_{j=1}^{n_i} \alpha \lambda_i(t_{ij}) \exp\{-\alpha \Lambda(t_{ij})\} \right) \cdot \left(\exp\{-\alpha \Lambda(t)\} \right)^{l_0 - \sum_{i=1}^m n_i}, \tag{3}$$

for any sequence of elements $(t_{i1}, \cdots, t_{in_i}, i = 1, 2, \cdots, m)$ satisfying $0 < t_{i1} < \cdots < t_{in_i} \le t$, $i = 1, 2, \cdots, m$. On the other hand, from Theorem 4,

$$\mathbb{P}(N_i(t) = n_i,\ i = 1, 2, \cdots, m) = \binom{l_0}{n_1, n_2, \cdots, n_m, l_0 - \sum_{i=1}^m n_i}$$
$$\times \prod_{i=1}^m \left(\alpha \int_0^t \lambda_i(x) \exp\{-\alpha \Lambda(x)\} dx \right)^{n_i} \left(\exp\{-\alpha \Lambda(t)\} \right)^{l_0 - \sum_{i=1}^m n_i}. \tag{4}$$

By combining (3) and (4), we have the desired result. □

Now the marginal process of the MEGPP will be characterized.

Theorem 9. *Let* $\{\mathbf{N}(t),\ t \ge 0\}$ *be MEGPP*$(\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, l_0)$. *The marginal process* $\{N_i(t),\ t \ge 0\}$ *is EGPP*$(\tilde{\lambda}_i(t), \alpha, l_0)$, *where*

$$\tilde{\lambda}_i(t) = \frac{\lambda_i(t) \exp\{-\alpha \Lambda(t)\}}{1 - \alpha \int_0^t \lambda_i(x) \exp\{-\alpha \Lambda(x)\} dx}, \qquad i = 1, 2, \cdots, m.$$

Proof. For the proof, the stochastic intensity functions of the marginal processes $\{N_i(t),\ t \ge 0\}$, $i = 1, 2, \cdots, m$, of MEGPP$(\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, l_0)$ will be derived. Denote by $\nu_t^{(i)}$ the stochastic intensity of $\{N_i(t),\ t \ge 0\}$, $i = 1, 2, \cdots, m$. Then $\nu_t^{(1)}$ can be specified as

$$\nu_t^{(1)} \equiv \lim_{\Delta t \to 0} \frac{\mathbb{P}\left(N_1(t, t + \Delta t) = 1 | \mathcal{H}_{t-}^{(1)} \right)}{\Delta t}$$
$$= \mathbb{E}_{\left(\mathcal{H}_{t-}^{(2)}, \mathcal{H}_{t-}^{(3)}, \cdots, \mathcal{H}_{t-}^{(m)} | \mathcal{H}_{t-}^{(1)} \right)} \left[\lim_{\Delta t \to 0} \frac{\mathbb{P}\left(N_1(t, t + \Delta t) = 1 | \mathcal{H}_{t-}^{(1)}, \mathcal{H}_{t-}^{(2)}, \cdots, \mathcal{H}_{t-}^{(m)} \right)}{\Delta t} \right],$$

where, from Definition 3,

$$\lim_{\Delta t \to 0} \frac{\mathbb{P}\left(N_1(t, t + \Delta t) = 1 | \mathcal{H}_{t-}^{(1)}, \mathcal{H}_{t-}^{(2)}, \cdots, \mathcal{H}_{t-}^{(m)} \right)}{\Delta t} = \left(-\alpha \left(\sum_{i=1}^m N_i(t-) \right) + \alpha l_0 \right) \lambda_1(t).$$

Thus, $\nu_t^{(1)}$ is obtained by

$$\nu_t^{(1)} = \left(-\alpha \left(N_1(t-) + \sum_{i=2}^m \mathbb{E}\left[N_i(t-) | \mathcal{H}_{t-}^{(1)} \right] \right) + \alpha l_0 \right) \lambda_1(t).$$

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We will now derive the conditional distribution of $\left(N_2(t-)|\mathcal{H}_{t-}^{(1)}\right)$; the other conditional distributions of $\left(N_i(t-)|\mathcal{H}_{t-}^{(1)}\right)$, $i=3,4,\cdots,m$, can be obtained similarly. From (3), the joint distribution of $(T_{11}, T_{12}, \cdots, T_{1N_1(t-)}, N_1(t-), N_2(t-))$ is obtained by

$$\frac{l_0!}{n_2!(l_0-n_1-n_2)!}\cdot\left(\prod_{j=1}^{n_1}\alpha\lambda_1(t_{1j})\exp\{-\alpha\Lambda(t_{1j})\}\right)\cdot\left(\int_0^t\alpha\lambda_2(x)\exp\{-\alpha\Lambda(x)\}dx\right)^{n_2}$$
$$\times\left(\sum_{i=3}^m\int_0^t\alpha\lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx+\exp\{-\alpha\Lambda(t)\}\right)^{l_0-n_1-n_2}.$$

Then, by similar arguments as those described in the proof of Theorem 2, it can be shown that the conditional distribution of $(N_2(t-)|T_{11}, T_{12}, \cdots, T_{1N_1(t-)}, N_1(t-))$ does not depend on $(T_{11}, T_{12}, \cdots, T_{1N_1(t-)})$, and accordingly,

$$(N_2(t-)|T_{11}, T_{12}, \cdots, T_{1N_1(t-)}, N_1(t-))\stackrel{D}{=}(N_2(t-)|N_1(t-)),$$

where '$\stackrel{D}{=}$' stands for equality in distribution. From Theorem 4,

$$\mathbb{P}(N_1(t)=n_1)=\binom{l_0}{n_1}\left(\alpha\int_0^t\lambda_1(x)\exp\{-\alpha\Lambda(x)\}dx\right)^{n_1}$$
$$\times\left(\sum_{i=2}^m\alpha\int_0^t\lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx+\exp\{-\alpha\Lambda(t)\}\right)^{l_0-n_1},$$

$$\mathbb{P}(N_1(t)=n_1, N_2(t)=n_2)=\binom{l_0}{n_1, n_2, l_0-n_1-n_2}\left(\alpha\int_0^t\lambda_1(x)\exp\{-\alpha\Lambda(x)\}dx\right)^{n_1}$$
$$\times\left(\alpha\int_0^t\lambda_2(x)\exp\{-\alpha\Lambda(x)\}dx\right)^{n_2}\left(\sum_{i=3}^m\alpha\int_0^t\lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx+\exp\{-\alpha\Lambda(t)\}\right)^{l_0-n_1-n_2},$$

and thus

$$\mathbb{P}(N_2(t)=n_2|N_1(t)=n_1)=\binom{l_0-n_1}{n_2}$$
$$\times\left(\frac{\alpha\int_0^t\lambda_2(x)\exp\{-\alpha\Lambda(x)\}dx}{\sum_{i=2}^m\alpha\int_0^t\lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx+\exp\{-\alpha\Lambda(t)\}}\right)^{n_2}$$
$$\times\left(\frac{\sum_{i=3}^m\alpha\int_0^t\lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx+\exp\{-\alpha\Lambda(t)\}}{\sum_{i=2}^m\alpha\int_0^t\lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx+\exp\{-\alpha\Lambda(t)\}}\right)^{l_0-n_1-n_2},$$

$n_2=0,1,\cdots,l_0-n_1$. Therefore,

$$\mathbb{E}[N_2(t-)|\mathcal{H}_{t-}^{(1)}]=\mathbb{E}[N_2(t-)|N_1(t-)]$$
$$=(l_0-N_1(t-))\frac{\alpha\int_0^t\lambda_2(x)\exp\{-\alpha\Lambda(x)\}dx}{\sum_{i=2}^m\alpha\int_0^t\lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx+\exp\{-\alpha\Lambda(t)\}},$$

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and, symmetrically,

$$\mathbb{E}\big[N_i(t-)|\mathcal{H}_{t-}^{(1)}\big] = \mathbb{E}[N_i(t-)|N_1(t-)] = (l_0 - N_1(t-))\frac{\alpha\int_0^t \lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx}{\sum_{i=2}^m \alpha\int_0^t \lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx + \exp\{-\alpha\Lambda(t)\}},$$

$i = 3, 4, \cdots, m$. Therefore,

$$\begin{aligned}\nu_t^{(1)} &= \left(-\alpha\left(N_1(t-) + \sum_{i=2}^m \mathbb{E}\Big[N_i(t-)|\mathcal{H}_{t-}^{(1)}\Big]\right) + \alpha l_0\right)\lambda_1(t)\\ &= \left(-\alpha\left(N_1(t-) + \sum_{i=2}^m \mathbb{E}\Big[N_i(t-)|N_1(t-)\Big]\right) + \alpha l_0\right)\lambda_1(t)\\ &= (-\alpha N_1(t-) + \alpha l_0)\frac{\lambda_1(t)\exp\{-\alpha\Lambda(t)\}}{\sum_{i=2}^m \alpha\int_0^t \lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx + \exp\{-\alpha\Lambda(t)\}}.\end{aligned}$$

By noting that

$$\begin{aligned}&\alpha\int_0^t \lambda_1(x)\exp\{-\alpha\Lambda(x)\}dx + \sum_{i=2}^m \alpha\int_0^t \lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx\\ &= \int_0^t \alpha\lambda(x)\exp\{-\alpha\Lambda(x)\}dx = 1 - \exp\{-\alpha\Lambda(t)\},\end{aligned} \tag{5}$$

we have the desired result. The stochastic intensity $\nu_t^{(i)}$, $i = 2, 3, \cdots, m$, can be obtained symmetrically. □

From the relationship in (5),

$$\tilde{\lambda}_1(t) = \frac{\lambda_1(t)\exp\{-\alpha\Lambda(t)\}}{\sum_{i=2}^m \alpha\int_0^t \lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx + \exp\{-\alpha\Lambda(t)\}},$$

and

$$\int_0^t \tilde{\lambda}_1(x)dx = -\frac{1}{\alpha}\ln\left(\sum_{i=2}^m \alpha\int_0^t \lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx + \exp\{-\alpha\Lambda(t)\}\right).$$

As

$$\alpha\int_0^t \lambda_1(x)\exp\{-\alpha\Lambda(x)\}dx + \sum_{i=2}^m \alpha\int_0^t \lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx + \exp\{-\alpha\Lambda(t)\} = 1, \qquad \forall\, t > 0,$$

we have

$$0 < \sum_{i=2}^m \alpha\int_0^\infty \lambda_i(x)\exp\{-\alpha\Lambda(x)\}dx + \exp\{-\alpha\Lambda(\infty)\} < 1,$$

whether $\Lambda(\infty) = \infty$ or $\Lambda(\infty) < \infty$. This implies that the marginal process $\{N_i(t),\ t \geq 0\}$ of MEGPP($\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, l_0$) is an incomplete EGPP. Thus, applying Corollary 1, it can be characterized as the mixture of the complete EGPPs.

Proposition 5. *Let* $\{N_i(t),\ t \geq 0\}$, $i = 1, 2, \cdots, m$, *be the marginal processes of MEGPP*($\lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t), \alpha, l_0$). *Let* L_i *be a binomial random variable:*

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$L_i \sim \text{Bin}\left(l_0, \alpha \int_0^\infty \lambda_i(x) \exp\{-\alpha\Lambda(x)\}dx\right)$, $i = 1, 2, \cdots, m$. *For* $t > 0$, *let*

$$\phi_i(t) \equiv \frac{\lambda_i(t) \exp\{-\alpha\Lambda(t)\}}{\alpha \int_t^\infty \lambda_i(x) \exp\{-\alpha\Lambda(x)\}dx}.$$

If $\{N_i^*(t),\ t \geq 0\}$ *satisfies* $\{N_i^*(t),\ t \geq 0\}|L_i \sim \text{EGPP}(\phi_i(t), \alpha, L_i)$, *then it is stochastically equivalent to* $\{N_i(t),\ t \geq 0\}$, $i = 1, 2, \cdots, m$.

Proof. Observe that $\tilde{\Lambda}_i(t) \equiv \int_0^t \tilde{\lambda}_i(s)ds$ is given by

$$\tilde{\Lambda}_i(t) = -\frac{1}{\alpha} \ln\left(1 - \alpha \int_0^t \lambda_i(x) \exp\{-\alpha\Lambda(x)\}dx\right),$$

and

$$\exp\{-\alpha\tilde{\Lambda}_i(t)\} = 1 - \alpha \int_0^t \lambda_i(x) \exp\{-\alpha\Lambda(x)\}dx.$$

Furthermore, we have

$$\exp\{-\alpha\tilde{\Lambda}_i(\infty)\} = 1 - \alpha \int_0^\infty \lambda_i(x) \exp\{-\alpha\Lambda(x)\}dx.$$

Then, by applying Corollary 1, we have the desired result. □

5. Some remarks for reliability applications

In reliability, various repair types have been developed based on different counting process models, and they have been applied to maintenance modeling and optimization. See [24], [26], and [27] for surveys on various practical maintenance models. More theoretical and sophisticated models have also been suggested in the literature (see, e.g., [20, 2, 4, 5, 6, 12, 1, 21]).

The two important basic repair types which have been most frequently applied in reliability are 'perfect' and 'minimal' repairs. By a perfect repair, the system is returned to a state that is *as good as new*, whereas by a minimal repair, the system is returned to a state that is *as bad as old* (see [13] and Section 5 of [8]). The failure process of a repairable system following the perfect repair process is modeled by the renewal process, and that following the minimal repair process is modeled by the NHPP.

As mentioned in Section 5 of [8], a counting process model specifies the corresponding repair type. The GPP can define a new repair type which is worse than minimal repair (see Section 5 of [8] and [17]), in the sense that the state of the system after the repair is worse than that just before the failure. On the other hand, if we define the counting process $\{N(t),\ t \geq 0\}$ as the failure process of a repairable system, where $N(t)$ represents the total number of failures of the system by time t, the stochastic intensity model of the EGPP in Definition 2 implies that the state of the system after the repair is better than that just before the failure, and, in this sense, the EGPP can be used to model a repair process which is better than the minimal repair process. More specifically, if we assume that the failure process $\{N(t),\ t \geq 0\}$ follows EGPP($\lambda(t)$, α, l_0), then this implies that, on each system failure, a repair which is better than the minimal repair is applied until $(l_0 - 1)$th failure. Note that, as mentioned earlier, in the EGPP, no event occurs after the occurrence of the l_0th event, whereas, in practice, the failure

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of the system recurs without limit. Thus, from the l_0th failure, the minimal repair process can be employed. From the counting-process point of view, the whole failure process in this case can be viewed as the combination of an EGPP and an NHPP, where the NHPP starts at a random time T_{l_0} which is the occurrence time of the l_0th event in EGPP($\lambda(t)$, α, l_0). Thus, under this type of repair process, before the l_0th failure of the system, the system state after the repair is better than after a minimal repair, and from l_0th failure, the system is minimally repaired. Interpreting this type of repair process from a practical point of view, in the initial stage of system operation, the reliability state of the system after a repair can be improved compared with the state just before the failure, but, after this initial stage, repairs cannot improve the reliability of the system, only restore the system to the as-bad-as-old condition. Employing this type of repair process, various maintenance models could be developed.

Appendix A. Proof of Theorem 1

Following similar procedures to those described in the proof of Theorem 1 in [8], the joint distribution of $(T_i, i = 1, 2, \cdots, N(t), N(t))$ is given by

$$f_{T_i,1\le i\le N(t),N(t)}(t_1, t_2, \cdots, t_n, n) = l_0^n \cdot \left(\prod_{i=1}^{n} \alpha\lambda(t_i)\exp\{-\alpha\Lambda(t_i)\}\right) \cdot \exp\{-(l_0 - n)\alpha\Lambda(t)\},$$

where $0 \le t_1 \le t_2 \le \cdots \le t_n \le t$, $n \le l_0$. Then, by integrating out the t_i, $i = 1, 2, \cdots, n$, the marginal distribution of $N(t)$ can be obtained as

$$\begin{aligned}\mathbb{P}(N(t) = n) &= \int_0^t \cdots \int_0^{t_3} \int_0^{t_2} f_{T_i,1\le i\le N(t),N(t)}(t_1, t_2, \cdots, t_n, n)dt_1dt_2\cdots dt_n \\ &= \binom{l_0}{n} (1 - \exp\{-\alpha\Lambda(t)\})^n (\exp\{-\alpha\Lambda(t)\})^{l_0-n}, \qquad n \le l_0. \end{aligned} \tag{6}$$

By Proposition 1, given $N(u_1) = n_1$, the process $\{N^{(u_1)}(t), \ t \ge 0\}$ is EGPP($\lambda(u_1 + t)$, α, $l_0 - n_1$). Then, by modifying the corresponding parameters in (6), we have

$$\begin{aligned}&\mathbb{P}(N(u_2) - N(u_1) = n_2 | N(u_1) = n_1) \\ &= \binom{l_0 - n_1}{n_2} (1 - \exp\{-\alpha(\Lambda(u_2) - \Lambda(u_1))\})^{n_2} \\ &\quad \times (\exp\{-\alpha(\Lambda(u_2) - \Lambda(u_1))\})^{l_0-n_1-n_2}, \ n_2 = 0, 1, \cdots, l_0 - n_1. \end{aligned} \tag{7}$$

The marginal distribution of $N(u_2) - N(u_1)$ can now be obtained as

$$\begin{aligned}\mathbb{P}(N(u_2) - N(u_1) = n_2) &= \sum_{n_1=0}^{l_0-n_2} \binom{l_0 - n_1}{n_2} (1 - \exp\{-\alpha(\Lambda(u_2) - \Lambda(u_1))\})^{n_2} \\ &\quad \times (\exp\{-\alpha(\Lambda(u_2) - \Lambda(u_1))\})^{l_0-n_1-n_2} \\ &\quad \times \binom{l_0}{n_1} (1 - \exp\{-\alpha\Lambda(u_1)\})^{n_1} (\exp\{-\alpha\Lambda(u_1)\})^{l_0-n_1} \\ &= \binom{l_0}{n_2} (\exp\{-\alpha\Lambda(u_1)\} - \exp\{-\alpha\Lambda(u_2)\})^{n_2} (1 - \exp\{-\alpha\Lambda(u_1)\} + \exp\{-\alpha\Lambda(u_2)\})^{l_0-n_2}. \end{aligned}$$

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Combining (6) and (7), $\mathbb{P}(N(u_1)=n_1, N(u_2)-N(u_1)=n_2)$ is given by

$$\mathbb{P}(N(u_1)=n_1, N(u_2)-N(u_1)=n_2)=\binom{l_0}{n_1}(1-\exp\{-\alpha\Lambda(u_1)\})^{n_1}(\exp\{-\alpha\Lambda(u_1)\})^{l_0-n_1}$$
$$\times\binom{l_0-n_1}{n_2}(1-\exp\{-\alpha(\Lambda(u_2)-\Lambda(u_1))\})^{n_2}(\exp\{-\alpha(\Lambda(u_2)-\Lambda(u_1))\})^{l_0-n_1-n_2}. \quad (8)$$

From the Markov property of the EGPP and Proposition 1,

$$\mathbb{P}(N(u_3)-N(u_2)=n_3|N(u_1)=n_1, N(u_2)-N(u_1)=n_2)$$
$$=\binom{l_0-n_1-n_2}{n_3}(1-\exp\{-\alpha(\Lambda(u_3)-\Lambda(u_2))\})^{n_3}$$
$$\times(\exp\{-\alpha(\Lambda(u_3)-\Lambda(u_2))\})^{l_0-n_1-n_2-n_3}. \quad (9)$$

By applying the above procedure recursively, for $j=4, 5, \cdots, r$, we can obtain

$$\mathbb{P}(N(u_j)-N(u_{j-1})=n_j|N(u_1)=n_1, N(u_2)-N(u_1)$$
$$=n_2, \cdots, N(u_{j-1})-N(u_{j-2})=n_{j-1}). \quad (10)$$

Finally, by combining (8)–(10),

$$\mathbb{P}(N(u_i)-N(u_{i-1})=n_i,\ i=1, 2, \cdots, r)$$
$$=\prod_{i=1}^{r}\left[\binom{l_0-\sum_{k=1}^{i-1}n_k}{n_i}(1-\exp\{-\alpha(\Lambda(u_i)-\Lambda(u_{i-1}))\})^{n_i}\right.$$
$$\left.\times(\exp\{-\alpha(\Lambda(u_i)-\Lambda(u_{i-1}))\})^{l_0-\sum_{k=1}^{i}n_k}\right], \qquad 0\le n_1+n_2+\cdots+n_r\le l_0,$$

where $\sum_{k=1}^{i-1}n_k\equiv 0$ when $i=1$. This can then be reduced to the desired result by simple algebra. □

Appendix B. Proof of Theorem 2

As in [8], p. 1167, define

$$\mathcal{H}_{[0,u)}\equiv\mathcal{H}_{u-}=\{N(u-), T_1, T_2, \cdots, T_{N(u-)}\}$$

and

$$\mathcal{H}_{[u,u+t)}=\left\{N((u+t)-)-N(u-), T_1^{(u)}, T_2^{(u)}, \cdots, T^{(u)}_{(N((u+t)-)-N(u-))}\right\}.$$

Note that

$$\lambda_t^u\equiv\lim_{\Delta t\to 0}\frac{\mathbb{P}\left(N^{(u)}(t, t+\Delta t)=1|\mathcal{H}_{[u,u+t)}\right)}{\Delta t}$$
$$=\mathbb{E}_{\mathcal{H}_{[0,u)}|\mathcal{H}_{[u,u+t)}}\left[\lim_{\Delta t\to 0}\frac{\mathbb{P}\left(N^{(u)}(t, t+\Delta t)=1|\mathcal{H}_{[0,u)}, \mathcal{H}_{[u,u+t)}\right)}{\Delta t}\right], \quad (11)$$

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where $\mathbb{E}_{\mathcal{H}_{[0,u)}|\mathcal{H}_{[u,u+t)}}[\cdot]$ represents the expectation with respect to the conditional distribution of $(\mathcal{H}_{[0,u)}|\mathcal{H}_{[u,u+t)})$ and

$$\lim_{\Delta t\to 0}\frac{\mathbb{P}\left(N^{(u)}(t,t+\Delta t)=1|\mathcal{H}_{[0,u)},\mathcal{H}_{[u,u+t)}\right)}{\Delta t}$$
$$=\lambda_{u+t}=\Big(-\alpha\Big(N(u-)+[N((u+t)-)-N(u-)]\Big)+\alpha l_0\Big)\lambda(u+t),$$

which contains only $N(u-)$ among $\mathcal{H}_{[0,u)}$. Thus, the expectation in (11) now becomes

$$\lambda_t^u=\Big(-\alpha\Big(\mathbb{E}[N(u-)|\mathcal{H}_{[u,u+t)}]+[N((u+t)-)-N(u-)]\Big)+\alpha l_0\Big)\lambda(u+t). \tag{12}$$

Denote by n_1, n_2, and $t_j^{(u)}$ the realizations of $N(u-)$, $N((u+t)-)-N(u-)$, and $T_j^{(u)}$, respectively. Then, by Proposition 1, the conditional joint distribution of

$$\Big(T_1^{(u)},T_2^{(u)},\cdots,T^{(u)}_{(N((u+t)-)-N(u-))},N((u+t)-)-N(u-)|N(u-)\Big)$$

is given by

$$(l_0-n_1)^{\underline{n_2}}\cdot\left(\prod_{j=1}^{n_2}\alpha\lambda\big(t_j^{(u)}\big)\exp\big\{-\alpha\Lambda\big(t_j^{(u)}\big)\big\}\right)$$
$$\times\exp\{-(l_0-n_1-n_2)\alpha\Lambda(u+t)\}\exp\{(l_0-n_1)\alpha\Lambda(u)\},$$

$u\le t_1^{(u)}\le t_2^{(u)}\le\cdots\le t_{n_2}^{(u)}<u+t$, $n_2=0,1,\cdots,l_0-n_1$. Then, using Theorem 1(i), the joint distribution of

$$\Big(T_1^{(u)},T_2^{(u)},\cdots,T^{(u)}_{(N((u+t)-)-N(u-))},N((u+t)-)-N(u-),N(u-)\Big)$$

and that of

$$\Big(T_1^{(u)},T_2^{(u)},\cdots,T^{(u)}_{(N((u+t)-)-N(u-))},N((u+t)-)-N(u-)\Big)$$

can be obtained. Now, combining these joint distributions, the conditional distribution of

$$\Big(N(u-)|T_1^{(u)},T_2^{(u)},\cdots,T^{(u)}_{(N((u+t)-)-N(u-))},N((u+t)-)-N(u-)\Big)$$

is obtained by

$$\binom{l_0-n_2}{n_1}\left(\frac{1-\exp\{-\alpha\Lambda(u)\}}{1-\exp\{-\alpha\Lambda(u)\}+\exp\{-\alpha\Lambda(u+t)\}}\right)^{n_1}$$
$$\times\left(\frac{\exp\{-\alpha\Lambda(u+t)\}}{1-\exp\{-\alpha\Lambda(u)\}+\exp\{-\alpha\Lambda(u+t)\}}\right)^{l_0-n_1-n_2},$$

$n_1=0,1,\cdots,l_0-n_2$, and, from this,

$$\mathbb{E}[N(u-)|T_1^{(u)},T_2^{(u)},\cdots,T^{(u)}_{(N((u+t)-)-N(u-))},N((u+t)-)-N(u-)]$$
$$=[l_0-(N((u+t)-)-N(u-))]\left(\frac{1-\exp\{-\alpha\Lambda(u)\}}{1-\exp\{-\alpha\Lambda(u)\}+\exp\{-\alpha\Lambda(u+t)\}}\right).$$

Therefore, from (12),

$$\lambda_t^u=\Big(-\alpha[N((u+t)-)-N(u-)]+\alpha l_0\Big)\frac{\lambda(u+t)\exp\{-\alpha\Lambda(u+t)\}}{1-\exp\{-\alpha\Lambda(u)\}+\exp\{-\alpha\Lambda(u+t)\}}.$$

□

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Appendix C. Proof of Theorem 3

From the proof of Theorem 1, the joint distribution of $(T_1, T_2, \cdots, T_{N(t)}, N(t))$ is given by

$$(l_0)^{\underline{n}} \cdot \left(\prod_{i=1}^{n} \alpha\lambda(t_i) \exp\{-\alpha\Lambda(t_i)\} \right) \cdot \exp\{-(l_0 - n)\alpha\Lambda(t)\},$$

$0 < t_1 \le t_2 \le \cdots \le t_n \le t$, and

$$\mathbb{P}(N(t) = n) = \binom{l_0}{n} (1 - \exp\{-\alpha\Lambda(t)\})^n \, (\exp\{-\alpha\Lambda(t)\})^{l_0 - n}.$$

Thus, the conditional joint distribution of $(T_1, T_2, \cdots, T_{N(t)} | N(t) = n)$ is obtained by

$$n! \prod_{i=1}^{n} \left(\frac{\alpha\lambda(t_i) \exp\{-\alpha\Lambda(t_i)\}}{1 - \exp\{-\alpha\Lambda(t)\}} \right), \qquad 0 < t_1 \le t_2 \le \cdots \le t_n \le t.$$

Then, defining

$$\phi(t, u) \equiv \frac{\lambda(u + t) \exp\{-\alpha\Lambda(u + t)\}}{1 - \exp\{-\alpha\Lambda(u)\} + \exp\{-\alpha\Lambda(u + t)\}},$$

and using Theorem 2, the joint conditional arrival time distribution of

$$\left(T_1^{(u)}, T_2^{(u)}, \cdots, T^{(u)}_{(N(u+t)-N(u))} | N(u + t) - N(u) = n \right)$$

is obtained by

$$n! \prod_{i=1}^{n} \left(\frac{\alpha\phi\left(t_i^{(u)} - u, u\right) \exp\left\{ -\alpha \int_0^{t_i^{(u)} - u} \phi(s, u) ds \right\}}{1 - \exp\{-\alpha \int_0^t \phi(s, u) ds\}} \right)$$

$$= n! \prod_{i=1}^{n} \left(\frac{\alpha\lambda\left(t_i^{(u)}\right) \exp\left\{ -\alpha\Lambda\left(t_i^{(u)}\right) \right\}}{\exp\{-\alpha\Lambda(u)\} - \exp\{-\alpha\Lambda(u + t)\}} \right), \qquad u < t_1^{(u)} \le t_2^{(u)} \le \cdots \le t_n^{(u)} \le u + t.$$

By replacing $u + t$ by v, we obtain the desired result. □

Appendix D. Proof of Theorem 5

By Proposition 3(ii), it suffices to show the result in Theorem 5 for the univariate process. Thus, letting $\{N(t),\ t \ge 0\}$ be EGPP$(\lambda(t), \alpha, l_0)$ with $\Lambda(\infty) < \infty$ and assuming the same condition on the random variable L, it will be shown that, if $\{N^*(t),\ t \ge 0\}$ satisfies $\{N^*(t),\ t \ge 0\} | L \sim$ EGPP$(\phi(t), \alpha, L)$, then it is stochastically equivalent to $\{N(t),\ t \ge 0\}$, where

$$\phi(t) \equiv \frac{\lambda(t) \exp\{-\alpha\Lambda(t)\}}{\exp\{-\alpha\Lambda(t)\} - \exp\{-\alpha\Lambda(\infty)\}}, \qquad t \ge 0.$$

Define $\mathcal{H}^*_{t-}$ as the history of the process $\{N^*(t),\ t \ge 0\}$ in [0,t):

$$\mathcal{H}^*_{t-} = \{N^*(t-), T_1^*, T_2^*, \cdots, T^*_{N^*(t-)}\},$$

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where T_i^* is the time from 0 to the time of the arrival of the ith event in $\{N^*(t),\ t \geq 0\}$, $0 \leq T_1^* \leq T_2^* \leq \cdots$. We will now derive the stochastic intensity λ_t^* of $\{N^*(t),\ t \geq 0\}$ and compare it with λ_t in Definition 2. Observe that

$$\lambda_t^* = \lim_{\Delta t \to 0} \frac{\mathbb{P}\left(N^*(t, t+\Delta t) = 1 | \mathcal{H}_{t-}^*\right)}{\Delta t} = \mathbb{E}_{(L|\mathcal{H}_{t-}^*)}\left[\lim_{\Delta t \to 0} \frac{\mathbb{P}\left(N^*(t, t+\Delta t) = 1 | \mathcal{H}_{t-}^*, L\right)}{\Delta t}\right],$$

where $\mathbb{E}_{(L|\mathcal{H}_{t-}^*)}[\cdot]$ stands for the expectation with respect to the conditional distribution of $(L|\mathcal{H}_{t-}^*)$ and

$$\lim_{\Delta t \to 0} \frac{\mathbb{P}\left(N^*(t, t+\Delta t) = 1 | \mathcal{H}_{t-}^*, L\right)}{\Delta t} = \Big(-\alpha N^*(t-) + \alpha L\Big)\phi(t).$$

Thus, $\lambda_t^* = \big(-\alpha N^*(t-) + \alpha \mathbb{E}[L|\mathcal{H}_{t-}^*]\big)\phi(t)$.

Denote by n, t_i, and l the realizations of $N^*(t-)$, T_i^*, and L, respectively. From the proof of Theorem 1, the conditional joint distribution of $\left(T_1^*, T_2^*, \cdots, T_{N^*(t-)}^*, N^*(t-)|L\right)$ is given by

$$(l)^{\underline{n}} \cdot \left(\prod_{i=1}^{n} \alpha\phi(t_i) \exp\{-\alpha\Phi(t_i)\}\right) \cdot \exp\{-(l-n)\alpha\Phi(t)\},$$

where $\Phi(t) \equiv \int_0^t \phi(s)ds$, $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < t$, $0 \leq n \leq l$. Then it can be shown that the conditional distribution of $\left(L|T_1^*, T_2^*, \cdots, T_{N^*(t-)}^*, N^*(t-)\right)$ is obtained by

$$\binom{l_0 - n}{l - n} \left(\frac{\exp\{-\alpha\Lambda(t)\} - \exp\{-\alpha\Lambda(\infty)\}}{\exp\{-\alpha\Lambda(t)\}}\right)^{l-n} \left(\frac{\exp\{-\alpha\Lambda(\infty)\}}{\exp\{-\alpha\Lambda(t)\}}\right)^{l_0-l},$$

$n \leq l \leq l_0$. Then,

$$\mathbb{E}[L|\mathcal{H}_{t-}^*] = N^*(t-) + (l_0 - N^*(t-))\left(\frac{\exp\{-\alpha\Lambda(t)\} - \exp\{-\alpha\Lambda(\infty)\}}{\exp\{-\alpha\Lambda(t)\}}\right),$$

and finally,

$$\lambda_t^* = \big(-\alpha N^*(t-) + \alpha \mathbb{E}[L|\mathcal{H}_{t-}^*]\big)\phi(t) = (-\alpha N^*(t-) + \alpha l_0)\lambda(t),$$

which completes the proof. □

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