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Normal and anomalous explosion mechanisms for nonlinear instabilities in three-wave coupling

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It is proved that a system of three-wave coupling with weak damping and the same sign of the coupling coefficients can be completely described by the Weierstrass function. On the basis of the bifurcation theory of the Weierstrass function that we have developed, criteria are found for distinguishing three phases of the three-wave coupling, namely the stable phase and the excited explosive and auto-explosive phases. Two kinds of explosion mechanisms in the three-wave coupling are found and described. The mechanism for the excited explosive phase is normal because trigger energy is required in such a mechanism and the threshold λth is proportional to δ^2 (δ is the frequency mismatch), whereas the anomalous mechanism for the auto-explosive phase does not need any trigger – only proper changes of internal parameters to make the absolute invariant J less than 27 are required. In addition, the repetition period of the explosive instability is given. Finally, the effects of weak damping on the bifurcation structure of the explosive instability in three-wave interactions are analysed.

1. Introduction

Nonlinear phenomena were known a century and a half ago (Faraday 1831; Rayleigh 1883, 1887). After World War II, the development of material sciences and electronics led to a great interest in nonlinear theory and its development. For 20 years and more, a variety of basic nonlinear phenomena in optics (Armstrong *et al.* 1970; Bullough 1977) and plasma physics (Davidson 1972; Nishikawa and Liu 1976; Weiland *et al.* 1977; Kaup *et al.* 1979; Galeev and Sagdeev 1979; Wang and Li 1991) have been widely investigated. The great interest in the latter has mostly been stimulated by research on controlled thermo nuclear fusion.

Explosive instabilities are of a complicated nonlinear nature. They are not restricted to three-wave interactions. The Rundle mode of an earthquake (Rundle 1989; Wu 1994) is an example. With the help of the free-energy function (Wu 1994) $f = -pu - a_0 - a_1 u^2 - a_2 u^4$, the Rundle mode of an earthquake may be described by the following equation:

$$\left(\frac{du}{dt}\right)^2 = 2(pu + a_0 + a_1 u^2 + a_2 u^4), \tag{1}$$

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where u represents shear dislocation or tension dislocation, and p, a_0, a_1 and a_2 are constants. Equation (1) can be transformed into the Weierstrass equation (Wang and Guo 1989)

$$\left(\frac{dy}{dt}\right)^2 = 4y^3 - g_2 y - g_3, \tag{2}$$

where

$$g_2 = 4a_0 a_2 + \tfrac{1}{3}a_1^2,$$

$$g_3 = \frac{1}{216}\begin{vmatrix} 12a_0 & 3p & 2a_1 \\ 3p & 2a_1 & 0 \\ 2a_1 & 0 & 12a_2 \end{vmatrix}$$

are invariants (see e.g. Akhiezer 1948). The solution of (1) can be expressed in terms of the Weierstrass function. We have found that the Weierstrass function has a bifurcation structure around a bifurcation point (Li *et al*. 1995). Hence the Rundle mode of earthquake should have a bifurcation structure, and so the study of this problem may be valuable in earthquake prediction.

As processes of three-wave coupling are basic for the understanding of complicated nonlinear plasma phenomena, they have attracted much attention (Sagdeev and Galeev 1969; Davidson 1972; Weiland *et al*. 1977; Kaup *et al*. 1979; Pfirsch 1990; Stenflo 1994). Three-wave interactions have been investigated extensively in the literature. These investigations fall roughly into two categories.

(i) Both spatial variation and temporal evolution are considered. As a result of three-wave interaction, singular spikes (Bers 1975; Kaup 1976; Rieman *et al*. 1977) or solitons (Kaup *et al*. 1979) can develop under specific conditions.

(ii) The spatial dependence is ignored, and attention is concentrated on the time evolution. The basic equations governing the processes of three-wave coupling can be written as (see e.g. Weiland *et al*. 1977)

$$\frac{da_j}{dt} + (\nu_j - i\omega_j)\, a_j = C_j\, a_k^*\, a_l^* \quad (k \neq l \neq j), \tag{3}$$

where $a_j (j = 1, 2, 3)$ represents the electrical component of the jth wave and is written as

$$a_j(t) = a_j^0(t) \exp[i(\omega_j t + \phi_j t + \phi_j)]. \tag{4}$$

Here we assume

$$\sum_{j=1}^{3} \phi_j = 0, \tag{5}$$

and the asterisk denotes the complex conjugate; C_j $(j = 1, 2, 3)$ are the coupling coefficients, and ν_j $(j = 1, 2, 3)$ is the damping coefficient of the jth wave. In the special case where dissipation is ignored, the Manley–Rowe relations can be derived from (3),

$$\frac{|a_1|^2}{C_1} - \frac{|a_2|^2}{C_2} = \text{const}, \quad \frac{|a_2|^2}{C_2} - \frac{|a_3|^2}{C_3} = \text{const}, \tag{6}$$

and C_j $(j = 1, 2, 3)$ may be chosen as real if the condition (5) is satisfied (Weiland *et al*. 1977). Equations (6) represent the conservation of energy; there are two different cases.

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(*a*) The coupling coefficients are not of the same sign. In this case the solutions of (3) are all finite, and may be expressed in terms of Jacobi elliptic functions (Armstrong *et al.* 1962; Sagdeev and Galeev 1969; Davidson 1972; Nishikawa and Liu 1976; Weiland *et al.* 1977; Wang and Li 1991).

(*b*) The coupling coefficients are the same sign. In this special case the amplitudes of three waves may grow in step in accordance with the Manley–Rowe relations. This is due to the fact that in this case one or two negative-energy waves take part in the three-wave coupling, and the amplitudes of negative-energy waves will grow as their wave energy declines (Landau and Lifshitz 1960). The solution corresponds to the explosive instability (Rosenbluth *et al.* 1968; Davidson 1972; Kaup *et al.* 1979).

The system of three-wave interactions has been studied by many authors. However, previous solutions for explosive instabilities were mostly asymptotic and not exact (see e.g. Weiland *et al.* 1977). A formal solution in terms of the Weierstrass function was obtained by Pfirsch (1990) for the explosive instability in three-wave interactions. The threshold condition for the explosive instability have been analysed qualitatively with the aid of the potential in the Weierstrass equation. However, just as in the study of the Rundle mode of an earthquake, it is difficult to perform a complete and exact analysis using the free energy because the most important features of the explosive instabilities such as the structures of the three phases, the conditions for phase transitions and the quantitative formula of the repetition period cannot be found by means of the traditional procedure. These characteristics can be fully elucidated as in this paper with the bifurcation theory of the Weierstrass function (Li *et al.* 1995).

In the case of weak dissipation, it will be shown that (3) with the same signs of coefficients can also be solved analytically. These are presented in Sec. 2 of this paper. In order to explore the bifurcation structure of explosive instability in the three-wave interaction, we first deal with the case $\nu = 0$, and the damping effect will be discussed in Sec. 5. It will be shown that the solution can be expressed in terms of the Weierstrass function. Then the bifurcation structure of the Weierstrass function and the expression for the repetition period of the explosive instability are presented in Sec. 3. On the basis of the bifurcation structure of the Weierstrass function, normal and anomalous explosion mechanisms in three-wave coupling are expounded and criteria distinguishing three phases are given in Sec. 4. In Sec. 5 the bifurcation characteristics of explosive instabilities in three-wave interactions are summarized and put into context. In the case of weak dissipation, the solution for explosive instability in three-wave coupling is found, and the damping effect, which mainly makes the explosive phases decay into the stable phase, is analysed further.

2. The analytical solution for three-wave interactions with the same signs of coupling coefficients

Let $(\omega_j, \mathbf{k}_j)$ $(j = 1, 2, 3)$ be three waves in a plasma, interacting with each other according to plasma dynamics, satisfying the relations

$$\sum_{j=1}^{3} \omega_j = \delta, \quad \sum_{j=1}^{3} \mathbf{k}_j = 0, \tag{7}$$

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where δ represents the mismatch of three frequencies and is very small, $|\delta| \ll |\omega_j|$ $(j = 1, 2, 3)$. In this section we consider the case of weak dissipation, and ignore the spatial dependence of wave amplitudes, which are governed by (3).

In this paper we restrict attention to the case in which C_j $(j = 1, 2, 3)$ are of the same sign. Without loss of generality, we assume C_j $(j = 1, 2, 3)$ to be positive real. A transformation for unification of the coupling coefficients can be made as follows:

$$b_j = \frac{a_j(t)}{C_j^{1/2}} \quad (j = 1, 2, 3). \tag{8}$$

Then (3) can be rewritten as

$$\frac{db_j}{dt} + (\nu_j - i\omega_j)\, b_j = \alpha b_k^* b_l^* \quad (j = 1, 2, 3) \tag{9}$$

where

$$\alpha = (C_1 C_2 C_3)^{1/2}. \tag{10}$$

Some algebraic manipulations used by Pfirsch (1990) in dealing with a dissipationless case can be modified for treating the case of weak dissipation. Putting

$$f = \alpha^* b_1 b_2 b_3, \tag{11}$$

(9) can be reduced to the form

$$\frac{db_j}{dt} + (\nu_j - i\omega_j)\, b_j = \frac{f^*}{b_j^*} \quad (j = 1, 2, 3). \tag{12}$$

For the sake of convenience, we assume that $\nu_1 = \nu_2 = \nu_3 = \nu$. From (12), we obtain

$$b_j b_j^* = F + F^* + \lambda_j = 2F_R + \lambda_j \quad (j = 1, 2, 3), \tag{13}$$

where

$$F(t) = \exp(-2\nu t) \int_0^t \exp(2\nu t')\, f(t')\, dt', \tag{14}$$

$\lambda_j = |b_j(0)|^2$ and $F_R = \mathrm{Re}\, F(t)$. Equation (13) indicates that in the case with dissipation $|b_j|^2$ $(j = 1, 2, 3)$ have the same functional dependence on time; this implies that, in the case of the same sign of the c_j, $|a_j|^2$ $(j = 1, 2, 3)$ grow in step, and an explosive instability can still occur under some conditions. In contrast, for the case with different signs of c_j, for example $c_j c_l > 0$ and $c_k < 0$ $(j \neq k \neq l)$, the phase difference between b_k and a_k is $\frac{1}{2}\pi$, and the phase differences both between b_j and a_j and between b_l and a_l are 0. Therefore the phase difference between a_k and a_j or a_k and a_l is $\frac{1}{2}\pi$. This means that $|a_k|^{-2}$ decreases as $|a_j|^2$ and $|a_l|^2$ grow, and this is exactly consistent with the Manley–Rowe relations. Hence the analysis in Sec. 2 is also suitable for the case with different signs of coupling coefficients, but is not so convenient as the usual method (cf. e.g. Sagdeev and Galeev 1969).

From (12) and (13), we obtain

$$\frac{db_j}{dt} = \left(i\omega_j - \nu_j + \frac{f^*}{2F_R + \lambda_j}\right) b_j \quad (j = 1, 2, 3). \tag{15}$$

Multiplying (15) by $\alpha^* b_k b_l$ and then summing over j, we find

$$\frac{df}{dt} = \sum_{j=1}^{3} \left(i\omega_j + \frac{f^*}{2F_R + \lambda_j}\right) f - 3\nu f. \tag{16}$$

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Equation (16) may be resolved into real and imaginary parts. As a result, we have

$$\frac{df_R}{dt} = \sum_{j=1}^{3} \frac{ff^*}{2F_R + \lambda_j} - \delta f_I - 3\nu f_R, \tag{17}$$

$$\frac{df_I}{dt} = \delta f_R - 3\nu f_I. \tag{18}$$

After some manipulation, assuming $\nu t \ll 1$, from (17) and (18), we obtain

$$ff^* = |\alpha|^2 \lambda_1 \lambda_2 \lambda_3 \prod_{j=1}^{3} \left(1 + \frac{2F_R}{\lambda_j}\right)(1 - 6e_2 \nu t), \tag{19}$$

$$\left(\frac{dF_R}{dt}\right)^2 = 2|\alpha|^2 (1 - 6e_2 \nu t) \sum_{j=1,\, k \neq l \neq j}^{3} \int (2F_R + \lambda_k)(2F_R + \lambda_l)\, dF_R$$

$$- \delta^2 F_R^2 (1 + e_1 \nu t) - 2\delta f_I(0) F_R - 4\nu F_R f_R - 6\nu \int f_R^2 \, dt', \tag{20}$$

where $0 < e_i < 1 (i = 1, 2)$. Here, provided that $\nu t \ll 1$, T is the repetition period of the nonlinear instability (see Sec. 3). Then we can estimate from (20) and (19) that the time dependence of F_R is basically the same as the case of $\nu = 0$ within one period, but the value of F_R slowly decreases from a period to the next, owing to damping.

Equation (20) is approximately integrable in the case of weak dissipation. In order to explore more deeply the characteristics of the explosive instability in three-wave coupling, we first consider the case of where damping is ignored; the damping effect will be discussed in Sec. 5.

Let $\nu = 0$; then we obtain from (20) a differential equation for $F_R(t)$:

$$\left(\frac{dF_R}{dt}\right)^2 = \tfrac{2}{3} A F_R^3 + B F_R^2 + 2C F_R + f_R^2(0), \tag{21}$$

where

$$\left.\begin{aligned} A &= 12|\alpha|^2, \\ B &= -\delta^2 + 4|\alpha|^2 \sum_{j=1}^{3} \lambda_j, \\ C &= -\delta f_I(0) + |\alpha|^2 (\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1), \\ f_R(0) &= |\alpha| (\lambda_1 \lambda_2 \lambda_3)^{1/2} \cos\beta \quad (\pi \geqslant \beta \geqslant 0), \\ f_I(0) &= |\alpha| (\lambda_1 \lambda_2 \lambda_3)^{1/2} \sin\beta. \end{aligned}\right\} \tag{22}$$

Equation (13) shows that one can study $F_R(t)$ in place of $|b_j|^2$ in the analysis of temporal behaviour. Furthermore, with the transformation (Wang and Guo 1989)

$$F_R(t) = \frac{6}{A} y(t) - \frac{B}{2A}, \tag{23}$$

(21) becomes

$$\left(\frac{dy}{dt}\right)^2 = 4y^3 - g_2 y - g_3. \tag{24}$$

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It is well known that this is the standard equation for the Weierstrass function $y(t, g_2, g_3)$ (Wang *et al.* 1989). Here g_2 and g_3 are two invariants of the Weierstrass function, and can be expressed as follows (see e.g. Akhiezer 1948)

$$g_2 = \tfrac{1}{12}(B^2 - 4AC),$$

$$g_3 = \frac{1}{216}\begin{vmatrix} 0 & A & B \\ A & B & 3C \\ B & 3C & 6f_R^2(0) \end{vmatrix}. \tag{25}$$

In the subsequent sections, the bifurcation structure of F_R and the repetition period of the explosive instability will be investigated on the basis of (22)–(25).

3. The bifurcation structure of the Weierstrass function and its period

Before proceeding with the present discussion, the main results concerning the bifurcation properties of the Weierstrass function (Li *et al.* 1995) are briefly reviewed as follows.

(i) Both g_2 and g_3 are relative invariants, while $J \equiv g_2^3/g_3^2$ is an absolute invariant of the Weierstrass function (see Akhiezer 1948);

$$y(t', J, J) \equiv (g_2/g_3)\, y(t, g_2, g_3)$$

(where $t' = (g_3/g_2)^{1/2}\, t$) is a new Weierstrass function, which determines the main characteristics of the Weierstrass function $y(t, g_2, g_3)$.

(ii) J ranges from $-\infty$ to ∞. The state with $J = 27$ is the bifurcation point of the Weierstrass function, around which there are three phases: the stable phase, the excited explosive phase and the auto-explosive phase.

(iii) The system is in the auto-explosive phase for $J < 27$. When $J \geqslant 27$, it is in the stable phase or the excited explosive phase, depending on the value of $y(0) \equiv \frac{1}{12}B$. It is in the excited explosive phase if

$$\tfrac{1}{12}B \geqslant (y_j)_{\max}; \tag{26}$$

it is in the stable phase if

$$(y_j)_{\min} \leqslant \tfrac{1}{12}B \leqslant (y_j)_{\mathrm{med}}, \tag{27}$$

where $(y_j)_{\min}$, $(y_j)_{\mathrm{med}}$ and $(y_j)_{\max}$ denote the minimum, median and maximum among the following three roots of the polynomial on the right-hand side of (24) respectively:

$$y_j = (\tfrac{1}{3}g_2)^{1/2} \cos[u + \tfrac{2}{3}\pi(j-1)] \quad (j = 1, 2, 3), \tag{28}$$

where

$$u = \tfrac{1}{3}\arccos\left[\mathrm{sign}(g_3)\left(\frac{27}{J}\right)^{1/2}\right] \tag{29}$$

and $\mathrm{sign}\,(x) = x/|x|$ $(x \neq 0)$, with $\mathrm{sign}(0) = 0$.

(iv) The excitation energy E needed for the phase transition from the stable phase to the excited explosive phase is given by

$$E = \frac{|g_3|}{2}\left[\left(\frac{J}{27}\right)^{1/2} + \mathrm{sign}(g_3)\right]. \tag{30}$$

(v) When $J = 27$, both the excited explosive phase and the auto-explosive phase are in a degenerate state, and (30) becomes

$$E = \begin{cases} g_3 & (g_3 > 0), \\ 0 & (g_3 \leqslant 0). \end{cases} \tag{31}$$

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Equation (31) indicates that the transition from the stable phase to the excited explosive phase is either of first order (for $g_3 > 0$) or second order (for $g_3 \leqslant 0$).

The main results reviewed above will lay the theoretical foundations of Secs 4 and 5. With the view to facilitating the later discussion, we cite formulae for the double periods (τ_1, τ_2) of the Weierstrass function (Li *et al.* 1995):

$$(\tau_2)^2 = \frac{2\pi^2(\theta_2^4+\theta_3^4)(\theta_3^4+\theta_4^4)(\theta_4^4-\theta_2^4)g_2}{9(\theta_2^8+\theta_3^8+\theta_4^8)g_3}, \tag{32}$$

$$\tau_1 = i\tau_2|\ln q|/\pi,$$

where θ_2, θ_3 and θ_4 are the last three of the theta functions θ_j $(j = 1, 2, 3, 4)$ and can be expressed as

$$\left.\begin{aligned} \theta_2(q) &= 2(q)^{1/4}(1+q^2+q^6+q^{12}+\ldots) \\ \theta_3(q) &= 1+2q+2q^4+2q^9+\ldots, \\ \theta_4(q) &= 1-2q+2q^4-2q^9+\ldots \quad . \end{aligned}\right\} \tag{33}$$

The relationships between g_2 and g_3 and the theta functions are (Wang and Guo 1989)

$$\left.\begin{aligned} g_2 &= \frac{2}{3}\left(\frac{\pi}{\tau_2}\right)^4(\theta_2^8+\theta_3^8+\theta_4^8), \\ g_3 &= \frac{4}{27}\left(\frac{\pi}{\tau_2}\right)^6(\theta_2^4+\theta_3^4)(\theta_3^4+\theta_4^4)(\theta_4^4-\theta_2^4). \end{aligned}\right\} \tag{34}$$

As can be seen from (25), g_2 and g_3 are determined by the initial values λ_j $(j = 1, 2, 3)$. The expression for J is found to be

$$J(q) \equiv \frac{g_2^3}{g_3^2} = \frac{27(\theta_2^8+\theta_3^8+\theta_4^8)^3}{2[(\theta_2^4+\theta_3^4)(\theta_3^4+\theta_4^4)(\theta_4^4-\theta_2^4)]^2}, \tag{35}$$

where the value of J can also be determined by the initial values λ_j $(j = 1, 2, 3)$. Thus q can be calculated from (33) and (35). Then, making use of (25), (35), (33) and (32), the double periods τ_1 and τ_2 can be calculated.

In addition, the explicit form for $J(q)$ can be written as (Madelung 1964)

$$J(q) = \left(\frac{1}{12}+20\sum_{n=1}^{\infty}\frac{n^3q^{2n}}{1-q^{2n}}\right)^3 \Big/ \left(\frac{1}{216}-\frac{7}{3}\sum_{n=1}^{\infty}\frac{n^5q^{2n}}{1-q^{2n}}\right)^2. \tag{36}$$

We can see from (36) that $J = 27$ at $q = 0$, J increases from 27 to $+\infty$ as q varies from 0 to 0.0432, and J decreases from 27 to $-\infty$ as q^2 varies from 0 to $-1/22.9$ (q being imaginary). In the light of the nature of the Weierstrass function, its singular points lie on the real axis of the complex plane (Wang and Guo 1989), and the solution of (24) is explosive only if q is real or purely imaginary. In both situations, the explicit expressions for the real repetition period of the explosive instabilities can be found as follows:

(i) For real q (i.e. $J > 27$, the excited explosive phase)

$$T^2 = \begin{cases} 2\pi^2\dfrac{g_2}{g_3}\theta_3^4(q)\left(\dfrac{1+k'^4-k'^2}{3J}\right)^{1/2} & (g_3 > 0), \quad (37) \\ -2(\ln q)^2\dfrac{g_2}{g_3}\theta_3^4(q)\left(\dfrac{1+k'^4-k'^2}{3J}\right)^{1/2} & (g_3 \leqslant 0), \quad (38) \end{cases}$$

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where $J \equiv g_2^3/g_3^2$ can be determined from the initial conditions (see (25)), and J, k' and q are related by

$$J = \frac{27[1+k'^4+(1-k'^2)^2]^3}{2[(2-k'^2)(1+k'^2)(2k'^2-1)]^2}, \tag{39}$$

$$\left.\begin{aligned} a &= \frac{1-k'^{1/2}}{2(1+k'^{1/2})}, \\ q &= a+2a^5+15a^9+\ldots \quad . \end{aligned}\right\} \tag{40}$$

(ii) For purely imaginary q (i.e. $J \leqslant 27$, the auto-explosive phase)

$$T^2 = \begin{cases} 2\pi^2 \dfrac{|g_2|}{g_3}\left(\dfrac{1-240q_m^2+2160q_m^4}{3J}\right)^{1/2} & (g_3 > 0), \quad (41) \\ 4(\ln q_m)^2 \left|\dfrac{g_2}{g_3}\right|\left(\dfrac{1-240q_m^2+2160q_m^4}{3J}\right)^{1/2} & (g_3 < 0), \quad (42) \end{cases}$$

where $q_m^2 = -q^2$, $J \equiv g_2^3/g_3^2$ is determined from the initial conditions (see (25)), and the relation between J and q_m^2 is

$$J = \frac{27(1-240q_m^2+2160q_m^4)^3}{(1+528q_m^2)^2(1-24q_m^2+24q_m^4)^2}. \tag{43}$$

4. Two kinds of explosive mechanism and phase criteria

Since the three-wave coupling can be completely described by $F_R(t)$ (cf. (13)), the system of three-wave interaction also has three phases (cf. (23) and Sec. 3). In this section we explore the criteria for distinguishing the three phases and two kinds of explosive mechanism for nonlinear instabilities are expounded.

The discriminant of the right-hand side of (24) is (cf. Wang and Guo 1989)

$$\Delta(\delta) = \tfrac{1}{16}[g_2^3(\delta) - 27g_3^2(\delta)]. \tag{44}$$

Obviously, $\Delta > 0$ and $\Delta < 0$ correspond to $J > 27$ and $J < 27$ respectively. In addition, when $\nu = 0$ but the frequency mismatch $\delta \neq 0$, on the basis of (18) and some curves of $f_R(t)$ (Pfirsch, 1990), it is apparent that $f_I(t)$ changes with t, and the instant when $f_I = 0$ can be found and chosen as the origin of time. Thus, without loss of generality, $\delta f_I(0)$ in (22) may be taken to be zero. Then it can be proved that

$$\Delta(0) \geqslant 0 \qquad \text{(with equality for } \lambda_1 = \lambda_2 = \lambda_3), \tag{45}$$

$$\Delta'(0) \equiv \Delta'(\delta)\,|_{\delta=0} = 0, \tag{46}$$

$$\Delta''(0) \equiv \Delta''(\delta)\,|_{\delta=0} = \alpha^{10}\lambda^5 P(\mu,\rho), \tag{47}$$

where

$$\lambda_1 = \lambda, \qquad \lambda_2 = \mu\lambda, \qquad \lambda_3 = \rho\lambda,$$

$$\begin{aligned} P(\mu,\rho) = \tfrac{1}{9}[&-2+10(\mu+\rho)+5.5(\mu^2+\rho^2)+58\mu\rho+52(\mu^2\rho+\mu\rho^2) \\ &+5.5(\mu^3+\rho^3)+10(\mu^4+\rho^4)+59\mu^2\rho^2+63.5(\mu\rho^3+\mu^3\rho) \\ &+5.5(\mu^2\rho^3+\mu^3\rho^2)+4.5(\mu^4\rho+\mu\rho^4)-2(\mu^5+\rho^5)]. \end{aligned} \tag{48}$$

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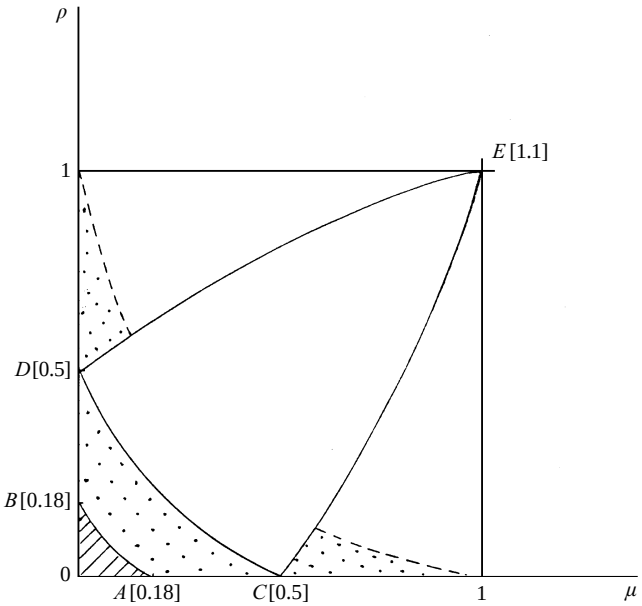


Figure 1. Several distinct regions of initial parameters (μ, ρ) for occurrence of explosive instabilities in the three-wave coupling. The curve *CED* represents $g_3 = 0$; the area *CEDBOAC* is for $g_3 < 0$, and the rest of the square is for $g_3 > 0$. The diagonally shaded area represents the region $P(\mu, \rho) < 0$, in which the anomalous explosive mechanism exists. Areas shaded with dots are regions where the non-resonant excited explosive phase ($\delta \neq 0$) can be triggered. The blank area in the square is the region where only the resonant excited explosive phase ($\delta = 0$) can be excited.

In Fig. 1 the separatrix AB is defined by $P(\mu, \rho) = 0$, by which two regions in which $P(\mu, \rho) > 0$ and $P(\mu, \rho) < 0$ respectively are separated: $P(\mu, \rho) < 0$ (i.e. $\Delta''(0)$ is negative), in the diagonally shaded area outside of which is the region where $\Delta''(0)$ is positive. In view of the fact that $\Delta(0) > 0$ and since $\Delta(\delta)$ increases with $|\delta|$ in the region where $P(\mu, \rho) > 0$, the stability of the stable phase changes with $|\delta|$ with the same pattern (cf. (30)). Therefore this region represents the normal state of the system. In contrast, the stability of the stable phase decreases with $|\delta|$ in the diagonally shaded area. Hence this area represents the anomalous state of the system, in which there exists an anomalous mechanism for the explosive instability in three-wave coupling.

4.1. The normal explosive mechanism and the criteria for the stable phase and the excited explosive phase

On the basis of (26), we find another condition for the excited explosive phase

$$\cos\left\{\frac{1}{3}\arccos\left[\operatorname{sign}(g_3)\left(\frac{27}{J}\right)^{1/2}\right]\right\} < \frac{4\alpha^2\lambda(1+\mu+\rho)-\delta^2}{2\{[4\alpha^2\lambda(1+\mu+\rho)-\delta^2]^2-48\alpha^4\lambda^2(\mu+\rho+\mu\rho)\}^{1/2}}. \quad (49)$$

Both the basic condition

$$P(\mu, \rho) > 0 \quad (50)$$

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and (49) represent the criteria for the excited explosive phase. Although (49) is rather complicated, it can be replaced by the following approximate inequalities:

$$\frac{\delta^2}{4\alpha^2\{1+\mu+\rho-[3(\mu+\rho+\mu\rho)]^{1/2}\}} < \lambda < \frac{\delta^2}{2\alpha^2\{2(1+\mu+\rho)-\{[17-\mathrm{sign}(g_3)]\,(\mu+\rho+\mu\rho)\}^{1/2}\}}, \quad (51)$$

$$2(1+\mu+\rho) > \{[17-\mathrm{sign}(g_3)]\,(\mu+\rho+\mu\rho)\}^{1/2}. \quad (52)$$

Thus (50)–(52) represent the criteria for the excited explosive phase in the case of $\delta \neq 0$ ((53) for $\delta = 0$)).

We see from the criteria (50)–(52) that there is a threshold of λ for exciting the explosive phase, and it is proportional to δ^2. This conclusion coincides exactly with the discussion at the beginning of this subsection. All of these characterize the features of the normal explosive mechanism for nonlinear instability. It should be noted that, as a special case of our results, the threshold condition $\lambda_1\lambda_2\lambda_3 \approx \delta^6$ obtained by Pfirsch (1990) is consistent with our conclusion.

Next, it is obvious that in the region where

$$2(1+\mu+\rho) < \{[17-\mathrm{sign}(g_3)]\,(\mu+\rho+\mu\rho)\}^{1/2} \quad (53)$$

only the resonant explosive phase ($\delta = 0$) can be excited. This result is attributed to the fact that the effect of frequency entrainment (Minorsky, 1947) no longer seems to be effective in this region. Two curves (the dashed lines) defined by $1+\mu+\rho = 2(\mu+\rho+\mu\rho)^{1/2}$ (for $g_3 > 0$) and the curve CD for $2(1+\mu+\rho) = 3[2(\mu+\rho+\mu\rho)]^{1/2}$ (for $g_3 < 0$) are drawn in Fig. 1.

Finally, the conditions for the stable phase fall into two cases: $\delta \neq 0$ and $\delta = 0$.

In the case of $\delta \neq 0$, the criterion for the stable phase is (53) or the following inequalities:

$$\lambda > \frac{\delta^2}{2\alpha^2\{2(1+\mu+\rho)-\{[17-\mathrm{sign}(g_3)]\,(\mu+\rho+\mu\rho)\}^{1/2}\}}, \quad (54)$$

or

$$\lambda < \frac{\delta^2}{4\alpha^2\{1+\mu+\rho-[3(\mu+\rho+\mu\rho)]^{1/2}\}} \quad (55)$$

in three regions of (μ,ρ) (shaded with dots in Fig. 1) defined by both (50) and (52).

In the case of $\delta = 0$ the system of three-wave coupling is stable in the whole region defined by (50), except for the case in which the excitation energy exceeds the threshold given by (30) in the area defined by (53).

4.2. An anomalous explosive mechanism and the criterion for the auto-explosive phase

The diagonally shaded area

$$P(\mu,\rho) < 0 \quad (56)$$

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may be further separated into two regions: $\Delta(\delta) > 0$, i.e. where

$$\delta^2 < \frac{\Delta(0)}{\alpha^{10}\lambda^5|P(\mu,\rho)|}, \tag{57}$$

and $\Delta(\delta) < 0$, i.e. where

$$\delta_2^2 > \delta^2 > \frac{\Delta(0)}{\alpha^{10}\lambda^5|P(\mu,\rho)|}. \tag{58}$$

Here $\pm\delta_2$ is the second pair of roots of the equation $\Delta(\delta) = 0$. Note that the region $P(\mu,\rho) < 0$ is inside the area where $g_3 < 0$ (cf. Fig. 1). In the region limited by (57), the state of system is still in the excited explosive phase, but the threshold λth is smaller than that in the case of $P(\mu,\rho) > 0$, and decreases with δ^2 because both of $g_3 < 0$ and J decreases with δ^2 (cf. (30)).

The criteria for the auto-explosive phase are (56) and (58) (i.e. $\Delta(\delta) < 0$), and no stable state can exist. The auto-explosive phase represents a special nonlinear instability induced by the self-excitation of the system. All of the above shows that the physical mechanism of the auto-explosive phase is special and anomalous, because only an appropriate change in the absolute invariant J can lead to a transition from the stable phase to the auto-explosive one. In contrast, external excitation energy is necessary for the excited explosion, since it is governed by the normal explosive mechanism.

It is evident from (58) that no resonant three-wave coupling can occupy the auto-explosive phase. The resonant three-wave interaction can only occupy the stable phase in the region where $P(\mu,\rho) > 0$ and the blank regions of the excited explosive phase (Fig. 1) if enough excitation energy is provided.

5. Discussion and conclusions

There are several salient conclusions that can be drawn from the results of the preceding analysis.

(i) When damping is negligible, the system of three-wave coupling with the same sign of the coupling coefficients can be described completely by the Weierstrass function $y(t, g_2, g_3)$, the basic properties of which are determined by its absolute invariant J. Around the bifurcation point $J = 27$ there are three phases.

(ii) The bifurcation structure of the explosive instability and both normal and anomalous explosive mechanisms for the nonlinear instabilities have been investigated in detail in this paper. The results are in agreement with the bifurcation theory of Poincaré (Minorsky 1947) and the theory of sudden change (Sturrock 1966). An auto-explosive state is very frequently associated with an excited explosive state. Resonant three-wave coupling is an only exception, in which there is no auto-explosive state (see Sec. 4). But one cannot affirm that no auto-explosive state could occur in non-resonant three-wave interactions.

(iii) $\frac{1}{2}g_2$ and $\frac{1}{2}g_3$ are similar to a gravitational force and a barrier respectively in (24). Equation (30) is a nonlinear barrier when $J \geqslant 27$. When $J < 27$, the nonlinear term dominates the system, so that the auto-explosive phase appears. This result has developed an idea of Sturrock (1966).

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(iv) The excited explosive phase can be triggered from the stable phase. One of the most important properties of the excited explosive phase is that $\lambda\text{th} \sim \delta^2$, which arises from the fact that g_3 grows with δ^2 in the regions defined by (50) and (52). For this reason, the explosive mechanism for the excited explosive phase is a normal one.

(v) Some of the anomalous properties in the region where $P(\mu, \rho) < 0$ are mainly due to the fact that $g_3 < 0$ there; therefore no trigger energy is required for a transition from the stable phase to the degenerate state of two explosive ones when $J = 27$ (cf. (31)). For the occurrence of the auto-explosive state, only an increase of a dispersion degree among λ_j $(j = 1, 2, 3)$ to make J less than 27 is needed. This special mechanism of explosive instability coincides with another idea of Sturrock (1966).

(vi) The repetition periods T of the explosive instabilities have been presented. The results are also valuable for other explosive processes governed by the Weierstrass equation.

(vii) Combining (14) and (21), we can solve (20) in the following way. We put

$$f_R(t) = h(t) f_{RO}(t), \tag{59}$$

with

$$\exp(-3\nu t) \leqslant h(t) \leqslant \exp(-\nu t), \tag{60}$$

where the left-hand side of (60) corresponds to the case of $t \ll T$ and the right-hand side to $t \to T$ as the explosion occurs. $f_{RO}(t) \equiv dF_{RO}/dt$ is the solution of (21). Obviously, the main properties of the explosive instability remain valid when $\nu T \ll 1$, but values of f_R and F_R slowly decrease from one period to the next owing to damping, so that the initial values λ_j also decrease (see (11)). Thus it is apparent that for $J > 27$ the excited explosive phase will decay back to the stable phase, because B (see (22)) decreases until (26) is no longer satisfied. In addition, if $J < 27$, it can be shown that J will gradually increase to 27 as the λ_j $(j = 1, 2, 3)$ die away owing to damping (see (25) and (22)), so that the auto-explosive phase will decay into the stable one. In the aforementioned two cases, the singular points of the solutions all move off the real axis of the complex plane (Li *et al.* 1995).

For the case in which the condition $\nu T \ll 1$ is broken, the effect of damping must be completely taken into consideration, and the system becomes more complicated. A further study of this issue is required.

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