Math. Proc. Camb. Phil. Soc. (1992), **112**, 309
Printed in Great Britain

Shape-preserving polynomial approximation in $C[-1, 1]$

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(*Received* 19 *August* 1991; *revised* 17 *December* 1991)

Abstract

A recent result of the first two authors has bridged the gap between the Timan-type pointwise estimate and the norm estimate for polynomial approximation in $C[-1, 1]$. Here we show that, when considering the second order modulus of smoothness, the aforementioned result holds with polynomials that preserve monotonicity, convexity and the values of the function at the end points of the interval $[-1, 1]$. This yields a generalization of results by Telyakovskii, Gopengauz, DeVore, Yu and the third author. Moreover, the construction here does not depend on the construction for the non-constrained approximation by polynomials.

1. *Introduction and the main result*

When approximating a function, it is often important to preserve some of its features. For instance, it does not make sense to approximate a density function by elements that may take negative values. Here, we will approximate monotone and convex functions on $[-1, 1]$ by polynomials and so, naturally, we prefer that the approximating polynomials will be monotone and convex respectively. Moreover, we will obtain an approximating polynomial that will coincide with f at ± 1.

We denote the class of polynomials of degree n by Π_n. The well-known Timan-type estimate for pointwise approximation of $f \in C[-1, 1]$ by a polynomial $P_n \in \Pi_n$ is given by

$$|f(x) - P_n(x)| \leqslant C(r)\, \omega^r(f, n^{-1}(n^{-1} + \sqrt{(1-x^2)})) \quad (-1 \leqslant x \leqslant 1), \tag{1·1}$$

where

$$\omega^r(f, t) = \sup_{0 \leqslant h \leqslant t} \|\Delta_h^r f(x)\|_{C[-1,1]} \tag{1·2}$$

and

$$\Delta_h^r f(x) = \begin{cases} \sum_{k=0}^{r} \binom{r}{k} (-1)^k f(x + (\frac{1}{2}r - k)h) & (x \pm \frac{1}{2}rh \in [-1, 1]) \\ 0 \quad \text{otherwise.} \end{cases} \tag{1·3}$$

Restricted forms of this result appear in the standard texts on approximation by

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Lorentz [7] and Timan [8]. The inequality (1·1) is Y. Brudnyi's extension of Timan's result. This, we trust, will be the standard form of the Timan-type estimate (see [4], 7·1) and will appear in that form in the much-awaited text on approximation by DeVore and Lorentz.

In 1985, it was shown by DeVore and Yu [2] that for a function $f \in C[-1,1]$ which is monotone, one can choose a monotone polynomial $P_n \in \Pi_n$ such that

$$|f(x)-P_n(x)| \leqslant C\omega^2\left(f, \frac{\sqrt{(1-x^2)}}{n}\right) \quad (-1 \leqslant x \leqslant 1). \tag{1·4}$$

Later, in 1986, Leviatan [5] showed that when f is convex one can choose a convex polynomial $P_n \in \Pi_n$ for which (1.4) is valid.

The norm estimate for the approximation of a function by polynomials was obtained by Ditzian and Totik [4], chapter 7. It was shown for $f \in L_p[-1,1]$, $1 \leqslant p \leqslant \infty$, that there exists $P_n \in \Pi_n$ such that

$$\|f-P_n\|_{L_p[-1,1]} \leqslant C(p,r)\,\omega^r_\phi(f, n^{-1})_p, \quad \phi(x) = \sqrt{(1-x^2)} \tag{1·5}$$

where the new measure of smoothness $\omega^r_\psi(f,t)_p$ (see [4]) is given by

$$\omega^r_\psi(f,t)_p = \sup_{0<h\leqslant t} \|\Delta^r_{h\psi} f\|_{L_p[-1,1]} \tag{1·6}$$

where ψ is a function of x and

$$\Delta^r_{h\psi} f(x) = \begin{cases} \sum_{k=0}^{r} \binom{r}{k} (-1)^k f(x+(\frac{1}{2}r-k)\,h\psi(x)) & (x \pm \tfrac{1}{2}r\,h\psi(x) \in [-1,1]) \\ 0 \quad \text{otherwise.} \end{cases} \tag{1·7}$$

It is worth noting that for $0 < p < 1$, (1·5) was recently established by DeVore, Leviatan and Yu [1]. However, here we restrict our attention to $p = \infty$ and refer to (1·5) only for that case. We will write $\omega^r_\psi(f,t)$ for $\omega^r_\psi(f,t)_\infty$.

For $r = 2$, and $f \in C[-1,1]$, Leviatan [5, 6] proved that (1·5) (for $p = \infty$) is valid with $P_n \in \Pi_n$ preserving monotonicity or convexity.

For $f \in C[-1,1]$ and $0 \leqslant \lambda \leqslant 1$, it was recently shown by Ditzian and Jiang [3] that there exists $P_n \in \Pi_n$ such that

$$|f(x)-P_n(x)| \leqslant C(r,\lambda)\,\omega^r_{\phi^\lambda}(f, n^{-1}(n^{-1}+\sqrt{(1-x^2)})^{1-\lambda}) \quad (-1 \leqslant x \leqslant 1), \tag{1·8}$$

and that result bridged the gap between (1·1) (the case $\lambda = 0$) and (1·5) (the case $\lambda = 1$).

We strengthen the above result for $r = 2$ in the following theorem which is the main result of this paper.

THEOREM 1.1. *For $f(x) \in C[-1,1]$, $0 \leqslant \lambda \leqslant 1$ and $\phi(x) = \sqrt{(1-x^2)}$, there exists $P_n \in \Pi_n$ with $P_n(\pm 1) = f(\pm 1)$ such that*

$$|f(x)-P_n(x)| \leqslant C(\lambda)\,\omega^2_{\phi^\lambda}(f, n^{-1}\phi(x)^{1-\lambda}) \quad (-1 \leqslant x \leqslant 1), \tag{1·9}$$

with $C(\lambda)$ independent of n and f. Moreover, if f is monotone or convex, or both, P_n can be chosen to be monotone or convex or both, respectively. The choice of P_n may be made independent of λ.

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2. *Proof of the main result*

In this section, we will prove the main result of the paper pending the proof of a Jackson-type inequality, which will be proved in stages in the next two sections. The proof here is constructive; that is, for $f \in C[-1,1]$, we will construct a sequence of polynomials $L_n(f,x)$ where L_n are linear operators on f such that

A $L_n(f,x) \in \Pi_{4n}$,
B $L_n(f,x)$ is monotone whenever $f(x)$ is monotone,
C $L_n(f,x)$ is convex whenever $f(x)$ is convex,
D $L_n(f,\pm 1) = f(\pm 1)$,
E $\|L_n(f,\cdot)\|_{C[-1,1]} \leqslant C\|f\|_{C[-1,1]}$,

and

F $|f(x)-L_n(f,x)| \leqslant C(\lambda)(\phi(x)^{1-\lambda}/n)^2 \|\phi^{2\lambda} f''\|_\infty$ for $f \in C^2[-1,1]$.

It should be emphasized that we may use $L_n(f,x) \in \Pi_{4n}$ instead of $L_n(f,x) \in \Pi_n$ as P_n may be chosen to be $L_{[n/4]}(f,x)$ and

$$\omega^2_{\phi^\lambda}(f,4t) \leqslant C\omega^2_{\phi^\lambda}(f,t). \tag{2·1}$$

The properties B, C, and D are the shape-preserving properties.

We now use the equivalence (see [**4**], chapter 2)

$$\omega^2_{\phi^\lambda}(f,t) \sim K_{\phi^\lambda,2}(f,t^2) \equiv \inf_{g \in C^2[-1,1]} (\|f-g\| + t^2\|\phi^{2\lambda} g''\|). \tag{2·2}$$

Hence there exists a function g_t satisfying

$$t^2\|\phi^{2\lambda} g_t''\| \leqslant C\omega^2_{\phi^\lambda}(f,t) \tag{2·3}$$

and

$$\|f-g_t\| \leqslant C\omega^2_{\phi^\lambda}(f,t). \tag{2·4}$$

We choose $t = \phi(x)^{1-\lambda}/n$ and apply E, F and the linearity of $L_n(f,x)$ to prove (1·9). Specifically,

$$\begin{aligned} |f(x)-L_n(f,x)| &\leqslant |f(x)-g_t(x)| + |g_t(x)-L_n(g_t,x)| + |L_n(f-g_t,x)| \\ &\leqslant (1+\|L_n\|)\|f-g_t\| + C(\lambda)(\phi(x)^{1-\lambda}/n)^2\|\phi^{2\lambda} g_t''\| \\ &\leqslant C\omega^2_{\phi^\lambda}(f,\phi(x)^{1-\lambda}/n). \end{aligned}$$

Hence the construction L_n satisfying A–F is sufficient to complete the proof.

Following [**2**] we define

$$L_n(f,x) = f(-1) + \sum_{j=0}^{n-1} s_j(R_j(x)-R_{j+1}(x)) \tag{2·5}$$

where

$$s_j = \frac{f(\xi_{j+1})-f(\xi_j)}{\xi_{j+1}-\xi_j}, \quad \xi_j = 1-R_j(1) \tag{2·6}$$

and the polynomial $R_j(x)$ (depending of n) is defined by

$$R_j(x) = \int_{-1}^{x} r_j(u)\,du, \quad r_j(\cos t) = \int_{-((n-j)/n)\pi}^{((n-j)/n)\pi} K_n(t+u)\,du \tag{2·7}$$

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where $K_n(t)$ is given by

$$K_n(t) = \lambda_n \left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t} \right)^8, \quad \int_{-\pi}^{\pi} K_n(t)\, dt = 1. \tag{2·8}$$

The operator $L_n(f,x)$ is obviously linear. It is clear from (2·5–2·8) that $L_n(f,x)$ maps $C[-1,1]$ into Π_{4n}. As $R_0(x) = 1+x$, $R_n(x) = 0$ and $R_j(x)$ are strictly increasing in j for $x > -1$, we have $-1 = \xi_0 < \ldots < \xi_n = 1$. The identity (2·5) implies $L_n(f, \pm 1) = f(\pm 1)$. Also (2·5) can be rewritten as

$$L_n(f,x) = f(-1) + s_0 R_0(x) + \sum_{j=1}^{n-1} (s_j - s_{j-1}) R_j(x). \tag{2·9}$$

For a monotone $f(x)$, the s_j have the same sign and since (2·7) implies that $R_j(x) - R_{j+1}(x)$ is increasing in x, (2·5) implies that $L_n(f,x)$ is monotone too. For a convex function f, the $s_j - s_{j-1}$ have the same sign and by (2·7) for $0 < t < \pi$, we have (see [5], p. 474)

$$R_j''(x) = r_j'(x) = \frac{-1}{\sin t} \frac{d}{dt} \int_{-((n-j)/n)\pi}^{((n-j)/n)\pi} K_n(t+u)\, du$$

$$= \frac{-\lambda_n}{\sin t} \left(\sin \frac{nt + (n-j)\pi}{2} \right)^8 \left(\left(\sin \left(\frac{t}{2} + \frac{n-j}{2n} \pi \right) \right)^{-8} - \left(\sin \left(\frac{t}{2} - \frac{n-j}{2n} \pi \right) \right)^{-8} \right) \geqslant 0.$$

Hence $L_n(f,x)$ is convex. In the last several lines, we summarized some of the known properties of $L_n(f,x)$ investigated by DeVore and Yu [2] and Leviatan [5]. DeVore and Yu [2], p. 330 also proved the boundedness of $L_n(f,x)$, i.e. property E. Therefore, as we know properties A–E of $L_n(f,x)$, we will complete the proof of Theorem 1·1 when we prove the Jackson-type estimate, namely, property F of $L_n(f,x)$. This will be done in the next two sections.

Remark 2·1. The polynomial $L_n(f,x)$ was constructed in a way that did not depend on λ, the monotonicity of f or the convexity of f. The construction here is independent of the construction in [3].

3. *A Jackson inequality for an intermediate operator*

The goal of this section is to prove the Jackson-type estimate for an intermediary operator $S_n(f,x)$ defined below in (3·4).

We first recall some properties of the ξ_j defined by (2·6) as shown by DeVore and Yu [2] (with slightly different ξ_j). (See also [5], p. 473, lemma A.)

LEMMA 3·1. *Let* $\delta_j = n^{-1} \sin((j/n)\pi) + n^{-2}$, *for* $j = 0, \ldots, n$. *Then*

$$C^{-1}\delta_j \leqslant \xi_{j+1} - \xi_j \leqslant C\delta_j \quad \text{for } j = 0, \ldots, n-1, \tag{3·1}$$

$$C^{-1}\delta_j \leqslant \delta_{j+1} \leqslant C\delta_j \quad \text{for } j = 0, \ldots, n-1, \tag{3·2}$$

and $$\xi_{j+1} - \xi_j \leqslant C(\sqrt{(1-u^2)})/n \quad \text{for} \quad \xi_j \leqslant u \leqslant \xi_{j+1} \quad \text{and} \quad j = 1, \ldots, n-2. \tag{3·3}$$

We define the intermediary operator $S_n(f,x)$ by

$$S_n(f,x) = f(-1) + s_0(1+x) + \sum_{j=1}^{n-1} (s_j - s_{j-1})\, \phi_j(x) \tag{3·4}$$

where $\phi_j(x) = (x - \xi_j)_+$ and both ξ_i and s_j are given by (2·6).

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THEOREM 3·2. *For* $f \in C^2[-1,1]$, $0 \leqslant \lambda \leqslant 1$, $\phi(x) = \sqrt{(1-x^2)}$ *and* $S_n(f,x)$ *defined by* (3·4), *we have*

$$|f(x)-S_n(f,x)| \leqslant C\frac{\phi(x)^{2-2\lambda}}{n^2}\|\phi^{2\lambda}f''\|_{C[-1,1]}. \tag{3·5}$$

Proof. Following [6], we have for $\xi_j \leqslant x \leqslant \xi_{j+1}$, $0 \leqslant j \leqslant n-1$,

$$|f(x)-S_n(f,x)| = \frac{(x-\xi_j)(\xi_{j+1}-x)}{\xi_{j+1}-\xi_j}\left|\frac{1}{\xi_{j+1}-x}(f(\xi_{j+1})-f(x))-\frac{1}{x-\xi_j}(f(x)-f(\xi_j))\right|$$

$$\leqslant \frac{1}{\xi_{j+1}-\xi_j}\left(\int_{\xi_j}^{x}(u-\xi_j)(\xi_{j+1}-x)|f''(u)|\,du+\int_{x}^{\xi_{j+1}}(\xi_{j+1}-u)(x-\xi_j)|f''(u)|\,du\right). \tag{3·6}$$

The above implies

$$|f(x)-S_n(f,x)| \leqslant (\xi_{j+1}-\xi_j)^2 \max_{\xi_j \leqslant u \leqslant \xi_{j+1}} \phi(u)^{-2\lambda}\|\phi^{2\lambda}f''\|_\infty \tag{3·7}$$

which, combined with (3·3) for $\xi_j \leqslant u, x \leqslant \xi_{j+1}$, yields (3·5) for $\xi_j \leqslant x \leqslant \xi_{j+1}$ and $1 \leqslant j \leqslant n-2$.

For $-1 \leqslant x < \xi_1$, we use (3·6) to write

$$|f(x)-S_n(f,x)| \leqslant \frac{1}{1+\xi_1}\left((\xi_1-x)\int_{-1}^{x}(1+u)^{1-\lambda}\,du+(1+x)\int_{x}^{\xi_1}\frac{\xi_1-u}{(1+u)^\lambda}\,du\right)\|\phi^{2\lambda}f''\|_\infty$$

$$\leqslant \frac{1}{1+\xi_1}((1+\xi_1)(1+x)^{2-\lambda}+(1+x)^{1-\lambda}(\xi_1-x)^2)\,\|\phi^{2\lambda}f\|_\infty$$

$$\leqslant 2(1+\xi_1)\,\phi(x)^{2-2\lambda}\,\|\phi^{2\lambda}f''\|_\infty. \tag{3·8}$$

Now (3·1) for $j=0$ implies $1+\xi_1 \leqslant C_1 n^{-2}$, which yields (3·5) for $-1 \leqslant x \leqslant \xi_1$. The estimate (3·5) for $\xi_{n-1} \leqslant x \leqslant 1$ can be treated similarly. ∎

4. *The Jackson inequality for* $L_n(f,x)$

In this section, we prove the Jackson inequality for $L_n(f,x)$ given by (2·2). That is, we show that property F of Section 2 is valid, and hence we complete the proof of Theorem 1·1.

Following earlier papers, we define

$$d_{j,n}(t) = \max(1, n \operatorname{dist}(t, \{-(j\pi/n), (j\pi/n)\})). \tag{4·1}$$

The following lemma is crucial for our proof.

LEMMA 4·1. *For* $0 \leqslant \lambda \leqslant 1$, $R_j(x)$ *defined by* (2·7), $\phi_j(x) = (x-\xi_j)_+$, $d_{j,n}(t)$ *given by* (4·1) *and* $x = \cos t$ *for* $0 \leqslant t \leqslant \pi$, *we have*

$$|\phi_j(x)-R_j(x)| \leqslant Cn^{-1}\phi(x)^{2(1-\lambda)}(\sin(j/n)\pi)^{2\lambda-1}d_{n-j,n}(t)^{-5}. \tag{4·2}$$

Proof. We first use the estimate

$$|\phi_j(x)-R_j(x)| \leqslant Cn^{-1}\min\{\sin t, \sin(j/n)\pi\}\,d_{n-j,n}(t)^{-5} \tag{4·3}$$

obtained by combining [2], lemma 5 and [6], lemma 6. For $2\lambda-1=\alpha \geqslant 0$, (4·3)

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implies (4·2) since $\min\{a,b\} \leqslant a^{1-\alpha}b^{\alpha}$ for $0 \leqslant a,b$ and $0 \leqslant \alpha \leqslant 1$. For $\sin t \geqslant \sin(j/n)\pi$, (4·3) implies (4·2) even if $2\lambda-1<0$. Therefore, to complete the proof of (4·2), we assume that $-1 \leqslant 2\lambda-1<0$ and $\sin t < \sin(j/n)\pi$ in which case $0 \leqslant t \leqslant \pi$ implies that either $t \in (0, \min(((n-j)/n)\pi, (j/n)\pi)) \equiv I_j'$ or $t \in (\max(((n-j)/n)\pi, (j/n)\pi), \pi) \equiv I_j''$. Since $\phi_j(\pm 1) = R_j(\pm 1)$, we have

$$\phi_j(x)-R_j(x) = \int_{-1}^{x} (\phi_j'(u)-R_j'(u))\,du = -\int_{x}^{1} (\phi_j'(u)-R_j'(u))\,du \tag{4·4}$$

and we use $\int_{-1}^{x}$ and $\int_{x}^{1}$ for the estimate on I_j'' and I_j', respectively. As the proofs for $t \in I_j'$ and $t \in I_j''$ are essentially the same, we will give the estimate only for $t \in I_j'$. An important tool in the proof is the following estimate by DeVore and Yu [2], (2·12b):

$$|\phi_j'(x)-R_j'(x)| \leqslant C d_{n-j,n}(t)^{-7}. \tag{4·5}$$

Writing $x = \cos t$, $t \in I_j'$ and $-1 \leqslant 2\lambda-1<0$, we have

$$|\phi_j(x)-R_j(x)| = \left|\int_{x}^{1} (\phi_j'(u)-R_j'(u))\,du\right| \leqslant C\int_{0}^{t} d_{n-j,n}(v)^{-7}\sin v\,dv$$

$$\leqslant C(\sin(j/n)\pi)^{2\lambda-1}\int_{0}^{t} \sin v(\sin(j/n)\pi)^{1-2\lambda} d_{n-j,n}(v)^{-7}\,dv \equiv J_j,$$

and

$$(\sin(j/n)\pi)^{1-2\lambda} = (\sin((n-j)/n)\pi)^{1-2\lambda} \leqslant (\sin v + |v-((n-j)/n)\pi|)^{1-2\lambda}$$

$$\leqslant (\sin v)^{1-2\lambda} + |v-((n-j)/n)\pi|^{1-2\lambda}. \tag{4·6}$$

Hence, it is sufficient to estimate J_j' and J_j'' formed from J_j above by replacing $(\sin(j/n)\pi)^{1-2\lambda}$ by $(\sin v)^{1-2\lambda}$ and $|v-((n-j)/n)\pi|^{1-2\lambda}$, respectively.

We observe that for $\alpha \geqslant \beta > 1$ and $t < ((n-j)/n)\pi$,

$$\int_{0}^{t} d_{n-j,n}(v)^{-\alpha}\,dv \leqslant C(\alpha)\,n^{-1} d_{n-j,n}(t)^{-\alpha+1} \leqslant C(\alpha)\,n^{-1} d_{n-j,n}(t)^{-\beta+1}. \tag{4·7}$$

We now write, using (4·7) with $\alpha = 7$ and $\beta = 6$,

$$J_j' \leqslant C(\sin(j/n)\pi)^{2\lambda-1}\int_{0}^{t} (\sin v)^{2-2\lambda} d_{n-j,n}(v)^{-7}\,dv$$

$$\leqslant C_1(\sin((n-j)/n)\pi)^{2\lambda-1}(\sin t)^{2-2\lambda}\int_{0}^{t} d_{n-j,n}(v)^{-7}\,dv$$

$$\leqslant C_1 n^{-1}(\sin((n-j)/n)\pi)^{2\lambda-1}(\sin t)^{2-2\lambda} d_{n-j,n}(t)^{-5}.$$

Next use (4·7) with $\alpha = 7+2\lambda-1$ and $\beta = 6$ and the inequality

$$|v-((n-j)/n)\pi| \leqslant n^{-1} d_{n-j,n}(v)$$

to estimate J_j'' as follows:

$$J_j'' = C(\sin(j/n)\pi)^{2\lambda-1}\int_{0}^{t} |v-((n-j)/n)\pi|^{1-2\lambda} d_{n-j,n}(v)^{-7}\sin v\,dv$$

$$= C(\sin(j/n)\pi)^{2\lambda-1} n^{2\lambda-2} d_{n-j,n}(t)^{-5}\sin t.$$

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On the other hand an elementary estimate of J_j'' yields

$$J_j'' \leqslant C(\sin(j/n)\pi)^{2\lambda-1} t n^{2\lambda-1} d_{n-j,n}(t)^{-5} \sin t.$$

Combining the above, we have

$$J_j'' \leqslant C(\sin(j/n)\pi)^{2\lambda-1}(\sin t)\, n^{2\lambda-1} d_{n-j,n}(t)^{-5} \min(t, 1/n)$$

and since $\min(t, n^{-1}) \leqslant t^{1-2\lambda} n^{-2\lambda}$ (for $0 \leqslant 1-2\lambda \leqslant 1$) and $t \leqslant \frac{1}{2}\pi \sin t$ (for $t \in [0, \frac{1}{2}\pi]$), the proof is complete. ∎

We now prove the main result of this section.

THEOREM 4·2. *For $f \in C^2[-1,1]$, $0 \leqslant \lambda \leqslant 1$, $\phi(x) = \sqrt{(1-x^2)}$ and $L_n(f,x)$ given by (2·5), we have*

$$|f(x) - L_n(f,x)| \leqslant C \frac{\phi(x)^{2-2\lambda}}{n^2} \|\phi^{2\lambda} f''\|_{C[-1,1]}. \tag{4·8}$$

Proof. Applying Theorem 3·2, it is sufficient to show that $|L_n(f,x) - S_n(f,x)|$ is bounded by the expression on the right-hand side of (4·8). By virtue of (2·9), (3·4) and Lemma 4·1, we may write

$$\begin{aligned} |L_n(f,x) - S_n(f,x)| &\leqslant \sum_{j=1}^{n-1} |s_j - s_{j-1}|\, |R_j(x) - \phi_j(x)| \\ &\leqslant C n^{-1} \phi(x)^{2(1-\lambda)} \sum_{j=1}^{n-1} |s_j - s_{j-1}| (\sin(j/n)\pi)^{2\lambda-1} d_{n-j,n}(t)^{-5} \end{aligned} \tag{4·9}$$

where $x = \cos t$.

We now use the Taylor formula to estimate $|s_j - s_{j-1}|$ as follows:

$$\begin{aligned} |s_j - s_{j-1}| &= \left| \frac{f(\xi_{j+1}) - f(\xi_j)}{\xi_{j+1} - \xi_j} - \frac{f(\xi_j) - f(\xi_{j-1})}{\xi_j - \xi_{j-1}} \right| \\ &\leqslant \int_{\xi_{j-1}}^{\xi_j} \frac{u - \xi_{j-1}}{\xi_j - \xi_{j-1}} |f''(u)|\, du + \int_{\xi_j}^{\xi_{j+1}} \frac{\xi_{j+1} - u}{\xi_{j+1} - \xi_j} |f''(u)|\, du. \end{aligned}$$

For $j = 2, \ldots, n-2$, we use Lemma 3·1 (first (3·3) then (3·1) and (3·2)) to obtain

$$\begin{aligned} |s_j - s_{j-1}| &\leqslant (|\xi_j - \xi_{j-1}| \max_{\xi_{j-1} \leqslant u \leqslant \xi_j} \phi(u)^{-2\lambda} + |\xi_{j+1} - \xi_j| \max_{\xi_j \leqslant u \leqslant \xi_{j+1}} \phi(u)^{-2\lambda}) \|\phi^{2\lambda} f''\| \\ &\leqslant C n^{-2\lambda}(|\xi_j - \xi_{j-1}|^{1-2\lambda} + |\xi_{j+1} - \xi_j|^{1-2\lambda}) \|\phi^{2\lambda} f''\| \\ &\leqslant C_1 n^{-2\lambda} \delta_j^{1-2\lambda} \leqslant C_2 n^{-1} (\sin(j/n)\pi)^{1-2\lambda}. \end{aligned}$$

For the last inequality we used the estimate $n^{-1} \sin(j/n)\pi \leqslant \delta_j \leqslant C n^{-1} \sin(j/n)\pi$ for $1 \leqslant j \leqslant n-1$.

For $j = 1$ (and similarly for $j = n-1$), we apply Lemma 3·3 to write

$$\begin{aligned} |s_1 - s_0| &\leqslant \left(\frac{1}{1+\xi_1} \int_{-1}^{\xi_1} (u+1)^{1-\lambda}\, du + \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} (\xi_2 - u)\, \phi(u)^{-2\lambda}\, du \right) \|\phi^{2\lambda} f''\| \\ &\leqslant C_3((\xi_1 + 1)^{1-\lambda} + n^{-1}(\sin(1/n)\pi)^{1-2\lambda}) \|\phi^{2\lambda} f''\|. \end{aligned}$$

Using Lemma 3·3 once more, we obtain

$$|s_1 - s_0| \leqslant C_4 n^{-1} (\sin(1/n)\pi)^{1-2\lambda} \|\phi^{2\lambda} f''\|.$$

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Combining the estimate which we achieved, i.e.,

$$|s_j - s_{j-1}| \leqslant Cn^{-1}(\sin(j/n)\pi)^{1-2\lambda}\|\phi^{2\lambda}f''\| \tag{4·10}$$

with (4·9), we complete the proof by recalling (2·24) of [2], i.e.,

$$\sum_{j=1}^{n-1} d_{n-j,n}(t)^{-5} \leqslant A \tag{4·11}$$

with A independent of t and n. ∎

The first author is supported by NSERC (Canada) grant A-4816 and the third author is supported by BSF grant 89-00505. This paper constitutes part of the second author's Ph.D. thesis.

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https://doi.org/10.1017/S0305004100070997 Published online by Cambridge University Press