Probability in the Engineering and Informational Sciences, **22**, 2008, 415–429. Printed in the U.S.A.
doi:10.1017/S0269964808000247

WARM-UP PERIODS IN SIMULATION CAN BE DETRIMENTAL

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The question of how long to run a discrete event simulation before data collection starts is an important issue when estimating steady-state performance measures such as average queue lengths. By using experiments based on numerical (nonsimulation) methods published elsewhere, we shed light on this question. Our experiments indicate that no initialization phase should be used when starting in state with a reasonable high equilibrium probability. Delaying data collection is only justified if the starting state is highly unlikely, and data collection should start as soon as a system enters a state with reasonably high probability.

1. INTRODUCTION

Discrete-event simulation is one of the most popular tools in operations research, and the practitioner should, therefore, be familiar with the issues arising in this context. One of these issues relates to the so-called start-up problem when estimating equilibrium expectations by simulation, such as the expected number of jobs in a queuing network, the expected cost in an inventory system, and related issues [1, pp. 403ff,12,13,14, pp. 409ff]. As is well known, the starting state will contaminate the estimators of these expectations, and one way to avoid this is to collect data only after a so-called warm-up period or burn-in period. We investigate this issue and we demonstrate that if one starts in an initial state that has a reasonably high equilibrium probability, there should not be a warm-up period. Although this conclusion might surprise some readers, it nicely dovetails on the findings of others, in particular the work of Madansky [16] and Wilson and Prisker [27]. Of course, it might not be convenient or even not practical to start the simulation in a state with a high equilibrium probability. For example, even in such a simple system as the $GI/G/c$ queue with a high number of servers,

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the equilibrium probability of the empty state is very low. For the $M/M/50$ system, for instance, it is around 10E-18. Still, starting in the empty state might be convenient because, otherwise, there is a large effort to initialize the different remaining service times of the servers properly.

In this article we use a Markov modeling approach, an approach that allows one to formulate systems of reasonable complexity, yet obtain results by numerical methods. By numerical methods, we understand in this article, for want of a better name, nonsimulation methods. Practitioners should consider these methods for their investigations, especially since there are a number of tools available on the Internet making this approach feasible [2,20]. Most of these tools are based on stochastic Petrinets or on stochastic activity nets, which are similar. Unfortunately, the computational effort of numerical methods increases exponentially with the number of state variables, and simulation eventually requires less computer time. One can, however, gain much insight from relatively small models and in this case, numerical methods are preferable.

To do our studies, we actually need extensions to the classical Markov-modeling tools, tools that are not implemented yet in the standard packages, although means of time averages are typically available, but variances of time averages are not, even though [22] did a study in this regard. The most efficient methods to find the variances required are described in [5,9]. However, there are a number of other relevant works, including [4,6,8,10,11,18,23–25]. A review of articles that appeared before 1975 is given in [19]. In the experiments of this paper, we use the methods described in [5,9]. These have also been used in [15,17,21].

In the following sections, we will compare the different starting states for a number of systems and calculate the mean-squared error (MSE) for these different initial states. This approach was also taken by others, in particular by Wilson and Pritsker [27]. In addition to varying the initial states, Wilson and Prisker also experimented with warm-up periods of different lengths. Here, we deal with the warm-up periods by starting with the equilibrium probabilities. In simulation terms, this would be equivalent to selecting the starting state randomly, with the choice of the starting state based on the equilibrium distribution of the system. This is equivalent to running the system for a very long time before stating data collection.

When evaluating the quality of estimators, we follow [27] by concentrating on the bias, the variance, and the MSE. This seems reasonable enough. However, some of our results show that the MSE might be misleading. Unfortunately, at the moment, we have no methods to numerically calculate a potentially better measure, such as the mean absolute error.

2. MARKOV MODELING

Our modeling tool is the Markov event system (MES) with rewards. In an MES, one has a number of state variables, such as the number of elements in different waiting lines, states of servers, such as the type of customers in service, and so on. The state

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variables are changed by events, such as arrivals, completions of services, and so on. The events change the system in a predictable way. Every event has a certain rate, and this rate depends only on the present state. In other words, the system is Markovian and it can be converted into a Markov chain. In this respect, an MES is similar to a generalized stochastic Petri net.

In the MES, the states are given by a set of state variables. For instance, if there are two lines in sequence in the system, we can denote each state by (X_1, X_2), where X_1 is the number of customers in line 1 and X_2 the number in line 2. If X_1 ranges from zero to N_1 and X_2 ranges from zero to N_2, there are $(N_1 + 1)(N_2 + 1)$ states, which can be readily enumerated. In this article we assume that the number of states is finite. In each state, certain events can occur, such as arrivals to the first queue, departures from the first queue with a simultaneous arrival to the second queue, and departures from the system. Once all events and their effect on the different states are known, the transition matrix of the underlying Markov chain can easily be constructed. When doing this, it is convenient to give each state a unique identification number.

Of course, the number of states of this Markov chain increases exponentially with the number of state variables; hence, using Markov chain theory for systems with many state variables leads to extremely large transition matrixes. On the other hand, the matrixes are sparse, because the number of entries in each row is limited by the number of events that can occur in the state in question.

In systems with many state variables, the identification numbers of the states are of no interest, and to extract information about state variables, one introduces rewards. Specifically, for every state i, a reward r_i per time unit is introduced. For instance, if the state variable of interest is the length X_1 of the first line in a tandem queuing system, one sets r_i to the value X_1 has in state i. Similarly, if one is interested in the number in a system consisting of two queues, r_i can be set to the value $X_1 + X_2$ happens to have in state i. Hence, if the set of all possible states and their identification numbers are given as

State #	1	2	3	4	5	6	7	8	9
(X_1, X_2)	(0, 0)	(0, 1)	(0, 2)	(1, 0)	(1, 1)	(1, 2)	(2, 0)	(2, 1)	(2, 2)

and if one needs to know the values of X_1 for the different states, one sets $r_1 = r_2 = r_3 = 0$, $r_4 = r_5 = r_6 = 1$, and $r_7 = r_8 = r_9 = 2$. In a similar fashion, one can find rewards to extract, say, $X_1 + X_2$. If one is interested if $X_1 = 0$, one sets $r_1 = r_2 = r_3 = 1$, and $r_i = 0$ otherwise. Hence, rewards allow for a great flexibility in modeling.

Generally, we use the symbol $R(t)$ for the reward rate at time t; that is, if $X(t)$, the state at time t, is equal to i, then $R(t) = r_{X(t)} = r_i$. Note that $R(t)$ is typically not Markovian. In fact, it is a common trick to make a non-Markovian process Markovian by adding so-called supplementary variables to the system. In this way, most discrete-event systems can be converted into Markovian systems by adding enough supplementary variables.

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3. EQUILIBRIUM EXPECTATIONS AND THEIR ESTIMATION

Steady-state simulations are typically done to estimate equilibrium expectations. Specifically, the simulator might want to find the reward rate in equilibrium; that is, if $R(t)$ converges in distribution to R as $t \to \infty$, he/she might want to estimate $\mathrm{E}(R)$. For doing this, simulators typically use time averages; that is,

$$\bar{R}(T) = \frac{1}{T}\int_0^T R(t)\,dt.$$

Here, time 0 indicates the start of data collection. The simulation might start before time 0. The main question, of course, is how close the estimator $\bar{R}(T)$ is to the expectation $\mathrm{E}(R)$. To judge this, we use (see also [26]) the bias, the variance and the MSE of $\bar{R}(T)$. These measures are defined respectively as

$$\begin{aligned}
\mathrm{B}(\bar{R}(T)) &= \mathrm{E}(\bar{R}(T)) - \mathrm{E}(R),\\
\mathrm{Var}(\bar{R}(T)) &= \mathrm{E}((\bar{R}(T) - \mathrm{E}(\bar{R}(T))^2,\\
\mathrm{MSE}(\bar{R}(T)) &= \mathrm{E}((\bar{R}(T) - \mathrm{E}(R))^2\\
&= \mathrm{Var}(\bar{R}(T)) + \mathrm{B}(\bar{R}(T))^2. \qquad \textbf{(1)}
\end{aligned}$$

We also use μ_R for $\mathrm{E}(R)$. To find these measures numerically, we use the method described in [5], a method based on randomization, also known as uniformization or Jensen's method [7]. As shown in [5], this method has the same computational complexity as randomization. A version of this method was programmed in Excel VBA and used for our experiments. Results were typically obtained in seconds, even though VBA is an interpreter. Of course, for large systems, one would use a compiler, which is much faster, but for our investigations, speed was not a problem. Our experiments showed that in order for $\mathrm{Var}(\bar{R}(T))$ to lead to an acceptable precision of the estimator, T must be large, and for large values of T, one must confront the possibility of rounding errors. For this reason, we used double precision. In order to increase the precision further, we also replaced r_i by $r_i - \mu_R$ when calculating $\mathrm{Var}(\bar{R}(T))$. This change obviously does not affect the result. The accuracy of the results was checked for the case of the $M/M/1$ queue by employing a different algorithm, an algorithm based on eigenvalues.

It is known that as $T \to \infty$, $T\,\mathrm{Var}(\bar{R}(T)$ approaches a limit V (see [23–25] and references therein). A simple method to find V was given in [4]. This method has been adapted later in [22]. We discovered later that V can be found from a set of equations that closely resemble the equilibrium equations. A summary of the method can be found in [9].

We also note that in Markov chains, the bias $\mathrm{B}(T)$ converges to BT as $T \to \infty$. Here, B can be found from a set of equations similar to the ones used to find the variance. This method is described in Appendix A. The derivation is actually similar

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to the one used for finding V. Once V and B are known, the MSE becomes

$$\begin{aligned} \mathrm{MSE}(\bar{R}(T)) &= \mathrm{Var}(\bar{R}(T)) + \mathrm{B}(\bar{R}(T))^2 \\ &= \frac{V}{T} + \frac{B^2}{T^2} + o\left(\frac{1}{T}\right) \\ &= \frac{V}{T} + o\left(\frac{1}{T}\right). \end{aligned} \tag{2}$$

Hence, the bias term goes to zero with the the square of T and soon becomes negligible. Of interest, of course, is how fast $T\,\mathrm{Var}(\bar{R}(T))$ and $\mathrm{B}(T)$ converges to VT, respectively BT. For this reason, we programed the algorithm determining V and B.

4. A ONE-SERVER QUEUE

In our first experiment, we considered the $M/M/1/10$ queue with arrival rate $\lambda = 0.8$ and $\mu = 1$, and we varied the initial states from 0 to 4. We also selected the initial state randomly, using the equilibrium distribution. In the later case, the system is in equilibrium right from the start. The state identification number and the reward are both equal to the number in system. The results are given in Table 1. To check if our programs work correctly, we also calculated the variances for both $T = 100$ and $T = 1000$ using eigenvalues [18]; the values are given in Table 2. As one can see, the numbers agree with a precision of six digits. This excludes any significant error, including errors due to rounding. We return now to the discussion of Table 1. We notice that starting in state 0 yields the lowest MSE, a result confirming the findings of Madansky [16] and Wilson and Pritsker [27]. We note that the starting state (the number in the system) does not really matter, and, in fact, the differences in the MSE are only in the third digit. We also see that the bias of $\bar{R}(T)$ is very close to $B/T = B/1000$, and the variance of $\bar{R}(T)$ differs from V/T by at most 0.0043. These results might not be too surprising. What is surprising, however, is that selecting the starting value according to the equilibrium probabilities yields the largest MSE; that

TABLE 1. Measures for the Time Average of the Number in an $M/M/1/10$ System; $\lambda = 0.8$, $\mu = 1$, $T = 1000$, $\mathrm{E}(R) = 2.966$, $\mathrm{Var}(R) = 7.561$, $V = 174.54$

$R(0)$	$\mathrm{B}(\bar{R}(T))$	$\mathrm{Var}(\bar{R}(T))$	B	MSE
Equil	0.0000	0.1725	0.0	0.1725
0	−0.0273	0.1703	27.3	0.1710
1	−0.0236	0.1705	23.6	0.1710
2	−0.0165	0.1709	16.5	0.1711
3	−0.0064	0.1714	6.4	0.1715
4	0.0061	0.1721	−6.1	0.1721

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TABLE 2. Comparing Jensen's Method with an Eigenvalue Method

	MSE($\bar{R}$(100))		MSE($\bar{R}$(1000))	
R(0)	Jensen	Eigenvalues	Jensen	Eigenvalues
Equil	1.538217	1.538217	0.172471	0.172471
0	1.391293	1.391293	0.170997	0.170997
1	1.392831	1.392831	0.171013	0.171013
2	1.404940	1.404940	0.171135	0.171135
3	1.437879	1.437879	0.171466	0.171466
4	1.500904	1.500904	0.172098	0.172098

is, a warm-up period actually increases rather than decreases MSE. This is completely counterintuitive, and we will, therefore, discuss this further in the next section.

We also note that run lengths below 1000 will not yield an E(R) with acceptable accuracy. For $T = 1000$, the standard deviation when starting in state 0 is $\sqrt{0.1703} = 0.4135$, and this is almost 14% of the expectation 2.966. If one wants to find E(R) at a precision $\epsilon = 0.1$ E(R) $= 0.2966$, one can use

$$T \approx \frac{z_{\alpha/2}^2 V}{\epsilon^2}.$$

For a 95% confidence level, one obtains $T = 7620$. Hence, even for this simple model, long runs are required.

From Table 1, one concludes that if V is to be estimated, then starting in equilibrium is best, because 1000 Var($\bar{R}$(1000)) is closest to $V = 174.5$. In fact, when starting in equilibrium, one finds 1000 Var($\bar{R}$(1000)) $= 172.5$, whereas for the starting values 0, 1, and 2, one obtains 172.1. Note that even when starting in equilibrium, one does not obtain V exactly, and this to be expected for mathematical reasons as indicated in Appendix 3.

5. STOCHASTIC VERSUS DETERMINISTIC SYSTEMS

To appreciate the significance of the findings given in Table 1, it is necessary to understand the difference between deterministic systems and stochastic systems. This difference is not apparent when considering graphs like the one given in Figure 1, which shows how the expectation of both $R(t)$ and $\bar{R}(t)$ converges toward the equilibrium value of 2.966. Indeed, this could as well be the graph involving a deterministic system. To clarify the distinction between the two types of system, let us first define deterministic and stochastic systems. According to Ziegler [28], in deterministic systems, one has a set of state variables that jointly determine the future of the system. In analogy, one could define a stochastic system as a system in which there is a set of variables, the state variables, which jointly determine the future distributions of all system variables—in particular, the distributions of all state variables. For instance,

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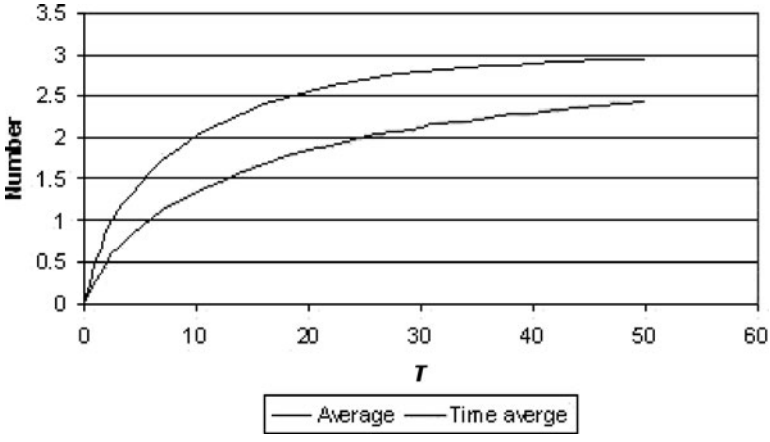


FIGURE 1. Convergence of E(R) toward its equilibrium.

in the $M/M/1$ queue, there is only one state variable—the number in the system—and this variable determines all future distributions of the system. It follows from the definition that once the deterministic system passes through the same state i twice, the future of the system after the second pass through i replicates exactly what happens after the first pass through state i. In fact, if any state other than i is visited between two visits of state i, the system is periodic and, consequently, has no steady state at all. If the system is to ever reach a steady state, no state can ever be visited twice unless the system is in equilibrium. Stochastic systems, on the other hand, can visit the same state many times even if their underlying distributions reach a limiting or steady-state distribution. In fact, in the $M/M/1$ queue, the system spends most of its time in state 0, but to obtain equilibrium expectations, one obviously has to visit other states in between. If a state is visited several times, then the future distributions are exactly replicated according to the definition of a stochastic system. For instance, after each visit to state 0, the $M/M/1$ queue regenerates. If one starts in state 0, then the distributions of the system are replicated once state 0 is reached again, and the question arises of why not have a warm-up period the second time this state is reached. Without a satisfactory answer to this question, the practice of using burn-in phases becomes questionable.

Of course, starting in a state that is never visited again, or a state visited with very low probability, is certainly not a very good state to start a simulation. If, for some reason, one starts in such a state, a warm-up period is reasonable, but this period should end as soon as a better state is reached. In this case, the idea is no longer that some measure, such as the expected number in the system, is close to its equilibrium, but the idea is merely to find a good state for starting data collection. This would explain the success of the rule of Conway [3] that says that one should end the warm-up period and start data collection as soon as the statistic collected is neither a maximum nor a minimum. At this point, one typically has left the region of very unlikely initial states.

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In this article, we assume that the quality of a state in which to start the simulation is measured by its equilibrium probability. This is justified by noting that for long initialization periods, one will be most likely in a state with a high equilibrium probability when data collection starts. Alternatively, one could start in a state that is frequently visited. The question of which is better would be a further topic of research and is beyond the scope of this article. We note, however, that in the case of $M/M/1$ queues of any buffer size, the number of visits to state 1 exceeds the number of visits to state 0. To see this, note that each visit to state 0 can be matched with a visit to state 1, but there are some visits to state 1 that cannot be matched. However, according to Table 1, state 0 has the lowest MSE.

The observation in Table 1 that starting in equilibrium results in the highest variance and the highest MSE is highly counterintuitive and needs further explanation. From a formal point of view, one can easily prove that starting in one of several states chosen at random will not minimize the variance. To see this, assume that the initial state i is chosen with probability $\pi_i(0)$ and that given one starts in state i, the bias and the variance of $\bar{R}(T)$ are $\mathrm{B}_i(\bar{R}(T))$ and $\mathrm{Var}_i(\bar{R}(T))$, respectively. The law of total expectation and total variance now allow us to find the unconditional bias and the unconditional variance $\mathrm{B}(\bar{R}(T))$ and $\mathrm{Var}(\bar{R}(T))$, respectively. One has

$$\begin{aligned}\mathrm{B}(\bar{R}(T)) &= \sum_i \pi_i(0) B_i(\bar{R}(T)),\\ \mathrm{Var}(\bar{R}(T)) &= \sum_i \pi_i(0)\mathrm{Var}_i(\bar{R}(T)) + \sum_i \pi_i(0)(B_i(\bar{R}(T)) - B(\bar{R}(T)))^2. \qquad \textbf{(3)}\end{aligned}$$

From this equation, one sees immediately that if $\mathrm{Var}_i(\bar{R}(T))$ is minimized for $i = i^*$, then $\mathrm{Var}_{i^*}(\bar{R}(T))$ is less than $\mathrm{Var}(\bar{R}(T))$. Hence, starting in state i^* gives a lower variance for $\bar{R}(T)$ than selecting the starting in state i with probability $\pi_i(0)$. The same is true when selecting the starting state i with the equilibrium probability π_i; that is, when starting with the probabilities that the system only reaches after a long initialization period.

Note that (3) holds for any Markov chain, no matter how large, and it holds also if supplementary variables are present. Of course, supplementary variables can be continuous, but they can be approximated by variables with a phase-type distribution. Since phase-type variables are dense on the positive axis, this approximation can be as close as one desires. Moreover, (3) can be rewritten such that it holds for continuous variables as well. To do that, one has to use multiple integrals, one for each continuous variable, in addition to the sum in (3). This seems to suggest that our results remain valid for non-Markovian systems.

Of course, using the variance might be misleading because when starting, the variance is certainly smaller when choosing the initial state deterministically rather than randomly: A deterministic start yields, by definition, a variance of 0 at time 0. For this reason, the MSE is the better measure. However, as indicated by (2), the difference between the variance and the MSE become negligible as T increases. For instance, in

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Table 1, the bias when starting in state 0 is −0.0273 and the square of this is 0.0007, which is insignificant.

The fact that having a warm-up period long enough to reach a statistical equilibrium will not yield the lowest possible MSE is counterintuitive and it throws some doubt on whether this measure should be replaced by, say, the mean absolute deviation. In fact, the variance, which is close to the MSE for large enough T, is very strongly influenced by extreme values. To show this, we increased the buffer size in the example of Table 1 from 10 to 15. Doing this increases the expectation by 19.2%, but when starting in equilibrium, $\mathrm{Var}(\bar{R}(T))$ increases by 88.5% and the standard deviation increases by 69.9%; that is, both the variance and the standard deviation increase much more than the expectation. Hence, using the MSE might be problematic. On the other hand, one needs the variance when dealing with the Normal distribution, which becomes relevant because for large T, $\bar{R}(T)$ approaches the Normal distribution.

6. MULTISERVER QUEUES

In multiserver queues, the equilibrium probability for the empty state as well as the number of visits to the empty state decreases when increasing the number of servers, provided the traffic intensity is held constant. In such a queuing system, it is also easy to find the state with the highest equilibrium probability: If the arrival rate is λ, the service rate is μ, and the number of servers is c, then the most likely state i is reached when $\lambda/\mu < i$ for the last time. This follows from the fact that if P_i is the equilibrium probability to be in state i, $P_i = P_{i-1}\lambda/(i\mu)$. Of course, $i < c$ if the system has an equilibrium.

Hence, consider the $M/M/10/30$ queue and the $M/M/50/60$ queue. The results are given in Tables 3 and 4. In both systems, we keep the traffic intensity ρ equal to 0.8 and $\mu = 1$. Hence, in the $M/M/10/30$ queue, where $c = 10$, we set $\lambda = 8$, whereas in the $M/M/50/60$ queue, we set $\lambda = 40$. If $c = 10$, the most likely states are the states with seven or eight elements in the system (both states have the same probability). We see that the initial state with the lowest MSE is 5, not 7. However, using state 7 as the initial state will not increase the MSE by much. We were expecting that state 0

TABLE 3. Bias, Mean, and Variance in an $M/M/10/30$ System; $\lambda = 8$, $\mu = 1$, $T = 1000$, $V = 1416.1$, $\mathrm{E}(R) = 9.5406$

	$R(0)$	Bias	B	Var	MSE
Equil	—	0.0000	0.0	1.3627	1.3627
Min var	0	−0.1734	173.4	1.2870	1.3171
Min MSE	5	−0.1107	110.7	1.2974	1.3097
Max prob	7	−0.0817	81.7	1.3043	1.3110
c	10	−0.0262	26.2	1.3221	1.3228

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Table 4. The $M/M/50/60$ Queue; $\lambda = 40$, $\mu = 1$, $V = 4454.5$, $T = 1000$, $\mathrm{E}(R) = 40.16$, $\mathrm{Var}(R) = 42.78$

	$R(0)$	Bias	B	Var	MSE
Equil	—	0.0000	0	4.218	4.218
Min Var	0	−2.0358	−1873.9	3.948	8.093
Max Prob	39	−0.0667	−66.7	4.091	4.095
c	50	0.5257	525.7	4.165	4.441

would really be a poor starting state, but this did not turn out to be the case. In fact, starting in equilibrium results in an even higher MSE.

We now turn to the $M/M/50/60$ queue. The results for this queue are given in Table 4. In this case, starting in state 0 is not a good idea because the MSE is 8.093, as opposed to 4.218 when selecting the starting state according to the equilibrium probabilities of the process. However, the best starting state is when there are 39 customers in the system. This is also the state with the highest probability. From Table 4, one also notices that starting with $c = 50$ customers also give reasonable results. This clearly indicates that the exact starting state is not very important. To confirm this, we also calculated the MSE for a starting state of 35, and we obtained 4.145, which is still quite good. However, starting with the system empty almost doubles the MSE. If, for some reason, starting in state 0 is the most convenient starting state, then one definitely wants to use a warm-up period. By using a warm-up period of 50, one already reduces the MSE to 4.680, and for a warm-up period of 100, the MSE is 4.248, which is almost equal to the situation when starting in equilibrium. The reason why state 0 yields such a high value for MSE is that its equilibrium probability is very low: It is only 4.17E-18! Note, however, that even when starting with an idle system, the variance is still lower than in the other cases. This shows that the variance can be misleading for small T.

7. TANDEM QUEUES

To show how to deal with systems having more than one state variable, we use two types of tandem queue as examples. In both models, arrivals are Poisson and service times are exponential. All arrivals first visit server 1, and after the service is complete, they proceed to server 2. Both servers have finite buffers and the buffer sizes are $N_1 = 10$ and $N_2 = 5$, respectively. Customers balk if the first buffer is full. The two models differ with regard to what happens when the second buffer is full. In the first model, customers finding the second buffer full leave the system, whereas in the second model, they block server 1, which means that server 1 stops working until the customer receiving service with server 2 leaves. The customer blocking the first server is considered part of the second buffer. The reward is given by the number in the system. The results of the nonblocking model are given in Table 5. We note that in this model, starting in the empty state yields the lowest variance but not the lowest

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TABLE 5. Nonblocking Tandem Queue; $\lambda = 0.8$, $\mu_1 = \mu_2 = 1$, $N_1 = 10$, $N_2 = 5$, $V = 223.4$, $T = 1000$, $\mathrm{E}(X_1 + X_2) = 4.775$, $\mathrm{Var}(X_1 + X_2) = 9.499$

	State	Bias	Var	MSE
Equil	—	0.0000	0.2208	0.2208
Min Var	0, 0	−0.0418	0.2181	0.2198
Min MSE	0, 5	−0.02174	0.2183	0.2187

MSE. The starting state minimizing the MES is to have no customers in line 1 and five in line 2. However, the difference in the MSE is minor, and the MSE is higher if starting with the equilibrium distribution. We also note that, as earlier, the variance is close to V/T.

In the model with blocking, the situation is similar, except that the variance is higher. The results are given in Table 6.

8. AVERAGES AND SUMS

Suppose a system is simulated N times, with all replications starting in the same state, and the average of $\bar{R}(T)$ is taken over the replications. Hence, if $\bar{R}_i(T)$ is the result obtained in the ith replication, $i = 1, 2, \ldots, N$, then the overall average is

$$\bar{R}^*(T) = \frac{1}{N} \sum_{i=1}^{N} \bar{R}_i(T). \tag{4}$$

It is easy to see that in this case, $\mathrm{E}(\bar{R}^*(T)) = \mathrm{E}(\bar{R}(T))$ and $\mathrm{Var}(\bar{R}^*(T)) = \mathrm{Var}(\bar{R}(T))/N$. This implies that as N increases, the variance of the $\bar{R}^*(T)$ decreases; that is, the process becomes more deterministic. Clearly, the MSE becomes

$$\begin{aligned} \mathrm{MSE}(\bar{R}^*(T)) &= \mathrm{Var}(\bar{R}^*(T)) + \mathrm{B}(\bar{R}^*(T)) \\ &= \mathrm{Var}(\bar{R}(T))/N + \mathrm{B}(\bar{R}(T). \end{aligned}$$

TABLE 6. Blocking Tandem Queue; $\lambda = 0.8$, $\mu_1 = \mu_2 = 1$, $N_1 = 10$, $N_2 = 5$, $V = 482.6$, $T = 1000$, $\mathrm{E}(X_1 + X_2) = 6.182$, $\mathrm{Var}(X_1 + X_2) = 12.91$

	State	Bias	Var	MSE
Equil	—	0.0000	0.4734	0.4734
Min Var	0, 0	−0.0913	0.4657	0.4741
Min MSE	0, 5	−0.0053	0.4701	0.4702

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Hence, as N increases, the bias gains in weight when compared to the variance. This has to be taken into account when choosing the starting state and when considering a warm-up period.

In networks, we often deal with sums of state variables, such as the sum of customers in several queues. Of course, the bias of a sum is equal to the sum of biases, and if the random variables are independent, the variance of a sum is equal to the sum of variances. Since the bias is squared when calculating the MSE, the relative importance of the bias increases. Of course, when looking at queuing networks, the state variables are correlated, but unless there is a perfect positive correlation, the importance of the bias still increases relative to the variance when calculating the MSE. This, again, has implications when choosing the starting state or when considering a warm-up period. For instance, in the tandem queue with a loss, the best starting state when estimating the expected number in the system is not (0, 0) but (0, 5). Of course, in the case of tandem queues, the effect was minor, but in systems with 20 or 30 queues, it should be considered.

9. CONCLUSION

In this article, we used numerical tools suggested by us earlier to find the bias, the variance, and the MSE for several systems. This research suggests that in many cases, warm-up periods are of no use and should be set to zero. This is particularly true if the simulation starts in a high probability state. However, starting in such a state might be inconvenient, and in this case, a warm-up period can be used for the express purpose to reach a high probability state. We stated that this rule gives credence to the suggestion of Conway to start the simulation when the observed value is neither a maximum nor a minimum. We note here that if indeed it is only the starting state that matters, then the knowledge of such a state, possibly found by a pilot simulation, can be used for later runs.

Acknowledgments

This research was supported by the Natural Sciences and Engineering Research Council of Canada. I also thank Javad Tavakoli and his student Jennifer Hawrylo for their comments.

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https://doi.org/10.1017/S0269964808000247 Published online by Cambridge University Press

APPENDIX A

In this appendix, we show that the limiting bias factor B can be obtained by solving equations that are very similar to the equilibrium equations. Hence, let S be the state space, let a_{ij}, i, $j \in S$, be the entries of the incremental generator, let $\pi_i(t)$ be the probability that the system is in state i at time t, and let π_i the corresponding equilibrium probability. We will show that in order to find B, one first has to solve the following equations for D_i, with the condition that $\sum_{i\in S} D_i = 0$:

$$\pi_j - \pi(0) = \sum_{i\in S} D_i a_{ij}, \qquad j \in S. \tag{A.1}$$

B is now given by

$$B = \sum_{i\in S} r_i D_i.$$

When solving (A.1), we first ignore the condition that $\sum D_i = 0$, which yields a preliminary solution D_i^* satisfying only (A.1). Since the equations in this system are dependent, we have one degree of freedom, which allows us to set one D_i to zero. Once all D_i^* are found, one sets

$$D_i = D_i^* - \pi_i \sum_{j\in S} D_j.$$

To prove these results, let $d_i(t) = \pi_i(t) - \pi_i$; that is, the $d_i(t)$ are the deviations of the transient probabilities from steady state.

As is well known, $\pi_j'(t) = \sum_{i\in S} \pi_i(t) a_{ij}$. Clearly, $d_i'(t) = \pi_i'(t)$. Also, $\sum_{i\in S} \pi_i a_{ij} = 0$. It follows that

$$d_j'(t) = \sum_{i\in S} \pi_i(t) a_{ij} = \sum_{i\in s} (\pi_i(t) - \pi_i) a_{ij}.$$

Hence,

$$d_j'(t) = \sum_{i\in S} d_i(t) a_{ij}.$$

We now integrate this equation from zero to T and replace $\int_0^T d_i(t)\,dt$ by $D_i(T)$. This yields

$$d_j(T) - d_j(0) = \sum_{i\in S} D_i(T) a_{ij}.$$

As $T \to \infty$, $d_j(T) \to 0$. In fact, a simple argument involving spectral analysis shows that $d_i(t)$ approaches zero exponentially fast. The integral $\int_0^T d_i(t)\,dt$ therefore has a limit as $T \to \infty$, and we denote this limit by D_i. Hence,

$$-d_j(0) = \pi_j - \pi_j(0) = \sum_{i\in S} D_i a_{ij}.$$

The bias now is

$$T\,\mathrm{E}(\bar{R}(T) - R) = \mathrm{E}\left(\int_0^T (R(t) - R)\,dt\right) = \sum_{i\in S} r_i \int_0^T (\pi_i(t) - \pi_i)\,dt.$$

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As $T \to \infty$, the integral on the right converges to D_i, and we get, as claimed,

$$B = \sum_{i \in S} r_i D_i.$$

It is easy to verify that the $d_i(t)$ have a sum of zero and this property is inherited by $D_i(T)$ and D_i.

Finally, note that if D_i^* satisfies (A.1), so does $D_i^* + u\pi_i$, where u is an arbitrary constant. To see this, just add u time the equilibrium equation $0 = \sum_{i \in S} \pi_i a_{ij}$ to (A.1). If we set $u = \sum_{i \in S} D_i$, then we obtain

$$\sum_{i \in S} (D_i^* - u\pi_i) = 0,$$

which means the $D_i = D_i^* - u\pi_i$.

APPENDIX B

We briefly also describe the equations used for obtaining V, which is similar to the one above. First, following [9,23,24], we assume that the process has run long enough such that we are in equilibrium. Alternatively, we can assume that we are in equilibrium at time 0; that is, at time 0, the process has already run for a long time. We have

$$\mathrm{Var}(\bar{R}(T)) = \frac{2}{T} \int_0^T \int_0^{T-t} \mathrm{Cov}(R(t), R(t+\tau))\, d\tau\, dt.$$

Here, Cov stands for covariance. We now have, as $T \to \infty$,

$$\lim_{T \to \infty} T\, \mathrm{Var}(\bar{R}(T)) = 2 \int_0^\infty \mathrm{Cov}(R(0), R(t))\, dt.$$

Note that even when starting in equilibrium, this limit only holds for large enough T and only if the covariance decreases fast enough, say exponentially fast as in finite-state Markov chains. Hence, we have really two different limits: the first one to reach equilibrium and the second one to have a large enough T such that the tail of the covariance becomes negligible. This might explain why $T\,\mathrm{Var}(\bar{R}(T))$ approaches V much slower than $\mathrm{B}(\bar{R}(T))$ approaches B.

Under the above conditions, one finds V by first solving the following equations for W_i, $i \in S$, $\sum_{i \in S} W_i = 0$:

$$(\mu_R - r_j) = \sum_{i \in S} W_i a_{ij}.$$

One then has

$$V = 2 \sum_{i \in S} (r_i - \mu_R) W_i.$$

The derivation is quite similar to the one given above and will not be presented here. For similar arguments, see [4,6].

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