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TIME ALLOCATION, THE DYNAMICS OF INTERACTION, AND TECHNOLOGY ADOPTION

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Inspired by recent literature that approaches the dissemination of knowledge from a social interaction perspective, the article explores the dynamics of a prototypical optimal control growth problem structured upon the following features: (i) the model economy is populated by a large number of rational agents; (ii) each agent allocates time, optimally, among production and social interaction; (iii) knowledge spreads through the contact with others; (iv) the propagation of ideas follows two steps—in a first stage, interaction promotes the acquisition of theoretical knowledge and, in a second stage, it works as a catalyst for the successful implementation of the theory to practical productive uses; (v) interaction contributes not only to the diffusion of a given state of technical knowledge—it fosters, as well, the growth of ideas and techniques. The model allows for the endogenous determination of optimal trajectories concerning the allocation of time and the intensity of interaction; moreover, a long-term endogenous growth rate for the economy is derived, with optimal growth being essentially driven by the state of techniques and by the forces that shape the human interaction process.

Keywords: Social Interaction, Knowledge Diffusion, Optimal Time Allocation, Endogenous Growth, Basic and Applied Research

1. INTRODUCTION

Models of endogenous growth are typically designed as optimal control problems that are solved by a representative agent who acts with the goal of maximizing the respective intertemporal utility. In these models, technology evolves exogenously or, in the cases in which it advances in an endogenous mode, it is purposefully developed through carefully planned and implemented research and development (R&D) activities. In recent years, this perspective on growth has evolved, as many of the contributions mentioned in Section 2 allow to document. It is increasingly common to encounter in the literature studies where technological progress is

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essentially the result of the local interaction between individual units in the economy. Although in such settings knowledge spreads via direct contact among players, this does not signify that knowledge is a pure public good: a search effort is required, and resources allocated to the stimulus of interaction are resources that cannot be simultaneously employed to produce goods.

The mentioned class of models, from which the paper by Lucas and Moll (2014) is a prototypical example, takes a large number of individual agents with a similar behavior: they all pursue an optimal allocation of time between production of new goods and the search for partners with whom to interact. Evidently, there is an implicit trade-off in this behavior: it is the production of goods that, ultimately, will increase utility; however, without interaction agents will not have access to new ideas and new techniques and, consequently, will not be able to enhance the productivity with which they produce. Although apparently simple in its premises, a growth model that basically inquires about the allocation of time between production and search for new ideas may involve highly intricate dynamics. In the Lucas–Moll model, for instance, the conceived analytical structure gives place to a computationally demanding setup built upon a two-dimensional system of partial differential equations.

In this article, a growth model that approaches the optimal allocation of time across production and decentralized interaction is presented and the respective dynamics are characterized. The model economy is populated by a large number of individuals with an identical goal: to maximize expected utility via computation of the optimal degree of social contact. The setting is such that, differently from the Lucas–Moll setup, a trivial optimal control problem arises, with first-order optimality conditions taking the form of ordinary differential equations to which conventional techniques of dynamic analysis can be applied.

Despite the relatively simple and straightforward analytical treatment that the growth problem may receive, it is a theoretical structure prepared to simultaneously address other meaningful growth issues besides time allocation. Agents are split into categories, at a given initial date, and with a given probability may evolve from one category to the next as they contact with other individuals in the economic network. The categories that one considers in the context of the growth model are ignorance relatively to the state of technology, theoretical or basic knowledge, and applied knowledge. There is a sequential path that each individual eventually follows starting at the first state and ending in the third state. The described process is, in fact, a process of technology diffusion in which those who know nothing about the technology may first acquire the conceptual knowledge about it and, in a second stage, possibly evolve to a state where they will learn or acquire the skills that enables them to work with or manage the technology for practical use.

Although the diffusion of technology is an important feature of the model, it is not approached in a dissociated manner relatively to innovation. One considers that social interaction is also fundamental to amplify the technology frontier: those who know the technology, when meeting, are able to innovate and, thus, contribute to the progress of the existing techniques. The setup

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considers as well the possibility of knowledge obsolescence: those who are unable to keep up with technical progress suffer the risk of falling back to the initial compartment of ignorance relatively to the current state of the technology. For the agents whose knowledge becomes obsolete, the process of diffusion starts all over again: they need to meet someone who knows the theory and, then, they will eventually progress again to the implementation stage when finding, in a second moment, someone who can help putting the new ideas into action.

The combination of the previous elements gives rise to an optimal growth problem with interesting dynamic properties. The representative agent will maximize expected utility subject to a couple of constraints regarding the evolution of densities of individuals across technology states and to a constraint concerning the growth rate of technology. The control variable is the share of time allocated to interaction or the degree of connectivity of the agent with other members of the assumed population.[1]

Because every agent faces a same optimal control problem and acts optimally, they will all choose a same degree of connectivity with other players in the economy. This means that, necessarily, the interaction process is characterized by homogeneous mixing, i.e., each agent will select, randomly across the population, a same number of individuals with whom to interact, what facilitates the analytical treatment of the model, since one can exclude from the start complex network structures. The novelty here is that while homogeneously mixing populations are generally taken by assumption for simplicity purposes, in this case homogeneous mixing is an endogenous characteristic of the environment: there is no reason for agents solving an identical problem to choose different degrees of interaction.

A trivial optimal control analysis is applied to the specific context of the model under evaluation. The steady state and respective stability properties are assessed. One derives the steady-state growth rate of the economy, the long-term rate of innovation, the equilibrium allocation of time across activities, and the optimal connectivity degree selected by the representative agent. The analysis of stability unveils the existence of a saddle-path stable equilibrium, common to most optimal control growth analyses.

Typically, representative agent growth models focus on the trade-off between consumption and savings and, occasionally, also on the compromise between allocating time to labor or leisure. The central contribution of this paper consists in highlighting the main implications of bringing a third trade-off to the core of the growth analysis, namely the intertemporal choice agents make between spending time performing activities directly associated with the creation of value or, alternatively, spending time in establishing networks of social contact that allow to acquire and absorb ideas and knowledge that are required for the enhancement of productivity. The main result, for endogenous growth theory, consists in the computation of a positive and constant steady-state growth rate that is essentially determined by the forces that shape interaction (transition probabilities across knowledge states emerging from random meetings and the configuration of the

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function that associates time allocated to interaction with the share of successfully consummated interactions).

The remainder of the paper is organized as follows. Section 2 reviews the literature that is relevant in the context of the current study. In Section 3, the process of knowledge diffusion and knowledge creation is described; the respective dynamics are discussed. Section 4 attaches the knowledge dynamic rules to an optimal control problem of time allocation; optimality conditions are derived and the properties of the steady state are discussed. In Section 5, heterogeneity across agents is introduced; individual agents are split into "good" and "bad" communicators and the results of the model with this variant are compared with the ones of the benchmark case. Section 6 introduces capital accumulation into the analysis with a typical capital accumulation equation being added. Section 7 concludes.

2. THE LITERATURE: SIX RELEVANT PIECES

The optimal growth model explored in this paper integrates a series of features that typically appear in the literature dissociated from each other. The most relevant contributions relating these features are reviewed in the following six subsections.

2.1. Technology Diffusion and Technological Convergence

The diffusion of ideas, knowledge, and techniques is, undoubtedly, an economic process of primary importance. This point is highlighted in Comin and Mestieri (2013), who present a thorough survey on issues relating the measurement, the sources, and the implications of technology diffusion. A key observation is that the ability to engage in successful original research conducting to innovation in the production of new goods and services is restricted to a few individuals, institutions, and countries, while the vast majority of economic agents simply adopts already existing technologies. Adoption is, in itself, an engine of growth, because it allows for the dissemination of the ideas and techniques that raise efficiency in production.

Innovation and adoption are, in fact, two sides of a same coin: without innovation, growth dynamics will sooner or later be exhausted; however, without diffusion and adoption, the potential of wealth creation that research results disclose will never be fulfilled. The interplay between innovation and imitation and their joint contribution to productivity growth is a subject of intense research; some meaningful studies on this issue include Mukoyama (2003), Luttmer (2011, 2015), Davis and Hashimoto (2015), and König et al. (2016). In the analysis, we engage on in the following sections, innovation and imitation are also analyzed in an integrated perspective. The decentralized interaction that takes place in society is a driving diffusion force and, simultaneously, allows as well for the emergence of new ideas and production techniques.

Another fundamental issue concerning diffusion is that it is synonymous of heterogeneity in the timing of adoption; some agents are early adopters, while

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others adhere to the innovation only after a significant share of the interested population has done the same. Peyton-Young (2009) identifies various sources of heterogeneous adoption behavior, including differences in the access to information, differences in the desirability to conform, and distinct learning capabilities. As a result, three classes of diffusion models are explored by the mentioned author: diffusion by contagion, diffusion by social influence, and diffusion by social learning. Mukoyama (2004) also focuses his attention in the gradual adoption of technologies and associates this to the distribution of skills in the economy. The same author, Mukoyama (2006), emphasizes yet another source of sluggish technology adoption: firms learn from the experience of users, making diffusion to be contingent on the quality of the communication channels and on the strength of the demand. In the model to analyze in the next sections, adoption patterns are shaped by social contact; it is implicitly assumed that once an agent that ignores an idea enters in contact with someone that is knowledgeable, the first makes a transition, with a given probability to the second state. The mentioned probability is shaped by the strength of the forces underlying social influence and social learning.

As it happens with the current article, technological diffusion processes are often approached in the context of growth models and help in explaining patterns of growth of national economies. Early influential studies on this subject include Barro and Sala-i-Martin (1997) and Benhabib and Spiegel (2005), who provided the first coherent explanations on how imitation and adoption can be interpreted as the required devices to approach the neoclassical growth concept of convergence in the context of the endogenous growth setup. The recent literature on growth has recovered the analysis of catching-up or convergence processes, with some interesting models being proposed. Meaningful contributions in this field include Xue and Yip (2012), Benhabib et al. (2014), Perla and Tonetti (2014), and Stokey (2015).

2.2. Basic Vs. Applied Research

In the growth-interaction model developed in this paper, the distinction between theoretical knowledge and applied knowledge acquires special importance. In growth models, this distinction goes back to Nelson and Phelps (1966), who have recognized the need for separating between two definitions of technology. A variable A measures the level of technology in practice, i.e., the technological knowledge currently in use in the productive activity, and a variable T, the theoretical level of technology that corresponds to the level of technology under the hypothetical circumstance of completely instantaneous diffusion. This last entity is the stock of knowledge that exists in the economy and that awaits to be given a productive use.

The above distinction between knowledge in theory and knowledge in practice presupposes a hierarchy of types of knowledge. This hierarchical relation is debated and explained in the work by Gersbach et al. (2009). Basic research is

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embryonic and it precedes applied research in the sense that the first generates the ideas, theories, and prototypes that constitute the base over which applications with a commercial use are developed. Furthermore, a two-way spillover between the two types of knowledge exists: applied techniques cannot be implemented if there is no theoretical base in which they can be sustained on; however, in the opposite direction, the practical implementation of new concepts and ideas triggers the required environment for the creation of new theoretical knowledge. Thus, two fundamental features arising from distinguishing among theory and applications that are central in the analysis of technology creation and diffusion are hierarchy and interdependency.

Along the same lines, we encounter a series of other studies, namely Mokyr (2002), Morales (2004), Arts et al. (2012), Akcigit et al. (2013), Cozzi and Galli (2014), and Prettner and Werner (2014). These all distinguish between a conceptual knowledge base and applied productive knowledge or techniques. Basic research is typically nonexcludable, it has to be somehow supported by the society as a whole and it is a necessary input in the development of technologies with a direct productive use; these, in turn, are excludable and are typically protected by patents.

2.3. Technological Obsolescence

Technological obsolescence is an important feature of the growth model in this paper. It is assumed that only the most recent generation of technology is productive, i.e., in every period, when a new technology appears, the previous generation of technological knowledge becomes obsolete from a productivity point of view. A significant part of the agents that already use the technology will be able to adapt to the new generation but there is a fraction of production units that cannot keep up with the technological progress and become obsolete.

An important pioneer contribution in the field of knowledge obsolescence is Rosen (1975). This author attaches the notion of obsolescence to the passage of time. From time to time new knowledge emerges, making worthless previously available technologies. Typically, the new knowledge reveals that previous generations of ideas and techniques are incomplete or even inadequate, losing the economic value they had in the past. In this perspective, the distinction between obsolescence and depreciation becomes self-evident: while depreciation has to do with an intrinsic loss of value of the techniques or of the capital goods that employ such techniques, obsolescence arises as the result of the emergence of other, more productive, ideas, concepts, and ways to produce.

The notion that obsolescence is a process associated with the replacement of old generations of knowledge by new generations that are economically more efficient has given rise to various relevant theories of technological obsolescence, innovation, and growth. Among these contributions, one can emphasize the work of Fishman et al. (1993), Lee and Lee (1998), Bosworth and Jobome (2003), and Boucekkine et al. (2004). In every case, innovation leading to the creation of new, state-of-the-art, technologies will imply that old technologies become

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economically obsolete. Some of these studies suggest that obsolescence might be endogenous, as producers choose the optimal moment to innovate in order to discard previous ideas, techniques, or processes. The software industry is a good example of an activity where purposeful obsolescence is carefully planned to maximize economic gains.

It is also important to highlight the relation of the obsolescence phenomenon, as characterized above, with the Schumpeterian notion of creative destruction, which is a central notion of the growth theory since the work of Aghion and Howitt (1992). It is through creative destruction that new generations of technology replace the previous ones; in the current paper, this creative destruction exists, leaving behind those who cannot adapt to the use of new techniques. Technological obsolescence may be, as well, associated with the extent in which knowledge is misallocated [Raurich et al. (2015)].

2.4. Epidemic/Rumor Spreading Theory

Technology diffusion takes place, in the model to be characterized, essentially through social contact or social interaction. Whenever an agent that is ignorant about a technology meets someone holding theoretical knowledge, the first agent evolves to the second state with a given probability. The same type of metamorphosis occurs in the transition from the theoretical knowledge to the applied knowledge state. The specific process through which these transitions take place borrows from two related strands of literature, namely epidemic contagion and rumor spreading.

Contagious diseases propagate as individuals meet each other. The fundamental feature of a disease propagation process concerns the existence of different categories, classes or, as Pastor-Santorras et al. (2015) call them, compartments. The compartments commonly assumed in this type of models are the susceptible, the infected, and the recovered, and the two most popular versions of the model are the susceptible-infected-recovered (SIR) and the susceptible-infected-susceptible (SIS). Other compartments can be included (as the immune), but the main versions with the most common types of agents are sufficient to understand the spreading process.

The modeling of epidemic processes can be interpreted as a metaphor that explains how any type of propagation process in nature, society, or economics takes place. The same principles underlying the spreading of diseases can be applied to the propagation of information, cultural, or social norms, or knowledge and technology, as it is the case dissected in this paper. One of the most popular adaptations of the epidemic spreading model is the rumor propagation setup, initially conceptualized by Daley and Kendall (1964, 1965) and Maki and Thompson (1973). In the rumor propagation models, the typical categories of agents are ignorant, spreaders, and stiflers. When an ignorant establishes contact with a spreader of a rumor, the first becomes knowledgeable about the rumor and will then spread it as well, with a given probability. When a spreader meets another

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spreader or a stifler, she loses the interest in spreading the rumor and passes to the stifler compartment.

The main difference between epidemic spreading and rumor spreading is identified by Boccaletti et al. (2006). According to these authors, the ultimate goal of modeling the propagation of infectious diseases consists in conceiving and designing strategies that allow to control and eventually eradicate those diseases. In opposition, rumor spreading frameworks are prepared to encounter ways of disseminating as fast and efficiently as possible the rumor. Because rumor spreading is a social process and, frequently, the respective models involve the desire to speed up the underlying diffusion, these are closer to the analysis one intends to undertake in this paper. Simultaneously, this type of propagation process has received much attention in recent literature. Therefore, a few more considerations about it are in order.

Rumor propagation models have been extended, in the last few years, essentially in two directions. On one hand, several authors have sophisticated the basic model through the introduction of new categories of agents [Thompson et al. (2003), Huo et al. (2012), Zhao et al. (2012), Wang et al. (2013)]. The new categories include active and passive agents (active agents are those who have a high propensity to contact others and to spread rumors; passive agents have the opposite attitude), trusted and untrusted neighbors (the probability of spreading a rumor is obviously higher when a trusted source is contacted), agents with stronger or weaker remembering capabilities, i.e., agents with a low or high propensity to forget the rumor, and a new category of incubators, who are those that ear the rumor but spread it only in future dates. Adding these new compartments of agents sophisticates the original model and gives rise to new dynamic patterns; however, in its essence the qualitative behavior continues to be essentially the same: the rumor spreads as the interaction across agents in some network of relations occurs.

A second direction of fruitful discussion on rumor propagation models is the one that highlights the relevance of the structure or topology of the network of social interactions. The original model considers a homogeneously mixing population. In this case, the interactions occur randomly with anyone in the network regardless of geography, community ties, or other factors. This is an analytically convenient assumption, however, it fails to represent many relevant types of social relations. It is common to identify rumor propagation processes in scale-free networks [Nekovee et al. (2007)] or in small-world networks [Zanette (2002)]. Scale-free networks are those in which a large number of nodes is connected to a few other, but where some nodes acquire the characteristic of "hubs" being linked with a large number of other agents in the network. Small-world networks are highly clustered structures of social relations, where the distance between any two nodes is relatively small. Our technology diffusion model will ignore network structure issues, by taking homogeneous mixing. This allows for analytical tractability and does not go against the assumption normally taken in economics that markets are complete and, therefore, any agent in the network can establish an interaction with any other agent in the same social structure.[2]

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Both epidemics and rumors have been used before as elements of analysis in the economic literature. To cite just a few examples, one can highlight the study of Kosfeld (2005) relating the effects of rumors in market prices and market transactions and the analysis of Evangelatos and Carayannis (2014) and Gurley and Johnson (2017), who adopt an epidemiological perspective to approach innovation diffusion, specifically taking into account the examination of patent citations and the spread of ideas among economic scholars. Another influential study that borrows ideas from epidemiology, in the case to justify information propagation mechanisms that might generate business fluctuations at the macrolevel, is Angeletos and La'O (2013); agents are split into three categories, as in most of the epidemic and rumor spreading models, being, in this case, the uninformed, the exuberant, and the informed. These three categories mimic those from rumor spreading models; obviously, periods in which exuberance dominates, are those in which an economic boom takes place. As the exuberance vanishes and the respective wave of optimism fades away, the economic activity is constricted.

2.5. Knowledge Diffusion in Social Networks

In the beginning of this section, mechanisms of knowledge diffusion were addressed. In that occasion, one has ignored the structure of interaction, focusing attention on scenarios of completely random meetings within a large population. This type of "mean-field" approach is convenient from an analytical tractability point of view; however, as argued in Peyton-Young (2006), the topology of the network might be decisive for the type of observed diffusion and even to determine whether the diffusion process is well succeeded.

Most of the studies concerned with the diffusion of knowledge in social networks emphasize the role of the interaction with the closest neighbors. In Lamberson (2010), the adoption of a new technology is conditional on prior beliefs and on the experiences of friends and acquaintances (i.e., the neighbors in the social network). López-Pintado (2013), in turn, refers to influence networks; in his perspective, the network gains a dynamic nature as the result of how influential some agents are and how easily some other agents are influenced by the first. In König (2014), the pattern of innovation diffusion is determined by the degree of collaboration among firms, which, in turn, depends on the expectations of each firm about learning the new technology from the collaboration with others; in this last case, the network structure is shaped by the expectations of agents and a cyclical pattern in the intensity of collaboration is likely to emerge given the two-way influence between the propagation of technology and the formation of alliances across firms.

In Acemoglu et al. (2011), the linear threshold model initially proposed by Granovetter (1978) is recovered in order to inquire about the patterns of knowledge diffusion. In the mentioned model, a given agent adopts an innovation only if the fraction of the respective neighbors that have already adopted is above a given threshold. The conclusion of the authors is that under this type of environment technology might spread faster when the networks display small degrees of

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clustering. Another study worth mentioning in this context is the one by Fogli and Veldkamp (2012). In their article, the issue of how the structure of the social network influences the process of technology diffusion is directly addressed. The capital conclusion is that some types of social networks propagate new technologies faster or more efficiently than others. Some societies, in the words of the authors, have growth-inhibiting social structures that prevent knowledge from spreading.

While the above reasoning contains compelling arguments pointing to the relevance of the social structure, one can also support the opposite view, a view based on agent rationality and on a cost-benefit analysis that might lead the market participants to collect the information required to contact any other agent in the economic system. The compatibility between local interaction under specific network configurations and the comprehensiveness of competitive market relations is debated in the next subsection.

2.6. The Economic Rationale of Social Interactions

Defying the idea, that has become popular, that the science of complexity can be a useful allied of economics in explaining socioeconomic issues, Durlauf (2005, 2012) argues that there is no compelling evidence to support the view that the economic system is basically a complex macronetwork emerging from a multiplicity of simple decentralized local interaction processes. The argument is if the interactions are local then they are necessarily confined to a part of the market and, thus, markets are incomplete.

The issue is not market incompleteness per se. Economists are aware that markets are often incomplete and the models they propose frequently incorporate the main obstacles that prevent the fully efficient performance of markets, namely transaction costs, information asymmetries, coordination failures, and strategic complementarities. The problem with the complexity approach is that it imposes market incompleteness in an ad-hoc way; interactions are assumed to be local without any good reason. Under this view, in which missing markets arise arbitrarily, the complex network approach does not bring anything truly new to the conventional wisdom in economics, where the reasons for the lack of market completeness are, by now, well understood.

A counter-argument relatively to the above reasoning is the one presented in Manski (2000), who emphasizes that there are areas of the economic science where market relations are insufficient to provide a solid and consistent interpretation of the observable phenomena. One of such areas is precisely research and innovation; the sharing of ideas occurs many times beyond the limits of the market relations, as agents establish social contact. This argument has been used by several authors, including Lucas (2009), Staley (2011), and Lafond (2015), who have set the foundations for introducing social interaction in the analysis of growth, namely in the context of endogenous growth models.

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A prominent study that goes in the direction mentioned above is the one cited in the introduction: Lucas and Moll (2014). These authors develop a new model of endogenous growth, being the novel distinctive features essentially the following: (i) knowledge is partially rival, more precisely, it is rival in the short run because people need to exert effort to acquire it and nonrival in the long run since it does not lose value as it spreads across the population; (ii) agents solve an optimal control problem, where the control variable relates to the time allocated to two main activities: production of consumption goods and search for new, more productive, ideas. Search occurs through social contact, which involves random meetings. By assuming that meetings are random, the Durlauf's critique is avoided: markets are not necessarily incomplete; each agent can contact any other agent in the population, but she must incur in a search cost for the meeting to occur. A meeting is, in this context, interpreted as the observation of another individual's productivity; when the productivity of the partner of interaction is higher than that of the first agent, this adopts the ideas of the other.

3. KNOWLEDGE DIFFUSION: SPREADING IDEAS AND TURNING THEORY INTO ACTION

Consider an economy endowed, at a given date t, with the level of technological knowledge $A(t) > 0$. In this economy, three categories of agents coexist: those who are unaware of the existence of the technology (share x); those who possess theoretical knowledge about the technology, but lack the skills to successfully apply it to production (share y); and those who not only are aware of the potential of the technology but also have the skills required to manage and apply it (share z). The mentioned shares cover the whole universe of agents, i.e., $x(t)+y(t)+z(t) = 1, \forall t$. Agents placed in share z will be able to generate output, with a level of productivity that has correspondence on the technology index $A(t)$.

Agents possess the faculty of evolving across categories through social contagion or social interaction. Following Nekovee et al. (2007), it is assumed that the flow of transition from a given state x to a given state y is delivered by the continuous-time dynamic equation

$$\dot{x}(\phi, t) = -\lambda\phi x(\phi, t)\sum_{\phi'} P(\phi'|\phi)y(\phi', t). \quad (1)$$

In equation (1), $\lambda \in (0, 1)$ represents the probability with which an agent of type x migrates to state y, when meeting an agent that is already in possession of knowledge of type y. Parameter ϕ is a measure of the number of links emanating from a specific node in the initial state x to other nodes in the underlying social network; ϕ', in turn, corresponds to the connectivity degree of each of the agents in state y that potentially interact with the ϕ-degree agent in state x. As in Piqueira (2010), the degree of connectivity is formulated as being a percentage, i.e., $\phi \in [0, 1]$; when $\phi = 0$, the agent remains in isolation; in the opposite case, $\phi = 1$, the agent is fully connected with the entire network. Expression $P(\phi'|\phi)$ is known

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as the degree–degree correlation function and it represents the weight of each connectivity class in the interaction with agents located at the nodes that possess connectivity degree ϕ.

If the social network is homogeneous, i.e., if all agents possess an identical connectivity degree, then $\sum_{\phi'} P(\phi'|\phi) y(\phi', t) = y(t)$ and expression (1) simplifies to[3]

$$\dot{x}(t) = -\lambda \phi x(t) y(t). \tag{2}$$

In the specific environment of the model economy of this paper, the connectivity degree will not be exogenously given; instead, it will be the outcome of an intertemporal decision process. Moreover, because every agent faces the same optimization problem, ϕ will turn out to be identical across agents. Consequently, a homogeneously mixing population emerges endogenously and the connectivity degree will be approached as a time-dependent variable.

Consider, for now, that equation (2) represents the process of transition of ignorance to theoretical knowledge. Following the rumor propagation literature discussed in Section 2, let us also assume that the transition from the theoretical knowledge state to the implementation state occurs under the following rule: when an agent at state y interacts with another agent in state y or in state z, that agent progresses to state z with a probability $\sigma \in (0, 1)$.[4] The differential equation representing the progressive increase in share z as the interaction process unfolds is, then,

$$\dot{z}(t) = \sigma \phi(t) y(t) [y(t) + z(t)]. \tag{3}$$

Relatively to the density of agents holding the theoretical knowledge about A, this will change given the inflow from the ignorant state and the outflow to the applied knowledge state, i.e.,

$$\dot{y}(t) = \lambda \phi(t) x(t) y(t) - \sigma \phi(t) y(t) [y(t) + z(t)]. \tag{4}$$

System (2)–(4)–(3) is similar to the ignorant-spreader-stifler framework from the rumor spreading literature. In this setup, there exists multiple equilibria: first, there is a no-contagion equilibrium, for which, in the long run, all the agents are concentrated in the susceptible or ignorant state, $x^* = 1$. This equilibrium point is unstable and will be sustained only under the initial condition $[x(0), y(0), z(0)] = (1, 0, 0)$. When $x(0) \neq 1$, the long-run locus will be such that $y^* = 0$; the distribution of agents between x^* and z^* will depend on the specific values of parameters λ and σ, on the long-term connectivity degree ϕ^*, and also on the initial values of the three density variables. In this second case, the system rests in a bifurcation position.[5] The latter result is the nontrivial one and it is accomplished whenever there is at least one agent that is not ignorant or, in the current context, has a theoretical idea that can be used to implement productive technologies. As in an epidemic, where a single individual can be the source of a fast spreading disease, also in a knowledge diffusion setting a single scientist can launch the idea

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or ideas that will conduct to the successful implementation of new production technologies.

The technology diffusion setup will accommodate an additional feature, which will imply the existence of a unique nontrivial determinate equilibrium point. This feature relates to the definition of the growth rate of the technology level. It is assumed that technology grows at rate

$$\frac{\dot{A}(t)}{A(t)} = \gamma\phi(t)z(t)\left[y(t) + z(t)\right]. \tag{5}$$

Equation (5) translates the following rationale: if all agents are implementing the technology, $z(t) = 1$, and the network is fully connected, $\phi(t) = 1$, then the technology is assumed to grow at a maximum constant rate $\gamma > 0$. However, the state of technology will only evolve when those who already use the technology interact with someone who also uses the technology or, at least, is aware about its existence and usefulness. Obviously, this interaction effect has as much impact on technology growth as the stronger is the degree of connectivity.[6]

In the current framework, the growth of the stock of productive ideas implies technological obsolescence: the faster $A(t)$ grows, the larger will be the number of technology users who will not be able to keep up with the increased sophistication and fall back to the ignorant state. Consequently, at each period there is a percentage $\delta[\frac{\dot{A}(t)}{A(t)}]$ of agents in migration from share z back to share x; this percentage increases with a faster pace of technological progress, $\delta'(.) > 0$. Taking a linear obsolescence function

$$\delta\left[\frac{\dot{A}(t)}{A(t)}\right] = \delta\frac{\dot{A}(t)}{A(t)} = \delta\gamma\phi(t)z(t)\left[y(t) + z(t)\right], \quad \delta \in (0, 1). \tag{6}$$

Technological obsolescence, formalized via equation (6), yields a new outflow–inflow movement that reshapes the equations of motion (2) and (3). These are replaced by

$$\dot{x}(t) = -\lambda\phi(t)x(t)y(t) + \delta\gamma\phi(t)z(t)\left[y(t) + z(t)\right], \tag{7}$$

$$\dot{z}(t) = \sigma\phi(t)y(t)[y(t) + z(t)] - \delta\gamma\phi(t)z(t)\left[y(t) + z(t)\right]. \tag{8}$$

System (7)–(4)–(8) represents a process in which individual agents are systematically moving from one state to the next: an ignorant may become a holder of conceptual knowledge, with a given probability, and later an agent endowed with the ability to implement the technology; the agent may not keep up with technological progress and fall back to the first category, initiating the same process all over again. As one will confirm below, there is a unique steady-state point (x^*, y^*, z^*) besides the trivial equilibrium of perpetual ignorance. However, the composition of each share does not remain constant in the equilibrium locus; the

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constant steady state just translates the idea that for every unit that abandons one of the categories there is another individual agent that replaces the first.

The aggregate properties of the propagation system are straightforward to derive. The first result is associated with the characterization of the steady state.

PROPOSITION 1. *The technology diffusion model has two steady-state points: (i) the diffusion-free equilibrium (DFE),* $(x^*, y^*, z^*) = (1, 0, 0)$ *and (ii) the diffusion-endemic equilibrium (DEE),*

$$(x^*, y^*, z^*) = \left(\frac{\sigma}{\lambda + \sigma}, \frac{\lambda}{\lambda + \sigma} \frac{\delta\gamma}{\sigma + \delta\gamma}, \frac{\lambda}{\lambda + \sigma} \frac{\sigma}{\sigma + \delta\gamma} \right).$$

Proof. Take system (7)–(4)–(8) and solve $\left[\dot{x}(t), \dot{y}(t), \dot{z}(t)\right] = (0, 0, 0)$. ■

The designation of the equilibrium points in Proposition 1 is adapted from the rumor propagation literature [see Huo et al. (2012)], which uses, respectively, the denominations rumor-free equilibrium and rumor-endemic equilibrium. Note that the DEE steady-state levels of each of the category shares is contingent solely on the probabilities λ and σ, the parameter δ and the growth rate γ; initial conditions are not relevant, neither it is the degree of connectivity.

Under the DFE technology does not grow, while under the DEE the steady-state innovation rate is

$$g_A^* = \gamma \left(\frac{\lambda}{\lambda + \sigma} \right)^2 \frac{\sigma}{\sigma + \delta\gamma} \phi^*. \tag{9}$$

Equation (9) establishes a direct relation between the degree of connectivity among agents in the long run and the steady-state growth rate of technology: the more interconnected the economy is, the faster will be the pace of innovation, this, of course, under the DEE. Result (9) indicates as well that in the long-term technical knowledge grows at a constant rate as long as ϕ^* is also constant.

Next, one evaluates the stability properties of the equilibria. The following result is derived.

PROPOSITION 2. *The DFE is unstable and the DEE corresponds to a stable oscillating steady state. These results hold regardless of the dynamics of the connectivity degree.*

Proof. See Appendix A.

The instability of the DFE signifies that no technological diffusion occurs solely under condition $x(0) = 1$; this translates the notion that a single scientist holding a new theoretical concept is all that is required for an everlasting process of knowledge propagation. The stability of the other equilibrium will, in turn, imply that an initial state such that $x(0) \neq 1$ triggers a convergence process toward the point (x^*, y^*, z^*) given by the DEE, where all the three densities acquire nonzero values.

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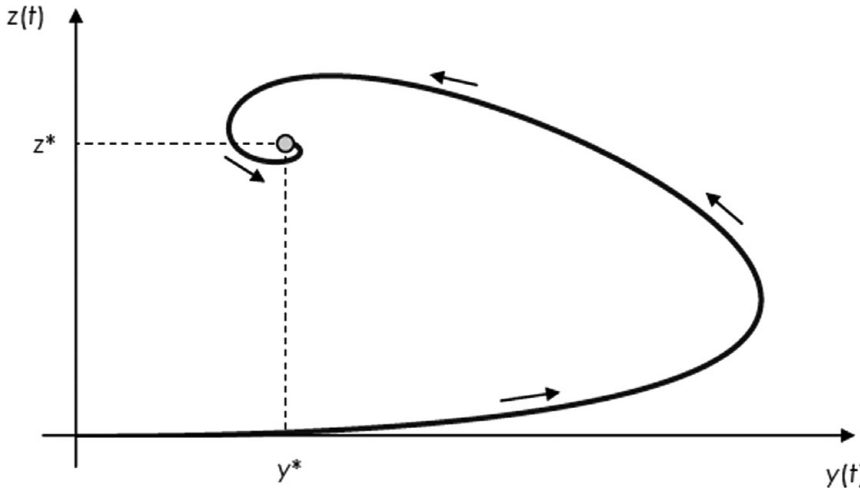


FIGURE 1. Transitional dynamics of the knowledge diffusion process.

Figure 1 displays the joint evolution of variables $y(t)$ and $z(t)$ in their oscillatory convergence path toward the long-run equilibrium, under the DEE. In the graphic, $y(0)$ is set to an infinitesimal positive value and $z(0)$ is set to zero, in order to illustrate that a very restrict set of ideas in their conceptual form is enough to ignite the knowledge diffusion process that will then unfold.

Until now, the technology diffusion process has been approached solely through the analysis of the state equations that characterize the evolution of agents across knowledge categories. Further insights are unveiled once one addresses the optimal choice concerning the allocation of time between production and social interaction. This is done in the next section.

4. OPTIMAL ALLOCATION OF TIME AND OPTIMAL CONNECTIVITY

In the model economy under analysis, agents face a same optimal control problem and behave alike. Hence, optimizing behavior can be characterized from the point of view of a representative agent.[7] The representative agent is the owner of a unit of time, per period t, that she wants to allocate between production and social interaction. A trade-off exists: production of consumption goods enhances utility, but it is social interaction that allows to collect the ideas that can be used to increase productivity.

Let $s(t)$ be the share of time allocated to social interaction. The production function of the agent is, in this case,

$$\widetilde{y}(t) = [1 - s(t)]\, A(t) z(t). \tag{10}$$

The agent produces an amount of goods identical to the level of technology weighted by the time spent in production and given the probability that she is

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in the possession of the knowledge required to operate the technology, i.e., the probability of belonging to share z.

For now, capital accumulation is ignored and a market-clearing condition imposes the coincidence between output and consumption, $\widetilde{y}(t) = c(t)$. Assuming an infinite horizon and an intertemporal discount rate $\rho > 0$, the representative agent maximizes the expected future flow of consumption utility by solving the optimal control problem

$$\underset{s(t)}{\text{Max}} \int_0^{+\infty} \mathbf{E}\{U[c(t)]\} \exp(-\rho t) dt. \tag{11}$$

The utility function, $U(.)$, belongs to the class of constant intertemporal elasticity of substitution utility functions, as commonly considered in optimal growth analyses. In order to simplify computation, a simple logarithmic objective function will be assumed.

Maximization problem (11) is a constrained optimization problem. A first set of constraints to which (11) is subject to corresponds to the rules of motion for the shares x, y, and z; a compact form of presenting these constraints comes from suppressing variable x from the analysis and to take the pair of equations in system (A.1) of Appendix A. Another dynamic constraint of the problem is the equation that furnishes the motion of the growth rate of the technology state, namely (5). Variables $y(t)$, $z(t)$, and $A(t)$ are the state variables of the problem, meaning that their initial values, $y(0)$, $z(0)$, $A(0)$ are given. In order to confine the analysis to the scenario of a single economically meaningful steady state (the DEE), one imposes the following condition upon the initial values of the share variables: $y(0) + z(0) > 0$. In the context of the optimal control problem one must also consider the constraints over the admissible values of the state variables: $0 \leq y(t) \leq 1$, $0 \leq z(t) \leq 1$, $A(t) \geq 0$ and on the control variable, $0 \leq s(t) \leq 1$.

To complete the characterization of the optimal control problem of the representative agent, one must establish a functional relation between the time allocated to interaction and successful interaction. Take function $\phi(t) = f[s(t)]$, $f : [0, 1] \rightarrow [0, 1]$. This will be a continuous and differentiable function with a positive first derivative and a negative second derivative: returns of the allocation of time to social interaction over connectivity are positive but diminishing; this is an apparently reasonable assumption, since time dedicated to the exchange of ideas with others will certainly be subject to decreasing returns. A functional form for $f(.)$ that fulfills the mentioned requisites and that will be convenient to carry out the analytical study of the model is the following: $\phi(t) = 1 - [1 - s(t)]^{\theta}$, with $\theta > 1$.[8,9]

The steady-state levels of shares y and z were determined in the preceding section independently from the optimization problem. By solving the problem one computes other relevant long-term values, namely the optimal degree of connectivity, the steady-state distribution of time between interaction and production, and

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also an explicit expression for the rate of innovation.[10] The long-term equilibrium will be unique, since we already have circumscribed the analysis to the DEE case.

PROPOSITION 3. *In the DEE, if inequality* $\theta \geq \frac{\rho}{\gamma\left(\frac{\lambda}{\lambda+\sigma}\right)^2 \frac{\sigma}{\sigma+\delta\gamma}}$ *is satisfied, then the following steady-state results hold:*

— *Optimal degree of connectivity:*

$$\phi^* = 1 - \frac{\rho}{\theta\gamma\left(\frac{\lambda}{\lambda+\sigma}\right)^2 \frac{\sigma}{\sigma+\delta\gamma}}.$$

— *Optimal growth rate of output and technology:*

$$g_{\tilde{y}}^* = g_A^* = \gamma\left(\frac{\lambda}{\lambda+\sigma}\right)^2 \frac{\sigma}{\sigma+\delta\gamma} - \frac{\rho}{\theta}.$$

— *Optimal share of time allocated to social interaction:*

$$s^* = 1 - \left[\frac{\rho}{\theta\gamma\left(\frac{\lambda}{\lambda+\sigma}\right)^2 \frac{\sigma}{\sigma+\delta\gamma}}\right]^{1/\theta}.$$

Proof. See Appendix B.

The inequality condition in Proposition 3 that places a lower bound on the value of parameter θ is required to guarantee nonnegative values of ϕ^*, g_A^*, and s^*. An inspection of the steady-state results shows that a more impatient representative agent (high ρ) will be an agent that allocates relatively less time to interaction and will also be an agent establishing a lower degree of connectivity with the economic network she belongs to. Additionally, the technological state and the economy will exhibit a lower long-term growth rate, i.e., impatience inhibits growth. Opposite results are found for parameter θ, which is a measure of the concavity in the relation between time spent in interaction and successful contact from the point of view of knowledge diffusion; a stronger concavity, that reflects a more efficient interaction, will compel the agent to spend additional time in the interaction activity, leading to a better interaction performance and stronger growth.

It is interesting to compare the steady-state growth of technology as first presented in Section 3, equation (9), with the corresponding result in Proposition 3. Expression (9) suggests that in a complete market, where all agents interact with each other, $\phi = 1$, the steady-state rate of innovation is $(g_A^*)_{\phi=1} = \gamma\left(\frac{\lambda}{\lambda+\sigma}\right)^2 \frac{\sigma}{\sigma+\delta\gamma}$; optimization of the allocation of time across production and interaction implies $g_A^* = (g_A^*)_{\phi=1} - \frac{\rho}{\theta}$. Because $\phi = 1$ can only be obtained if all the available time is allocated to interaction and this is not an admissible option, since it leaves no time left to produce goods to satisfy needs and create utility, the optimal result departs from the complete market ideal scenario by an amount $\frac{\rho}{\theta}$; (im)patience and the (in)efficiency of the interaction process are the two driving forces that may place

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the growth outcome closer or more apart from the ideal setting of an automatically completely connected economy network.

In Section 3, the stability of the system that had as endogenous variables $y(t)$ and $z(t)$ was evaluated. It was revealed that the local transitional dynamics of this pair of variables could be examined independently of the remainder of the model and that an outcome of stability holds regardless of the values of the relevant parameters. Now, one can complement the stability analysis by inquiring about the behavior of the technology variable in the vicinity of the steady state. This analysis requires the examination of the joint evolution of variables technology and respective shadow-price, $q_A(t)$. More precisely, one has to explore the qualitative behavior of the system of equations formed by the pair of normalized variables $A_n(t)$ and $q_{An}(t)$; these two variables are defined in Appendix B, in the proof of Proposition 3, and they basically represent detrended versions of the original variables, $A(t)$ and $q_A(t)$, such that their steady-state growth rates are equal to zero. For the pair of normalized variables, a saddle-path stability result is derived.

PROPOSITION 4. *In the diffusion-interaction model, the system formed by the pair of ordinary differential equations providing the rules of motion of variables $A_n(t)$ and $q_{An}(t)$ fulfills the requirements needed for an independent local stability analysis regardless from the other laws of motion associated to the same problem. In such system, a saddle-path stability relation between the two normalized variables exists, and one-dimensional stable and unstable trajectories are given by the following equations, respectively,*

$$q_{An}(t) = \left(2 + \frac{\sqrt{\theta}}{\rho}\right) q^*_{An} - \left(1 + \frac{\sqrt{\theta}}{\rho}\right) \frac{q^*_{An}}{A^*_n} A_n(t), \tag{12}$$

$$q_{An}(t) = \left(2 - \frac{\sqrt{\theta}}{\rho}\right) q^*_{An} + \left(\frac{\sqrt{\theta}}{\rho} - 1\right) \frac{q^*_{An}}{A^*_n} A_n(t). \tag{13}$$

Proof. See Appendix C.

The line representing the saddle-path stable trajectory in Proposition 4, (12), is a negatively sloped line. The slope of the unstable trajectory, (13), can be either positive or negative in theory; however, for a not absurdly large intertemporal discount rate it is surely positive. The dynamic behavior in the vicinity of the steady state of these two variables is depicted in Figure 2. The figure shows that for any initial pair $[A_n(0), q_{An}(0)]$ over the stable path, the two variables will converge in the direction of the steady state $\left(A^*_n, q^*_{An}\right)$ until this point is reached; as the value of one of the variables increases under this adjustment process, the value of the other variable falls. Although one cannot compute explicit expressions for A^*_n and q^*_{An} under the assumed setting, there is an important information one is knowledgeable about: when the convergence process toward the steady state is completed, the technology level will grow forever at a constant rate, namely the

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TABLE 1. Parameter values

λ	σ	δ	ρ	θ	γ_1	γ_2	γ_3
0.9	0.1	0.15	0.03	2.5	0.02	0.04	0.06

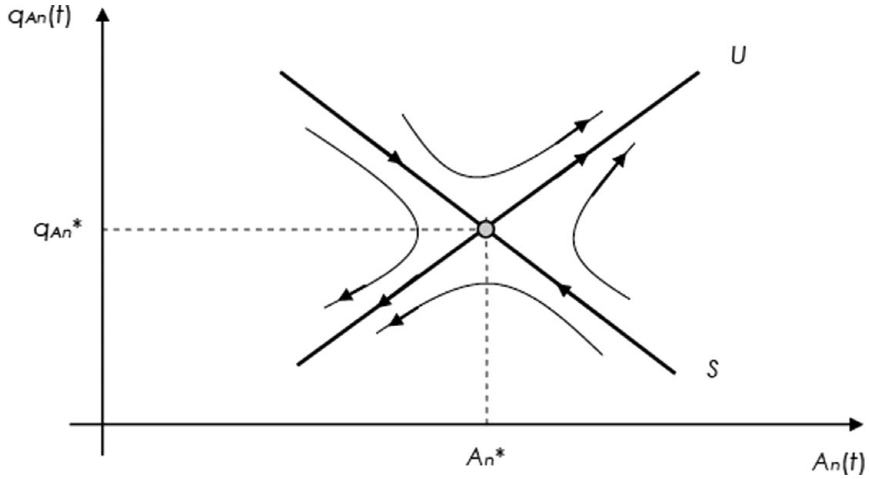


FIGURE 2. Phase diagram in the $A_n(t) - q_{An}(t)$ locus.

rate endogenously derived from the model's assumptions and that is presented in Proposition 3.

In order to conduct an illustration of the results obtained so far, consider a short numerical example. Selected parameter values are displayed in Table 1. The values chosen for probabilities λ and σ are the same as in one example of Huo et al. (2012) and translate the idea that it is relatively easy to acquire information about the existence and usefulness of technology through interaction but the same is not true with respect to the skills necessary to implement it. The rate of technological obsolescence is in conformity with the rate of average depreciation of R&D stocks mentioned in Bosworth and Jobome (2003); the value of θ is arbitrarily set at 2.5, and the intertemporal discount rate is the one adopted in Stokey (2015). Three alternative values are proposed for the potential growth rate γ.

Parameter values will be used to quantify the steady state. The DEE for the densities x, y, and z is, for each of the potential growth rates,

$$\gamma_1 : (x^*, y^*, z^*) = (0.1, 0.0262, 0.8738)\,,$$

$$\gamma_2 : (x^*, y^*, z^*) = (0.1, 0.0509, 0.8491)\,,$$

$$\gamma_3 : (x^*, y^*, z^*) = (0.1, 0.0743, 0.8257)\,.$$

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In this specific example, in the steady state there are 10% of the agents who are completely unaware of the latest technological improvement. The remaining population is associated to each of the states y and z, but the concrete percentages of allocation are contingent on the value of the potential rate of innovation; the higher the value of γ, the more agents will locate at state y in detriment of state z. In any case, parameter values are such that the steady-state share of agents who can manage the technology to productive use is the large majority.

The other meaningful steady-state values are those presented in Proposition 3. The constraint that is required to guarantee $\phi^* \geq 0$ is, in the illustration, equivalent to $\gamma \geq 0.0151$. This lower bound on the potential growth rate is respected for any of the three values of γ assumed in the example.

For the proposed parameter values, the following steady-state results are calculated:

(1) Optimal degree of connectivity:

$$\gamma_1 : \phi^* = 0.2370;\ \gamma_2 : \phi^* = 0.6074;\ \gamma_3 : \phi^* = 0.7309.$$

(2) Optimal growth rate:

$$\gamma_1 : g_A^* = 0.0037;\ \gamma_2 : g_A^* = 0.0186;\ \gamma_3 : g_A^* = 0.0326.$$

(3) Optimal share of time allocated to social interaction:

$$\gamma_1 : s^* = 0.7259;\ \gamma_2 : s^* = 0.6425;\ \gamma_3 : s^* = 0.5842.$$

The above results illustrate how an increase in the potential growth rate implies an increase in the steady-state degree of connectivity, an increase in the effective growth rate, and a fall in the time allocated to interaction. For instance, if the potential growth rate is 6%, the effective growth rate in the steady state will be 3.26%. This result is accomplished for a steady-state connectivity degree of 73.09%, i.e., every agent chooses to optimally contact that percentage of the whole population in search for new ideas, and this is done by employing 58.42% of the available time; the remaining 41.58% is dedicated to production.

Figure 3 represents the relation between the potential growth rate and each of the steady-state values ϕ^*, g_A^*, and s^*. The direct relation with the two first equilibrium values and the inverse relation with s^* is in this way also evident.

5. GOOD AND BAD COMMUNICATORS

In this section, an extended version of the diffusion-interaction framework is subject to analysis. This new setting allows for the existence of heterogeneity among agents. Two classes of agents are considered: a class of good communicators that are rational and optimize their behavior along the lines of the setup that was examined in the preceding section; and a class of agents endowed with poor communication skills, who will not only accomplish a lower connectivity degree when allocating the same time to interaction compared with the good communicators, but that will also be unable to optimize and will prefer to follow

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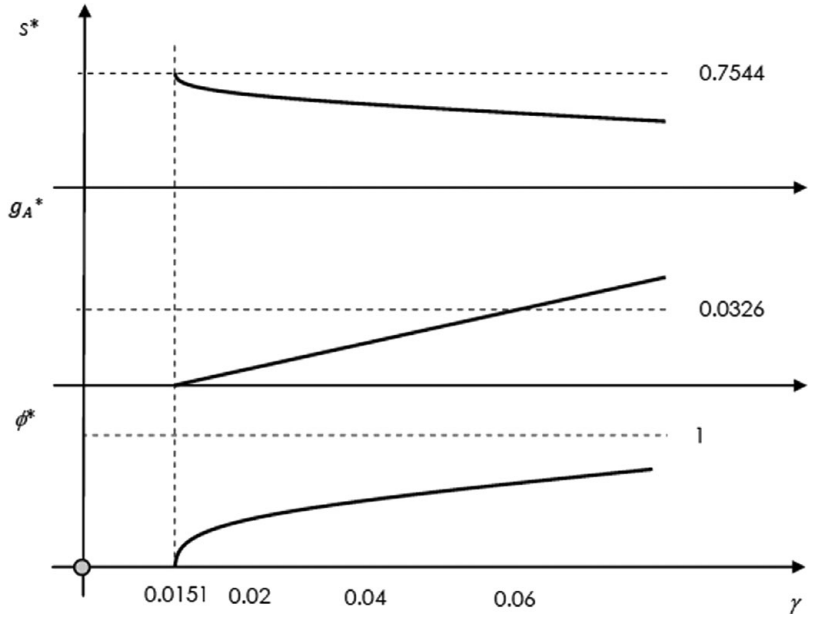


FIGURE 3. Steady-state results for different levels of γ.

a simple rule-of-thumb with respect to time allocation. This heuristic consists in allocating time by imitating the behavior of the other class.

Let $u_G \in (0, 1)$ be the share of good communicators in the economy and $u_B = 1 - u_G$ the share of bad communicators; these are exogenous constant values. Good communicators have a time allocation—interaction function similar to the one assumed in Section 4, i.e., $\phi_G(t) = 1-[1-s_G(t)]^\theta$. Bad communicators will achieve a lower connectivity degree when employing a same amount of time to interaction; taking parameter $\xi \in (0, 1)$, the time allocation-interaction function of bad communicators can be expressed under the form $\phi_B(t) = (1 - \xi)\{1 - [1 - s_B(t)]^\theta\}$. Figure 4 elucidates about the position of the two functions. The second class of agents accomplishes, for every time allocation level, a relatively lower degree of connectivity and, differently from what occurs with the good communicators, a full allocation of time to interaction will not signify the ability to contact with everyone else in the economic network.

As mentioned above, good communicators are also assumed perfectly endowed with the ability to optimize their behavior. In opposition, bad communicators will not display the capacity to choose an optimal path for the respective time allocation; these will imitate the behavior of the good communicators, choosing exactly the same trajectory of the time allocation share as the other class of agents. If, by assumption, $s_B(t) = s_G(t)$, then a constant ratio between connectivity degrees will hold irrespective of the time period, $\phi_B(t)/\phi_G(t) = 1 - \xi$.[11]

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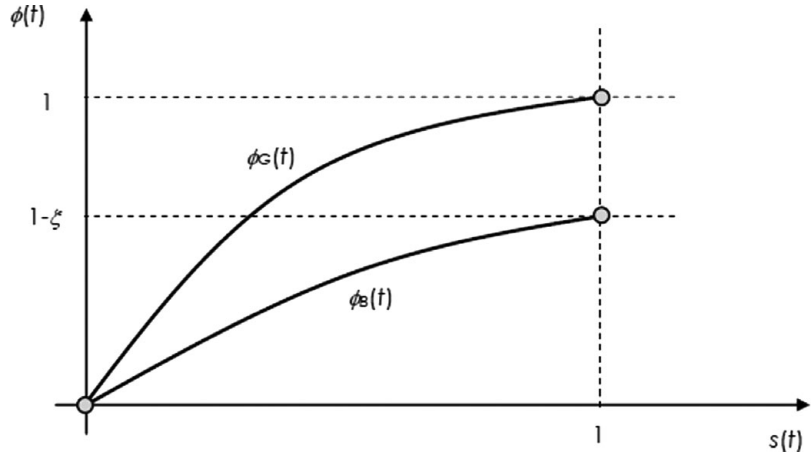


FIGURE 4. Connectivity functions for good and bad communicators.

Recover equation (1). The transition flow between states x and y continues to be represented as in that equation, but the degree–degree correlation functions must now weight the existence of the two classes, such that

$$\sum_{\phi'} P(\phi'|\phi)y(\phi', t) = \frac{\phi_B(t)u_B y_B(t) + \phi_G(t)u_G y_G(t)}{\overline{\phi}(t)} \tag{14}$$

with $\overline{\phi}(t) \equiv \phi_B(t)u_B + \phi_G(t)u_G$. Simplifying (14)

$$\sum_{\phi'} P(\phi'|\phi)y(\phi', t) = \frac{(1-\xi)(1-u_G)y_B(t) + u_G y_G(t)}{(1-\xi)(1-u_G) + u_G}. \tag{15}$$

Let the fraction in (15) be denoted by $Y(t)$ and define, as well,

$$Z(t) \equiv \frac{(1-\xi)(1-u_G)z_B(t) + u_G z_G(t)}{(1-\xi)(1-u_G) + u_G}. \tag{16}$$

Applying a same reasoning that led to (14) to the various transition processes that underlie the knowledge diffusion mechanism as characterized in Section 3, one can write, for the good communicators, the following differential equations, which describe the process of evolution across knowledge states:

$$\dot{x}_G(t) = -\lambda\phi_G(t)x_G(t)Y(t) + \delta\frac{\dot{A}(t)}{A(t)}, \tag{17}$$

$$\dot{y}_G(t) = \lambda\phi_G(t)x_G(t)Y(t) - \sigma\phi_G(t)y_G(t)\left[Y(t) + Z(t)\right], \tag{18}$$

$$\dot{z}_G(t) = \sigma\phi_G(t)y_G(t)\left[Y(t) + Z(t)\right] - \delta\frac{\dot{A}(t)}{A(t)}. \tag{19}$$

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In expressions (17) to (19), $x_G(t)$, $y_G(t)$, and $z_G(t)$ are, respectively, the densities of ignorant, holders of basic knowledge, and applied technicians in the class of good communicators; other variables and parameters in these expressions are as defined previously.

Relatively to the growth rate of technology, one continues to consider the same type of interaction process conducting to innovation as before, although share z that now establishes successful interaction with other agents in categories y and z will be an weighted average of agents in both good and bad communicator classes. The relevant expression is

$$\frac{\dot{A}(t)}{A(t)} = \gamma \phi_G(t) \left[(1-\xi)(1-u_G) z_B(t) + u_G z_G(t)\right] \left[Y(t) + Z(t)\right]. \tag{20}$$

For the class of bad communicators, a group of three equations similar to (17) to (19) is presentable, with $x_B(t)$, $y_B(t)$, and $z_B(t)$ representing, respectively, ignorant, theorists, and practitioners, endowed with poor communication skills:

$$\dot{x}_B(t) = -(1-\xi)\lambda \phi_G(t) x_B(t) Y(t) + \delta \frac{\dot{A}(t)}{A(t)}, \tag{21}$$

$$\dot{y}_B(t) = (1-\xi)\lambda \phi_G(t) x_B(t) Y(t) - (1-\xi)\sigma \phi_G(t) y_B(t) \left[Y(t) + Z(t)\right], \tag{22}$$

$$\dot{z}_B(t) = (1-\xi)\sigma \phi_G(t) y_B(t) \left[Y(t) + Z(t)\right] - \delta \frac{\dot{A}(t)}{A(t)}. \tag{23}$$

The two three-dimensional systems that characterize knowledge diffusion for both good and bad communicators can be reduced, as in the system of Section 3, to two-dimensional systems under the equalities $x_G(t) + y_G(t) + z_G(t) = 1$ and $x_B(t) + y_B(t) + z_B(t) = 1$. Undertaking the required transformation

$$\dot{y}_G(t) = \phi_G(t) \left\{\lambda \left[1 - y_G(t) - z_G(t)\right] Y(t) - \sigma y_G(t) \left[Y(t) + Z(t)\right]\right\}, \tag{24}$$

$$\dot{z}_G(t) = \phi_G(t) \left\{\sigma y_G(t) - \delta\gamma \left[(1-\xi)(1-u_G) z_B(t) + u_G z_G(t)\right]\right\} \left[Y(t) + Z(t)\right]. \tag{25}$$

and, for the class of bad communicators,

$$\dot{y}_B(t) = (1-\xi)\phi_G(t) \left\{\lambda \left[1 - y_B(t) - z_B(t)\right] Y(t) - \sigma y_B(t) \left[Y(t) + Z(t)\right]\right\}, \tag{26}$$

$$\dot{z}_B(t) = (1-\xi)\phi_G(t) \times \left\{\sigma y_B(t) - \frac{1}{1-\xi}\delta\gamma \left[(1-\xi)(1-u_G) z_B(t) + u_G z_G(t)\right]\right\} \left[Y(t) + Z(t)\right]. \tag{27}$$

The dynamic analysis of the diffusion-interaction model with two classes begins with the characterization of the steady state. As the last sections highlighted, the

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framework necessarily involves two steady states, the DFE and the DEE. The first continues to exist in the current specification of the framework, but given its economic irrelevance it will be ignored and attention will concentrate in the diffusion-endemic steady state.

A first set of steady-state results one derives by applying the set of conditions $(\dot{x}_G, \dot{y}_G, \dot{z}_G, \dot{x}_B, \dot{y}_B, \dot{z}_B) = \overline{0}$, with $\overline{0}$ a null vector, is the following:

$$x_G^* = (1-\xi)x_B^*, \tag{28}$$

$$y_G^* = (1-\xi)y_B^*, \tag{29}$$

$$z_G^* = 1-(1-\xi)\left(1-z_B^*\right). \tag{30}$$

Observe that $x_G^* < x_B^*$, $y_G^* < y_B^*$, $z_G^* > z_B^*$, $\forall \xi \in (0,1)$. These inequalities reveal the steady-state advantage that being a good communicator brings: good communicators, relatively to bad communicators, will be allocated, in a greater percentage, to the applied technology state, the one that matters from the productivity enhancement point of view.

Computation of steady-state values for each of the endogenous variables yields the following explicit equilibrium point:

$$x_G^* = [1-\xi(1-u_G)]\,\frac{\sigma}{\lambda+\sigma}, \tag{31}$$

$$x_B^* = \frac{1-\xi(1-u_G)}{1-\xi}\,\frac{\sigma}{\lambda+\sigma}, \tag{32}$$

$$y_G^* = [1-\xi(1-u_G)]\,\frac{\lambda}{\lambda+\sigma}\,\frac{\delta\gamma}{\sigma+\delta\gamma}, \tag{33}$$

$$y_B^* = \frac{1-\xi(1-u_G)}{1-\xi}\,\frac{\lambda}{\lambda+\sigma}\,\frac{\delta\gamma}{\sigma+\delta\gamma}, \tag{34}$$

$$z_G^* = [1-\xi(1-u_G)]\,\frac{\lambda}{\lambda+\sigma}\,\frac{\sigma}{\sigma+\delta\gamma} + \xi(1-u_G), \tag{35}$$

$$z_B^* = \frac{1-\xi(1-u_G)}{1-\xi}\,\frac{\lambda}{\lambda+\sigma}\,\frac{\sigma}{\sigma+\delta\gamma} - \frac{\xi}{1-\xi}u_G. \tag{36}$$

Note that one can recover the original model's steady state without agent heterogeneity by letting $\xi = 0$; in this case, it is straightforward to confirm that $x_G^* = x_B^* = \frac{\sigma}{\lambda+\sigma}$, $y_G^* = y_B^* = \frac{\lambda}{\lambda+\sigma}\frac{\delta\gamma}{\sigma+\delta\gamma}$, $z_G^* = z_B^* = \frac{\lambda}{\lambda+\sigma}\frac{\sigma}{\sigma+\delta\gamma}$.

To further explore the steady-state results, one must solve the optimal control problem of the representative good communicator agent. A problem similar to (11) should be taken. The production function of a good communicator is $\widetilde{y}_G(t) = [1-\phi(t)]^{1/\theta}A(t)z_G(t)$ and the problem is solved exactly under a same

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procedure as in Section 4.[12] Two results are particularly relevant; first the steady-state optimal connectivity degree and, second, the steady-state rate of innovation. For the connectivity degree, the outcome is

$$\phi_G^* = 1 - \frac{\rho}{\theta\gamma\left[(1-\xi)(1-u_G)z_B^* + u_G z_G^*\right](Y^* + Z^*)}, \tag{37}$$

which is equivalent to

$$\phi_G^* = 1 - \frac{\rho}{\theta\gamma\left[1-\xi(1-u_G)\right]\left(\frac{\lambda}{\lambda+\sigma}\right)^2 \frac{\sigma}{\sigma+\delta\gamma}}. \tag{38}$$

Observe, as well, that the average number of links, weighted by the number of individuals in each communication class is

$$(1-u_G)\phi_B^* + u_G\phi_G^* = 1 - \xi(1-u_G) - \frac{\rho}{\theta\gamma\left(\frac{\lambda}{\lambda+\sigma}\right)^2 \frac{\sigma}{\sigma+\delta\gamma}}. \tag{39}$$

The condition that guarantees the nonnegativeness of the steady-state connectivity degree is now more restrictive than in the first version of the model; it poses the following lower bound on the value of θ, $\theta \geq \frac{\rho}{\gamma[1-\xi(1-u_G)]\left(\frac{\lambda}{\lambda+\sigma}\right)^2 \frac{\sigma}{\sigma+\delta\gamma}}$. If one compares (39) with the equilibrium connectivity degree in the case of agent homogeneity, one realizes that the result in (39) is a lower value than the one presented in Proposition 3, as long as $\xi > 0$ and $0 < u_G < 1$. This is a logical result, given that in the current scenario a share of agents, $1 - u_G$, has poorer communication skills and does not exhibit the ability to solve the respective optimal control problem. This weaker connectivity will have a direct impact on the long-run growth of technology. By replacing (37) into a steady-state expression for (20), one derives

$$g_A^* = \left[1-\xi(1-u_G)\right]\gamma\left(\frac{\lambda}{\lambda+\sigma}\right)^2 \frac{\sigma}{\sigma+\delta\gamma} - \frac{\rho}{\theta}. \tag{40}$$

Comparing (40) with the equilibrium innovation rate in Proposition 3, one observes that there is a loss of growth directly associated with the existence of a communication bias that affects part of the population, relatively to the benchmark case.

The analysis of the heterogeneous communicators setup becomes complete with an evaluation of stability. Although one should expect qualitatively similar results to those obtained in the homogeneously mixing population case, the larger dimension of the underlying dynamic system introduces some computational obstacles that prevent obtaining simple outcomes. Appendix D addresses the stability issue.

6. THE GROWTH MODEL WITH CAPITAL ACCUMULATION

Capital accumulation is now added to the diffusion-interaction framework. In this setting, variable $k(t) \geq 0$ represents the stock of capital per unit of labor and

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$c(t) \geq 0$ defines per capita consumption. A Cobb–Douglas production function with Hicks-neutral technical progress is assumed, $\widetilde{y}(t) = [1 - s(t)]\, A(t) z(t) k(t)^{\alpha}$, with $\alpha \in (0, 1)$ the output-capital elasticity. The optimization problem that the representative agent faces is

$$\underset{s(t),c(t)}{\text{Max}} \int_0^{+\infty} \mathbf{E}\{\ln[c(t)]\} \exp(-\rho t) dt. \tag{41}$$

Problem (41) is subject to the following constraints: the dynamics of the population shares in system (A.1) of Appendix A, the growth rate of technology, (5), and the static constraint that associates time allocation to successful interaction. Besides these, another constraint must be included, namely the trivial capital accumulation differential equation,

$$\dot{k}(t) = \widetilde{y}(t) - c(t) - \delta_k k(t), \tag{42}$$

where $\delta_k \in (0, 1)$ is the capital depreciation rate. Initial values of the state variables $k(0)$, $y(0)$, $z(0)$, $A(0)$ are given, and the constraints on the values that the state and the control variables can assume continue to apply.

Steady-state results are derived after solving the optimal control problem. Again, the attention is centered in the DEE equilibrium.

PROPOSITION 5. *The steady-state optimal connectivity degree and the steady-state optimal innovation rate in the setting with capital accumulation are identical to those found in the original diffusion-interaction model. The stock of capital, consumption, and income grow, in the capital accumulation setup, at a higher rate than the rate of innovation,* $g_{\widetilde{y}}^* = g_k^* = g_c^* = \frac{1}{1-\alpha} g_A^*$.

Proof. See Appendix E.

The introduction of capital accumulation into the model does not change the main steady-state results: time allocation, degree of interaction, and rate of innovation are the same assuming or not a process of capital accumulation. Nevertheless, the shape of the production function implies that the growth rate of the economy is no longer identical to the rate of technological progress; this is the direct consequence of how the capital input and technology are combined to generate output.

Another relevant steady-state result that emerges in the capital accumulation setting is the equilibrium consumption-capital ratio. This ratio is defined in Appendix E and it is one of the endogenous variables of the dynamic system obtained in the process of solving the optimal control problem. The steady-state consumption-capital ratio is

$$\psi^* = \frac{1}{\alpha}\left[\gamma\left(\frac{\lambda}{\lambda+\sigma}\right)^2 \frac{\sigma}{\sigma+\delta\gamma} + \frac{\theta-1}{\theta}\rho + (1-\alpha)\delta_k\right]. \tag{43}$$

The steady-state level of consumption relatively to the accumulated stock of capital, as displayed in (43), benefits from a large number of agents acquainted

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with the technological state, from a high intertemporal discount rate and from a strong depreciation rate of capital.

Regarding stability, the analysis indicates the following,

PROPOSITION 6. *The DEE of the diffusion-interaction model with capital accumulation is saddle-path stable. In the three-dimensional space of the underlying dynamic system, a one-dimensional stable trajectory exists.*

Proof. See Appendix F.

The saddle-path stability result in Proposition 6 is compatible with local convergence in the direction of the steady state. This is so because, as detailed in Appendix F, the dynamic analysis considers, as endogenous variables, the consumption-capital ratio, a ratio involving variables technology and capital, and the product between technology and its co-state variable. Only the second of these aggregates is an exclusive combination of state variables, while the other two involve a control variable and a shadow-price; thus, these last ones can be initially adjusted in order for the system to follow the one-dimensional stable path that directs the economy toward the equilibrium position.

7. CONCLUSION

Optimal growth models generally search for a compromise between a parsimonious structure of analysis and a comprehensiveness of the subjects they can, at the same time, address. The model proposed in this paper consists in a relatively simple framework that, nonetheless, is rich enough to weight, in an integrated manner, a large variety of issues relating the growth performance of economies. The model is simple in the sense it can be formalized under the form of a typical optimal control problem subject to a small number of constraints, which acquire the shape of ordinary differential equations. It is comprehensive because it simultaneously approaches optimal intertemporal time allocation, local interaction among rational agents, technology diffusion and innovation, technological obsolescence, basic versus applied research, optimal connectivity, utility maximization, and output growth.

The process of knowledge diffusion is essential, in the framework under analysis, for the determination of the long-run equilibrium of the economy. Two equilibria exist, an unstable diffusion-free equilibrium, for which no meaningful growth dynamics can be identified, since the economy remains forever in a state of stagnation, and an endemic equilibrium, which is the nontrivial equilibrium that is important to dissect in this setting. The endemic steady state is stable, in what regards technology diffusion and, once the control variables are added to the analysis, a saddle-path equilibrium, common in this type of optimal growth models, emerges. The saddle-path stable steady state signifies, in this context, that the agent that solves the optimal control problem can place herself on the stable trajectory, in the direction of the long-term equilibrium once it controls

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for the values of the variables that she can manipulate. Control variables are, in the simplest version of the model, the time allocation share or, alternatively, the degree of connectivity in the social network, and in the version of the model that admits capital accumulation also the consumption variable.

Although the analysis begins by assuming that multiple agents exist and interact in a social network, the study of the model's dynamics may be reduced to the problem of a unique representative agent. This is so because all agents face exactly the same economic scenario and, thus, they will all act in the same manner. Every agent will choose, in the long run, under the endemic steady state, a same distribution of time between production and local interaction in search for new ideas, and a same degree of connectivity with other agents in the economy. Furthermore, the framework is able to endogenously derive a constant rate of growth, which decisively depends on the long-term shares of individuals with knowledge about the state-of-the-art technology, on the intertemporal discount rate and on the parameter that measures the elasticity between time allocation and connectivity across agents.

The designed model contributes to endogenous growth theory by highlighting the role of the production-interaction time allocation trade-off. Sustained long-term growth, observed in developed economies along the past decades, is undoubtedly the outcome of technological innovation and diffusion. Despite the nonrival essence of these growth determinants, though, the resources required for the generation and dissemination of knowledge, namely human intervention, are rival, because time spent sharing ideas and creating knowledge is necessarily time diverted from the creation of value and vice-versa. Any interpretation of the growth process that ignores this evidence overlooks an important part of the mechanics of growth.

NOTES

1. A direct relation between the time allocated to interaction and the degree of interaction that the agent establishes with her peers, at each time period, is assumed and, thus, one or the other variable can be taken, indistinctly, as the control variable of the problem.

2. See Section 2.6 for further details on the (in)compatibility between social interaction and market completeness.

3. Another way of looking at the proposed rule of motion is to associate it with the physics' law of mass action: the density of ignorant individuals falls, given the transition probability and the connectivity degree, as the result of the value of the product between the shares of ignorant and knowledgeable individuals. This product reflects the frequency of contacts between agents located in each category. Equations (3) and (4) are designed by applying the same type of reasoning.

4. This rule specifies that someone with a conceptual knowledge about a given technique will potentially evolve to the implementation stage through interaction with someone who already possesses the practical knowledge or with another theorist. The logic supporting the latter possibility is based on the supposition that contact and discussion about a technique between two individual agents endowed with the understanding of how the technology works will allow to discover how it can be implemented. This assumption is essential, as discussed below, to be feasible to have a couple of scientists as the trigger of the knowledge diffusion process: if there were no way through which theorists, by themselves, could reach a practical solution, no idea could ever start to be implemented.

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5. The notion of bifurcation is, in this concrete case, directly associated with the impossibility of finding a determinate solution for system $(\dot{x}, \dot{y}, \dot{z}) = (0, 0, 0)$.

6. One might argue that innovation, as expressed via differential equation (5), is represented in an oversimplified form. Our single focus is on the interaction among knowledgeable individuals, thus setting aside relevant economic processes, namely, in this case, the purposive R&D investment effort that, in reality, productive units systematically and pervasively undertake.

In fact, we are implicitly assuming that there is a constant flow of R&D investment that would generate a constant rate of technological change if no interaction dynamics were present. The emphasis on social interaction to justify innovation is in conformity with the following perspicacious argumentation by Lucas (2009, p.1):

It is widely agreed that the productivity growth of the industrialized economies is mainly an ongoing intellectual achievement, a sustained flow of ideas. Are these ideas the achievements of a few geniuses, Newton, Beethoven and a handful of others, viewed as external to the activities of ordinary people? Are they the product of a specialized research sector, engaged in the invention of patent-protected processes over which they have monopoly rights? Both images are based on important features of reality and both have inspired interesting growth theories, but neither seems to me central. What is central, I believe, is the fact that the industrial revolution involved the emergence (or rapid expansion) of a class of educated people, thousands - now many millions - of people who spend entire careers exchanging ideas, solving work-related problems, generating new knowledge.

7. This representative agent should be interpreted as an heterogeneous household, embodying the three individual types one is considering (those with no knowledge, those holding theoretical knowledge, and those holding the skills to implement the knowledge). Because only the last category of individuals participates in production, only these generate income. Therefore, there is an implicit intrahousehold transfer process that guarantees subsistence for all the elements integrating the household.

On a society wide interpretation, one could, alternatively, imagine that part of the productive sector income is channeled, through the fiscal system, to universities (that develop public basic science activities) and to nonactive people (who receive social security transfers). Redistribution matters are, nevertheless, overlooked in this framework.

8. Constraint $\theta > 1$ guarantees that diminishing returns hold for the assumed functional form of $f(.)$. Later in the analysis, an eventually more restrictive constraint on the value of θ emerges; such constraint is imposed in order to guarantee that condition $\phi(t) \geq 0$ is not violated.

9. Because of the functional dependency between time allocated to interaction and interaction outcome, the differential equations that constitute the constraints to the optimal control problem, namely (7), (4), (8), and (5), could be written taking variable $s(t)$ instead of $\phi(t)$. For instance, equation (5) could be replaced by $\frac{\dot{A}(t)}{A(t)} = \gamma f[s(t)]z(t)\,[y(t) + z(t)]$. The optimal control problem might then be solved for one or the other variable as control variable, with interchangeable results. We opt to work directly with the degree of interaction, although keeping in mind the straightforward relation between the two, as imposed by $f(.)$.

10. Note that, given production function (10), the growth rate of the economy coincides with the rate of innovation, in the steady state.

11. Other versions of the behavior of the two classes of agents could be considered, namely that they could be both optimizers, but this would turn the analytical study of the model impracticable. The simplifying assumption that allows for a constant ratio between connectivity degrees is convenient from an analytical tractability point of view.

12. One must follow the same steps as those taken in Appendix B.

13. Note that in the proposed problem, the necessary conditions for optimality are also sufficient conditions since the objective function is concave and the constraints of the problem form a convex set.

14. The trace of G has negative sign because the elements in the main diagonal are all negative. The determinant is positive and it corresponds to a relatively simple expression, namely, $Det(G) = (1-\xi)^2(1-u_G)\lambda\sigma(\frac{\delta\gamma}{\sigma+\delta\gamma})^2[(\lambda+\sigma)\delta\gamma+\sigma^2]$. The principal minors of order 2 of G are all positive

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and, thus, their sum is positive. The principal minors of order 3 of G are all negative and, consequently, their sum is negative. A patient calculus process allows to confirm the inequality of the Routh-Hurwitz criterion; we refrain to present it here given the huge amount of elements both terms of the inequality contain.

15. A same argument presented with the characterization of the framework without capital to justify the sufficiency of the necessary optimality conditions continues to hold in the scenario with capital accumulation.

16. Condition $Det\,(J_k) < 0$ holds for $\psi^* > 0$ and $\psi^* - \rho > 0$. The veracity of this second inequality is not self-evident, but it can be confirmed by noting that

$$\psi^* - \rho = \frac{1}{\alpha}\left[\gamma\left(\frac{\lambda}{\lambda+\sigma}\right)^2 \frac{\sigma}{\sigma+\delta\gamma} - \frac{\rho}{\theta} + (1-\alpha)\,(\rho+\delta_k)\right]$$

In the steady-state, condition (E.8) is equivalent to

$$\theta \geq \frac{\rho}{\gamma\left(\frac{\lambda}{\lambda+\sigma}\right)^2 \frac{\sigma}{\sigma+\delta\gamma}}$$

what implies that the difference between the two first terms within brackets is non-negative. Consequently, the whole expression is necessarily a positive amount for all admissible parameter values.

REFERENCES

Acemoglu, D., A. Ozdaglar, and E. Yildiz (2011) Diffusion of innovations in social networks. In E. Chong, J. Farrell, E. Camacho, and M. Polycarpou (eds.), *Proceedings of the 50th IEEE Conference on Decision and Control (CDC-ECC)*, pp. 2329–2334. Orlando, FL: Institute of Electrical and Electronics Engineers (IEEE).

Aghion, P. and P. Howitt (1992) A model of growth through creative destruction. *Econometrica* 60, 323–351.

Akcigit, U., D. Hanley, and N. Serrano-Velarde (2013) Back to Basics: Basic Research Spillovers, Innovation Policy and Growth. NBER working paper 19473, National Bureau of Economic Research.

Angeletos, G. M. and J. La'O (2013) Sentiments. *Econometrica* 81, 739–779.

Arts, S., B. Cassiman, and R. Veugelers (2012) Mind the Gap: Capturing Value from Basic Research: Boundary Crossing Inventors and Partnerships. CEPR discussion paper 9215.

Barro, R. J. and X. Sala-i-Martin (1997) Technological diffusion, convergence and growth. *Journal of Economic Growth* 1, 1–26.

Benhabib, J., J. Perla, and C. Tonetti (2014) Catch-up and fall-back through innovation and imitation. *Journal of Economic Growth* 19, 1–35.

Benhabib, J. and M. M. Spiegel (2005) Human capital and technology diffusion. In P. Aghion and S. N. Durlauf (eds.), *Handbook of Economic Growth*, vol. 1a, pp. 935–966. Amsterdam and San Diego: Elsevier, North-Holland.

Boccaletti, S., V. Latora, Y. Moreno, M. Chavez, and D. U. Hwang (2006) Complex networks: Structure and dynamics. *Physics Reports* 424, 175–308.

Bosworth, D. and G. Jobome (2003) The rate of depreciation of technological knowledge: Evidence from patent renewal data. *Economic Issues* 8, 59–82.

Boucekkine, R., C. Saglam, and T. Vallée (2004) Technology adoption under embodiment: A two-stage optimal control approach. *Macroeconomic Dynamics* 8, 250–271.

Comin, D. A. and M. Mestieri (2013) Technology Diffusion: Measurement, Causes and Consequences. NBER working paper 19052, National Bureau of Economic Research.

Cozzi, G. and S. Galli (2014) Sequential R&D and blocking patents in the dynamics of growth. *Journal of Economic Growth* 19, 183–219.

Daley, D. J. and D. G. Kendall (1964) Epidemics and rumors. *Nature* 204, 1118.

Daley, D. J. and D. G. Kendall (1965) Stochastic rumours. *Journal of the Institute of Mathematics and Its Applications* 1, 42–55.

https://doi.org/10.1017/S1365100517000633 Published online by Cambridge University Press

Davis, C. and K. I. Hashimoto (2015) R&D subsidies, international knowledge diffusion, and fully endogenous productivity growth. *Macroeconomic Dynamics* 19, 1816–1838.
Durlauf, S. N. (2005) Complexity and empirical economics. *Economic Journal* 115, F225–F243.
Durlauf, S. N. (2012) Complexity, economics and public policy. *Politics, Philosophy and Economics* 11, 45–75.
Evangelatos, N. and E. Carayannis (2014) Innovation diffusion: An epidemiological perspective. *International Journal of Social Ecology and Sustainable Development* 5, 22–30.
Fishman, A., N. Gandal, and O. Shy (1993) Planned obsolescence as an engine of technological progress. *Journal of Industrial Economics* 41, 361–370.
Fogli, A. and L. Veldkamp (2012) Germs, Social Networks and Growth. NBER working paper 18470, National Bureau of Economic Research.
Gersbach, H., G. Sorger, and C. Amon (2009) Hierarchical Growth: Basic and Applied Research. Working paper 0912, Department of Economics, University of Vienna.
Granovetter, M. (1978) Threshold models of collective behavior. *American Journal of Sociology* 83, 1420–1443.
Gurley, N. and D. K. N. Johnson (2017) Viral economics: An epidemiological model of knowledge diffusion in economics. *Oxford Economic Papers* 69, 320–331.
Huo, L., P. Huang, and C. X. Guo (2012) Analyzing the dynamics of a rumor transmission model with incubation. *Discrete Dynamics in Nature and Society* article ID 328151, 21 pages.
König, M. D. (2014) Technology Cycles in Dynamic R&D Networks. Working paper 178, Department of Economics, University of Zurich.
König, M. D., J. Lorenz, and F. Zilibotti (2016) Innovation vs. imitation and the evolution of productivity distributions. *Theoretical Economics* 11, 1053–1102.
Kosfeld, M. (2005) Rumours and markets. *Journal of Mathematical Economics* 41, 646–664.
Lafond, F. (2015) Self-organization of knowledge economies. *Journal of Economic Dynamics and Control* 52, 150–165.
Lamberson, P. J. (2010) Social learning in social networks. *B.E. Journal of Theoretical Economics* 10(1), article 36.
Lee, I. H. and J. Lee (1998) A theory of economic obsolescence. *Journal of Industrial Economics* 46, 383–401.
López-Pintado, D. (2013) Influence networks. *Games and Economic Behavior* 75, 776–787.
Lucas, R. E. (2009) Ideas and growth. *Economica* 76, 1–19.
Lucas, R. E. and B. Moll (2014) Knowledge growth and the allocation of time. *Journal of Political Economy* 122, 1–51.
Luttmer, E. G. J. (2011) Technology diffusion and growth. *Journal of Economic Theory* 147, 602–622.
Luttmer, E. G. J. (2015) Four Models of Knowledge Diffusion and Growth. Working paper 724, Federal Reserve Bank of Minneapolis.
Maki, D. P. and M. Thompson (1973) *Mathematical Models and Applications, with Emphasis on Social, Life, and Management Sciences.* Englewood Cliffs, NJ: Prentice-Hall.
Manski, C. F. (2000) Economic analysis of social interactions. *Journal of Economic Perspectives* 14, 115–136.
Mokyr, J. (2002) *The Gifts of Athena.* Princeton, NJ: Princeton University Press.
Morales, M. F. (2004) Research policy and endogenous growth. *Spanish Economic Review* 6, 179–209.
Mukoyama, T. (2003) Innovation, imitation and growth with cumulative technology. *Journal of Monetary Economics* 50, 361–380.
Mukoyama, T. (2004) Diffusion and innovation of new technologies under skill heterogeneity. *Journal of Economic Growth* 9, 451–479.
Mukoyama, T. (2006) Rosenberg's 'Learning by Using' and technology diffusion. *Journal of Economic Behavior and Organization* 61, 123–144.
Nekovee, M., Y. Moreno, G. Bianconi, and M. Marsili (2007) Theory of rumor spreading in complex social networks. *Physica A* 374, 457–470.

https://doi.org/10.1017/S1365100517000633 Published online by Cambridge University Press

Nelson, R. R. and E. S. Phelps (1966) Investment in humans, technological diffusion, and economic growth. *American Economic Review* 56, 69–75.
Pastor-Santorras, R., C. Castellano, P. Van Mieghem, and A. Vespignani (2015) Epidemic processes in complex networks. *Reviews of Modern Physics* 87, 925–979.
Perla, J. and C. Tonetti (2014) Equilibrium imitation and growth. *Journal of Political Economy* 122, 52–76.
Peyton-Young, H. (2006) The diffusion of innovations in social networks. In L. E. Blume and S. N. Durlauf (eds.), *The Economy as an Evolving Complex System III: Current Perspectives and Future Directions*, pp. 267–282. New York: Oxford University Press.
Peyton-Young, H. (2009) Innovation diffusion in heterogeneous populations: Contagion, social influence, and social learning. *American Economic Review* 99, 1899–1924.
Piqueira, J. R. C. (2010) Rumor propagation model: An equilibrium study. *Mathematical Problems in Engineering* Article ID 631357, 7 pages.
Prettner, K. and K. Werner (2014) Human Capital, Basic Research and Applied Research: Three Dimensions of Human Knowledge and their Differential Growth Effects. Center for European Governance and Economic Development Research Discussion Papers 186, Department of Economics, University of Goettingen.
Raurich, X., F. Sanchez-Losada, and M. Vilalta-Bufi (2015). Knowledge Misallocation and Growth. *Macroeconomic Dynamics* 19, 1540–1564.
Rosen, S. (1975) Measuring the obsolescence of knowledge. In F. T. Juster (ed.), *Education, Income and Human Behavior*, pp. 199–232. New York: Carnegie Foundation and Columbia University Press.
Staley, M. (2011) Growth and the diffusion of ideas. *Journal of Mathematical Economics* 47, 470–478.
Stokey, N. L. (2015) Catching up and falling behind. *Journal of Economic Growth* 20, 1–36.
Thompson, K., R. C. Estrada, D. Daugherty, and A. Cintron-Arias (2003) A Deterministic Approach to the Spread of Rumors. Dept. of Biological Statistics & Computational Biology, Cornell University, Technical report BU-1642-M.
Wang, Y. Q., X. Y. Yang, Y. L. Han, and X. A. Wang (2013) Rumor spreading model with trust mechanism in complex social networks. *Communications in Theoretical Physics* 59, 510–516.
Xue, J. and C. K. Yip (2012) Factor substitution and economic growth: A unified approach. *Macroeconomic Dynamics* 16, 625–656.
Zanette, D. H. (2002) Dynamics of rumor propagation on small-world networks. *Physical Review E* 65, 041908.
Zhao, L. J., J. J. Wang, Y. C. Chen, Q. Wang, J. J. Cheng, and H. X. Cui (2012) SIHR rumor spreading model in social networks. *Physica A* 391, 2444–2453.

APPENDIX A: PROOF OF PROPOSITION 2

Recover system (7)–(4)–(8). Under condition $x(t) + y(t) + z(t) = 1$, the system is presentable in the following condensed form:

$$\begin{cases} \dot{y}(t) = \phi(t)\{\lambda[1 - y(t) - z(t)]\, y(t) - \sigma y(t)[y(t) + z(t)]\} \\ \dot{z}(t) = \phi(t)\{\sigma y(t)[y(t) + z(t)] - \delta\gamma z(t)[y(t) + z(t)]\} \end{cases}. \tag{A.1}$$

The two-dimensional system of equations (A.1) can be evaluated, in terms of local dynamics, regardless from the motion of variable $\phi(t)$. The linearization of (A.1) in the

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vicinity of any of the two steady-state points yields the following matricial system:

$$\begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \phi^* \begin{bmatrix} \lambda - (\lambda+\sigma)(2y^* + z^*) & -(\lambda+\sigma)y^* \\ \sigma(2y^* + z^*) - \delta\gamma z^* & \sigma y^* - \delta\gamma\,(y^* + 2z^*) \end{bmatrix} \cdot \begin{bmatrix} y(t) - y^* \\ z(t) - z^* \end{bmatrix}. \tag{A.2}$$

For the DFE, the Jacobian matrix of system (A.2) is $J_{\text{DFE}} = \phi^* \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$. The eigenvalues of J_{DFE} are $\varepsilon_1 = \lambda$ and $\varepsilon_2 = 0$. In this case, there is an unstable dimension and a bifurcation dimension, meaning that no convergence in the direction of the DFE will take place; the DFE only holds if it coincides with the initial state of the system, case in which the system rests in this point forever.

For the DEE, one computes de Jacobian matrix $J_{\text{DEE}} = \frac{\lambda}{\lambda+\sigma}\phi^* \begin{bmatrix} -\delta\gamma & -\delta\gamma \\ \sigma & -\delta\gamma \end{bmatrix}$. The eigenvalues of J_{DEE} are the solution of

$$\varepsilon_1, \varepsilon_2 = \frac{\text{Tr}(J_{\text{DEE}})}{2} \pm \sqrt{\left[\frac{\text{Tr}(J_{\text{DEE}})}{2}\right]^2 - \text{Det}(J_{\text{DEE}})}. \tag{A.3}$$

with $\text{Tr}(J_{\text{DEE}})$ and $\text{Det}(J_{\text{DEE}})$ representing, respectively, the trace and the determinant of the Jacobian matrix. Observing that $\text{Tr}(J_{\text{DEE}}) = -2\frac{\lambda}{\lambda+\sigma}\delta\gamma\phi^*$ and $\text{Det}(J_{\text{DEE}}) = \frac{\lambda}{\lambda+\sigma}\delta\gamma(\sigma + \delta\gamma)\phi^*$, it is straightforward to obtain

$$\varepsilon_1, \varepsilon_2 = \frac{\lambda}{\lambda+\sigma}\left(-\delta\gamma \pm \sqrt{-\sigma\delta\gamma}\right)\phi^*. \tag{A.4}$$

For any connectivity degree larger than zero, the eigenvalues of J_{DEE} are a pair of conjugate complex roots with a negative real part. Therefore, the dynamics of the model in the vicinity of the DEE are characterized by a stable oscillating behavior. ■

APPENDIX B: PROOF OF PROPOSITION 3

The current-value Hamiltonian function associated with the optimal control problem characterized in Section 4 is

$$\begin{aligned} &H\left[s(t), y(t), z(t), A(t), q_y(t), q_z(t), q_A(t)\right] \\ &\quad = \ln\{[1 - s(t)]\,A(t)z(t)\} + q_y(t)\phi(t)X_y(t) + q_z(t)\phi(t)X_z(t) + q_A(t)\phi(t)X_A(t). \end{aligned} \tag{B.1}$$

In expression (B.1), variables $q_y(t)$, $q_z(t)$, and $q_A(t)$ are the co-state variables or shadow-prices associated with variables $y(t)$, $z(t)$, and $A(t)$, respectively. To simplify notation, one considers $X_y(t) \equiv \lambda\,[1 - y(t) - z(t)]\,y(t) - \sigma y(t)\,[y(t) + z(t)]$, $X_z(t) \equiv \sigma y(t)\,[y(t) + z(t)] - \delta\gamma z(t)\,[y(t) + z(t)]$, and $X_A(t) \equiv \gamma z(t)\,[y(t) + z(t)]$. Making use of the relation between time share and connectivity degree, the Hamiltonian function might

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be written under the form

$$H\left[\phi(t), y(t), z(t), A(t), q_y(t), q_z(t), q_A(t)\right] = \frac{1}{\theta}\ln\left[1-\phi(t)\right] + \ln\left[A(t)z(t)\right] + \phi(t)\left[q_y(t)X_y(t) + q_z(t)X_z(t) + q_A(t)X_A(t)\right]. \tag{B.2}$$

Applying Pontryagin's principle, first-order optimality conditions are as follows[13]:

$$\frac{\partial H}{\partial \phi(t)} = 0 \Rightarrow \phi(t) = 1 - \frac{1}{\theta\left[q_y(t)X_y(t) + q_z(t)X_z(t) + q_A(t)X_A(t)\right]}, \tag{B.3}$$

$$\dot{q_y}(t) = \rho q_y(t) - \frac{\partial H}{\partial y(t)}, \tag{B.4}$$

$$\dot{q_z}(t) = \rho q_z(t) - \frac{\partial H}{\partial z(t)}, \tag{B.5}$$

$$\dot{q_A}(t) = \rho q_A(t) - \frac{\partial H}{\partial A(t)}, \tag{B.6}$$

and the transversality condition

$$\lim_{t\to+\infty}\left[y(t)q_y(t) + z(t)q_z(t) + A(t)q_A(t)\right]\exp(-\rho t) = 0. \tag{B.7}$$

The connectivity degree must be nonnegative; thus, optimality condition (B.3) imposes a relevant constraint; this takes the form of a lower bound on the value of parameter θ:

$$\theta \geq \frac{1}{q_y(t)X_y(t) + q_z(t)X_z(t) + q_A(t)X_A(t)}. \tag{B.8}$$

Condition (B.8) is relevant only for $0 < q_y(t)X_y(t) + q_z(t)X_z(t) + q_A(t)X_A(t) < 1$; otherwise, the constraint that holds is $\theta > 1$.

One avoids the presentation of explicit expressions for (B.4) and (B.5) since the dynamic analysis can be pursued without them. On the contrary, it is relevant to compute the derivative in (B.6) and to present the differential equation for the dynamics of $q_A(t)$ under the form

$$\dot{q_A}(t) = \left\{\rho - \gamma\phi(t)z(t)\left[y(t) + z(t)\right]q_A(t)\right\} - \frac{1}{A(t)}. \tag{B.9}$$

Steady-state results can now be derived. Start by defining $v \equiv g_A^* = \gamma\phi^* z^*(y^* + z^*)$, such that a new normalized technology variable, of zero steady-state growth, is $A_n(t) \equiv \exp(-vt)A(t)$. The equation of motion for this new technology variable is

$$\dot{A_n}(t) = \left\{\gamma\phi(t)z(t)\left[y(t) + z(t)\right] - v\right\}A_n(t). \tag{B.10}$$

The inspection of the differential equation (B.9) allows to verify that the co-state variable of technology, $q_A(t)$, grows at a steady-state growth rate of $-v$; thus, defining $q_{An}(t) \equiv \exp(vt)q_A(t)$ we have, again, a variable with a steady-state constant value. Replacing $A(t)$ and $q_A(t)$ by the respective normalized variables in (B.9), an equivalent equation is obtained:

$$\dot{q_{An}}(t) = \left\{(\rho + v) - \gamma\phi(t)z(t)\left[y(t) + z(t)\right]\right\}q_{An}(t) - \frac{1}{A_n(t)}. \tag{B.11}$$

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The evaluation of (B.11) in the steady state allows for the determination of the following product between the technology level and the respective co-state variable, which is valid in the DEE that is being assumed,

$$A^*q_A^* = A_n^*q_{An}^* = \frac{1}{(\rho+v)-\gamma\phi^*z^*(y^*+z^*)} = \frac{1}{\rho}. \tag{B.12}$$

Next, recover optimality condition (B.3). Noting that $X_y^* = X_z^* = 0$, the mentioned condition, when evaluated in the steady state, corresponds to $\phi^* = 1 - \frac{1}{\theta q_A^* X_A^*} = 1 - \frac{1}{\theta q_A^* \gamma z^*(y^*+z^*)A^*}$. Resorting to (B.12), an explicit solution for the equilibrium level of ϕ^* is derived:

$$\phi^* = 1 - \frac{\rho}{\theta\gamma z^*(y^*+z^*)} = 1 - \frac{\rho}{\theta\gamma\left(\frac{\lambda}{\lambda+\sigma}\right)^2\frac{\sigma}{\sigma+\delta\gamma}}. \tag{B.13}$$

Inequality (B.8) is, in the steady state, equivalent to $\theta \geq \frac{\rho}{\gamma\left(\frac{\lambda}{\lambda+\sigma}\right)^2\frac{\sigma}{\sigma+\delta\gamma}}$, a condition that must hold in order to guarantee a nonnegative ϕ in the steady state. Replacing (B.13) into (9), the long-run rate of innovation displayed in the proposition is derived.

Relatively to the steady-state share of time allocated to interaction, this is obtained by observing that $s(t) = f^{-1}[\phi(t)] = 1 - [1-\phi(t)]^{1/\theta}$ and, thus, $s^* = 1 - (1-\phi^*)^{1/\theta}$. The substitution of ϕ^* by the corresponding value in (B.13) into this last expression yields the equilibrium value of s presented in the proposition. ■

APPENDIX C: PROOF OF PROPOSITION 4

An overall evaluation of the local dynamics of the diffusion-interaction growth model implies assuming a six-equation matricial system that involves, as endogenous variables, shares $y(t)$ and $z(t)$, their respective co-state variables, $q_y(t)$ and $q_z(t)$, and the technology normalized variables defined in Appendix B, $A_n(t)$ and $q_{An}(t)$. This set of variables possesses a common property: they all have constant steady-state values. The linearized system takes the form

$$\begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \\ \dot{A}_n(t) \\ \dot{q}_{An}(t) \\ \dot{q}_y(t) \\ \dot{q}_z(t) \end{bmatrix} = \widetilde{J}\begin{bmatrix} y(t)-y^* \\ z(t)-z^* \\ A_n(t)-A_n^* \\ q_{An}(t)-q_{An}^* \\ q_y(t)-q_y^* \\ q_z(t)-q_z^* \end{bmatrix}. \tag{C.1}$$

Matrix $\widetilde{J}$ represents the Jacobian of the system. The elements of the Jacobian will not be fully presented in explicit form because this is unnecessary for the characterization of the relevant dynamics. Nevertheless, five 2×2 submatrices of $\widetilde{J}$ are straightforward to compute and their computation is sufficient to understand the local dynamics of the technology variables. First, note that $\widetilde{J}_1 = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}$ is the Jacobian matrix for the pair of variables $y(t)$ and $z(t)$, which was computed when evaluating the DEE in Section 3.

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Second, regard that $\widetilde{J}_2 = \begin{bmatrix} j_{13} & j_{14} \\ j_{23} & j_{24} \end{bmatrix}$, $\widetilde{J}_3 = \begin{bmatrix} j_{15} & j_{16} \\ j_{25} & j_{26} \end{bmatrix}$, and $\widetilde{J}_6 = \begin{bmatrix} j_{35} & j_{36} \\ j_{45} & j_{46} \end{bmatrix}$ are null matrices, i.e., their elements are all zero. Finally, one can determine the elements of $\widetilde{J}_5 = \begin{bmatrix} j_{33} & j_{34} \\ j_{43} & j_{44} \end{bmatrix}$; submatrix $\widetilde{J}_5$ can be evaluated independently from the rest of the system, in the sense that the respective eigenvalues are the two eigenvalues of $\widetilde{J}$ that deliver the stability outcome for the pair of variables $A_n(t)$ and $q_{An}(t)$.

Taking into consideration that $\phi(t)$ is given by the expression in equation (B.3) of Appendix B, the required calculus leads to

$$\widetilde{J}_5 = \frac{\rho}{\theta} \begin{bmatrix} 1 & \frac{A_n^*}{q_{An}^*} \\ (\theta - 1)\frac{q_{An}^*}{A_n^*} & -1 \end{bmatrix}. \tag{C.2}$$

Trace and determinant of $\widetilde{J}_5$ are, respectively, $\mathrm{Tr}(\widetilde{J}_5) = 0$ and $\mathrm{Det}(\widetilde{J}_5) = -\frac{1}{\theta} < 0$. Values of trace and determinant imply that one of the eigenvalues of the matrix is negative and the other is positive, having symmetric values, $\varepsilon_2 = -\varepsilon_1$. Therefore, a saddle-path stable equilibrium exists in the transitional dynamics relation between $A_n(t)$ and $q_{An}(t)$. The computation of the eigenvalues is straightforward

$$\varepsilon_1, \varepsilon_2 = \frac{\mathrm{Tr}(\widetilde{J}_5)}{2} \pm \sqrt{\left[\frac{\mathrm{Tr}(\widetilde{J}_5)}{2}\right]^2 - \mathrm{Det}(\widetilde{J}_5)} = \pm\sqrt{\frac{1}{\theta}}. \tag{C.3}$$

The stable (S) and unstable (U) trajectories of the saddle-path relation can be analytically derived, via identification of the eigenvectors associated, respectively, with the negative and with the positive eigenvalues. The general expressions of S and U are, respectively,

$$q_{An}(t) - q_{An}^* = \frac{p_2^s}{p_1^s}\left[A_n(t) - A_n^*\right], \tag{C.4}$$

$$q_{An}(t) - q_{An}^* = \frac{p_2^u}{p_1^u}\left[A_n(t) - A_n^*\right], \tag{C.5}$$

with p_1^s, p_2^s, p_1^u, p_2^u the elements of the two eigenvectors. Straightforward computation transforms (C.4) and (C.5) into lines S and U as presented in the proposition. ■

APPENDIX D: STABILITY IN THE HETEROGENEOUS COMMUNICATORS SETTING

As in the agent homogeneity scenario, the system involving the dynamics of the shares of knowledge categories, in the current case the system composed by equations (24) to (27), is expected to exhibit full stability under a local analysis of the diffusion-endemic equilibrium. Furthermore, the technology system, concerning the technology variable and the respective shadow-price, is supposed to be saddle-path stable.

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The linearization of system (24) to (27) yields

$$\begin{bmatrix} \dot{y}_G(t) \\ \dot{y}_B(t) \\ \dot{z}_G(t) \\ \dot{z}_B(t) \end{bmatrix} = \frac{\lambda}{\lambda+\sigma}\phi_G^* G \begin{bmatrix} y_G(t)-y_G^* \\ y_B(t)-y_B^* \\ z_G(t)-z_G^* \\ z_B(t)-z_B^* \end{bmatrix}, \tag{D.1}$$

with $\frac{\lambda}{\lambda+\sigma}\phi_G^* G$ the respective Jacobian matrix. Matrix G is

$$G = \begin{bmatrix} -(1-u_G)\frac{\sigma^2}{\sigma+\delta\gamma} - \frac{(\lambda+\sigma)\delta\gamma}{\sigma+\delta\gamma} & (1-\xi)(1-u_G)\frac{\sigma^2}{\sigma+\delta\gamma} \\ u_G\frac{\sigma^2}{\sigma+\delta\gamma} & -(1-\xi)\left[u_G\frac{\sigma^2}{\sigma+\delta\gamma} + \frac{(\lambda+\sigma)\delta\gamma}{\sigma+\delta\gamma}\right] \\ \sigma & 0 \\ 0 & (1-\xi)\sigma \end{bmatrix}$$

$$\begin{bmatrix} -(\lambda+u_G\sigma)\frac{\delta\gamma}{\sigma+\delta\gamma} & -(1-\xi)(1-u_G)\frac{\sigma\delta\gamma}{\sigma+\delta\gamma} \\ -u_G\frac{\sigma\delta\gamma}{\sigma+\delta\gamma} & -(1-\xi)(1-u_G)\frac{\sigma\delta\gamma}{\sigma+\delta\gamma} \\ -u_G\delta\gamma & -(1-\xi)(1-u_G)\delta\gamma \\ -u_G\delta\gamma & -(1-\xi)(1-u_G)\delta\gamma \end{bmatrix}. \tag{D.2}$$

The stability of the system is approachable by examining the signs of the eigenvalues of matrix G. The large dimension of the matrix does not allow for a direct computation of such eigenvalues; however, a cumbersome and tedious computation process allows to verify that the Routh–Hurwitz stability criterion is satisfied, namely the sign of the trace of G is negative, $\mathrm{Tr}(G) < 0$, the sign of the sum of the principal minors of order 2 of G is positive, $M_2(G) > 0$, the sign of the sum of the principal minors of order 3 of G is negative, $M_3(G) < 0$, and the sign of the determinant of G is positive, $\mathrm{Det}(G) > 0$. Furthermore, condition $\mathrm{Tr}(G)M_2(G)M_3(G) > [M_3(G)]^2 + [\mathrm{Tr}(G)]^2\,\mathrm{Det}(G)$ is satisfied as well.[14]

Given the intractability of the general expressions, the presence of stability is illustrated with an example. Recover the parameter values already assumed in Table 1. Besides those, take $\gamma = 0.06$, $\xi = 0.1$, and $u_G = 0.5$. For these values, matrix G is

$$G = \begin{bmatrix} -0.12844 & 0.04128 & -0.00863 & -0.00372 \\ 0.04587 & -0.1156 & -0.00413 & -0.00372 \\ 0.1 & 0 & -0.0045 & -0.00405 \\ 0 & 0.09 & -0.0045 & -0.00405 \end{bmatrix}. \tag{D.3}$$

Trace, sums of principal minors, and determinant are, in this case, $\mathrm{Tr}(G) = -0.25259$, $M_2(G) = 0.01624$, $M_3(G) = -2.877\,5 \times 10^{-4}$, $\mathrm{Det}(G) = 2.861 \times 10^{-7}$. The condition of the Routh–Hurwitz criterion is satisfied: $\mathrm{Tr}(G)M_2(G)M_3(G) > [M_3(G)]^2 + [\mathrm{Tr}(G)]^2\,\mathrm{Det}(G) \Leftrightarrow 4.673\,1 \times 10^{-6} > 1.010\,5 \times 10^{-7}$. This confirms the stability of the diffusion-endemic equilibrium in this particular case. In the example, the eigenvalues of G can be numerically computed and they are $\varepsilon_1 = -0.00106$, $\varepsilon_2 = -0.02774$, $\varepsilon_3 = -0.05940$, $\varepsilon_4 = -0.16440$. The eigenvalues are all negative, what confirms the stability result.

A second stage of the stability analysis consists in evaluating the technology system that emerges from solving the optimal control problem. This system is composed by equation

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(20) and by the rule of motion for variable $q_A(t)$; this is,

$$\dot{q_A}(t) = \{\rho - \gamma\phi_G(t)\,[(1-\xi)(1-u_G)z_B(t) + u_G z_G(t)]\,[Y(t)+Z(t)]\}\,q_A(t) - \frac{1}{A(t)}. \tag{D.4}$$

Let $v_G = g_A^*$, as given by (40). A normalized version of the previous system, for $A_n(t) \equiv \exp(-v_G t)A(t)$ and $q_{An}(t) \equiv \exp(v_G t)q_A(t)$ is

$$\dot{A_n}(t) = \{\gamma\phi_G(t)\,[(1-\xi)(1-u_G)z_B(t) + u_G z_G(t)]\,[Y(t)+Z(t)] - v_G\}\,A_n(t), \tag{D.5}$$

$$\dot{q_{An}}(t) = \{(\rho + v_G) - \gamma\phi_G(t)\,[(1-\xi)(1-u_G)z_B(t) + u_G z_G(t)]\,[Y(t)+Z(t)]\}\,q_{An}(t) - \frac{1}{A_n(t)}. \tag{D.6}$$

The system of differential equations (D.5)–(D.6) leads to the exact same linearized matricial system as equations (B.10)–(B.11), for which the Jacobian matrix was denoted by $\widetilde{J}_5$. Therefore, the dynamics are exactly the same as in the original version of the model, i.e., saddle-path stability characterizes the dynamic relation between the technology variable and its co-state variable.

APPENDIX E: PROOF OF PROPOSITION 5

The diffusion-interaction model with capital accumulation is solvable through the application of the Pontryagin's principle to the optimal control problem characterized in the beginning of Section 6. The current-value Hamiltonian function is, for this problem,

$$\begin{aligned} H\left[\phi(t), c(t), k(t), y(t), z(t), A(t), q_k(t), q_y(t), q_z(t), q_A(t)\right] \\ = \ln[c(t)] + q_k(t)\left\{[1-\phi(t)]^{1/\theta}\,A(t)z(t)k(t)^{\alpha} - c(t) - \delta_k k(t)\right\} \\ + \phi(t)\left[q_y(t)X_y(t) + q_z(t)X_z(t) + q_A(t)X_A(t)\right]. \end{aligned} \tag{E.1}$$

In function (E.1), $q_k(t)$ is the co-state variable of capital; all the other variables, expressions, and parameters were already defined in previous parts of the analysis.

The list of first-order optimality conditions is the following[15]:

$$\frac{\partial H}{\partial c(t)} = 0 \Rightarrow q_k(t) = \frac{1}{c(t)}, \tag{E.2}$$

$$\frac{\partial H}{\partial \phi(t)} = 0 \Rightarrow \phi(t) = 1 - \left\{\frac{q_k(t)A(t)z(t)k(t)^{\alpha}}{\theta\left[q_y(t)X_y(t) + q_z(t)X_z(t) + q_A(t)X_A(t)\right]}\right\}^{\theta/(\theta-1)}, \tag{E.3}$$

$$\begin{aligned} \dot{q_k}(t) &= \rho q_k(t) - \frac{\partial H}{\partial k(t)} \Rightarrow \\ \dot{q_k}(t) &= \left\{(\rho+\delta_k) - \alpha\,[1-\phi(t)]^{1/\theta}\,A(t)z(t)k(t)^{-(1-\alpha)}\right\}q_k(t), \end{aligned} \tag{E.4}$$

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$$\dot{q_A}(t) = \rho q_A(t) - \frac{\partial H}{\partial A(t)} \Rightarrow$$

$$\dot{q_A}(t) = \{\rho - \gamma\phi(t)z(t)\,[y(t) + z(t)]\}\,q_A(t) - [1 - \phi(t)]^{1/\theta}\,z(t)k(t)^{\alpha}q_k(t). \quad \textbf{(E.5)}$$

Equations similar to (B.4) and (B.5) in Appendix B, duly adapted to the Hamiltonian under evaluation, are also optimality conditions, as well as the transversality condition.

The co-state variable $q_k(t)$ can be suppressed from the analysis, given equality (E.2); this implies that equation (E.4) is equivalent to an equation for the dynamics of the consumption variable,

$$\frac{\dot{c}(t)}{c(t)} = \alpha\,[1 - \phi(t)]^{1/\theta}\,A(t)z(t)k(t)^{-(1-\alpha)} - (\rho + \delta_k). \quad \textbf{(E.7)}$$

An important remark, already stated for the framework without capital, is that condition $\phi(t) \geq 0$ must be satisfied. In this case, given optimality condition (E.3), the following inequality is required to hold:

$$\theta \geq \frac{q_k(t)A(t)z(t)k(t)^{\alpha}}{q_y(t)X_y(t) + q_z(t)X_z(t) + q_A(t)X_A(t)}. \quad \textbf{(E.8)}$$

Once in the possession of the optimality conditions, equilibrium properties may be assessed. In the steady state, recalling that $X_y^* = X_z^* = 0$, and given (E.3),

$$\phi^* = 1 - \left[\frac{(k^*)^{\alpha}}{\theta\gamma(y^* + z^*)q_A^*c^*}\right]^{\theta/(\theta-1)}. \quad \textbf{(E.9)}$$

Note, from (E.9), that for the degree of connectivity to be constant in the steady state, then the ratio $\frac{(k^*)^{\alpha}}{q_A^*c^*}$ must also be constant. Besides an immutable $\phi(t)$, the steady state is also characterized by constant growth rates of consumption, g_c^*, and capital accumulation , g_k^*. Constant steady-state consumption implies, given (E.7), that $A^*\,(k^*)^{-(1-\alpha)}$ is constant; likewise g_k^* remains constant if the ratio $\frac{c^*}{k^*}$ is unchanged. From the above observations, one can characterize the steady state as the long-run outcome such that $c(t)$ and $k(t)$ grow at the same rate, $A(t)$ and $k(t)^{1-\alpha}$ grow at the same rate, and $q_A(t)$ and $k(t)^{-(1-\alpha)}$ also grow at an identical rate. Letting $v \equiv g_A^* = \gamma\phi^*z^*(y^* + z^*)$, as in Appendix B, it is true that $g_{q_A}^* = -v$ and $g_k^* = g_c^* = \frac{1}{1-\alpha}v$.

To obtain further insights on the dynamic analysis, consider the following definitions: $\psi(t) \equiv \frac{c(t)}{k(t)}$, $\zeta(t) \equiv \frac{A(t)}{k(t)^{1-\alpha}}$, and $\omega(t) \equiv A(t)q_A(t)$. These new variables share the property of steady-state zero growth. It is straightforward to compute differential equations for the time evolution of each of these variables; they are

$$\frac{\dot{\psi}(t)}{\psi(t)} = \psi(t) - (1 - \alpha)\,[1 - \phi(t)]^{1/\theta}\,z(t)\zeta(t) - \rho, \quad \textbf{(E.10)}$$

$$\frac{\dot{\zeta}(t)}{\zeta(t)} = \gamma\phi(t)z(t)\,[y(t) + z(t)] - (1 - \alpha)\,[1 - \phi(t)]^{1/\theta}\,z(t)\zeta(t) + (1 - \alpha)\,[\psi(t) + \delta_k], \quad \textbf{(E.11)}$$

$$\frac{\dot{\omega}(t)}{\omega(t)} = \rho - [1 - \phi(t)]^{1/\theta}\,z(t)\frac{\zeta(t)}{\psi(t)\omega(t)}. \quad \textbf{(E.12)}$$

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The last of these equations, (E.12), will allow for the explicit determination of the steady-state values of optimal connectivity and of the growth rates of technology and capital. For $g^*_\omega = 0$, it holds that $\frac{\zeta^*}{\psi^*\omega^*} = \frac{\rho}{(1-\phi^*)^{1/\theta}z^*}$. Next, recover (E.9) and observe that this equation can be expressed under the form $\phi^* = 1 - \left[\frac{\zeta^*}{\theta\gamma(y^*+z^*)\psi^*\omega^*}\right]^{\theta/(\theta-1)}$. Replacing, in this last expression, the previous ratio, one arrives to result

$$\phi^* = 1 - \frac{\rho}{\theta\gamma z^*(y^*+z^*)} = 1 - \frac{\rho}{\theta\gamma\left(\frac{\lambda}{\lambda+\sigma}\right)^2\frac{\sigma}{\sigma+\delta\gamma}}. \tag{E.13}$$

The steady-state connectivity degree (E.13) is exactly the same found in Proposition 3 for the model without capital accumulation. Consequently, the technology level must grow the same, as well, as in the simpler specification of the model, $g^*_A = \left(\frac{\lambda}{\lambda+\sigma}\right)^2\frac{\gamma\sigma}{\sigma+\delta\gamma} - \frac{\rho}{\theta}$, while capital and consumption grow at rate $g^*_k = g^*_c = \frac{1}{1-\alpha}\left[\left(\frac{\lambda}{\lambda+\sigma}\right)^2\frac{\gamma\sigma}{\sigma+\delta\gamma} - \frac{\rho}{\theta}\right]$. At last, observe that $g^*_{\tilde{y}} = g^*_A + \alpha g^*_k = g^*_k$, and thus the economy grows at the same steady-state rate as capital and consumption. ■

APPENDIX F: PROOF OF PROPOSITION 6

The model with capital accumulation involves seven endogenous variables: diffusion shares $y(t)$ and $z(t)$, the ratios defined in Appendix E, $\psi(t)$, $\zeta(t)$, and $\omega(t)$, and the co-state variables $q_y(t)$ and $q_z(t)$. The value of $\phi(t)$ is expressed as a combination of the above variables, under optimality condition (E.3). The dynamic behavior of the pair of variables $y(t)$ and $z(t)$ can be analyzed independently of the rest of the system, as in the model with absence of capital, and the analysis delivers the exact same results as in that model. Moreover, in the vicinity of the steady state, the dynamics of variables $\psi(t)$, $\zeta(t)$, and $\omega(t)$ is independent of the co-state variables $q_y(t)$ and $q_z(t)$, since the derivatives of each of the equations (E.10)–(E.12) with respect to variables $q_y(t)$ and $q_z(t)$, when evaluated in the steady state, are all zero. Therefore, the stability of the capital accumulation model can be assessed by examining the three-dimensional subsystem given by

$$\begin{bmatrix} \dot{\psi}(t) \\ \dot{\zeta}(t) \\ \dot{\omega}(t) \end{bmatrix} = J_k . \begin{bmatrix} \psi(t) - \psi^* \\ \zeta(t) - \zeta^* \\ \omega(t) - \omega^* \end{bmatrix}. \tag{F.1}$$

It is after the computation of the elements of the Jacobian matrix J_k that one will be able to evaluate the stability of the DEE in the current case. This computation is not straightforward since one should take into account that variable $\phi(t)$, present in the expressions of every equation in the system (E.10)–(E.12), is given by (E.3). The derivatives of expression (E.3) with respect to each of the variables $\psi(t)$, $\zeta(t)$, and $\omega(t)$, when evaluated in the steady state, deliver the results

$$\left.\frac{\partial\phi(t)}{\partial\psi(t)}\right|_{(\psi^*,\zeta^*,\omega^*)} = \frac{\theta}{\theta-1}(1-\phi^*)\frac{1}{\psi^*}, \tag{F.2}$$

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$$\left.\frac{\partial\phi(t)}{\partial\zeta(t)}\right|_{(\psi^*,\zeta^*,\omega^*)} = -\frac{\theta}{\theta-1}(1-\phi^*)\frac{1}{\zeta^*}, \tag{F.3}$$

$$\left.\frac{\partial\phi(t)}{\partial\omega(t)}\right|_{(\psi^*,\zeta^*,\omega^*)} = \frac{\theta}{\theta-1}(1-\phi^*)\frac{1}{\omega^*}. \tag{F.4}$$

With the above derivatives and proceeding with the relevant calculus, one arrives to the contents of matrix J_k:

$$J_k = \begin{bmatrix} \psi^* + (1-\alpha)\frac{\rho}{\theta-1}\psi^*\omega^* & -(1-\alpha)\frac{\theta\rho}{\theta-1}\frac{(\psi^*)^2\omega^*}{\zeta^*} & (1-\alpha)\frac{\rho}{\theta-1}(\psi^*)^2 \\ \frac{\rho}{\theta-1}\frac{\zeta^*}{\psi^*} + (1-\alpha)\frac{\rho}{\theta-1}\zeta^*\omega^* + (1-\alpha)\zeta^* & -\frac{\rho}{\theta-1} - (1-\alpha)\theta\frac{\rho}{\theta-1}\psi^*\omega^* & \frac{\rho}{\theta-1}\frac{\zeta^*}{\omega^*} + (1-\alpha)\frac{\rho}{\theta-1}\psi^*\zeta^* \\ \frac{\rho}{\theta-1}\frac{\omega^*}{\psi^*} & \frac{\rho}{\theta-1}\frac{\omega^*}{\zeta^*} & \frac{\rho}{\theta-1} \end{bmatrix}. \tag{F.5}$$

It is not feasible to derive tractable explicit expressions for the eigenvalues of matrix (F.5); however, the straightforward inspection of the values of the respective trace and determinant will allow for the qualitative characterization of transitional dynamics. Trace and determinant are, respectively, $\mathrm{Tr}(J_k) = \rho$ and $\mathrm{Det}(J_k) = -\alpha(\psi^* - \rho)\psi^*$. The trace is positive and the determinant, given the expression of ψ^* in equation (43), is, without ambiguity, a negative value.[16] Because the trace of a square matrix is the sum of its eigenvalues and the determinant corresponds to the product of all eigenvalues, only one possibility in terms of signs of the eigenvalues exists: one must be negative, while the other two will exhibit a positive sign. Thus, in the three-dimensional space defined by the variables under evaluation, a one-dimensional stable trajectory emerges. The system is saddle-path stable. ■

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