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# Idealizations of maximal Buchsbaum modules over a Buchsbaum ring

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## *Introduction*

Throughout this paper $A$ denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $M$ denotes a finitely generated $A$-module. Moreover $\mathrm{H}^i_{\mathfrak{m}}(*)$ stands for the $i$th local cohomology functor with respect to $\mathfrak{m}$ (cf. [**10**]). We refer to [**15**] for unexplained terminology.

To describe our study we firstly recall the principle of idealizations, which was introduced by Nagata in [**15**]. If we make the Cartesian product $A \times M$ into a commutative ring by giving it componentwise addition and the multiplication defined by $(a, x).(a', x') = (aa', ax' + a'x)$, we call it the idealization of $M$ and denote it by $A \ltimes M$. The idealization $A \ltimes M$ is again a Noetherian local ring with maximal ideal $\mathfrak{m} \times M$ and its Krull dimension equals $\dim A$. We know at least two interesting papers concerning idealizations: Reiten [**16**] showed by using the principle of idealizations that any Noetherian local ring possessing a Gorenstein module of rank 1 is a homomorphic image of a Gorenstein local ring, and Aoyama [**1**] studied the condition for idealizations to be quasi-Gorenstein, and used this for the first step (the reduction of the problem to the $(S_2)$ case) of the proof that any localization of the canonical module $K_A$ of $A$ at $\mathfrak{p} \in \mathrm{Supp}_A K_A$ is the canonical module of the local ring $A_{\mathfrak{p}}$.

The purpose of this paper is to clarify the condition for idealizations to be Buchsbaum, especially for idealizations of canonical modules. We recall the definition of a Buchsbaum ring, which was introduced by Stückrad and Vogel [**20**]. A Noetherian local ring $A$ with $\dim A > 0$ is said to be a Buchsbaum ring if the difference $l_A(A/\mathfrak{q}) - e_{\mathfrak{q}}(A)$ is an invariant of $A$ not depending on the choice of parameter ideal $\mathfrak{q}$ of $A$ (cf. Definition 1·1). Then our problem is stated as follows:

*Problem.* What is the condition for idealizations to be Buchsbaum?

In March 1982 this was posed to the author by Shiro Goto at Nihon University. He had already given us part of the answer to this problem in (2·8) of [**2**] and (4·6) of [**3**]; cf. also (2·5) in [**5**]. It seems that there are two appealing points in this study: the first one is that by applying the principle of idealizations we can get many nice examples of Buchsbaum rings; the second one is that as described below our method yields a new approach to estimating the minimal number of generators of the canonical module of a Buchsbaum ring.

In §1 we give the definition of a Buchsbaum module and recall several facts from the theory of Buchsbaum modules. After discussing them we prove a formula in Proposition 1·8 which we need later to prove Theorem 3·3. We discuss in §2 the basic behaviour of the idealizations which are Buchsbaum (see (2·3), (2·6) and (2·7)) and



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give examples satisfying such conditions. §3 is devoted to studying the conditions for idealizations of canonical modules to be Buchsbaum, and Theorem 3·3 is the main result of this paper. We also show that many Buchsbaum rings satisfy these conditions (see (3·11) and (3·12)). We finally deal with the quasi-Buchsbaum property (cf. (4·1) for the definition) of idealizations. After giving a generalization of the principle of idealizations, we describe the condition for idealizations to be quasi-Buchsbaum (see (4·3) and (4·6)).

## 1. *Preliminary results*

In this preliminary section, we shall give the definition of a Buchsbaum module and describe certain fundamental facts from the theory of Buchsbaum modules. Then we prove a technical formula in Proposition 1·8, which plays an important role in the proof of our main result, Theorem 3·3.

We begin with the following

*Definition* 1·1 ([**20**, **21**]). A finitely generated $A$-module $M$ with $\dim_A M > 0$ is said to be a *Buchsbaum* $A$-module if the difference

$$l_A(M/\mathfrak{q}M) - e_{\mathfrak{q}}(M)$$

is an invariant of $M$ not depending on the choice of parameter ideal $\mathfrak{q}$ of $M$, where $l_A(*)$ and $e_{\mathfrak{q}}(*)$ denote respectively the length and the multiplicity with respect to $\mathfrak{q}$ of an $A$-module. We denote this invariant of $M$ by $\mathrm{I}_A(M)$ (or simply $\mathrm{I}(M)$). The local ring $A$ is called a Buchsbaum ring if it is a Buchsbaum module over itself.

We shall here collect some fundamental facts from the theory of Buchsbaum modules which we shall need later: for more details see [**23**], and also [**7**, **8**].

(1) Let $N$ be any $A$-submodule of $M$. For a primary decomposition of $N$, say

$$N = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_A M/N} N(\mathfrak{p}),$$

we define the unmixed component, denoted $\mathrm{U}_A(N)$, of $N$ as follows:

$$\mathrm{U}_A(N) := \bigcap_{\dim_A A/\mathfrak{p} = \dim_A M/N} N(\mathfrak{p}).$$

With this notation we can describe several conditions equivalent to that in Definition 1·1. These are as follows:

(i) $M$ is a Buchsbaum $A$-module;

(ii) every system of parameters for $M$, say $a_1, \ldots, a_s$ (where $s := \dim_A M$), forms a *weak* $M$-sequence, that is, the equality

$$(a_1, \ldots, a_{i-1})M : a_i = (a_1, \ldots, a_{i-1})M : \mathfrak{m}$$

holds for all $i \in \{1, \ldots, s\}$;

(iii) every system of parameters $a_1, \ldots, a_s$ for $M$ forms a $d$-sequence, that is, the equality

$$(a_1, \ldots, a_{i-1})M : a_i a_j = (a_1, \ldots, a_{i-1})M : a_j$$

holds for all $i, j \in \{1, \ldots, s\}$ with $i \leqslant j$;

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(iii)′ every system of parameters $a_1, \ldots, a_s$ for $M$ satisfies

$$[(a_1, \ldots, a_{i-1})M : a_i] \cap \mathfrak{q}M = (a_1, \ldots, a_{i-1})M$$

for all $i \in \{1, \ldots, s\}$, where $\mathfrak{q} := (a_1, \ldots, a_s)$;

(iv) for any system of parameters $a_1, \ldots, a_s$ for $M$ the equality

$$\mathrm{U}_A((a_1, \ldots, a_{i-1})M) = (a_1, \ldots, a_{i-1})M : \mathfrak{m}$$

holds for all $i \in \{1, \ldots, s\}$.

For the equivalence of these conditions, see [**23**], §1 of chapter I, and also [**12**, **14**].

We can easily see that the notion of a Buchsbaum module is a generalization of the notion of a Cohen–Macaulay module: more precisely $M$ is a Cohen–Macaulay module if and only if it is a Buchsbaum module with $\mathrm{I}_A(M) = 0$.

(2) Assume that $M$ is a Buchsbaum $A$-module of dimension $s$. Then every local cohomology module $\mathrm{H}^i_{\mathfrak{m}}(M)$ $(i \neq s)$ of $M$ is annihilated by the maximal ideal $\mathfrak{m}$ of $A$, and the invariant $\mathrm{I}_A(M)$ is given in the following form:

$$\mathrm{I}_A(M) = \sum_{i=0}^{s-1} \binom{s-1}{i} l_A(\mathrm{H}^i_{\mathfrak{m}}(M)).$$

Therefore $\mathrm{H}^0_{\mathfrak{m}}(M) = 0 : \mathfrak{m} = 0 : a$ holds for any parameter $a \in \mathfrak{m}$ for $M$. So as $\mathrm{H}^0_{\mathfrak{m}}(M) \cap \mathfrak{q}M = (0)$ for any parameter ideal $\mathfrak{q}$ of $M$ (cf. (iii)′ above), $M/\mathrm{H}^0_{\mathfrak{m}}(M)$ is also a Buchsbaum $A$-module with $\mathrm{H}^i_{\mathfrak{m}}(M/\mathrm{H}^0_{\mathfrak{m}}(M)) \cong \mathrm{H}^i_{\mathfrak{m}}(M)$ for all $i > 0$, and thus we have

$$\mathrm{I}_A(M/\mathrm{H}^0_{\mathfrak{m}}(M)) = \mathrm{I}_A(M) - l_A(\mathrm{H}^0_{\mathfrak{m}}(M)).$$

Let $s \geqslant 2$ and let $a \in \mathfrak{m}$ be again any parameter for $M$. Then $M/aM$ is a Buchsbaum $A$-module too, and

$$\mathrm{H}^i_{\mathfrak{m}}(M/aM) \cong \mathrm{H}^i_{\mathfrak{m}}(M) \oplus \mathrm{H}^{i+1}_{\mathfrak{m}}(M) \quad \text{for} \quad 0 \leqslant i < s-1,$$

so we have $\mathrm{I}_A(M/aM) = \mathrm{I}_A(M)$; cf. [**23**], §2 of chapter I.

(3) If the given local ring $A$ is a homomorphic image of another local ring $R$, it is easy to see that $M$ is a Buchsbaum module over $A$ if and only if it is so over $R$. Let us denote by $\hat{A}$ (resp. $\hat{M}$) the completion of $A$ (resp. $M$) with respect to the $\mathfrak{m}$-adic topology. Then $M$ is a Buchsbaum module over $A$ if and only if $\hat{M}$ is a Buchsbaum module over $\hat{A}$. Combining these facts and Cohen's structure theorem for complete local rings we may sometimes assume that the given local ring $A$ is complete Gorenstein and $M$ is faithful (or $\dim_A M = \dim A$), if no loss of generality can be expected. (Cf. [**23**], (1·6) and (1·13) of chapter I.)

(4) As is well known the local cohomology functors $\mathrm{H}^i_{\mathfrak{m}}(*)$ with respect to a maximal ideal $\mathfrak{m}$ are naturally equivalent to the functors $\varinjlim_n \mathrm{Ext}^i_A(A/\mathfrak{m}^n, *)$ (cf. [**10**]). Thus we can define the canonical map

$$\phi^i_M : \mathrm{Ext}^i_A(A/\mathfrak{m}, M) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(M) \cong \varinjlim_n \mathrm{Ext}^i_A(A/\mathfrak{m}^n, M).$$

Concerning this map $\phi^i_M$, we have the following powerful criterion which determines whether a given module is Buchsbaum.

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(Surjectivity Criterion.) If the canonical map $\phi_M^i$ is surjective for all $i \neq \dim_A M$, then the $A$-module $M$ is Buchsbaum. Moreover if $A$ is a regular local ring the converse is also true. (Cf. [23], (2·10) and (2·16) of chapter I.)

Assume that $\mathfrak{m}.\mathrm{H}_{\mathfrak{m}}^t(M) = (0)$, where $t := \mathrm{depth}_A M$. Then it is easy to check that the map $\phi_M^t$ is surjective. (In fact one can show this by induction on $t$.) Combining this fact and the above surjectivity criterion, we conclude the following: if $\mathfrak{m}.\mathrm{H}_{\mathfrak{m}}^t(M) = (0)$ and $\mathrm{H}_{\mathfrak{m}}^i(M) = (0)$ for all $i \neq t$, $\dim_A M$, then $M$ is a Buchsbaum $A$-module. (Cf. [23], (2·12) of chapter I.)

(5) Notice that in general there is a finite system of generators of the maximal ideal $\mathfrak{m}$ such that any subsystem consisting of $d$ elements (where $d := \dim A$) forms a minimal reduction of $\mathfrak{m}$. In fact, we can construct such a system in the following way. Let $\mathrm{G} := \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ be the associated graded ring of $A$ and $\mathfrak{M} := \bigoplus_{n>0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ the unique graded maximal ideal of G. According to the argument of (1·9) of chapter I in [23], there exists a finite system $\{a_\lambda\}_{\lambda \in \Lambda}$ of elements in $A$ such that the initial forms of $a_\lambda$'s form a $\mathrm{G}_{\mathfrak{M}}$-basis of the maximal ideal $\mathfrak{M}.\mathrm{G}_{\mathfrak{M}}$ (in the sense of [23], (1·7) of chapter I). This means the system $\{a_\lambda\}_{\lambda \in \Lambda}$ satisfies the following two conditions:

($a$) it is a minimal system of generators of $\mathfrak{m}$;

($b$) every subsystem of $\{a_\lambda\}_{\lambda \in \Lambda}$ consisting of $d$ elements forms a minimal reduction of $\mathfrak{m}$.

Therefore we know that such a system $\{a_\lambda\}_{\lambda \in \Lambda}$ is just the required one. We call here a system which satisfies the conditions ($a$) and ($b$), a *basis of* $\mathfrak{m}$ *with reductions*. Using this notion we can describe further conditions equivalent to that in Definition 1·1. Namely, if $\dim_A M = \dim A = d$, then the following statements are equivalent to the conditions (i)–(iv) in Remark 1:

(v) every system $a_1, \ldots, a_d$ of parameters for $M$ such that $(a_1, \ldots, a_d)$ is a minimal reduction of $\mathfrak{m}$ forms a weak $M$-sequence in any order;

(vi) any subsystem of some basis of $\mathfrak{m}$ with reductions consisting of $d$ elements forms a weak $M$-sequence in any order;

(vii) every local cohomology module $\mathrm{H}_{\mathfrak{m}}^i(M)$ $(i \neq d)$ of $M$ is of finite length and the equality

$$l_A(M/\mathfrak{q}M) - e_{\mathfrak{q}}(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} l_A(\mathrm{H}_{\mathfrak{m}}^i(M))$$

holds for any minimal reduction $\mathfrak{q}$ of $\mathfrak{m}$;

(viii) every local cohomology module $\mathrm{H}_{\mathfrak{m}}^i(M)$ $(i \neq d)$ of $M$ is of finite length and there is a basis $\{a_\lambda\}_{\lambda \in \Lambda}$ of $\mathfrak{m}$ with reductions such that the equality

$$l_A(M/\mathfrak{q}_I M) - e_{\mathfrak{q}_I}(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} . l_A(\mathrm{H}_{\mathfrak{m}}^i(M))$$

holds for all $I \subset \Lambda$ with $\# I = d$, where $\mathfrak{q}_I := (a_\lambda \mid \lambda \in I)$. (See [23], (2·15) of chapter I and [26], (2·1) and (3·2).)

In order to describe very technical manipulations with (sub-)systems of parameters for Buchsbaum modules we introduce some useful notation. For a system of elements in $A$, say $a_1, \ldots, a_r$, we define the $A$-submodule $\Sigma(a_1, \ldots, a_r; M)$ of $M$ as follows:

$$\Sigma(a_1, \ldots, a_r; M) := \sum_{i=1}^{r} [(a_1, \ldots, \hat{a_i}, \ldots, a_r)M : a_i] + (a_1, \ldots, a_r)M.$$

When we set $\mathfrak{q} = (a_1, \ldots, a_r)$ or $\underline{a} = a_1, \ldots, a_r$ we also use the notation $\Sigma(\mathfrak{q}; M)$ or $\Sigma(\underline{a}; M)$. In the case $M = A$ we sometimes omit the letter $M$, writing $\Sigma(a_1, \ldots, a_r)$

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or $\Sigma(\mathfrak{q})$, etc., if no confusion can be expected. With this notation, the next results describe some fundamental facts about $\Sigma(a_1, \ldots, a_r; M)$ but not given explicit statements in other articles. First we have the following

LEMMA 1·2. *Let $a_1, a_2, \ldots, a_r$ be a subsystem of parameters for a Buchsbaum $A$-module $M$. Then for any positive integers $m_1, \ldots, m_r, n_1, \ldots, n_r$ the equality*

$$(a_1^{m_1+n_1}, \ldots, a_r^{m_r+n_r})M : a_1^{m_1} \ldots a_r^{m_r} = \Sigma(a_1^{n_1}, \ldots, a_r^{n_r}; M)$$

*holds.*

(See (4·7) in [**4**], lemma 23 in the appendix of [**23**] and (3·5) of [**26**]; see also (2·3) of [**8**] for the more general statement.)

*Proof.* We shall here prove this lemma by induction on $r$. The case $r = 1$ is obvious. Assume that $r \geqslant 2$ and that the assertion is true for $r-1$. Let $x$ be an element of $M$ such that

$$a_1^{m_1} \ldots a_r^{m_r}.x = y + a_r^{m_r+n_r}.z$$

with

$$y \in (a_1^{m_1+n_1}, \ldots, a_{r-1}^{m_{r-1}+n_{r-1}})M \quad \text{and} \quad z \in M.$$

Then we have

$$a_1^{m_1} \ldots a_{r-1}^{m_{r-1}}.x - a_r^{n_r}.z \in (a_i^{m_i+n_i} \mid 1 \leqslant i \leqslant r-1)M : \mathfrak{m},$$

by the weak sequence property (cf. Remark 1 above). Since $a_1, \ldots, a_{r-1}$ form a subsystem of parameters for $M$ we have

$$(a_i^{m_i+n_i} \mid 1 \leqslant i \leqslant r-1)M : \mathfrak{m} = \sum_J \prod_{j \in J} a_j^{m_j+n_j-1}.[(a_j \mid j \in J)M : \mathfrak{m}], \tag{1}$$

where $J$ runs through all subsets of $\{1, \ldots, r-1\}$. (See Goto's Lemma in chapter II of [**23**] (p. 139) – this can be easily shown by induction on the length of a sequence $a_1, \ldots, a_{r-1}$.) Hence

$$a_1^{m_1} \ldots a_{r-1}^{m_{r-1}}.x = a_r^{n_r}.z + \sum_J \prod_{j \in J} a_j^{m_j+n_j-1}.\beta_J \tag{2}$$

with $\beta_J \in (a_j \mid j \in J)M : \mathfrak{m}$. Put $\Lambda := \{1, \ldots, r-1\}$. Since $a_1, \ldots, a_{r-1}$ form a weak $M$-sequence in any order we see that

$$(a_j \mid j \in J)M : \mathfrak{m} = (a_j \mid j \in J)M : a_k$$

for each $J \neq \Lambda$ and $k \notin J$, and this means that the element $a_1 \ldots a_{r-1}.\beta_J$ belongs to $(a_j^2 \mid j \in J)M$ for each $J \neq \Lambda$. Multiplying both sides of (2) by $a_1 \ldots a_{r-1}$ implies

$$\prod_{i=1}^{r-1} a_i^{m_i+1}.(x - \prod_{j \in \Lambda} a_j^{n_j-1}.\beta_\Lambda) \in (a_1^{m_1+n_1+1}, \ldots, a_{r-1}^{m_{r-1}+n_{r-1}+1}, a_r^{n_r})M.$$

Notice that $a_1, \ldots, a_{r-1}$ is a subsystem of parameters for the Buchsbaum module $M/a_r^{n_r}M$ too, so we get by the hypothesis of induction on $r$ that

$$x - \prod_{j \in \Lambda} a_j^{n_j-1}.\beta_\Lambda \in \sum_{i=1}^{r-1} [(a_1^{n_1}, \ldots, \hat{a}_i^{n_i}, \ldots, a_{r-1}^{n_{r-1}}, a_r^{n_r})M : a_i^{n_i}] + (a_1^{n_1}, \ldots, a_r^{n_r})M.$$

Thus $x \in \Sigma(a_1^{n_1}, \ldots, a_r^{n_r}; M)$. The converse is clear and this completes the proof of Lemma 1·2.

Now let $l_1, \ldots, l_r$ be non-negative integers (not necessary positive) and let $u$ be an element of $M$ such that

$$a_1^{l_1} \ldots a_r^{l_r}.u \in \Sigma(a_1^{l_1+n_1}, \ldots, a_r^{l_r+n_r}; M).$$

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Then by Lemma 1·2 we have

$$\prod_{i=1}^{r} a_i^{l_i+1} \cdot u \in (a_1^{l_1+n_1+1}, \ldots, a_r^{l_r+n_r+1})M.$$

This means by Lemma 1·2 again that $u \in \Sigma(a_1^{n_1}, \ldots, a_r^{n_r}; M)$. The converse is obvious and therefore we have for any integers $l_1, \ldots, l_r \geqslant 0$ and $n_1, \ldots, n_r > 0$ the following equality:

$$\Sigma(a_1^{l_1+n_1}, \ldots, a_r^{l_r+n_r}; M) : a_1^{l_1} \ldots a_r^{l_r} = \Sigma(a_1^{n_1}, \ldots, a_r^{n_r}; M). \tag{3}$$

Let $a_1, a_2, \ldots, a_s$ be a system of parameters for $M$, where $s := \dim_A M$. As is well known, the local cohomology module $\mathrm{H}^i_{\mathfrak{m}}(M)$ is isomorphic to the direct limit $\varinjlim_n \mathrm{H}^i(\underline{a}^n; M)$ defined by the Koszul (co-)homology modules $\mathrm{H}^i(\underline{a}^n; M)$ of $M$ with respect to the systems $\underline{a}^n := a_1^n, \ldots, a_s^n$ $(n > 0)$, for each $i$; cf. [**10**, **11**]. In particular, for $i = s$ we have the following direct system:

$$M/(\underline{a})M \longrightarrow M/(\underline{a}^2)M \longrightarrow \ldots \longrightarrow \varinjlim_n M/(\underline{a}^n)M \cong \mathrm{H}^s_{\mathfrak{m}}(M),$$

where the map $M/(\underline{a}^n)M \longrightarrow M/(\underline{a}^{n+l})M$ for $n, l > 0$ is multiplication by the element $a_1^l \ldots a_s^l$.

Assume that $M$ is a Buchsbaum $A$-module. By Lemma 1·2 we have

$$\mathrm{Ker}\,(M/(\underline{a}^n)M \xrightarrow{a_1^l \ldots a_s^l} M/(\underline{a}^{n+l})M) = \Sigma(\underline{a}^n; M)/(\underline{a}^n)M$$

for any $n$ and $l$. This induces the new direct system as follows:

$$M/\Sigma(\underline{a}; M) \longrightarrow M/\Sigma(\underline{a}^2; M) \longrightarrow \ldots \longrightarrow \mathrm{H}^s_{\mathfrak{m}}(M),$$

where all the induced maps $M/\Sigma(\underline{a}^n; M) \longrightarrow M/\Sigma(\underline{a}^{n+l}; M)$ are injective (cf. (3)) and its direct limit coincides with $\mathrm{H}^s_{\mathfrak{m}}(M)$ too. Namely we have

LEMMA 1·3. *Let $a_1, a_2, \ldots, a_s$ be a system of parameters for a Buchsbaum $A$-module $M$ and put $\underline{a}^n := a_1^n, \ldots, a_s^n$ for integers $n > 0$. Then the $s$th local cohomology module $\mathrm{H}^s_{\mathfrak{m}}(M)$ of $M$ is given by the direct limit of the direct system consisting of $A$-modules $M/\Sigma(\underline{a}^n; M)$ and injective maps $a_1^l \ldots a_s^l : M/\Sigma(\underline{a}^n; M) \longrightarrow M/\Sigma(\underline{a}^{n+l}; M)$ for $n, l > 0$, namely*

$$\mathrm{H}^s_{\mathfrak{m}}(M) \cong \varinjlim_n M/\Sigma(\underline{a}^n; M).$$

Let $a_1, \ldots, a_r, a_{r+1}$ be a weak $M$-sequence in any order. Then we have

$$\begin{aligned}(a_1, \ldots, \hat{a_i}, \ldots, a_r)M : a_i &= (a_1, \ldots, \hat{a_i}, \ldots, a_r)M : \mathfrak{m} \\ &\subset (a_1, \ldots, a_r)M : \mathfrak{m} \\ &= (a_1, \ldots, a_r)M : a_{r+1}\end{aligned}$$

for all $i = 1, \ldots, r$. So there are inclusions

$$(a_1, \ldots, a_r)M \subset \Sigma(a_1, \ldots, a_r; M) \subset (a_1, \ldots, a_r)M : a_{r+1}. \tag{4}$$

Now assume that $M$ is a Buchsbaum $A$-module and $a_1, \ldots, a_r, a_{r+1}$ is a subsystem of parameters for $M$. Applying inductively Remark 2 after Definition 1·1 we have

$$l_A\left(\frac{(a_1, \ldots, a_r)M : a_{r+1}}{(a_1, \ldots, a_r)M}\right) = l_A(\mathrm{H}^0_{\mathfrak{m}}(M/(a_1, \ldots, a_r)M)) = \sum_{i=0}^{r} \binom{r}{i} l_A(\mathrm{H}^i_{\mathfrak{m}}(M)). \tag{5}$$

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On the other hand, using the proof of (3·6) of [**4**] we can establish the following equality:

$$l_A\left(\frac{\Sigma(a_1, \ldots, a_r; M)}{(a_1, \ldots, a_r)M}\right) = \sum_{i=0}^{r-1} \binom{r}{i} l_A(\mathrm{H}^i_{\mathfrak{m}}(M)). \tag{6}$$

Combining (5) and (6) (see also (4)) we obtain

LEMMA 1·4. *Let $a_1, \ldots, a_r, a_{r+1}$ $(r > 0)$ be a subsystem of parameters for a Buchsbaum $A$-module $M$. Then the following equality holds*:

$$l_A\left(\frac{(a_1, \ldots, a_r)M : a_{r+1}}{\Sigma(a_1, \ldots, a_r; M)}\right) = l_A(\mathrm{H}^r_{\mathfrak{m}}(M)).$$

COROLLARY 1·5. *Under the hypotheses of Lemma* 1·4, *if* $\mathrm{H}^r_{\mathfrak{m}}(M) = (0)$ *then one has*

$$(a_1, \ldots, a_r)M : a_{r+1} = \Sigma(a_1, \ldots, a_r; M).$$

For convenience we introduce one more piece of notation. Let $a_1, \ldots, a_r, a_{r+1}$ $(r > 0)$ be a system of elements in $A$. We define the $A$-submodule $\Sigma(a_1, \ldots, a_r | a_{r+1}; M)$ of $M$ as follows:

$$\Sigma(a_1, \ldots, a_r | a_{r+1}; M) := \sum_{i=1}^{r} [(a_1, \ldots, \hat{a_i}, \ldots, a_r, a_{r+1})M : a_i] + (a_1, \ldots, a_r, a_{r+1})M.$$

Put $M' := M/a_{r+1}M$. We easily see that $\Sigma(a_1, \ldots, a_r | a_{r+1}; M)$ coincides with the preimage of $\Sigma(a_1, \ldots, a_r; M')$ via the projection $M \longrightarrow M'$, namely

$$\Sigma(a_1, \ldots, a_r; M') = \Sigma(a_1, \ldots, a_r | a_{r+1}; M)/a_{r+1}M. \tag{7}$$

By our definition, moreover, it immediately follows that

$$\Sigma(a_1, \ldots, a_{r+1}; M) = \Sigma(a_1, \ldots, a_r | a_{r+1}; M) + [(a_1, \ldots, a_r)M : a_{r+1}]. \tag{8}$$

Concerning these modules we have the following

LEMMA 1·6 (cf. (3·5) in [**4**]). *Let $M$ be a Buchsbaum $A$-module of dimension $s \geqslant 2$ and let $a_1, \ldots, a_r, a_{r+1}$ $(1 \leqslant r < s)$ be a subsystem of parameters for $M$. Then the following equality holds*:

$$\Sigma(a_1, \ldots, a_r | a_{r+1}; M) \cap [(a_1, \ldots, a_r)M : a_{r+1}] = \Sigma(a_1, \ldots, a_r; M).$$

*Proof.* This is the same as the proof of (3·5) in [**4**]; see also the claim in the proof of (3·6) in [**4**].

As a consequence of Lemma 1·4 and Lemma 1·6 we have

COROLLARY 1·7. *Under the hypotheses of Lemma* 1·6 *one has*

$$l_A\left(\frac{\Sigma(a_1, \ldots, a_{r+1}; M)}{\Sigma(a_1, \ldots, a_r | a_{r+1}; M)}\right) = l_A(\mathrm{H}^r_{\mathfrak{m}}(M)).$$

*Therefore if* $\mathrm{H}^r_{\mathfrak{m}}(M) = (0)$ *one also has*

$$\Sigma(a_1, \ldots, a_{r+1}; M) = \Sigma(a_1, \ldots, a_r | a_{r+1}; M).$$

Now we are ready to give a technical formula, which is an evaluation of the lengths of some interesting modules and is used later to prove the main result of this article (Theorem 3·3).

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PROPOSITION 1·8. *Let $M$ be a Buchsbaum $A$-module of dimension $s$ and let $\underline{a} := a_1, \ldots, a_s$ be a system of parameters for $M$. Then the equality*

$$l_A\left(\frac{\sum_{i=1}^{s} a_1 \ldots \hat{a}_i \ldots a_s \cdot \Sigma(\underline{a}; M) + \Sigma(\underline{a}^2; M)}{\Sigma(\underline{a}^2; M)}\right) = l_A(M/\Sigma(\underline{a}; M)) + \sum_{i=1}^{s-1} \binom{s}{i-1} l_A(\mathrm{H}^i_{\mathfrak{m}}(M))$$

*holds, where $\underline{a}^2 := a_1^2, \ldots, a_s^2$.*

*Proof.* Let us write

$$\triangle(\underline{a}; M) := \sum_{i=1}^{s} a_1 \ldots \hat{a}_i \ldots a_s \cdot \Sigma(\underline{a}; M) + \Sigma(\underline{a}^2; M);$$

$$\square(\underline{a}; M) := \triangle(\underline{a}; M)/\Sigma(\underline{a}^2; M).$$

We shall prove the result by induction on $s$. Let $s = 1$ and put $a := a_1$. Then since

$$\triangle(a; M) = \Sigma(a; M) + \Sigma(a^2; M) = [0:a] + aM,$$

we have

$$\triangle(a; M)/\Sigma(a^2; M) = \{[0:a] + aM\}/\{[0:a] + a^2M\} \cong M/\Sigma(a; M);$$

cf. Lemma 1·2. Thus the case $s = 1$ is obvious.

Now let $s \geqslant 2$ and assume that the assertion holds for $s-1$. Put $M' := M/a_sM$. As $\underline{b} := a_1, \ldots, a_{s-1}$ forms a system of parameters for $M'$, we have the next formula by the hypothesis of induction on $s$:

$$l_A(\square(\underline{b}; M')) = l_A(M'/\Sigma(\underline{b}; M')) + \sum_{j=1}^{s-2} \binom{s-1}{j-1} l_A(\mathrm{H}^j_{\mathfrak{m}}(M')). \tag{9}$$

Let us denote by $\triangle'$ the preimage of $\triangle(\underline{b}; M')$ via the projection $M \longrightarrow M'$, i.e.

$$\triangle' := \sum_{i=1}^{s-1} a_1 \ldots \hat{a}_i \ldots a_{s-1} \cdot \Sigma(\underline{b} \mid a_s; M) + \Sigma(\underline{b}^2 \mid a_s; M),$$

where $\underline{b}^2 := a_1^2, \ldots, a_{s-1}^2$. By (7) we see that

$$\square(\underline{b}; M') \cong \triangle'/\Sigma(\underline{b}^2 \mid a_s; M). \tag{10}$$

Let us define the $A$-submodules $F_j$ $(0 \leqslant j \leqslant s-1)$ of $M$ inductively as follows:

$$F_0 := \sum_{i=1}^{s-1} a_1 \ldots \hat{a}_i \ldots a_s \cdot \{\sum_{l \neq s,i} [(a_1, \ldots, \hat{a}_l, \ldots, a_s)M : a_l]\} + a_1 \ldots a_s \cdot M + \Sigma(\underline{a}^2; M),$$

$$F_j := F_{j-1} + a_1 \ldots a_{s-1} \cdot [(a_1, \ldots, \hat{a}_j, \ldots, a_{s-1}, a_s)M : a_j].$$

Then clearly there are the inclusions

$$\Sigma(\underline{a}^2; M) \subset F_0 \subset F_1 \subset \ldots \subset F_{s-1} \subset \triangle(\underline{a}; M),$$

and moreover the following statements are true.

*Claim.* (1) $F_{s-1} = \triangle(\underline{a}; M)$.
(2) $F_0/\Sigma(\underline{a}^2; M) \cong \triangle'/\Sigma(\underline{b}^2, a_s; M)$.
(3) *For each $j = 1, \ldots, s-1$ we have*

$$F_j/F_{j-1} \cong [(a_1, \ldots, \hat{a}_j, \ldots, a_s)M : a_j]/\Sigma(a_1, \ldots, \hat{a}_j, \ldots, a_s; M).$$

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*Proof of Claim.* (1) First of all, by Lemma 1·2 we easily see

$$\Sigma(\underline{a}^2; M) \supset a_1 \dots \hat{a_i} \dots a_s . [(a_1, \dots, \hat{a_i}, \dots, a_s) M : a_i]$$

for any $i$. To obtain the assertion (1) we show that $F_{s-1}$ contains

$$a_1 \dots \hat{a_i} \dots a_s . [(a_1, \dots, \hat{a_j}, \dots, a_s) M : a_j]$$

for all $i, j$. By our definition of $F_{s-1}$ and by the above observation it is enough to check just the case $1 \leqslant i \leqslant s-1$ and $j = s$. We may further assume that $i = 1$. Let $x$ be an element of $M$ such that $a_s x = a_1 x_1 + \dots + a_{s-1} x_{s-1}$ with $x_k \in M$. Then we have

$$a_2 \dots a_s . x = a_2 \dots a_{s-1}(a_1 x_1 + \dots + a_{s-1} x_{s-1}) \equiv a_1 \dots a_{s-1} . x_1 \mod (\underline{a}^2) M.$$

As the element $x_1$ belongs to $(a_2, \dots, a_s) M : a_1$ we know that $a_1 \dots a_{s-1} . x_1$, and hence $a_2 \dots a_s . x$ itself, is contained in $F_{s-1}$.

(2) Notice that $a_s . \Sigma(\underline{b}^2 \mid a_s; M)$ is contained in $\Sigma(\underline{a}^2; M)$, by Lemma 1·2 again, and that, for any $i$, one has

$$a_1 \dots \hat{a_i} \dots a_s . (\underline{b}, a_s) M \equiv a_1 \dots a_s . M \mod (\underline{a}^2) M.$$

By these facts and the observation at the beginning of the proof of assertion (1) above we get

$$a_s . \triangle' \equiv F_0 \mod \Sigma(\underline{a}^2; M).$$

This implies that the map defined by

$$a_s : \triangle' \longrightarrow F_0 / \Sigma(\underline{a}^2; M)$$

is well defined and surjective. Clearly, by equation (3) after Lemma 1·2, the kernel of this map coincides with $\triangle' \cap \Sigma(\underline{b}^2, a_s; M)$, and thus to obtain assertion (2) it is enough to prove that $\Sigma(\underline{b}^2, a_s; M) \subset \triangle'$. By (8) however it suffices to show that $\triangle'$ contains $(\underline{b}^2) M : a_s$. Let $y$ be any element of $M$ such that $a_s . y \in (\underline{b}^2) M$. By the formula (1) there is an expression of $y$ as follows:

$$y = \Sigma a_I . y_I,$$

the sum being over all $I \subset \{1, \dots, s-1\}$, with $y_I \in (a_i \mid i \in I) M : \mathfrak{m}$, where we put $a_I := \prod_{i \in I} a_i$. For any $I$ such that $I \neq \{1, \dots, s-1\}$, the element $a_I . y_I$ belongs to $\Sigma(\underline{b}^2; M)$ (and hence $\Sigma(\underline{b}^2 \mid a_s; M)$), because

$$(a_i \mid i \in I) M : \mathfrak{m} = (a_i \mid i \in I) M : a_j$$

for some $j \in \{1, \dots, s-1\} \backslash I$. From the above expression for $y$, this element $y$ is contained in $a_1 \dots a_{s-1} . M + \Sigma(\underline{b}^2 \mid a_s; M)$. However $a_2 \dots a_{s-1} . \Sigma(\underline{b} \mid a_s; M)$ contains $a_1 \dots a_{s-1} . M$, and therefore we conclude that $y \in \triangle'$.

(3) Let $1 \leqslant j \leqslant s-1$. By the definition of $F_j$ the map given by multiplication by the element $a_1 \dots a_{s-1}$ such that

$$a_1 \dots a_{s-1} : [(a_1, \dots, \hat{a_j}, \dots, a_s) M : a_j] \longrightarrow F_j / F_{j-1}$$

is well defined and surjective. Thus to get the assertion (3) it is enough to show that the kernel of this map coincides with $\Sigma(a_1, \dots, \hat{a_j}, \dots, a_s; M)$. First, we have

$$F_0 \subset a_j a_s . M + \Sigma(\underline{a}^2; M). \tag{11}$$

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In fact, choose an element $z$ of $(a_1, \ldots, \hat{a}_k, \ldots, a_s)M : a_k$ for some $k$ such that $k \neq j, s$. Then we have

$$a_1 \ldots \hat{a}_j \ldots a_s . z = \prod_{l \neq j,k} a_l . a_k z \equiv a_1 \ldots \hat{a}_k \ldots a_s . w \mod (a_l^2 \mid l \neq j, k)M$$

for a suitable element $w$ of $M$. Hence by our choice of $k$ the element $a_1 \ldots \hat{a}_j \ldots a_s . z$ is contained in $a_j a_s . M + (\underline{a}^2)M$. This yields the inclusion (11). Now let $v \in (a_1, \ldots, \hat{a}_j, \ldots, a_s)M : a_j$ and assume that $a_1 \ldots a_{s-1} . v$ belongs to $F_{j-1}$. Using (11) we can express $a_1 \ldots a_{s-1} . v$ as follows:

$$a_1 \ldots a_{s-1} . v = a_1 \ldots a_{s-1} . \alpha + a_j a_s . \beta + \gamma,$$

with $\alpha \in \sum_{i=1}^{j-1} [(a_1, \ldots, \hat{a}_i, \ldots, a_s)M : a_i]$, $\beta \in M$ and $\gamma \in \Sigma(\underline{a}^2; M)$. Since we have $a_j \{a_1 \ldots \hat{a}_j \ldots a_{s-1} . (v - \alpha) - a_s . \beta\} \in \Sigma(\underline{a}^2; M)$, by equation (3) we have

$$a_1 \ldots \hat{a}_j \ldots a_{s-1} . (v - \alpha) - a_s . \beta \in \Sigma(a_1^2, \ldots, a_j, \ldots, a_s^2; M).$$

Notice that for any parameter ideal $\mathfrak{q}$ of $M$ the quotient $A$-module $\Sigma(\mathfrak{q}; M)/\mathfrak{q}M$ is annihilated by the maximal ideal $\mathfrak{m}$; cf. (ii) of Remark 1 after (1·1). Multiplying both sides of the above equality by the element $a_s$ we obtain

$$a_1 \ldots \hat{a}_j \ldots a_s . (v - \alpha) \in (a_1^2, \ldots, a_j, \ldots, a_s^2)M.$$

Considering this relation $\mathrm{mod}\, a_j M$, we see that $v - \alpha$ is contained in $\Sigma(a_1, \ldots, \hat{a}_j, \ldots, a_s \mid a_j; M)$ (cf. Lemma 1·2 and (7)) and hence by our choice of $\alpha$ we get

$$v \in \Sigma(a_1, \ldots, \hat{a}_j, \ldots, a_s \mid a_j; M).$$

Finally, applying Lemma 1·6, we conclude that the element $v$ is contained in $\Sigma(a_1, \ldots, \hat{a}_j, \ldots, a_s; M)$ and this completes the proof of the Claim.

Now we are ready to finish the proof of Proposition 1·8. By the Claim and Lemma 1·4 we immediately have

$$l_A(\square(\underline{a}; M)) = l_A(\triangle' / \Sigma(\underline{b}^2, a_s; M)) + (s-1) . l_A(\mathrm{H}_{\mathfrak{m}}^{s-1}(M)).$$

Notice that $M \supset \Sigma(\underline{b}^2, a_s; M) \supset \Sigma(\underline{b}^2 \mid a_s; M)$, and hence by (10) and Corollary 1·7 we know that

$$l_A(\triangle' / \Sigma(\underline{b}^2, a_s; M)) = l_A(\square(\underline{b}; M')) - l_A(\mathrm{H}_{\mathfrak{m}}^{s-1}(M)).$$

As $M'/\Sigma(\underline{b}; M') \cong M/\Sigma(\underline{b} \mid a_s; M)$ (cf. (7)) and as $M \supset \Sigma(\underline{a}; M) \supset \Sigma(\underline{b} \mid a_s; M)$ we obtain

$$l_A(M'/\Sigma(\underline{b}; M')) = l_A(M/\Sigma(\underline{a}; M)) + l_A(\mathrm{H}_{\mathfrak{m}}^{s-1}(M))$$

by (8) and Corollary 1·7 again. On the other hand we already know by Remark 2 after Definition 1·1 that

$$l_A(\mathrm{H}_{\mathfrak{m}}^j(M')) = l_A(\mathrm{H}_{\mathfrak{m}}^j(M)) + l_A(\mathrm{H}_{\mathfrak{m}}^{j+1}(M))$$

for $0 \leqslant j \leqslant s-2$. Combining these facts and (9) we finally get

$$\begin{aligned} l_A(\square(\underline{a}; M)) &= l_A(\square(\underline{b}; M')) + (s-2) . l_A(\mathrm{H}_{\mathfrak{m}}^{s-1}(M)) \\ &= l_A(M/\Sigma(\underline{a}; M)) + \sum_{i=1}^{s-1} \binom{s}{i-1} l_A(\mathrm{H}_{\mathfrak{m}}^i(M)). \end{aligned}$$

Therefore our Proposition 1·8 has been established.

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## 2. *Condition for idealizations to be Buchsbaum*

In this section we shall discuss the condition under which idealizations are Buchsbaum and give examples satisfying it.

The concept of idealization is the main tool in our study. We make the Cartesian product $A \times M$ into a commutative ring with respect to componentwise addition and multiplication defined by $(a, x).(a', x') = (aa', ax' + a'x)$. We call this the *idealization* of $M$ (over $A$) and denote it by $A \ltimes M$. The idealization $A \ltimes M$ is a Noetherian local ring with identity $(1, 0)$, its maximal ideal is $\mathfrak{m} \times M$, and its Krull dimension equals $\dim A$ (notice that $\operatorname{Spec}(A \ltimes M) \cong \operatorname{Spec}(A)$ via the map $\mathfrak{p} \times M \mapsto \mathfrak{p}$). There is a canonical projection $\rho : A \ltimes M \longrightarrow A$ defined by $\rho((a, x)) = a$ and a canonical inclusion $\sigma : A \longrightarrow A \ltimes M$ defined by $\sigma(a) = (a, 0)$. These maps are local homomorphisms and we can regard any $A$-module (resp. $A \ltimes M$-module) as an $A \ltimes M$-module (resp. $A$-module) by $\rho$ (resp. $\sigma$). Let $\epsilon : M \longrightarrow A \ltimes M$ be the canonical inclusion such that $\epsilon(x) = (0, x)$. Then we have an exact sequence of $A \ltimes M$-modules:

$$0 \longrightarrow M \xrightarrow{\epsilon} A \ltimes M \xrightarrow{\rho} A \longrightarrow 0.$$

Moreover we note that the structure of $A$-modules induced by the composition $\rho \circ \sigma$ coincides with the original one.

*Remark* 2·1. Let $\mathfrak{Q}$ be an ideal of $A \ltimes M$ and put $\mathfrak{q} := \rho(\mathfrak{Q})$. Then the following assertions hold.

(1) There is an exact sequence of $A \ltimes M$-modules such that

$$M/\mathfrak{q}M \xrightarrow{\epsilon'} A \ltimes M/\mathfrak{Q} \xrightarrow{\rho'} A/\mathfrak{q} \longrightarrow 0,$$

where $\rho'$ (resp. $\epsilon'$) is induced from $\rho$ (resp. $\epsilon$). If $\mathfrak{Q} = \mathfrak{q}(A \ltimes M)$, then the map $\epsilon'$ is injective.

(2) As $\mathfrak{Q}^n((0) \times M) = (0) \times \mathfrak{q}^n M$ one knows that $\mathfrak{q}^{n-1}(A \ltimes M) \supset \mathfrak{Q}^n \supset \mathfrak{q}^{n+1}(A \ltimes M)$ for any integer $n > 0$. Therefore $\mathfrak{Q}$ is $\mathfrak{m} \times M$-primary if and only if $\mathfrak{q}$ is $\mathfrak{m}$-primary. If this is the case and if $\dim_A M = \dim A$ one has $e_{\mathfrak{Q}}(A \ltimes M) = e_{\mathfrak{q}}(A) + e_{\mathfrak{q}}(M)$.

(3) Since the idealization $A \ltimes M$ can be regarded as an $A$-module $A \oplus M$ via the map $\sigma$, one has $\mathrm{H}^i_{\mathfrak{m} \times M}(A \ltimes M) \cong \mathrm{H}^i_{\mathfrak{m}}(A) \oplus \mathrm{H}^i_{\mathfrak{m}}(M)$ as $A$-modules for all $i \in \mathbb{Z}$. (As regards (3), see Lemma 4·2 below.)

Suppose that $A \ltimes M$ is a Buchsbaum ring. By Remark 2 after (1·1), all local cohomology modules $\mathrm{H}^i_{\mathfrak{m} \times M}(A \ltimes M)$ of $A \ltimes M$ for $0 \leqslant i < \dim A$ are annihilated by its maximal ideal $\mathfrak{m} \times M$. Hence by (3) of Remark 2·1 every local cohomology module $\mathrm{H}^i_{\mathfrak{m}}(M)$ of $M$ is annihilated by the maximal ideal $\mathfrak{m}$ for $0 \leqslant i < \dim A$. So concerning the Krull dimension of $M$ there are only two possibilities: either $\dim_A M = 0$ (and moreover in this case the $A$-module $M$ must be annihilated by $\mathfrak{m}$ because $\mathrm{H}^0_{\mathfrak{m}}(M) = M$), or $\dim_A M = \dim A$. For the case $\dim_A M = 0$ it was proved by Goto in (2·8) of [2] that $A \ltimes M$ is a Buchsbaum ring if and only if $A$ is a Buchsbaum ring and $M$ is annihilated by $\mathfrak{m}$. Therefore our interest is in the case when $\dim_A M = \dim A$. Then (1) and (2) of Remark 2·1 imply that $A$ and $M$ are Buchsbaum (cf. the proof of (2·8) of [2]).

We make the following remark: if a Buchsbaum $A$-module $M$ satisfies the equality $\dim_A M = \dim A$, then we say that $M$ is a *maximal* Buchsbaum $A$-module; see [7], in

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which it was proved that if $A$ is a regular local ring there are only finitely many indecomposable isomorphism classes of maximal Buchsbaum $A$-modules. Our problem can be reformulated thus: what is the condition for idealizations of maximal Buchsbaum modules over a Buchsbaum ring to become Buchsbaum? This section is devoted to determining this condition and discussing the behaviour of modules which satisfy it.

For the rest of this section, let us assume that $A$ is a Buchsbaum ring of dimension $d$ and $M$ is a maximal Buchsbaum $A$-module, and for simplicity we further assume that the residue field $A/\mathfrak{m}$ is infinite.

LEMMA 2·2. *Let $\mathfrak{q} = (a_1, \ldots, a_d)$ be a parameter ideal of $A$. Then the following are equivalent.*

(1) $\Sigma(\mathfrak{q}).M = \mathfrak{q}.M$.

(2) $[(a_1, \ldots, \hat{a}_i, \ldots, a_d) : a_i].M = (a_1, \ldots, \hat{a}_i, \ldots, a_d).M$ *for every* $i \in \{1, \ldots, d\}$.

(3) $\Sigma(\mathfrak{q}_I).M = \mathfrak{q}_I.M$ *for all* $I \subset \{1, \ldots, d\}$, *where we put* $\mathfrak{q}_I := (a_i \mid i \in I)$.

(4) $a_1, a_2, \ldots, a_d$ *form a weak* $A \ltimes M$*-sequence in any order.*

(5) $l_{A \ltimes M}(A \ltimes M/\mathfrak{Q}) - e_{\mathfrak{Q}}(A \ltimes M) = \mathrm{I}(A) + \mathrm{I}_A(M)$ *holds for every parameter ideal* $\mathfrak{Q}$ *of* $A \ltimes M$ *such that* $\rho(\mathfrak{Q}) = \mathfrak{q}$.

*Proof.* Recall that every system of parameters for $M$ forms a d-sequence on $M$ (cf. Remark 1 after (1·1)), and thus from (4) in §1 we obtain

$$\Sigma(\mathfrak{q}_I).M \cap \mathfrak{q}M \subset [\mathfrak{q}_I : a_j].M \cap \mathfrak{q}M \subset [\mathfrak{q}_I M : a_j] \cap \mathfrak{q}M = \mathfrak{q}_I M$$

for all $I \subsetneqq \{1, \ldots, d\}$ and $j \notin I$, where we put $\mathfrak{q}_I := (a_i \mid i \in I)$. Hence the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are obvious. By the definition of a weak sequence we easily obtain (3) $\Leftrightarrow$ (4).

For the proof of the implication (1) $\Rightarrow$ (5) we first establish the following claim.

*Claim.* $(0) \times M \cap \mathfrak{Q} = (0) \times \mathfrak{q}M$.

In fact, let $v$ be an element of $M$ and assume that $(0, v)$ is contained in $\mathfrak{Q}$. Then we can express $(0, v)$ as follows:

$$(0, v) = \sum_{i=1}^{d} (a_i, x_i).(b_i, y_i) \quad \text{with} \quad (b_i, y_i) \in A \ltimes M,$$

where we put $\mathfrak{Q} = ((a_1, x_1), \ldots, (a_d, x_d))$. Hence we get

$$0 = \sum_{i=1}^{d} a_i b_i; \tag{12}$$

$$v = \sum_{i=1}^{d} a_i y_i + \sum_{i=1}^{d} b_i x_i. \tag{13}$$

By (12) each $b_i$ is contained in $\Sigma(\mathfrak{q})$, hence by our assumption (1) we have $b_i x_i \in \mathfrak{q}M$ for all $i$. This implies that $v$ belongs to $\mathfrak{q}M$ by (13). The converse is obvious since $(0) \times \mathfrak{q}M = \mathfrak{Q}.((0) \times M)$.

By the claim we have the exact sequence

$$0 \longrightarrow M/\mathfrak{q}M \longrightarrow A \ltimes M/\mathfrak{Q} \longrightarrow A/\mathfrak{q} \longrightarrow 0.$$

Hence we see that

$$l_{A \ltimes M}(A \ltimes M/\mathfrak{Q}) = l_A(A/\mathfrak{q}) + l_A(M/\mathfrak{q}M),$$

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and thus

$$l_{A\ltimes M}(A\ltimes M/\mathfrak{Q})-e_{\mathfrak{Q}}(A\ltimes M)=\{l_A(A/\mathfrak{q})-e_{\mathfrak{q}}(A)\}+\{l_A(M/\mathfrak{q}M)-e_{\mathfrak{q}}(M)\}$$
$$=\mathrm{I}(A)+\mathrm{I}_A(M).$$

It remains to prove the implication (5) $\Rightarrow$ (1). It is enough to show that

$$[(a_1,\ldots,a_{d-1}):a_d].M\subset\mathfrak{q}M.$$

Let $c$ be an element of $A$ such that

$$a_d c=a_1c_1+\ldots+a_{d-1}c_{d-1}$$

with each $c_i\in A$, and let $z$ be any element of $M$. Then there is an expression of the element $(0, cz)$ as follows:

$$(0,cz)=\sum_{i=1}^{d-1}(a_i,0).(-c_i,0)+(a_d,z).(c,0). \tag{14}$$

Consider the ideal $\mathfrak{Q}'=((a_1,0),\ldots,(a_{d-1},0),(a_d,z))$ of $A\ltimes M$. Clearly $\rho(\mathfrak{Q}')=\mathfrak{q}$ and $\mathfrak{Q}'$ is a parameter ideal of $A\ltimes M$ (cf. (2) of Remark 2·1). Our assumption (5) implies

$$l_{A\ltimes M}(A\ltimes M/\mathfrak{Q}')=l_A(A/\mathfrak{q})+l_A(M/\mathfrak{q}M)$$

by (2) of (2·1). Therefore the sequence of $A\ltimes M$-modules

$$0\longrightarrow M/\mathfrak{q}M\xrightarrow{\epsilon}A\ltimes M/\mathfrak{Q}'\longrightarrow A/\mathfrak{q}\longrightarrow 0$$

must be exact. The equation (14) and the injectivity of $\epsilon$ imply that the element $cz$ is contained in $\mathfrak{q}M$.

Now the condition under which idealizations are Buchsbaum is given by the following proposition; this was essentially shown in [9] (also see [5], (2) of (2·5)).

Proposition 2·3. *The following statements are equivalent.*

(1) $A\ltimes M$ *is a Buchsbaum ring.*
(2) $\Sigma(\mathfrak{q}).M=\mathfrak{q}M$ *holds for any parameter ideal* $\mathfrak{q}$ *of* $A$.
(3) $\Sigma(\mathfrak{q}).M=\mathfrak{q}M$ *holds for any minimal reduction* $\mathfrak{q}$ *of* $\mathfrak{m}$.

*Proof.* By Lemma 2·2 the equivalence (1) $\Leftrightarrow$ (2) is already proven and the implication (2) $\Rightarrow$ (3) is obvious. So we will show (3) $\Rightarrow$ (1). By Remark 5 after Definition 1·1 we can find a system of generators of the maximal ideal $\mathfrak{m}\times M$ of $A\ltimes M$, say $\{(a_\lambda,x_\lambda)\}_{\lambda\in\Lambda}$, such that every subsystem of it consisting of $d$ elements forms a minimal reduction of the maximal ideal $\mathfrak{m}\times M$. By (2) of (2·1), the system $\{a_\lambda\}_{\lambda\in\Lambda}$ generates the maximal ideal $\mathfrak{m}$ of $A$, and we moreover see that every subsystem of $\{a_\lambda\}_{\lambda\in\Lambda}$ consisting of $d$ elements forms a minimal reduction of $\mathfrak{m}$. Let $\mathfrak{Q}$ be an ideal of $A\ltimes M$ generated by $d$ arbitrary elements of this system $\{(a_\lambda,x_\lambda)\}_{\lambda\in\Lambda}$. Then our assertion (3) implies that

$$l_{A\ltimes M}(A\ltimes M/\mathfrak{Q})-e_{\mathfrak{Q}}(A\ltimes M)=\sum_{i=0}^{d-1}\binom{d-1}{i}.l_A(\mathrm{H}^i_{\mathfrak{m}\times M}(A\ltimes M)),$$

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by Lemma 2·2 again. Combining this fact and Remark 5 after (1·1) again, we conclude that $A \ltimes M$ is a Buchsbaum ring, and this finishes the proof of (2·3).

As consequences of (2·3) we have the following corollaries.

COROLLARY 2·4. (1) *If $M$ (resp. $A$) is Cohen–Macaulay, then the idealization $A \ltimes M$ is a Buchsbaum ring.*

(2) *$A$ is Cohen–Macaulay if and only if every idealization of a maximal Buchsbaum module over $A$ is a Buchsbaum ring.*

*Proof.* Assume that $M$ (resp. $A$) is Cohen–Macaulay. Then obviously $\Sigma(\mathfrak{q}; M)$ (resp. $\Sigma(\mathfrak{q}).M) = \mathfrak{q}M$ for any parameter ideal $\mathfrak{q}$ of $A$. Clearly then by (2·3) $A \ltimes M$ is a Buchsbaum ring. On the other hand, considering the idealization of $A$ itself, if $A \ltimes A$ is a Buchsbaum ring then we have $\Sigma(\mathfrak{q}).A = \mathfrak{q}$ for any parameter ideal $\mathfrak{q}$ of $A$, by (2·3) again, and hence $A$ is Cohen–Macaulay.

COROLLARY 2·5. *Suppose that $A \ltimes M$ is a Buchsbaum ring. Then the following statements are true:*

(i) $\mathrm{I}(A \ltimes M) = \mathrm{I}(A) + \mathrm{I}_A(M)$;

(ii) $\mathrm{H}^0_{\mathfrak{m}}(A).M = (0)$ *and* $(A/\mathrm{H}^0_{\mathfrak{m}}(A)) \ltimes M$ *is also a Buchsbaum ring, $M$ being regarded as a module over* $A/\mathrm{H}^0_{\mathfrak{m}}(A)$;

(iii) $(0) \times M \cap \mathfrak{Q}^n = (0) \times \mathfrak{q}^n M$ *for every parameter ideal $\mathfrak{Q}$ of $A \ltimes M$ and every $n > 0$, where* $\mathfrak{q} := \rho(\mathfrak{Q})$;

(iv) *let $\mathfrak{Q}$, $\mathfrak{q}$ be as in* (iii), *and let*

$$\mathrm{R}_{\mathfrak{q}}(M) := \bigoplus_{n \geq 0} \mathfrak{q}^n M \quad \text{and} \quad \mathrm{G}_{\mathfrak{q}}(M) := \bigoplus_{n \geq 0} \mathfrak{q}^n M/\mathfrak{q}^{n+1} M$$

*denote the Rees module and the associated graded module of $M$ with respect to $\mathfrak{q}$, respectively; similarly define $\mathrm{R}_{\mathfrak{Q}}(A \ltimes M)$, $\mathrm{G}_{\mathfrak{Q}}(A \ltimes M)$, $\mathrm{R}_{\mathfrak{q}}(A)$ and $\mathrm{G}_{\mathfrak{q}}(A)$. Then there are two exact sequences of graded $\mathrm{R}_{\mathfrak{Q}}(A \ltimes M)$-modules*

$$0 \longrightarrow \mathrm{R}_{\mathfrak{q}}(M) \longrightarrow \mathrm{R}_{\mathfrak{Q}}(A \ltimes M) \longrightarrow \mathrm{R}_{\mathfrak{q}}(A) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{G}_{\mathfrak{q}}(M) \longrightarrow \mathrm{G}_{\mathfrak{Q}}(A \ltimes M) \longrightarrow \mathrm{G}_{\mathfrak{q}}(A) \longrightarrow 0.$$

*Proof.* (i) This is obvious by (1·2). (ii) As $0 : a = \mathrm{H}^0_{\mathfrak{m}}(A)$ for any parameter $a \in \mathfrak{m}$ for $A$ (cf. Remark 2 after (1·1)) we have $\mathrm{H}^0_{\mathfrak{m}}(A).M = (0)$ by (3) of (2·2). The assertion is then clear by (2·3). (iii) Put $\mathfrak{Q} = (f_1, \ldots, f_d)$, and let $f_i = (a_i, x_i)$ for each $i$. Clearly $\mathfrak{q} = (a_1, \ldots, a_d)$, and each power $\mathfrak{Q}^n$ has the following generators:

$$\mathfrak{Q}^n = ((F_1(\underline{a}), y_1), \ldots, (F_t(\underline{a}), y_t));$$

where the $F_i(\underline{X})$ are all the monomials of degree $n$ in

$$\mathbb{Z}[X_1, \ldots, X_d], \quad t := \binom{d+n-1}{n}$$

and the $y_i$ are suitable elements in $\mathfrak{q}^{n-1}M$. Now let $z$ be any element of $M$ such that $(0, z) \in \mathfrak{Q}^n$. Then we have

$$(0, z) = \sum_{i=1}^{t} (c_i, z_i).(F_i(\underline{a}), y_i)$$

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with $(c_i . z_i) \in A \ltimes M$. From this we get

$$0 = \sum_{i=1}^{t} c_i . F_i(\underline{a}), \tag{15}$$

$$z = \sum_{i=1}^{t} c_i . y_i + \sum_{i=1}^{t} F_i(\underline{a}) . z_i. \tag{16}$$

Recall that every system of parameters for a Buchsbaum ring $A$ forms a d-sequence on $A$ (cf. Remark 1 after (1·1)). Hence according to theorem 3·1 of [13] the symmetric algebra $\mathrm{Sym}_A(\mathfrak{q})$ and the Rees algebra $\mathrm{R}_{\mathfrak{q}}(A)$ are isomorphic to each other via the canonical map. This implies that each element $c_i$ is contained in $\Sigma(\mathfrak{q})$ by (15), and by (2·3) we have $c_i . y_i \in \Sigma(\mathfrak{q}) . \mathfrak{q}^{n-1} M = \mathfrak{q}^n M$. From (16) we conclude that $z$ belongs to $\mathfrak{q}^n M$. Finally assertion (iv) comes immediately from (iii).

We denote by $\mu_A(M)$ the minimal number of generators of an $A$-module $M$. Let $\mathfrak{q}$ be a minimal reduction of $\mathfrak{m}$. Then clearly $\mu_A(M) = l_A(M/\mathfrak{m}M) \leqslant l_A(M/\mathfrak{q}M)$. As $M$ is a maximal Buchsbaum $A$-module and $\mathfrak{q}$ is a minimal reduction of $\mathfrak{m}$ we have

$$\mu_A(M) \leqslant e_A(M) + \mathrm{I}_A(M),$$

where $e_A(M)$ (or simply $e(M)$) denotes the multiplicity of $M$ with respect to the maximal ideal $\mathfrak{m}$. In the rest of this section, we describe some interesting connections with our study of idealizations in the case when $\mu_A(M) = e_A(M) + \mathrm{I}_A(M)$. First we recall a definition: if $A$ is a Buchsbaum ring, we say that $A$ has *minimal multiplicity* if the equality

$$e(A) = 1 + \sum_{i=1}^{d-1} \binom{d-1}{i-1} l_A(\mathrm{H}^i_{\mathfrak{m}}(A))$$

holds. In general the term on the right-hand side of this equality is a lower bound for the multiplicity $e(A)$ of a Buchsbaum ring $A$, and the condition that $A$ has minimal multiplicity is equivalent to saying that $\mathfrak{m} = \Sigma(\mathfrak{q})$ holds for some (and hence every) minimal reduction $\mathfrak{q}$ of $\mathfrak{m}$; see §4 of [4]. We have the following

Proposition 2·6. *If the equality $\mu_A(M) = e_A(M) + \mathrm{I}_A(M)$ holds, then the idealization $A \ltimes M$ is a Buchsbaum ring. If $A$ has minimal multiplicity the converse is also true.*

*Proof.* Choose a minimal reduction $\mathfrak{q}$ of $\mathfrak{m}$. The equality $\mu_A(M) = e_A(M) + \mathrm{I}_A(M)$ implies $\mathfrak{m}M = \mathfrak{q}M$, therefore we obviously get $\Sigma(\mathfrak{q}) . M = \mathfrak{q}M$. Combining this fact and (2·3) we see $A \ltimes M$ is a Buchsbaum ring. Now assume that $A$ has minimal multiplicity. Then by (2·3) the Buchsbaum property of $A \ltimes M$ means that $\mathfrak{m}M = \mathfrak{q}M$, and hence we clearly have the required equality $\mu_A(M) = e_A(M) + \mathrm{I}_A(M)$.

Let us denote by $v(A)$ the embedding dimension of $A$, i.e. $v(A) := \mu_A(\mathfrak{m})$. We say that a Buchsbaum ring $A$ has *maximal embedding dimension* if the following equality holds:

$$v(A) = e(A) + d - 1 + \mathrm{I}(A).$$

Note that in general the term on the right-hand side of this equality is an upper bound for the embedding dimension $v(A)$ of a Buchsbaum ring $A$, and the condition that $A$ has maximal embedding dimension is equivalent to saying that $\mathfrak{m}^2 = \mathfrak{q}\mathfrak{m}$ holds for some (and hence every) minimal reduction $\mathfrak{q}$ of $\mathfrak{m}$; see [3].

Proposition 2·7. *$A \ltimes M$ is a Buchsbaum ring with maximal embedding dimension if and only if $A$ has this property and the equality $\mu_A(M) = e_A(M) + \mathrm{I}_A(M)$ holds.*

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*Proof.* Suppose that $A$ has maximal embedding dimension and suppose that $\mu_A(M) = e_A(M) + \mathrm{I}_A(M)$. By (2·6) $A \ltimes M$ is a Buchsbaum ring. As $v(A \ltimes M) = v(A) + \mu_A(M)$ and as $e(A \ltimes M) = e(A) + e_A(M)$ we get

$$v(A \ltimes M) = e(A \ltimes M) + d - 1 + \mathrm{I}(A \ltimes M);$$

see also (i) of (2·5). The converse is shown in the same way.

In the rest of this section we shall give several examples satisfying the equivalent conditions of Proposition 2·3.

*Example* 2·8 ([9]). $A \ltimes \mathfrak{m}$ is a Buchsbaum ring.

In fact let $\underline{a} := a_1, \ldots, a_d$ be a system of parameters for $A$. Choose an element $x$ of $A$ such that $a_d x = a_1 x_1 + \ldots + a_{d-1} x_{d-1}$ with each $x_i \in A$. Since the parameters $a_i$ are analytically independent, the elements $x_i$ and $x$ belong to the maximal ideal $\mathfrak{m}$. Hence we know

$$(a_1, \ldots, a_{d-1}) : a_d = (a_1, \ldots, a_{d-1})\mathfrak{m} :_{\mathfrak{m}} a_d.$$

As the maximal ideal $\mathfrak{m}$ is a maximal Buchsbaum $A$-module (see (2·4) of [2]) we have

$$(a_1, \ldots, a_{d-1})\mathfrak{m} :_{\mathfrak{m}} a_d = (a_1, \ldots, a_{d-1})\mathfrak{m} :_{\mathfrak{m}} \mathfrak{m},$$

and hence

$$[(a_1, \ldots, a_{d-1}) : a_d].\mathfrak{m} \subset (a_1, \ldots, a_{d-1})\mathfrak{m},$$

namely $\Sigma(\underline{a}).\mathfrak{m} = (\underline{a})\mathfrak{m}$. Therefore by (2·3) $A \ltimes \mathfrak{m}$ is a Buchsbaum ring.

*Example* 2·9. Assume that $d \geqslant 2$ and $\mathrm{H}^p_{\mathfrak{m}}(A) = (0)$ for all $p \neq 1, d$. Let $E_i$ $(i \geqslant 1)$ be the $i$th syzygy module of the residue field $A/\mathfrak{m}$ over $A$; namely if $F.$ is the minimal free resolution of $A/\mathfrak{m}$ over $A$ we define

$$E_i := \mathrm{Im}\,(F_i \longrightarrow F_{i-1}) \quad \text{for all} \quad i \geqslant 1.$$

Then

(i) every $E_i$ satisfies the surjectivity criterion (see Remark 4 after (1·1)) with $\mathrm{H}^p_{\mathfrak{m}}(E_i) = (0)$ for all $p = 0,\ i+1, \ldots, d-1$, therefore $E_i$ is a maximal Buchsbaum $A$-module;

(ii) $A \ltimes E_i$ is a Buchsbaum ring for all $i \geqslant 1$.

*Proof.* Recall that for any finitely generated $A$-module $E$ such that $\mathfrak{m}.\mathrm{H}^t_{\mathfrak{m}}(E) = (0)$, where $t := \mathrm{depth}_A E$, the canonical map

$$\phi^t_E : \mathrm{Ext}^t_A(A/\mathfrak{m}, E) \longrightarrow \mathrm{H}^t_{\mathfrak{m}}(E)$$

is always surjective; see Remark 4 after (1·1) again.

We prove (i) by induction on $i$. Let $i = 1$. Looking at the exact sequence of $A$-modules

$$0 \longrightarrow \mathfrak{m} = E_1 \longrightarrow A \longrightarrow A/\mathfrak{m} \longrightarrow 0$$

and using our standard assumption on $A$, we get that $\mathrm{H}^p_{\mathfrak{m}}(\mathfrak{m}) = (0)$ for all $p \neq 1, d$ and $\mathfrak{m}.\mathrm{H}^1_{\mathfrak{m}}(\mathfrak{m}) = (0)$ from the Buchsbaum property of $\mathfrak{m}$; see (2·8). Hence, by the above observation, the $A$-module $\mathfrak{m}\,(= E_1)$ satisfies the required condition. Let $i \geqslant 2$ and assume that $E_{i-1}$ satisfies the surjectivity criterion (i.e. the canonical map

$$\phi^p_{i-1} : \mathrm{Ext}^p_A(A/\mathfrak{m}, E_{i-1}) \longrightarrow \mathrm{H}^p_{\mathfrak{m}}(E_{i-1})$$

is surjective for all $p \neq d$) and $\mathrm{H}^p_{\mathfrak{m}}(E_{i-1}) = (0)$ for $p = 0,\ i, \ldots, d-1$. Considering the exact sequence of $A$-modules

$$0 \longrightarrow E_i \longrightarrow F_{i-1} \xrightarrow{\delta} E_{i-1} \longrightarrow 0 \tag{17}$$

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we obtain a commutative diagram with an exact bottom row for each $p$ as follows:

$$\begin{array}{ccccc} \mathrm{Ext}_A^{p-1}(A/\mathfrak{m}, E_{i-1}) & \longrightarrow & \mathrm{Ext}_A^{p}(A/\mathfrak{m}, E_i) & & \\ \downarrow{\scriptstyle \phi_{i-1}^{p-1}} & & \downarrow{\scriptstyle \phi_i^p} & & \\ \mathrm{H}_{\mathfrak{m}}^{p-1}(E_{i-1}) & \longrightarrow & \mathrm{H}_{\mathfrak{m}}^{p}(E_i) & \longrightarrow & \mathrm{H}_{\mathfrak{m}}^{p}(F_{i-1}). \end{array} \tag{18}$$

Let $2 \leqslant p \leqslant d-1$. Since $\mathrm{H}_{\mathfrak{m}}^{p}(F_{i-1}) = (0)$ by our assumption on $A$ and since $\phi_{i-1}^{p-1}$ is surjective by our hypothesis on $E_{i-1}$ we conclude that $\phi_i^p$ is surjective. Next we consider the remaining cases $p = 0, 1$. As $\mathrm{H}_{\mathfrak{m}}^{0}(E_i) \subset \mathrm{H}_{\mathfrak{m}}^{0}(F_{i-1})$ by (18) and as $\mathrm{H}_{\mathfrak{m}}^{0}(F_{i-1}) = (0)$ by our assumption on $A$, we know that $\mathrm{H}_{\mathfrak{m}}^{0}(E_i) = (0)$. Moreover, recalling the hypothesis on $E_{i-1}$ that $\mathrm{H}_{\mathfrak{m}}^{0}(E_{i-1}) = (0)$, we have $\mathrm{H}_{\mathfrak{m}}^{1}(E_i) \subset \mathrm{H}_{\mathfrak{m}}^{1}(F_{i-1})$ from (18) again. Then as $\mathrm{H}_{\mathfrak{m}}^{1}(F_{i-1})$ is a direct sum of copies of $\mathrm{H}_{\mathfrak{m}}^{1}(A)$ we have $\mathfrak{m}.\mathrm{H}_{\mathfrak{m}}^{1}(E_i) = (0)$. Thus, by the observation at the beginning again, the map $\phi_i^1$ is surjective too. The rest of the assertion (i) is shown in the same way.

To prove (ii), by (2·8) we may assume that $i \geqslant 2$. Let $a_1, a_2, \ldots, a_d$ be a system of parameters for $A$. We make the following

*Claim.* $[(a_j):\mathfrak{m}].E_i = a_j E_i$ for $j = 1, \ldots, d$.

In fact, let $b$ be an element of $A$ such that $b \in (a_j):\mathfrak{m}$ and $e$ an element of $E_i$. Write $e = (e_1, \ldots, e_\beta)$ in $F_{i-1}$, where $\beta := \mathrm{rank}_A F_{i-1}$. By the minimality of the resolution $F$. we see that each $e_k$ is contained in the maximal ideal $\mathfrak{m}$. By our choice of $b$ we have $be_k = a_j f_k$ with $f_k \in A$ for $k \in \{1, \ldots, \beta\}$. Then put $f := (f_1, \ldots, f_\beta)$ in $F_{i-1}$. Clearly $be = a_j f$. As $e$ belongs to the kernel of $\delta$ we have $a_j \delta(f) = 0$. As $a_j$ is a parameter for $A$ we know that $0:a_j = \mathrm{H}_{\mathfrak{m}}^{0}(A) = (0)$, and thus the element $a_j$ is a non-zero divisor on $F_{i-2}$ (and hence on $E_{i-1}$). Therefore $\delta(f) = 0$, i.e. $f \in E_i$. This implies that the element $be$ is contained in $a_j E_i$.

Applying Corollary 1·5 inductively, the standard assumption on $A$ implies the following equality:

$$\Sigma(a_1, \ldots, a_d) = [(a_1):\mathfrak{m}] + \ldots + [(a_d):\mathfrak{m}].$$

Then by the claim we have

$$\Sigma(a_1, \ldots, a_d).E_i = a_1 E_i + \ldots + a_d E_i,$$

hence $A \ltimes E_i$ is a Buchsbaum ring by (2·3).

*Example* 2·10 ((2) of (4·6) in [3]). Suppose that $A$ is a regular local ring. Then every maximal Buchsbaum $A$-module $M$ satisfies the equality $\mu_A(M) = e_A(M) + \mathrm{I}_A(M)$, and hence the idealization $A \ltimes M$ is a Buchsbaum ring with maximal embedding dimension (cf. (2·7)).

*Example* 2·11. (1) Let $A$ be as in (2·9) and assume that it has minimal multiplicity; see (4·11) of [4] for the existence of such Buchsbaum rings. Then every syzygy module $E_i$ with $i \geqslant 1$ of the residue field $A/\mathfrak{m}$ over $A$ is a maximal Buchsbaum $A$-module which satisfies the equality

$$\mu_A(E_i) = e_A(E_i) + \mathrm{I}_A(E_i),$$

by (2·6) and (ii) of (2·9).

(2) If $A$ has maximal embedding dimension, then the maximal ideal $\mathfrak{m}$ is a

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maximal Buchsbaum $A$-module such that the equality $\mu_A(\mathfrak{m}) = e_A(\mathfrak{m}) + \mathrm{I}_A(\mathfrak{m})$ holds. Hence $A \ltimes \mathfrak{m}$ is a Buchsbaum ring with maximal embedding dimension. by (2·7) and (2·8).

Further examples can be found in the next section.

## 3. *Idealizations of canonical modules*

In this section, applying (2·3) we shall determine the condition for idealizations of canonical modules of maximal Buchsbaum modules to be Buchsbaum.

For simplicity, throughout this section let us assume that the given local ring $A$ is complete with respect to the $\mathfrak{m}$-adic topology and the residue field $A/\mathfrak{m}$ is infinite. We begin with the definition of the canonical modules, cf. [**10**, **11**]; more details can be found in [**1**].

*Definition* 3·1. For a finitely generated $A$-module $M$ we define the *canonical module* $K_M$ of $M$ (over $A$) as follows:

$$K_M := \mathrm{Hom}_A(\mathrm{H}^s_{\mathfrak{m}}(M), \mathrm{E}_A(A/\mathfrak{m})),$$

where $s := \dim_A M$ and $\mathrm{E}_A(A/\mathfrak{m})$ denotes the injective envelope of $A/\mathfrak{m}$ over $A$. We also write $M^D$ instead of $K_M$.

The canonical module $K_M$ of $M$ is again a finitely generated $A$-module such that $\dim_A K_M = \dim_A M$ and it satisfies the condition $(S_2)$; recall that $M$ satisfies $(S_2)$ if

$$\mathrm{depth}_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \geqslant \min\{2, \dim_{A_{\mathfrak{p}}} M_{\mathfrak{p}}\}$$

for all $\mathfrak{p} \in \mathrm{Supp}_A M$.

*Remark* 3·2. Suppose that $\mathrm{H}^i_{\mathfrak{m}}(M)$ is of finite length for all $i \neq s$. Then the following statements are true.

(1) $K_M$ is a Buchsbaum $A$-module if $M$ is. (See the remark after theorem 4·9 in chapter II of [**23**].)

(2) There is an $A$-isomorphism

$$\mathrm{H}^i_{\mathfrak{m}}(K_M) \cong \mathrm{Hom}_A(\mathrm{H}^{s+1-i}_{\mathfrak{m}}(M), \mathrm{E}_A(A/\mathfrak{m}))$$

for every $i \in \{2, \ldots, s-1\}$ and there is an exact sequence of $A$-modules

$$0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow M^{DD} \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(M) \longrightarrow 0;$$

cf. Korollar (3·1·3) in [**18**].

(3) If $\mathfrak{q}$ is a parameter ideal of $M$, then it is also a parameter ideal of $K_M$ and the equality $e_{\mathfrak{q}}(K_M) = e_{\mathfrak{q}}(M)$ holds; cf. lemma 2·2 of [**24**].

Now we shall prove our main result, which gives the condition for idealizations of canonical modules to be Buchsbaum.

THEOREM 3·3. *Let $A$ be a Buchsbaum ring of dimension $d$ and $M$ a maximal Buchsbaum $A$-module. Then the following are equivalent*:

(1) *$A \ltimes K_M$ is a Buchsbaum ring*;

(2) *$\Sigma(\mathfrak{q}) \cdot \Sigma(\mathfrak{q}; M) = \mathfrak{q} \cdot \Sigma(\mathfrak{q}; M)$ holds for every minimal reduction $\mathfrak{q}$ of $\mathfrak{m}$*:

(3) *$\Sigma(\underline{a}^2; M) : \Sigma(\mathfrak{q}) = \Sigma(\underline{a}^2; M) : \mathfrak{q}$ holds for every minimal reduction $\mathfrak{q} = (a_1, \ldots, a_d)$ of $\mathfrak{m}$, where $\underline{a}^2 := a_1^2, \ldots, a_d^2$.*

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As consequences of our theorem we obtain the following corollaries.

COROLLARY 3·4. *Let $A$, $M$ be as in Theorem* 3·3, *and assume that $d \geqslant 2$ and that the equality*

$$e_A(M) = \sum_{i=1}^{d-1} \binom{d-1}{i-1} l_A(\mathrm{H}^i_{\mathfrak{m}}(M))$$

*holds. Then $A \ltimes M$ is a Buchsbaum ring if and only if $A \ltimes K_M$ is. When this is the case, $\mu_{A \ltimes M}(K_{A \ltimes M}) = \mu_A(K_A) + \mu_A(K_M)$.*

COROLLARY 3·5. *If a Buchsbaum ring $A$ satisfies $\mu_A(K_A) \geqslant e_A(K_A) + \mathrm{I}_A(K_A) - 1$, then $A \ltimes K_A$ is a Buchsbaum ring.*

Until (3·9) we assume that $A$ is a Buchsbaum ring of dimension $d$ and that $M$ is a maximal Buchsbaum $A$-module, and we let $\mathfrak{q} = (a_1, \ldots, a_d)$ be a parameter ideal of $A$.

The following result comes immediately from theorem 5·21 of [**19**] and (2·2).

LEMMA 3·6. *Put $U_i := (a_1, \ldots, \hat{a}_i, \ldots, a_d) : a_i$. Then the following conditions are equivalent*:

(1) $\Sigma(\mathfrak{q}).K_M = \mathfrak{q}.K_M$;

(2) $U_i.K_M = (a_1, \ldots, \hat{a}_i, \ldots, a_d).K_M$ *for all* $i \in \{1, \ldots, d\}$;

(3) $[0 : \Sigma(\mathfrak{q})]_{\mathrm{H}^d_{\mathfrak{m}}(M)} = [0 : \mathfrak{q}]_{\mathrm{H}^d_{\mathfrak{m}}(M)}$;

(4) $[0 : U_i]_{\mathrm{H}^d_{\mathfrak{m}}(M)} = [0 : (a_1, \ldots, \hat{a}_i, \ldots, a_d)]_{\mathrm{H}^d_{\mathfrak{m}}(M)}$ *for all* $i \in \{1, \ldots, d\}$.

PROPOSITION 3·7. *Suppose that* $\Sigma(\mathfrak{q}).K_M = \mathfrak{q}.K_M$. *Then* $\Sigma(\mathfrak{q}).\Sigma(\mathfrak{q}; M) = \mathfrak{q}.\Sigma(\mathfrak{q}; M)$.

*Proof.* Let us write

$$U_i := (a_1, \ldots, \hat{a}_i, \ldots, a_d) : a_i \quad \text{and} \quad U_i(M) := (a_1, \ldots, \hat{a}_i, \ldots, a_d)M : a_i.$$

First we show that

$$U_i.U_i(M) \subset (a_1, \ldots, \hat{a}_i, \ldots, a_d).\Sigma(\mathfrak{q}; M)$$

for each $i \in \{1, \ldots, d\}$. In fact it is enough to consider just the case $i = d$. Let $b$ be any element of $A$ such that $b \in U_d$ and $x$ any element of $M$ such that $x \in U_d(M)$. By our choice of $b \in \mathfrak{m}$ and $x$, we have

$$bx = a_1 x_1 + \ldots + a_{d-1} x_{d-1},$$

$$a_d b = a_1 b_1 + \ldots + a_{d-1} b_{d-1}$$

with $x_j \in M$ and $b_j \in U_j$ $(1 \leqslant j \leqslant d-1)$. Hence for each $j$ we see that

$$a_1 \ldots a_d.x_j = a_1 \ldots \hat{a}_j \ldots a_d.a_j x_j = a_1 \ldots \hat{a}_j \ldots a_d.(bx - \sum_{k \neq j} a_k x_k)$$

$$\equiv a_1 \ldots \hat{a}_j \ldots a_{d-1}.a_d bx \equiv a_1 \ldots a_{d-1}.b_j x \mod (a_1^2, \ldots, a_{d-1}^2)M.$$

Since $b_j \in \mathfrak{m}$ the element $b_j x$ belongs to $(a_1, \ldots, a_{d-1})M$ by our choice of $x$ again, and hence from the above observation we have $a_1 \ldots a_d.x_j \in (a_1^2, \ldots, a_d^2)M$. This implies that $x_j \in \Sigma(\mathfrak{q}; M)$.

Next we show that

$$U_i.U_j(M) \subset (a_1, \ldots, \hat{a}_j, \ldots, a_d).\Sigma(\mathfrak{q}; M)$$

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for all $i \neq j$. In fact we may assume that $i = 1$ and $j = d$. Let $c$ be any element of $U_1$ and $y$ any element of $U_d(M)$. As $c$ belongs to $\mathfrak{m}$ we have

$$cy = a_1 y_1 + \ldots + a_{d-1} y_{d-1} \quad \text{with} \quad y_l \in M.$$

Let $k$ be an integer such that $1 \leqslant k \leqslant d-1$ and put

$$z := a_1 \ldots \hat{a}_k \ldots a_d . y.$$

Then we easily see that $a_j z \equiv 0 \bmod (\underline{a}^2) M$, where $\underline{a}^2 := a_1^2, \ldots, a_d^2$, for $j = 2, \ldots, d$. Consider the canonical map $\phi : M/(\underline{a}^2) M \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(M)$ (see the remark before Lemma 1·3). We see from the previous observation that $\phi(z \bmod (\underline{a}^2) M)$ is annihilated by the ideal $(a_2, \ldots, a_d)$. Now applying the assumption that $\Sigma(\mathfrak{q}) . K_M = \mathfrak{q} . K_M$, we see that $\phi(z \bmod (\underline{a}^2) M)$ is annihilated by $U_1$ (cf. (2) of (3·6)). Hence as $c \in U_1$ we have for $n \gg 0$

$$0 \equiv a_1^n \ldots a_d^n . cz \equiv a_1^{n+1} \ldots a_d^{n+1} . y_k \mod (\underline{a}^{n+2}) M.$$

Therefore we obtain $y_k \in \Sigma(\mathfrak{q}; M)$ by (1·2), and we have proved Proposition 3·7.

To discuss the converse of (3·7) we require the key proposition in this section:

PROPOSITION 3·8. *Via the canonical map* $M/\Sigma(\underline{a}^2; M) \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(M)$ *given in Lemma* 1·3 *one has an* $A$*-isomorphism*

$$\mathrm{Hom}_A(A/\mathfrak{q}, \mathrm{H}^d_{\mathfrak{m}}(M)) \cong \frac{\left\{ \sum_{i=1}^{d} a_1 \ldots \hat{a}_i \ldots a_d . \Sigma(\mathfrak{q}; M) \right\} + \Sigma(\underline{a}^2; M)}{\Sigma(\underline{a}^2; M)}$$

*where* $\underline{a}^2 := a_1^2, \ldots, a_d^2$.

*Proof.* By Proposition 1·8 it is enough to show that

$$l_A(\mathrm{Hom}_A(A/\mathfrak{q}, \mathrm{H}^d_{\mathfrak{m}}(M))) = l_A(M/\Sigma(\mathfrak{q}; M)) + \sum_{i=1}^{d-1} \binom{d}{i-1} l_A(\mathrm{H}^i_{\mathfrak{m}}(M)).$$

By Matlis duality we immediately obtain $l_A(\mathrm{Hom}_A(A/\mathfrak{q}, \mathrm{H}^d_{\mathfrak{m}}(M))) = l_A(K_M/\mathfrak{q}K_M)$. Note that $K_M$ is a maximal Buchsbaum $A$-module by (1) of (3·2), and hence we know by (3) of (3·2) that

$$l_A(K_M/\mathfrak{q}K_M) = l_A(M/\mathfrak{q}M) - \mathrm{I}_A(M) + \mathrm{I}_A(K_M).$$

Looking at $M \supset \Sigma(\mathfrak{q}; M) \supset \mathfrak{q}M$ we get

$$l_A(\mathrm{Hom}_A(A/\mathfrak{q}, \mathrm{H}^d_{\mathfrak{m}}(M))) = l_A(M/\Sigma(\mathfrak{q}; M)) + l_A(\Sigma(\mathfrak{q}; M)/\mathfrak{q}M) - \mathrm{I}_A(M) + \mathrm{I}_A(K_M).$$

Combining this and the following equations:

$$\mathrm{I}_A(K_M) = \sum_{i=2}^{d-1} \binom{d-1}{i} . l_A(\mathrm{H}^{d+1-i}_{\mathfrak{m}}(M)),$$

$$\mathrm{I}_A(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} . l_A(\mathrm{H}^i_{\mathfrak{m}}(M)),$$

$$l_A(\Sigma(\mathfrak{q}; M)/\mathfrak{q}M) = \sum_{i=0}^{d-1} \binom{d}{i} . l_A(\mathrm{H}^i_{\mathfrak{m}}(M)),$$

(cf. (2) of (3·2), Remark 2 after (1·1) and equation (6)) we finally obtain the required equality and this finishes the proof of (3·8).

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COROLLARY 3·9. *Under the hypotheses of* (3·8) *one has*

$$\mu_A(K_M) = l_A(\Sigma(\underline{a}^2; M) : \mathfrak{m}/\Sigma(\underline{a}^2; M)); l_A(K_M/\mathfrak{q}.K_M) = l_A(\Sigma(\underline{a}^2; M) : \mathfrak{q}/\Sigma(\underline{a}^2; M)).$$

*Proof of Theorem* 3·3. The implication (1) $\Rightarrow$ (2) is given by (3·7) and (2·3).

(2) $\Rightarrow$ (3). Let $\mathfrak{q} := (a_1, \ldots, a_d)$ be a minimal reduction of $\mathfrak{m}$. By the assumption (2) we obviously get the following inclusion:

$$\Sigma(\mathfrak{q}).\left(\sum_{i=1}^{d} a_1 \ldots \hat{a_i} \ldots a_d . \Sigma(\mathfrak{q}; M)\right) = \sum_{i=1}^{d} a_1 \ldots \hat{a_i} \ldots a_d . \mathfrak{q} . \Sigma(\mathfrak{q}; M)$$
$$\subset a_1 \ldots a_d . \Sigma(\mathfrak{q}; M) + (\underline{a}^2) M \subset \Sigma(\underline{a}^2; M).$$

This implies by (3·8) that $\Sigma(\underline{a}^2; M) : \mathfrak{q} \subset \Sigma(\underline{a}^2; M) : \Sigma(\mathfrak{q})$, whence the required equality follows at once.

(3) $\Rightarrow$ (1). By (3·8) we already know that $\mathrm{Hom}_A(A/\mathfrak{q}, \mathrm{H}^d_{\mathfrak{m}}(M))$ is isomorphic to $\mathrm{Hom}_A(A/\mathfrak{q}, M/\Sigma(\underline{a}^2; M))$. Hence the assumption (3) shows that

$$[0 : \mathfrak{q}]_{\mathrm{H}^d_{\mathfrak{m}}(M)} = [0 : \Sigma(\mathfrak{q})]_{\mathrm{H}^d_{\mathfrak{m}}(M)},$$

and by (3·6) we therefore have $\Sigma(\mathfrak{q}).K_M = \mathfrak{q}.K_M$. Finally by (2·3) we conclude that $A \ltimes K_M$ is a Buchsbaum ring and this completes the proof of Theorem 3·3.

Before proving the corollaries we remark that in general when $\dim_A M = \dim A$ one has the following inequalities:

$$\mu_A(K_A) + \mu_A(K_M) \geqslant \mu_{A \ltimes M}(K_{A \ltimes M}) \geqslant \mu_A(K_M). \tag{19}$$

In fact, according to Korollar 5·14 in Vortrag 5 of [11], we have an $A \ltimes M$-isomorphism such that $K_{A \ltimes M} \cong \mathrm{Hom}_A(A \ltimes M, K_A)$. It is easy to see that there is an $A$-isomorphism

$$\mathrm{Hom}_A(A \ltimes M, K_A) \xrightarrow{\cong} \mathrm{Hom}_A(M, K_A) \oplus K_A$$

defined by $\alpha \mapsto (\alpha \circ \epsilon, \alpha((1, 0)))$ for each $\alpha \in \mathrm{Hom}_A(A \ltimes M, K_A)$. Applying in a similar way the argument described in the next section (see the remark before (4·2)) we can make this $A$-module $\mathrm{Hom}_A(M, K_A) \oplus K_A$ into an $A \ltimes M$-module, say

$$\mathrm{Hom}_A(M, K_A) \overset{\phi}{\ltimes} K_A,$$

with respect to the $A$-linear map $\phi : M \otimes_A \mathrm{Hom}_A(M, K_A) \longrightarrow K_A$ such that $\phi(x \otimes f) = f(x)$ for each $x \in M$ and $f \in \mathrm{Hom}_A(M, K_A)$. We have therefore

$$K_{A \ltimes M} \cong \mathrm{Hom}_A(M, K_A) \overset{\phi}{\ltimes} K_A$$

as $A \ltimes M$-modules, and

$$(\mathfrak{m} \times M).K_{A \ltimes M} \cong \mathfrak{m}.\mathrm{Hom}_A(M, K_A) \times (\mathfrak{m}.K_A + \mathrm{Im}(\phi)).$$

As $K_M \cong \mathrm{Hom}_A(M, K_A)$ we obtain

$$\begin{aligned} \mu_{A \ltimes M}(K_{A \ltimes M}) &= l_{A \ltimes M}(K_{A \ltimes M}/(\mathfrak{m} \times M).K_{A \ltimes M}) \\ &= l_A(K_M/\mathfrak{m}.K_M) + l_A(K_A/(\mathfrak{m}.K_A + \mathrm{Im}(\phi))) \\ &= \mu_A(K_M) + \mu_A(K_A) - l_A((\mathfrak{m}.K_A + \mathrm{Im}(\phi))/\mathfrak{m}.K_A). \end{aligned}$$

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Since $\mu_A(K_A) \geqslant l_A((\mathfrak{m}.K_A + \mathrm{Im}(\phi))/\mathfrak{m}.K_A)$ we finally get the required inequalities (19). Moreover if the map $\phi$ is surjective then $\mu_{A \ltimes M}(K_{A \ltimes M}) = \mu_A(K_M)$ holds.

*Proof of the Corollaries.* Let $\mathfrak{q} = (a_1, \ldots, a_d)$ be a minimal reduction of $\mathfrak{m}$.

For (3·4), arguing as in the proof of theorem 4·1 of [**4**] we have

$$e_{\mathfrak{q}}(M) = l_A(M/\Sigma(\mathfrak{q};M)) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} l_A(\mathrm{H}^i_{\mathfrak{m}}(M)).$$

As $e_A(M) = e_{\mathfrak{q}}(M)$ the assumption implies $l_A(M/\Sigma(\mathfrak{q};M)) = 0$, whence $M = \Sigma(\mathfrak{q};M)$. Thus $\Sigma(\mathfrak{q}).M = \mathfrak{q}M$ if and only if $\Sigma(\mathfrak{q}).\Sigma(\mathfrak{q};M) = \mathfrak{q}.\Sigma(\mathfrak{q};M)$, therefore our equivalence follows from (3·3) and (2·3). Next, assume that $A \ltimes M$ is a Buchsbaum ring. First, we will prove the following equality:

$$[\Sigma(\underline{a}^2):\mathfrak{m}] \times [\Sigma(\underline{a}^2;M):\mathfrak{m}] = \Sigma(\underline{a}^2;A \ltimes M):\mathfrak{m} \times M. \tag{20}$$

In fact, from our assumption we see that $\Sigma(\mathfrak{q}).M = \mathfrak{q}M$ by (2·3), and hence

$$\begin{aligned}[\Sigma(\underline{a}^2):\mathfrak{m}].M &\subset [\Sigma(\underline{a}^2):\mathfrak{q}].M \\ &\equiv \left\{\sum_{i=1}^{d} a_1 \ldots \hat{a}_i \ldots a_d.\Sigma(\mathfrak{q})\right\}.M \\ &\equiv a_1 \ldots a_d.M \equiv a_1 \ldots a_d.\Sigma(\mathfrak{q};M) \equiv (0)\end{aligned}$$

mod $\Sigma(\underline{a}^2;M)$ by (3·8). This implies that the $A \ltimes M$-module in the left-hand side of (20) is annihilated by the maximal ideal $\mathfrak{m} \times M$. Then the left-hand side of (20) is contained in the right-hand side and the converse is obvious, so we get the required equality. This equality and (3·9) yield the last assertion of (3·4).

For the proof of (3·5) we first remark that

$$[\Sigma(\underline{a}^2):\mathfrak{m}] + a_1 \ldots a_d.A \subset \Sigma(\underline{a}^2):\mathfrak{q}.$$

As $\mu_A(K_A) = l_A(\Sigma(\underline{a}^2):\mathfrak{m}/\Sigma(\underline{a}^2))$ and as $e_A(K_A) + \mathrm{I}_A(K_A) = l_A(\Sigma(\underline{a}^2):\mathfrak{q}/\Sigma(\underline{a}^2))$ by (3·9), our standard assumption means that one has only two cases: either $\Sigma(\underline{a}^2):\mathfrak{m} \ni a_1 \ldots a_d$ or the equality $\Sigma(\underline{a}^2):\mathfrak{q} = [\Sigma(\underline{a}^2):\mathfrak{m}] + a_1 \ldots a_d.A$ holds. In the first case, by equation (3) we know that $\mathfrak{m} \subset \Sigma(\mathfrak{q})$, and hence $\mathfrak{m} = \Sigma(\mathfrak{q})$. Thus

$$\Sigma(\mathfrak{q}).\Sigma(\mathfrak{q}) = \mathfrak{m}.\Sigma(\mathfrak{q}) \subset \mathfrak{q} \cap \mathfrak{m}^2 \subset \mathfrak{q}\mathfrak{m} = \mathfrak{q}.\Sigma(\mathfrak{q}),$$

therefore by (3·3) $A \ltimes K_A$ is a Buchsbaum ring. In the second case, one has $\Sigma(\underline{a}^2):\mathfrak{q} = \Sigma(\underline{a}^2):\Sigma(\mathfrak{q})$, whence $A \ltimes K_A$ is a Buchsbaum ring, by (3·3) again.

In the rest of this section we shall give examples which satisfy the equivalent conditions of Theorem 3·3.

*Example* 3·10 ([**9**]). Let $(R, \mathfrak{n})$ be a complete Cohen–Macaulay local ring of dimension $d > 0$ and $E$ a maximal Buchsbaum $R$-module. Put $A := R \ltimes E$. Then $A \ltimes K_A$ is a Buchsbaum ring.

In fact, by (2·4), obviously $A$ is a Buchsbaum ring. Let $\mathfrak{q} = (a_1, \ldots, a_d)$ be a minimal reduction of $\mathfrak{m}$ and put $a_i = (r_i, e_i)$ for each $i$. Then it is easy to check that

$$\Sigma(\mathfrak{q}) = \mathfrak{q} + ((0) \times \Sigma(\underline{r};E)),$$

where $\underline{r} := r_1, \ldots, r_d$. Since $((0) \times E)^2 = (0)$ we immediately get that $\Sigma(\mathfrak{q}).\Sigma(\mathfrak{q}) = \mathfrak{q}.\Sigma(\mathfrak{q})$, and therefore by (3·3) $A \ltimes K_A$ is a Buchsbaum ring.

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By (4·11) of [4] we can construct Buchsbaum rings which satisfy the equivalent conditions of Theorem 3·3 and, moreover, whose local cohomology modules have given arbitrary lengths.

*Example* 3·11. Let $(R, \mathfrak{n})$ be a complete regular local ring with $\dim R = d > 0$ and let $E_i$ $(0 \leqslant i \leqslant d)$ denote the $i$th syzygy module of the residue field $R/\mathfrak{n}$ over $R$; in particular $E_0 = R/\mathfrak{n}$ and $E_d = R$. Let $h_0, \ldots, h_d$ be non-negative integers and put

$$A := R \ltimes (E_0^{h_0} \oplus \ldots \oplus E_d^{h_d}).$$

Then, by (2) of (4·11) in [4] and by (2·11), $A$ is a Buchsbaum ring with maximal embedding dimension and with $l_A(\mathrm{H}^i_{\mathfrak{m}}(A)) = h_i$ for $i = 0, \ldots, d-1$. Moreover the following are true:

(i) $A \ltimes K_A$ is a Buchsbaum ring, and if $h_0 = h_1 = 0$ it is also quasi-Gorenstein (i.e. $K_B \cong B$, where $B := A \ltimes K_A$);

(ii) $\mu_A(K_A) = e_A(K_A) + \mathrm{I}_A(K_A) - 1$ if $h_d > 0$.

In fact, by (3·10) (also see (2·11)) and by (2·11) in [1], we get the assertion (i). Put $E := \bigoplus_{i=0}^{d} E_i^{h_i}$. If $h_d > 0$, then the canonical map $E \otimes \mathrm{Hom}_R(E, R) \longrightarrow R$ is surjective, and hence by the claim after the proof of Theorem 3·3 we get

$$\mu_A(K_A) = \mu_R(K_E) = e_R(K_E) + \mathrm{I}_R(K_E) = e_A(K_A) + \mathrm{I}_A(K_A) - 1.$$

Most known Buchsbaum rings satisfy the equivalent conditions of Theorem 3·3. In fact we have

*Example* 3·12. Let $S = \bigoplus_{n \geqslant 0} S_n$ be a homogeneous Noetherian graded ring over an infinite field $k$ with $\dim S = d > 0$, $S = S_0[S_1]$ and $S_0 = k$, and let $\mathfrak{M}$ denote the unique homogeneous maximal ideal of $S$, i.e. $\mathfrak{M} = \bigoplus_{n > 0} S_n$. For a graded $S$-module $E = \bigoplus_{n \in \boldsymbol{Z}} E_n$, we write $[E]_n$ for the homogeneous component of $E$ with degree $n$, i.e. $[E]_n = E_n$, and $E(r)$ denotes the graded $S$-module given by the shifting of $E$ of order $r$, i.e. $[E(r)]_n = E_{r+n}$ for each $n$.

Let $r > 0$ be an integer and suppose that $S$ satisfies the following conditions:

(*a*) $[\mathrm{H}^i_{\mathfrak{M}}(S)]_n = (0)$ $(n \neq r-i)$ for $i = 0, \ldots, d-1$;

(*b*) $[\mathrm{H}^d_{\mathfrak{M}}(S)]_n = (0)$ for all $n > r-d$.

(See §5 of [5], §3 of [21] and [22] for examples of such rings.) Put $A := (S_{\mathfrak{M}})^{\wedge}$, the completion of $S_{\mathfrak{M}}$. Then we have the following statements.

(i) $A \ltimes K_A$ is a Buchsbaum ring if $r \geqslant 2$; cf. [9].

(ii) Suppose $[\mathrm{H}^d_{\mathfrak{M}}(S)]_{r-d} = (0)$. Then the equality

$$\mu_A(K_A) = e_A(K_A) + \mathrm{I}_A(K_A) - l_S(S/\Sigma(\underline{x}) : \mathfrak{M})$$

holds for any linear system of parameters $\underline{x}$ for $S$.

In fact choose a linear system of parameters $\underline{x} := x_1, \ldots, x_d$ for $S$. We claim that

$$\Sigma(\underline{x}) = (\underline{x}) + [\Sigma(\underline{x})]_r, \quad \text{and} \quad \Sigma(\underline{x}) \supset S_n \quad \text{for all} \quad n > r.$$

As $[\mathrm{H}^0_{\mathfrak{M}}(S/(x_1, \ldots, \hat{x_i}, \ldots, x_d))]_n = (0)$ for $n \neq r$ by our assumption (*a*), it follows for each $i$ that

$$[(x_1, \ldots, \hat{x_i}, \ldots, x_d) : x_i/(x_1, \ldots, \hat{x_i}, \ldots, x_d)]_n = (0) \quad (n \neq r).$$

By considering the inclusion $(S/\Sigma(\underline{x}))(d) \longrightarrow \mathrm{H}^d_{\mathfrak{M}}(S)$ (see (1·3)), the assumption (*b*) implies $[S/\Sigma(\underline{x})]_n = (0)$ for all $n > r$. Now let $u, v$ be homogeneous elements of $\Sigma(\underline{x})$

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of degree $r$. Clearly $uv$ is contained in $(\underline{x})$, say $uv = x_1 w_1 + \ldots + x_d w_d$ with $w_i \in S_{2r-1}$. If $r \geqslant 2$ we have $2r-1 > r$, and hence each element $w_i$ belongs to $\Sigma(\underline{x})$ from the above claim. Therefore $\Sigma(\underline{x}).\Sigma(\underline{x}) = (\underline{x}).\Sigma(\underline{x})$, and by (3·3) $A \ltimes K_A$ is a Buchsbaum ring.

Next we evaluate

$$l_S([0:(\underline{x})]_{S/\Sigma(\underline{x}^2)}) - l_S([0:\mathfrak{M}]_{S/\Sigma(\underline{x}^2)}). \tag{21}$$

To do this we look at the following inclusions (cf. (1·3)):

$$(S/\Sigma(\underline{x}))(d) \longrightarrow (S/\Sigma(\underline{x}^2))(2d) \longrightarrow \mathrm{H}^d_{\mathfrak{M}}(S). \tag{22}$$

The assumption that $[\mathrm{H}^d_{\mathfrak{M}}(S)]_{r-d} = (0)$ implies by the above claim that $[\Sigma(\underline{x})]_r = S_r$ and $\Sigma(\underline{x}) = (\underline{x}) + S_r$. Thus by (3·8) we have

$$\Sigma(\underline{x}^2):(\underline{x}) \equiv x_1 \ldots x_d.S + \left\{\sum_{i=1}^{d} x_1 \ldots \hat{x}_i \ldots x_d.S_r\right\} \mod \Sigma(\underline{x}^2). \tag{23}$$

As

$$S_1.S_r \subset \Sigma(\underline{x}) \cap S_{r+1} = (\underline{x}) \cap S_{r+1}$$

we see that

$$\left\{\sum_{i=1}^{d} x_1 \ldots \hat{x}_i \ldots x_d.S_r\right\}.S_1 \equiv x_1 \ldots x_d.S_r \equiv x_1 \ldots x_d.\Sigma(\underline{x}) \equiv (0) \mod \Sigma(\underline{x}^2),$$

and so we get

$$\left\{\sum_{i=1}^{d} x_1 \ldots \hat{x}_i \ldots x_d.S_r\right\} \subset \Sigma(\underline{x}^2):\mathfrak{M}.$$

So by (22) and (23) we can evaluate (21) as follows:

$$\begin{aligned} l_S([0:(\underline{x})]_{S/\Sigma(\underline{x}^2)}) - l_S([0:\mathfrak{M}]_{S/\Sigma(\underline{x}^2)}) &= l_S([0:(\underline{x})]_{S/\Sigma(\underline{x})}) - l_S([0:\mathfrak{M}]_{S/\Sigma(\underline{x})}) \\ &= l_S(S/\Sigma(\underline{x}):\mathfrak{M}). \end{aligned}$$

Therefore we obtain by (3·9) that

$$\mu_A(K_A) = l_A(K_A/(\underline{x})K_A) - l_S(S/\Sigma(\underline{x}):\mathfrak{M}).$$

Since $l_A(K_A/(\underline{x}K_a) = e_A(K_A) + I_A(K_A)$ we finally have the required equality of (ii).

*Example* 3·13. Suppose that a Buchsbaum ring $A$ is quasi-Gorenstein (i.e. $K_A \cong A$) and not Cohen–Macaulay. Then by (2) of (2·4) $A \ltimes K_A$ is not a Buchsbaum ring.

To close this section we shall deal with idealizations of the $S_2$-fications. For a finitely generated $A$-module $E$ such that $l_A(\mathrm{H}^i_{\mathfrak{m}}(E))$ is finite for all $i \neq \dim_A E$ and $\mathrm{depth}_A E > 0$, we call the $A$-module $E^{DD}$ (see Definition 3·1) the *$S_2$-fication* of $E$ and denote it by $\tilde{E}$ (a more general argument can be found in §5 of [**8**]). Then recalling Remark 3·2 we get the following observations:

($a$) there is an exact sequence of $A$-modules such that

$$0 \longrightarrow E \longrightarrow \tilde{E} \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(E) \longrightarrow 0;$$

($b$) if $E$ is a Buchsbaum $A$-module then so is $\tilde{E}$, and when this is the case one has $\mathrm{I}_A(\tilde{E}) = \mathrm{I}_A(E) + (\dim_A E - 1).l_A(\mathrm{H}^1_{\mathfrak{m}}(E))$;

($c$) $e_A(\tilde{E}) = e_A(E)$.

As a consequence of (3·3) we immediately have the following result.

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COROLLARY 3·14. *Under the hypotheses of Theorem* 3·3 *let us assume that* $\operatorname{depth}_A M > 0$. *Then* $A \ltimes \tilde{M}$ *is a Buchsbaum ring if and only if*

$$\Sigma(\mathfrak{q}).\Sigma(\mathfrak{q}; K_M) = \mathfrak{q}.\Sigma(\mathfrak{q}; K_M)$$

*holds for every minimal reduction* $\mathfrak{q}$ *of* $\mathfrak{m}$.

*Example* 3·15. Let $(R, \mathfrak{n})$ be a complete Buchsbaum ring of dimension $d > 0$. Put $A := R \ltimes K_R$. Then $A \ltimes K_A$ is a Buchsbaum ring if and only if $K_R$ is a Cohen–Macaulay $R$-module, i.e. if $\mathrm{H}^i_{\mathfrak{n}}(R) = (0)$ for $i = 2, \ldots, d-1$.

First we shall prove the 'only if' part. By (ii) of (2·5), we may assume that depth $R > 0$. Let $\mathfrak{r}$ be a minimal reduction of $\mathfrak{n}$. Suppose that $A \ltimes K_A$ is a Buchsbaum ring. By (3·3), we have $\Sigma(\mathfrak{r}A).\Sigma(\mathfrak{r}A) = \mathfrak{r}A.\Sigma(\mathfrak{r}A)$, so in particular $\Sigma(\mathfrak{r}).\Sigma(\mathfrak{r}; K_R) = (\mathfrak{r}).\Sigma(\mathfrak{r}; K_R)$. This implies by (3·14) that $R \ltimes \tilde{R}$ is a Buchsbaum ring. Applying (4·5) described in the next section, we conclude that $K_R$ is Cohen–Macaulay. The converse is obvious by (2·4).

## 4. *Condition for idealizations to be quasi-Buchsbaum*

In this section we discuss the condition for idealizations to be quasi-Buchsbaum. In particular, when the given local ring and module are Buchsbaum, we can give much sharper conditions; such conditions appear in [**9**]. Throughout this section $A$ is a Noetherian local ring of dimension $d > 0$ and $M$ is a finitely generated $A$-module.

*Definition* 4·1 ([**23**, **25**]). We say that $M$ is a *quasi-Buchsbaum* $A$-module if every local cohomology module $\mathrm{H}^i_{\mathfrak{m}}(M)$ $(i \neq \dim_A M)$ is annihilated by the maximal ideal $\mathfrak{m}$. If the local ring $A$ is a quasi-Buchsbaum module over itself, then it is said to be a quasi-Buchsbaum ring.

According to [**25**] $M$ is a quasi-Buchsbaum $A$-module if and only if some (and hence every) system of parameters, say $a_1, a_2, \ldots, a_s$ $(s := \dim_A M)$, for $M$ contained in $\mathfrak{m}^2$ forms a weak $M$-sequence; i.e. the equality

$$(a_1, \ldots, a_{i-1})M : a_i = (a_1, \ldots, a_{i-1})M : \mathfrak{m}$$

holds for $i = 1, \ldots, s$. In [**6**] there were many examples of quasi-Buchsbaum rings which are not Buchsbaum and whose local cohomology modules have the required lengths.

To describe our condition under which idealizations are quasi-Buchsbaum, we extend the concept of the idealization. Given an $A$-linear map $\phi : M \otimes_A E \longrightarrow F$ we can make the Cartesian product $E \times F$ into an $A \ltimes M$-module with respect to componentwise addition and multiplication defined by

$$(a, x).(e, f) = (ae, af + \phi(x \otimes e)).$$

We denote this $A \ltimes M$-module by $E \overset{\phi}{\ltimes} F$. By general facts on additive and linear functors (cf. corollary 6·2 in [**10**]), there is a canonical $A$-homomorphism

$$\sigma^i_M : M \otimes_A \mathrm{H}^i_{\mathfrak{m}}(A) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(M) \tag{24}$$

for each $i \in \mathbb{Z}$.

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LEMMA 4·2. $\mathrm{H}^i_{\mathfrak{m}\times M}(A\ltimes M)\cong \mathrm{H}^i_{\mathfrak{m}}(A)\overset{\sigma^i_M}{\ltimes}\mathrm{H}^i_{\mathfrak{m}}(M)$ *as* $A\ltimes M$*-modules for all* $i\in\mathbb{Z}$.

*Proof.* The multiplication $A\ltimes M\xrightarrow{(a,x)}A\ltimes M$, where $(a,x)\in A\ltimes M$, can be expressed by a $2\times 2$ matrix

$$\begin{pmatrix} a.\mathrm{id}_A & 0 \\ \delta_x & a.\mathrm{id}_M \end{pmatrix},$$

whose entries are $A$-homomorphisms, where $\delta_x: A\longrightarrow M$ is given by $\delta_x(c)=cx$. Since $\mathrm{H}^i_{\mathfrak{m}\times M}(*)\cong \mathrm{H}^i_{\mathfrak{m}}(*)$ as connected sequences of functors from the category of all $A$-modules to itself, the multiplication $\mathrm{H}^i_{\mathfrak{m}\times M}(A\ltimes M)\xrightarrow{(a,x)}\mathrm{H}^i_{\mathfrak{m}\times M}(A\ltimes M)$ must be expressed as follows:

$$\begin{pmatrix} a.\mathrm{id}_{\mathrm{H}^i_{\mathfrak{m}}(A)} & 0 \\ \mathrm{H}^i_{\mathfrak{m}}(\delta_x) & a.\mathrm{id}_{\mathrm{H}^i_{\mathfrak{m}}(M)} \end{pmatrix}.$$

This is just the same as in our assertion.

This leads immediately to the next result.

PROPOSITION 4·3. *The idealization* $A\ltimes M$ *is a quasi-Buchsbaum ring if and only if the following four conditions are fulfilled*:

(i) $A$ *is a quasi-Buchsbaum ring*;
(ii) $M$ *is a quasi-Buchsbaum* $A$*-module*;
(iii) $\dim_A M=\dim A$;
(iv) *the map* $\sigma^i_M$ *defined in* (24) *is a zero map for every* $i\neq d$.

COROLLARY 4·4. *Let* $A$ *be a quasi-Buchsbaum ring and* $M$ *a quasi-Buchsbaum* $A$*-module with* $\dim_A M=\dim A$. *If for each* $i\in\{0,\ldots,d-1\}$ *either* $\mathrm{H}^i_{\mathfrak{m}}(A)=(0)$ *or* $\mathrm{H}^i_{\mathfrak{m}}(M)=(0)$, *then* $A\ltimes M$ *is a quasi-Buchsbaum ring.*

COROLLARY 4·5. *Let* $A$ *be a complete quasi-Buchsbaum ring with* $\operatorname{depth} A>0$. *Then the following are equivalent*:

(1) $A\ltimes\tilde{A}$ *is a Buchsbaum ring*;
(2) $A\ltimes\tilde{A}$ *is a quasi-Buchsbaum ring*;
(3) $\mathrm{H}^i_{\mathfrak{m}}(A)=(0)$ *for* $i=2,\ldots,d-1$.

*When these conditions hold,* $\tilde{A}$ *is a Cohen–Macaulay* $A$*-module.*

*Proof.* The implication (1) $\Rightarrow$ (2) is obvious. Consider the following commutative diagram for $i\geqslant 2$:

$$\begin{array}{ccc} A\otimes_A \mathrm{H}^i_{\mathfrak{m}}(A) & \xrightarrow[\cong]{\sigma^i_A} & \mathrm{H}^i_{\mathfrak{m}}(A) \\ \downarrow & & \downarrow\cong \\ \tilde{A}\otimes_A \mathrm{H}^i_{\mathfrak{m}}(A) & \xrightarrow{\sigma^i_{\tilde{A}}} & \mathrm{H}^i_{\mathfrak{m}}(\tilde{A}). \end{array}$$

If $A\ltimes\tilde{A}$ is a quasi-Buchsbaum ring, then by (4·3) all the maps $\sigma^i_{\tilde{A}}$ $(i\neq d)$ are zero, and hence it follows that $\mathrm{H}^i_{\mathfrak{m}}(A)=(0)$ for $i=2,\ldots,d-1$. Therefore we get (2) $\Rightarrow$ (3). The assertion (3) implies that $A$ is a Buchsbaum ring (cf. Remark 4 after (1·1))

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and $H^i_{\mathfrak{m}}(\tilde{A}) = (0)$ for all $i \neq d$, and hence by (2·4) $A \ltimes \tilde{A}$ is a Buchsbaum ring, so that (3) $\Rightarrow$ (1).

However, if both $A$ and $M$ are Buchsbaum, we have a much simpler condition under which idealizations are quasi-Buchsbaum:

PROPOSITION 4·6 ([9]). *Let $A$ be a Buchsbaum ring and $M$ a maximal Buchsbaum $A$-module. Then $A \ltimes M$ is a quasi-Buchsbaum ring if and only if there is a parameter ideal $\mathfrak{q}$ of $A$ such that $\Sigma(\mathfrak{q}).M = \mathfrak{q}.M$. When this is the case, $\Sigma(\mathfrak{q}).M = \mathfrak{q}.M$ holds for every parameter ideal $\mathfrak{q}$ of $A$ which is contained in $\mathfrak{m}^2$.*

*Proof.* First we prove the 'if' part. Let $\mathfrak{q} = (a_1, \ldots, a_d)$ be a parameter ideal of $A$ such that $\Sigma(\mathfrak{q}).M = \mathfrak{q}.M$. We claim that this implies $\Sigma(\underline{a}^2).M = (\underline{a}^2).M$, where $\underline{a}^2 := a_1^2, \ldots, a_d^2$. In fact $\Sigma(\underline{a}^2)$ satisfies

$$\Sigma(\underline{a}^2) = \sum_{I \subsetneq \{1,\ldots,d\}} a_I.[\mathfrak{q}_I : \mathfrak{m}],$$

where $a_I := \prod_{i \in I} a_i$ and $\mathfrak{q}_I := (a_i \mid i \in I)$ for each $I$, and in particular $a_\phi := 1$ and $\mathfrak{q}_\phi := (0)$ (see (3·4) in [4], and also equation (1) in the proof of (1·2)). The assumption that $\Sigma(\mathfrak{q}).M = \mathfrak{q}.M$ implies that $[\mathfrak{q}_I : \mathfrak{m}].M = \mathfrak{q}_I.M$ by (2·2), and hence

$$\Sigma(\underline{a}^2).M = \sum_I a_I.\mathfrak{q}_I M = (\underline{a}^2).M.$$

Thus, invoking (2·2) again, we see that $a_1^2, \ldots, a_d^2$ form a weak $A \ltimes M$-sequence which is contained in $(\mathfrak{m} \times M)^2$, whence $A \ltimes M$ is a quasi-Buchsbaum ring. The 'only if' part and the last assertion are shown similarly.

Theorem 3·3 and (4·6) above also yield the following result.

COROLLARY 4·7. *Let $A$ be a complete Buchsbaum ring and $M$ a maximal Buchsbaum $A$-module. Then $A \ltimes K_M$ is a quasi-Buchsbaum ring if and only if there exists a parameter ideal $\mathfrak{q}$ of $A$ such that $\Sigma(\mathfrak{q}).\Sigma(\mathfrak{q}; M) = \mathfrak{q}.\Sigma(\mathfrak{q}; M)$; and when this is the case, then $\Sigma(\mathfrak{q}).\Sigma(\mathfrak{q}; M) = \mathfrak{q}.\Sigma(\mathfrak{q}; M)$ for every parameter ideal $\mathfrak{q}$ of $A$ which is contained in $\mathfrak{m}^2$.*

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