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ON A CLASS OF GENERALIZED MARSHALL–OLKIN BIVARIATE DISTRIBUTIONS AND SOME RELIABILITY CHARACTERISTICS

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We consider here a general class of bivariate distributions from reliability point of view, and refer to it as generalized Marshall–Olkin bivariate distributions. This class includes as special cases the Marshall–Olkin bivariate exponential distribution and the class of bivariate distributions studied recently by Sarhan and Balakrishnan [25]. For this class, the reliability, survival, hazard, and mean residual life functions are all derived, and their monotonicity is discussed for the marginal as well as the conditional distributions. These functions are also studied for the series and parallel systems based on this bivariate distribution. Finally, the Clayton association measure for this bivariate model is derived in terms of the hazard gradient.

1. INTRODUCTION

In their pioneering paper, Marshall and Olkin [22] developed a bivariate exponential distribution in the following way. Suppose a system consists of two components. The system is subjected to three exponential independent shocks U_1, U_2, and U_3 with failure rates λ_1, λ_2, and λ_{12}. Component 1 fails when either of shock 1 or shock 3 occurs, while Component 2 fails when either of shock 2 or shock 3 occurs. Thus, the random life length X_1 of Component 1 is such that $X_1 = \min(U_1, U_3)$, while the random life length of Component 2 is such that $X_2 = \min(U_2, U_3)$. Evidently, the joint survival function of (X_1, X_2) is given by

$$\overline{F}_{X_1,X_2}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2) \tag{1.1}$$

$$= \exp[-\lambda_1 x_1 + \lambda_2 x_2 + \lambda_{12} \max(x_1, x_2)] \tag{1.2}$$

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for $x_1, x_2 > 0, \lambda_1, \lambda_2, \lambda_{12} > 0$. The survival function in (1.1) is the so-called Marshall–Olkin bivariate exponential distribution; see Kotz, Balakrishnan, and Johnson [17] and Balakrishnan and Lai [2].

Recently, Sarhan and Balakrishnan [25] used the above construction by taking the distributions of U_1 and U_2 as generalized exponential distributions having cumulative distribution functions

$$F_{U_1}(u_1) = (1 - e^{-u_1})^{\theta_1}, \quad u_1 > 0, \theta_1 > 0$$

and

$$F_{U_2}(u_2) = (1 - e^{-u_2})^{\theta_2}, \quad u_2 > 0, \theta_2 > 0.$$

For various properties of generalized exponential distributions, one may refer to Gupta and Kundu [11,12]. Some bivariate generalized exponential distributions and associated inferential issues have been discussed by Kundu and Gupta [18,19]. Multivariate extensions of Sarhan and Balakrishnan's [25] model have been studied by Franco and Vivo [7] and Franco, Kundu, and Vivo [6]. While these authors have assumed U_1 andU_2 to have identical baseline exponential distribution, we consider here U_1 and U_2 to have general distributions.

In the present work, we assume that U_1, U_2, and U_3 have any continuous reliability functions $\overline{G}_1(t), \overline{G}_2(t)$, and $\overline{G}_3(t)$, respectively. Then, the survival function of (X_1, X_2) is given by

$$\overline{F}_{X_1,X_2}(x_1, x_2) = \overline{G}_1(x_1)\overline{G}_2(x_2)\overline{G}_3(\max(x_1, x_2)), \quad x_1 \geq 0, x_2 \geq 0. \tag{1.3}$$

We refer to this distribution of (X_1, X_2) as the Generalized Marshall–Olkin (GMO) bivariate distribution. Clearly, GMO is the most general continuous bivariate distribution retaining the feature of three independent shocks, one representing the common cause of failure acting on a two-component system and another two shocks acting separately on the two components in an independent manner.

In this paper, we study some properties of reliability, hazard, and mean residual life functions of the marginal and conditional distributions of the marginal and conditional distributions of the GMO distribution in (1.3). We also study these functions for the series and parallel systems based on this general bivariate distribution.

2. JOINT RELIABILITY FUNCTION OF GMO DISTRIBUTION

In this section, we show that the reliability function of the GMO distribution can be decomposed as a mixture of bivariate absolutely continuous and singular reliability functions. For convenience, we shall write $\overline{F}(x_1, x_2)$ instead of $\overline{F}_{X_1,X_2}(x_1, x_2)$.

THEOREM 2.1: *For all $x_1, x_2 \geq 0$, we have*

$$\overline{F}(x_1, x_2) = \theta \overline{F}_s(x_1, x_2) + (1 - \theta)\overline{F}_{ac}(x_1, x_2), \tag{2.1}$$

where

$$\theta = \int_0^\infty \overline{G}_1(u)\overline{G}_2(u)dG_3(u),$$

$$\overline{F}_s(x_1, x_2) = \overline{G}_1(\max(x_1, x_2))\overline{G}_2(\max(x_1, x_2))\overline{G}_3(\max(x_1, x_2))$$

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and

$$\overline{F}_{ac}(x_1,x_2)=\begin{cases}\overline{F}_s(x_1,x_2), & x_1=x_2,\\ \overline{G}_1(x_1)\overline{G}_3(x_1)\left\{\frac{\overline{G}_2(x_2)-\theta\overline{G}_2(x_1)}{1-\theta}\right\}, & x_1>x_2,\\ \overline{G}_2(x_2)\overline{G}_3(x_2)\left\{\frac{\overline{G}_1(x_1)-\theta\overline{G}_1(x_2)}{1-\theta}\right\}, & x_1<x_2.\end{cases}$$

PROOF: For all $x_1,x_2\geq 0$, we can express

$$\begin{aligned}\overline{F}(x_1,x_2)&=P(X_1>x_1,X_2>x_2)\\&=P(U_1>x_1,U_2>x_2,U_3>\max(x_1,x_2))\\&=P(U_1>x_1,U_2>x_2,U_3>\max(x_1,x_2)|U_3<\min(U_1,U_2))\\&\quad\times P(U_3<\min(U_1,U_2))\\&\quad+P(U_1>x_1,U_2>x_2,U_3>\max(x_1,x_2)|U_3>\min(U_1,U_2))\\&\quad\times P(U_3>\min(U_1,U_2))\\&=\theta\overline{F}_s(x_1,x_2)+(1-\theta)\overline{F}_{ac}(x_1,x_2),\end{aligned}$$

where θ is

$$\begin{aligned}\theta&=P(U_3<\min(U_1,U_2))\\&=\int_0^\infty P(\min(U_1,U_2>u)dG_3(u)\\&=\int_0^\infty \overline{G}_1(u)\overline{G}_2(u)dG_3(u).\end{aligned}$$

Also,

$$\begin{aligned}\overline{F}_s(x_1,x_2)&=P(U_1>x_1,U_2>x_2,U_3>\max(x_1,x_2)|U_3<\min(U_1,U_2))\\&=P(U_1>x_1,U_2>x_2,U_3>\max(x_1,x_2)|U_1>U_3,U_2>U_3)\\&=P(U_1>\max(x_1,x_2),U_2>\max(x_1,x_2),U_3>\max(x_1,x_2))\\&=\overline{G}_1(\max(x_1,x_2))\overline{G}_2(\max(x_1,x_2))\overline{G}_3(\max(x_1,x_2)),\end{aligned}$$

and

$$\begin{aligned}\overline{F}_{ac}(x_1,x_2)&=\frac{\overline{F}(x_1,x_2)-\theta\overline{F}_s(x_1,x_2)}{1-\theta}\\&=\frac{1}{1-\theta}\{\overline{G}_1(x_1)\overline{G}_2(x_2)\overline{G}_3(\max(x_1,x_2))\\&\quad-\theta\,\overline{G}_1(\max(x_1,x_2))\overline{G}_2(\max(x_1,x_2))\overline{G}_3(\max(x_1,x_2))\}.\end{aligned}$$

This completes the proof of the theorem. ■

3. JOINT DENSITY FUNCTION OF GMO DISTRIBUTION

Due to the presence of the singular component $\overline{F}_s$ in (2.1), the GMO distribution does not have a density with respect to the two-dimensional Lebesgue measure even when the

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reliability functions $\overline{G}_1, \overline{G}_2$ and $\overline{G}_3$ are absolutely continuous. However, it does have a density with respect to the measure described below.

Let μ, μ_s and μ_{ac} be the measures such that

$$\mu([x_1,\infty)\times[x_2,\infty)) = \overline{F}(x_1,x_2),$$
$$\mu_s([x_1,\infty)\times[x_2,\infty)) = \overline{F}_s(x_1,x_2)$$

and

$$\mu_{ac}([x_1,\infty)\times[x_2,\infty)) = \overline{F}_{ac}(x_1,x_2).$$

Then, μ_{ac} is absolutely continuous with respect to the two-dimensional Lebesgue measure ν_2 so that

$$\begin{aligned}\overline{F}_{ac}(x_1,x_2) &= \mu_{ac}([x_1,\infty)\times[x_2,\infty)) \\ &= \int_{x_1}^{\infty}\int_{x_2}^{\infty}\left\{\frac{\partial^2}{\partial u\partial v}\overline{F}_{ac}(u,v)\right\}dudv.\end{aligned}$$

To handle μ_s, let $B^* = \{(x_1,x_2)\in R^2 | x_1 = x_2\}$ and $(A\cap B^*)_p$ be the projection of $A\cap B^*$ onto the x_1-plane. In the case when $A = [x_1,\infty)\times[x_2,\infty)]$, we have

$$(A\cap B^*)_p = [\max(x_1,x_2),\infty].$$

Now, let ν be a measure such that ν is absolutely continuous with respect to one-dimensional Lebesgue measure ν_1 and

$$\begin{aligned}\overline{F}_s(x_1,x_2) &= \mu_s([x_1,\infty)\times[x_2,\infty)) \\ &= \nu((A\cap B^*)_p), A = [x_1,\infty)\times[x_2,\infty) \\ &= \nu([\max(x_1,x_2),\infty)) \\ &= \int_{\max(x_1,x_2)}^{\infty}\left\{\frac{d}{du}\overline{F}_s(u,u)\right\}du \\ &= \int_{\max(x_1,x_2)}^{\infty}\left\{\frac{d}{du}\overline{G}_1(u)\overline{G}_2(u)\overline{G_3}(u)\right\}du.\end{aligned}$$

Note that $\nu_2(B^*) = 0$ and, hence $\mu_{ac}(B^*) = 0$. On the other hand, by using $(B^*)^{\mathrm{c}}$ to denote the complement of B^*, we find

$$\mu_s((B^*)^{\mathrm{c}} = \nu(((B^*)^{\mathrm{c}}\cap B^*)_p) = \nu(\phi) = 0,$$

where ϕ denotes the empty set. Thus, the measures μ_{ac} and μ_s are mutually singular. Since

$$\mu = \theta\mu_s + (1-\theta)\mu_{ac},$$

it then follows that the GMO distribution has density function (with respect to μ) given by

$$\begin{aligned}f(x_1,x_2) &= \theta\left\{\frac{d}{dx_1}\overline{F}_s(x_1,x_2)I_{B^*}(x_1,x_2)\right\} \\ &\quad + (1-\theta)\left\{\frac{\partial^2}{\partial x_1\partial x_2}\overline{F}_{ac}(x_1,x_2)\right\}\{1-I_{B^*}(x_1,x_2)\},\end{aligned}$$

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where $I_{B^*}(x_1, x_2)$ is the indicator function of B^*,

$$\frac{d}{dx_1}\overline{F}_s(x_1, x_1) = \frac{d}{dx_1}\{\overline{G}_1(x_1)\overline{G}_2(x_1)\overline{G}_3(x_1)\}$$

and

$$\frac{\partial^2}{\partial x_1 \partial x_2}\overline{F}_{ac}(x_1, x_2) = 0 \quad \text{if } x_1 = x_2.$$

It is easy to observe that

$$\int_{[x_1,\infty)}\int_{[x_2,\infty)} f(u, v)d\mu = \overline{F}_{X_1,X_2}(x_1, x_2)$$

for all x_1, x_2.

Our results here are quite general and are valid for any distributions for U_1, U_2 and U_3. However, we shall focus on the generalized exponential distributions for U_1 and U_2 and an exponential distribution for U_3, as done earlier by Sarhan and Balakrishnan [25].

4. RELIABILITY CHARACTERISTICS OF MARGINALS

In this section, we study various reliability characteristics of the marginal distributions.

4.1. Preliminaries

Let T be a non-negative random variable denoting the life length of a component having cumulative distribution function (c.d.f.) $F(t)$ with $F(0) = 0$ and probability density function (p.d.f.) $f(t)$. Then, the failure rate of T is given by $r(t) = f(t)/R(t)$, where $R(t) = 1 - F(t) = \overline{F}(t)$ is the reliability function of T. Let us assume that $f(t)$ is continuous and twice differentiable on $(0,\infty)$.

Let $h : R_+ \to R_+$ be a real-valued differentiable function. Then, $h(t)$ is said to be

(1) *increasing* if $h'(t) > 0$ for all t, and is denoted by I;
(2) *decreasing* if $h'(t) < 0$ for all t, and is denoted by D;
(3) *bathtub shaped* if $h'(t) < 0$ for $t \in (0, t_0)$, $h'(t_0) = 0$, $h'(t) > 0$ for $t > t_0$, and is denoted by B;
(4) *upside down bathtub shaped* if $h'(t) > 0$ for$t \in (0, t_0)$, $h'(t_0) = 0$, $h'(t) < 0$ for $t > t_0$, and is denoted by U.

For various properties of classes of distributions with hazard rates of these forms, interested readers may refer to Barlow, Marshall, and Proschan [4], Barlow and Proschan [3], Gupta [8], Gupta and Warren [13], and Marshall and Olkin [23].

4.2. Failure Rate Characteristics

The marginal reliability function of X_i is easily seen to be

$$\overline{F}_{X_i}(x_i) = \overline{G}_i(x_i)\overline{G}_3(x_i), \quad i = 1, 2,$$

and the corresponding p.d.f. is given by

$$f_{X_i}(x_i) = g_i(x_i)\overline{G}_3(x_i) + \overline{G}_i(x_i)g_3(x_i), \quad i = 1, 2,$$

where $g_i(x_i)$ is the pdf of $U_i (i = 1, 2)$ and $g_3(x_i)$ is the pdf of U_3.

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The failure rate of X_i can be expressed as

$$h_{X_i}(t) = h_{U_i}(t) + h_{U_3}(t), \quad i = 1, 2,$$

where $h_{U_i}(t)$ is the failure rate of $U_i (i = 1, 2)$ and $h_{U_3}(t)$ is the failure rate of U_3.

We shall now suppose that U_3 has an exponential distribution with failure rate θ_0, as done earlier by Sarhan and Balakrishnan [25]. Then, we have

$$h_{X_i}(t) = h_{U_i}(t) + \theta_0, \quad i = 1, 2. \tag{4.1}$$

Thus, in this case, the monotonicity of the failure rate of X_i solely depends on the monotonicity of the failure rate of U_i.

Gupta, Gupta, and Gupta [8] discussed the monotonicity of the failure rate for the Lehmann class of distributions, which includes exponentiated Weibull, exponentiated exponential, and exponentiated gamma distributions as special cases.

Suppose a random variable U has a Weibull distribution with reliability function

$$\overline{F}_U(t) = e^{-t^\alpha}, \quad t > 0, \quad \alpha > 0.$$

Then, the exponentiated Weibull distribution with reliability function $\{\overline{F}_U(t)\}^\theta$, which includes the exponentiated exponential distribution as a special case, has the following failure rate characteristics:

(a) IFR if $\alpha \geq 1$, $\alpha\theta \geq 1$;
(b) DFR if $\alpha \leq 1$, $\alpha\theta \leq 1$;
(c) *Type B* if $\alpha > 1$, $\alpha\theta < 1$;
(d) *Type U* if $\alpha < 1$, $\alpha\theta > 1$;
(e) IFR if $\alpha = 1$, $\theta > 1$ (which is the exponentiated exponential distribution);
(f) DFR if $\alpha = 1$, $\theta < 1$ (which is the exponentiated exponential distribution).

4.3. Mean Residual Life Function

The mean residual life function (MRLF) of a random variable X is defined as

$$\mu(t) = E(X - t | X > t) = \frac{\int_t^\infty \overline{F}(u)du}{\overline{F}(t)},$$

where $\overline{F}(t)$ is the reliability function of X. Then, corresponding to the model in (4.1), we have the MRLF of X_i as

$$\begin{aligned} \mu_{X_i}(t) &= \frac{\int_t^\infty \overline{G}_i(u)e^{-\theta_0 u}du}{\overline{F}_{X_i}(t)} \\ &= \frac{1}{\theta_0} - \frac{\frac{1}{\theta_0}\int_t^\infty e^{-\theta_0 u} f_{U_i}(u)du}{\overline{G}_{U_i}(t)e^{-\theta_0 t}}, \quad i = 1, 2. \end{aligned} \tag{4.2}$$

In particular, we have

$$E(X_i) = \frac{1}{\theta_0} - \frac{1}{\theta_0}\int_0^\infty e^{-\theta_0 u} f_{U_i}(u)du, \quad i = 1, 2.$$

We now consider two examples. The second example deals with the generalized exponential distributions for U_1 and U_2 while the first example deals with the class of linear

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hazard rate functions which has been studied by various authors including Bain [1], Sen and Bhattacharya [27], Lin, Wu, and Balakrishnan [20], and Sarhan and Kundu [26].

Example 4.1: Suppose U_i has linear failure rate given by $h_i(t) = a_i + b_i t (i = 1, 2)$ and U_3 has a constant failure rate θ_0. Then,

$$\begin{aligned}\mu_{X_i}(t) &= \frac{\int_t^\infty e^{-\theta_0 u} e^{-a_i u - b_i u^2/2}}{e^{-(\theta_0 + a_i t) - b_i t^2/2}} \\ &= \frac{1 - \Phi(\sqrt{b_i}t + (\theta_0 + a_i)/\sqrt{b_i})}{\sqrt{b_i}\phi((\sqrt{b_i}t + (\theta_0 + a_i)/\sqrt{b_i})}, \quad i = 1, 2. \end{aligned} \tag{4.3}$$

This gives

$$E(X_i) = \frac{1 - \Phi((\theta_0 + a_i)/\sqrt{b_i})}{\sqrt{b_i}\phi((\theta_0 + a_i)/\sqrt{b_i})}, \quad i = 1, 2, \tag{4.4}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of a standard normal variable.

Example 4.2: If U_1 and U_2 have generalized exponential distributions and U_3 has constant failure rate θ_0, then the MRLF is given by

$$\begin{aligned}\mu_{X_i}(t) &= \frac{\int_t^\infty e^{-\theta_0 u}[1 - (1 - e^{-u})^{\theta_i}]du}{\overline{F}_{X_i}(t)} \\ &= \frac{(1/\theta_0)e^{-\theta_0 t} - \beta_{e^{-t}}(\theta_0, \theta_i + 1)}{e^{-\theta_0}[1 - (1 - e^{-t})^{\theta_i}}, \quad i = 1, 2, \end{aligned} \tag{4.5}$$

where

$$\beta_u(a, b) = \int_0^u v^{a-1}(1 - v)^{b-1} dv$$

is the incomplete beta function.

This gives

$$E(X_i) = \frac{1}{\theta_0} - \beta(\theta_0, \theta_i + 1), \quad i = 1, 2, \tag{4.6}$$

where $\beta(a, b)$ is the complete beta function.

The monotonicity of the MRLF can be studied by using the following result of Gupta and Akman [10].

THEOREM 4.1: *Suppose X is a random variable with failure rate $r(t)$ and MRLF $\mu(t)$. Then:*

(a) If $r(t)$ is IFR, $\mu(t)$ is DMRL (decreasing mean residual life);

(b) If $r(t)$ is DFR, $\mu(t)$ is IMRL (increasing mean residual life);

(c) Suppose $r(t) \in B$ and $\mu = \mu(0)$. Then,

(i) if $\mu r(0) \leq 1$, then $\mu(t) \in D$;

(ii) if $\mu r(0) > 1$, then $\mu(t) \in U$;

(d) Suppose $r(t) \in U$ and $\mu = \mu(0)$. Then,

(i) if $\mu r(0) \geq 1$, then $\mu(t) \in I$;

(ii) if $\mu r(0) < 1$, then $\mu(t) \in B$.

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5. RELIABILITY CHARACTERISTICS OF CONDITIONALS

5.1. Conditional Reliability Functions and Failure Rates

The survival function of X_1, given $X_2 > x_2$, is given by

$$\overline{F}_{X_1|(X_2>x_2)}(x_1|X_2 > x_2) = \frac{\overline{G}_1(x_1)\overline{G}_3(x_1 \vee x_2)}{\overline{G}_2(x_2)},$$

where $x_1 \vee x_2 = \max(x_1, x_2)$. The corresponding conditional failure rate is given by

$$h_{X_1|(X_2>x_2)}(x_1, x_2) = \begin{cases} -\dfrac{\partial}{\partial x_1} \ln \overline{G}_1(x_1) = h_{U_1}(x_1), & x_1 < x_2, \\ -\dfrac{\partial}{\partial x_1}[\ln \overline{G}_1(x_1) + \ln \overline{G}_3(x_1)] = h_{U_1}(x_1) + h_{U_3}(x_1), & x_1 > x_2. \end{cases}$$

Similarly, the conditional survival function of X_2, given $X_1 > x_1$, is given by

$$\overline{F}_{X_2|(X_1>x_1)}(x_2|X_1 > x_1) = \frac{\overline{G}_2(x_2)\overline{G}_3(x_1 \vee x_2)}{\overline{G}_3(x_1)},$$

and the corresponding conditional failure rate is

$$h_{X_2|(X_1>x_1)}(x_1, x_2) = \begin{cases} h_{U_2}(x_2) + h_{U_3}(x_2), & x_1 < x_2, \\ h_{U_2}(x_2), & x_1 > x_2. \end{cases} \tag{5.1}$$

Thus, we have the hazard components to be

$$\begin{aligned} &(h_{U_1}(x_1), h_{U_2}(x_2) + h_{U_3}(x_2)), \quad x_1 < x_2, \\ &(h_{U_1}(x_1) + h_{U_3}(x_1), h_{U_2}(x_2)), \quad x_1 > x_2. \end{aligned}$$

The monotonicity of the conditional failure rate of X_1, given $X_2 > x_2$, depends on the monotonicity of the failure rates of U_1 and U_3. Similarly, the monotonicity of the conditional failure rate of X_2, given $X_1 > x_1$, depends on the monotonicity of the failure rates of U_2 and U_3.

5.2. Monotonicity of the Failure Rate of X_1 Given $X_2 > x_2$, as a Function of x_2

To study the monotonicity of the failure rate of X_1, given $X_2 > x_2$, as a function of x_2, we use the notions of TP_2 and RR_2 defined as follows.

Definition 5.1: *A function $f(x, y)$ is said to be* $\mathrm{TP}_2(RR_2)$ *if*

$$f(x, y)f(u, v) - f(x, v)f(u, y) \geq (\leq)0 \quad \textit{for all } x < u \textit{ and } y < v.$$

The $\mathrm{TP}_2(\mathrm{RR}_2)$ property is a notion of positive (negative) dependence and is the strongest of all dependence notions in the literature; see, for example, Joe [15] for a comprehensive discussion on various dependence notions and their inter-relationships.

Theorem 5.1: *The joint reliability function $\overline{F}_{X_1,X_2}(x_1, x_2)$ has the* TP_2 *property.*

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PROOF: We readily have

$$\overline{F}_{X_1,X_2}(x_1,x_2) = \int_0^\infty P(\min(U_1,u_3) > x_1, \min(U_2,u_3) > x_2) dG_3(u_3),$$

where $G_3(.)$ is the distribution function of U_3. Now,

$$P(\min(U_1,u_3) > x_1) = \overline{G_1}(x_1) I(x_1,u_3),$$

where

$$I(x_1,u_3) = \begin{cases} 0, & x_1 \geq u_3, \\ 1, & x_1 < u_3. \end{cases}$$

Since the kernel $I(x_1,u_3)$ is TP_2, $P(\min(U_1,u_3) > x_1)$ is TP_2 in x_1 and u_3. Similarly, we have

$$P(\min(U_2,u_3) > x_2) = \overline{G_2}(x_2) I(x_2,u_3),$$

where

$$I(x_2,u_3) = \begin{cases} 0, & x_2 \geq u_3, \\ 1, & x_2 < u_3; \end{cases}$$

Since $I(x_2,u_3)$ is TP_2, $P(\min(U_2,u_3) > x_2)$ is TP_2 in x_2 and u_3. Hence, by the composition formula (see, e.g., Karlin [16]), we obtain $\overline{F}_{X_1,X_2}(x_1,x_2)$ to have the TP_2 property. ■

The above established TP_2 property helps us to determine the monotonicity of the failure rate of X_1, given $X_2 > x_2$, as a function of x_2. For this purpose, we need the following lemma due to Shaked [28].

LEMMA 5.1: *If* $\overline{F}_{X_1,X_2}(x_1,x_2)$ *is* $\mathrm{TP}_2(RR_2)$, *the conditional failure rate of* X_1, *given* $X_2 > x_2$, *is decreasing (increasing) in* x_2.

Using this lemma, we immediately observe that the failure rate of X_1, given $X_2 > x_2$, is decreasing in x_2. A similar statement can, of course, be made regarding the other hazard component.

Remark 5.1: The TP_2 property of $\overline{F}_{X_1,X_2}(x_1,x_2)$ also implies that $Cov(X_1,X_2) > 0$. For some other implications, one may refer to Shaked [28].

Remark 5.2: The results for the conditional distribution of X_2, given $X_1 > x_1$, can be similarly obtained.

Remark 5.3: The results for the conditional survival functions reveal that the results for the conditional distribution of X_i, given $X_j (i,j = 1,2)$, are the same as those for the conditional distribution of X_i, given $X_j > x_j (i,j = 1,2)$.

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5.3. Mean Residual Life Function

The MRLF of X_1, given $X_2 > x_2$, is given by

$$\mu_{X_1|(X_2>x_2)}(t) = \begin{cases} \dfrac{\int_t^{x_2} \overline{G}_1(u)du}{\overline{G}_1(t)} + \dfrac{\int_{x_2}^{\infty} \overline{G}_1(u)\overline{G}_3(u)du}{\overline{G}_1(t)\overline{G}_3(x_2)}, & 0 \le t < x_2, \\ \dfrac{\int_t^{\infty} \overline{G}_1(u)\overline{G}_3(u)du}{\overline{G}_1(t)\overline{G}_3(t)}, & t \ge x_2. \end{cases} \tag{5.2}$$

In particular, we have the mean of X_1, given $X_2 > x_2$, as

$$\mu_{X_1|(X_2>x_2)} = \frac{\overline{G}_3(x_2)\int_0^{x_2} \overline{G}_1(u)du + \int_{x_2}^{\infty} \overline{G}_1(u)\overline{G}_3(u)du}{\overline{G}_3(x_2)}. \tag{5.3}$$

We shall now present two examples.

Example 5.1: Suppose U_3 has an exponential distribution with failure rate θ_0 and U_1 has a linear failure rate given by $h_1(t) = a_1 + b_1 t$. In this case,

$$\mu_{X_1|(X_2>x_2)}(t) = \frac{\Phi(\sqrt{b_1}x_2 + a_1/\sqrt{b_1}) - \Phi(\sqrt{b_1}t + a_1/\sqrt{b_1})}{\sqrt{b_1}\phi(\sqrt{b_1}t + a_1/\sqrt{b_1})} + \frac{1 - \Phi(\sqrt{b_1}x_2 + (a_1+\theta_0)/\sqrt{b_1})}{\sqrt{b_1}\phi(\sqrt{b_1}t + (a_1+\theta_0)/\sqrt{b_1})}, \quad 0 \le t \le x_2,$$

and

$$\mu_{X_1|(X_2>x_2)}(t) = \frac{1 - \Phi(\sqrt{b_1}t + (a_1+\theta_0)/\sqrt{b_1})}{\sqrt{b_1}\phi(\sqrt{b_1}t + (a_1+\theta_0)/\sqrt{b_1})}, \quad t \ge x_2. \tag{5.4}$$

The mean of $X_1|(X_2 > x_2)$ can be easily obtained from above.

Example 5.2: If U_3 has an exponential distribution with failure rate θ_0 and U_1 has the generalized standard exponential distribution, we have the MRLF as

$$\mu_{X_1|(X_2>x_2)}(t) = \frac{\int_t^{x_2}\{1-(1-e^{-u})^{\theta_1}\}du}{\{1-(1-e^{-t})^{\theta_1}\}} + \frac{\int_{x_2}^{\infty}\{1-(1-e^{-u})^{\theta_1}\}e^{-\theta_0 u}du}{\{1-(1-e^{-t})^{\theta_1}\}e^{-\theta_0 x_2}}, \quad 0 \le t < x_2,$$

and

$$\mu_{X_1|(X_2>x_2)}(t) = \frac{\int_t^{\infty}\{1-(1-e^{-u})^{\theta_1}\}e^{-\theta_0 u}du}{\{1-(1-e^{-t})^{\theta_1}\}e^{-\theta_0 t}}, \quad t > x_2.$$

In this case, we have the mean of X_1, given $X_2 > x_2$, as

$$\begin{aligned} \mu_{X_1|(X_2>x_2)} &= \int_0^{x_2}\{1-(1-e^{-u})^{\theta_1}\}du + \frac{1}{e^{-\theta_0 x_2}}\int_{x_2}^{\infty}\{1-(1-e^{-u})^{\theta_1}\}e^{-\theta_0 u}du \\ &= \int_0^{x_2}\{1-(1-e^{-u})^{\theta_1}\}du + \frac{1}{e^{-\theta_0 x_2}}\left\{\left(\frac{1}{\theta_0}\right)e^{-\theta_0 x_2} - \beta_{e^{-x_2}}(\theta_0, \theta_0+1)\right\}, \end{aligned}$$

where

$$\beta_u(a,b) = \int_0^u v^{a-1}(1-v)^{b-1}dv$$

is the incomplete beta function, as before.

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6. ASSOCIATION MEASURE

6.1. Definition

In the context of bivariate survival models induced by frailties, Oakes [24] studied the association measure

$$\theta(x_1, x_2) = \frac{\overline{F}\ \overline{F_{12}}}{\overline{F_1}\overline{F_2}},$$

where $\overline{F} = \overline{F}(x_1, x_2)$ is the joint reliability function, $\overline{F}_{12} = (\partial^2/\partial x_1 \partial x_2)\overline{F}(x_1, x_2), \overline{F}_1 = (\partial/\partial x_1)\overline{F}(x_1, x_2)$, and $\overline{F_2} = (\partial/\partial x_2)\overline{F}(x_1, x_2)$; see also Clayton [5] who presented the above association measure, derived from the Cox model, in a study of the association between the life spans of fathers and their sons.

It can be easily seen that

$$\theta(x_1, x_2) = \frac{h(x_1|X_2 = x_2)}{h_1(x_1, x_2)}.$$

If X_1, X_2 denote the ages at death of a man and his father, then the numerator is the hazard rate for son at x_1 given that the father died at age x_2, while the denominator is the hazard rate for son at age x_1 given that the father lived past x_2. Also,

$$h(x_1|X_2 = x_2) = -\overline{F}_{12}/\overline{F}_2 \quad \text{and} \quad h_1(x_1, x_2) = -\overline{F_1}/\overline{F}.$$

6.2. Derivation of $\theta(x, y)$

From the formulas given earlier, it can be verified that

$$\begin{aligned}\frac{\partial^2}{\partial x_1 \partial x_2} \ln \overline{F}(x_1, x_2) &= \frac{\overline{F}_1\ \overline{F}_2}{\overline{F}^2}(\theta - 1) \\ &= h_1(x_1, x_2)h_2(x_1, x_2)(\theta - 1).\end{aligned}$$

This gives

$$-\frac{\partial}{\partial x_2} h_1(x_1, x_2) = h_1(x_1, x_2)h_2(x_1, x_2)(\theta - 1),$$

and consequently

$$\theta(x_1, x_2) = 1 - \frac{\frac{\partial}{\partial x_2} h_1(x_1, x_2)}{h_1(x_1, x_2)h_2(x_1, x_2)} = 1 - \frac{\frac{\partial}{\partial x_1} h_2(x_1, x_2)}{h_1(x_1, x_2)h_2(x_1, x_2)}$$

by symmetry.

This readily shows that the two hazard components jointly determine $\theta(x_1, x_2)$.

Remark 6.1: From the hazard components of the model we have considered, we observe that the value of $\theta(x_1, x_2)$ is 1 for $x_1 \neq x_2$. However, when $x_1 = x_2$, $\theta(x_1, x_2)$ does not exist due to the fact that $h_1(x_1, x_2)$ does not exist when $x_1 = x_2$.

6.3. Effects of the Measure and Some Applications

The deviation of the ratio of these hazards from 1 characterizes the strength of mutual dependence of respective life spans. The stronger the dependence between X_1 and X_2 is,

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the larger the value of $\theta(x_1, x_2)$ is. If $\theta(x_1, x_2)$ decreases to 1 when x_1 and x_2 tend to $+\infty$, then the dependence of the pair (X_1, X_2) is at least DTP$(0, 1)$ or DTP$(1, 0)$. This means that the conditional hazard of X_2, given $X_1 = x_1$, and the conditional hazard of X_1, given $X_2 = x_2$, decrease in x_1 and x_2, respectively; see Shaked [28] for the definitions of DTP$(0, 1)$ and DTP$(1, 0)$.

Clayton [5] proposed this measure by assuming that the association arises because the two members of a pair share some common influence and not because one event influences the other. Thus, $\theta(x_1, x_2)$ explains an association between two non-negative survival times with continuous joint distribution by their common dependence on an unobserved random variable. This unobserved random variable is the so-called frailty or environmental effect; see Oakes [24] , Manatunga and Oakes [21], and Gupta and Kirmani [14] for more details on frailty models.

7. RELIABILITY CHARACTERISTICS OF SERIES AND PARALLEL SYSTEMS

Let $T_1 = \min(X_1, X_2)$ and $T_2 = \max(X_1, X_2)$ be the lifetimes of series and parallel systems based on components (X_1, X_2). In this section, we study some reliability characteristics of these two systems.

7.1. Series System

The survival function of T_1 is given by

$$\overline{F}_{T_1}(t) = \overline{F}_{X_1,X_2}(t, t) = \overline{G}_1(t)\overline{G}_2(t)\overline{G}_3(t). \tag{7.1}$$

Note that Eq. (3.9) of Franco and Vivo [7] can be deduced from (7.1).

From (7.1), we have the failure rate of T_1 as

$$\begin{aligned} h_{T_1}(t) = -\frac{d}{dt}\ln \overline{F}_{T_1}(t) &= h_{U_1}(t) + h_{U_2}(t) + h_{U_3}(t) \\ &= h_U(t) + h_{U_3}(t), \end{aligned}$$

where $h_U(t) = h_{U_1}(t) + h_{U_2}(t)$ is the failure rate of a series system with independent components U_1 and U_2.

Now, suppose U_i has a generalized exponential distribution with parameter θ_i (for $i = 1, 2$) and U_3 has an exponential distribution with failure rate θ_0. Then, we simply have

$$h_{T_1}(t) = h_{U_1}(t) + h_{U_2}(t) + \theta_0,$$

and from this we immediately have the following implications:

(1) If $\theta_1 > 1$ and $\theta_2 > 1$, $h_{U_1}(t)$ and $h_{U_2}(t)$ are both increasing, and so $h_{T_1}(t)$ is increasing;

(2) If $\theta_1 < 1$ and $\theta_2 < 1$, $h_{U_1}(t)$ and $h_{U_2}(t)$ are both decreasing, and so $h_{T_1}(t)$ is decreasing;

(3) If $\theta_1 > 1$ and $\theta_2 < 1$ (or vice versa), we have

$$h'_{T_1}(t) = h'_{U_1}(t) + h'_{U_2}(t),$$

and so the change point of the failure rate of T_1 is the solution of the equation

$$h'_{U_1}(t) + h'_{U_2}(t) = 0.$$

Since $h'_{U_1}(t) > 0$ and $h'_{U_2}(t) < 0$, the turning point is unique.

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Next, we have the MRLF of T_1 as

$$\mu_{T_1}(t) = \frac{\int_t^\infty \overline{G}_1(u)\overline{G}_2(u)\overline{G}_3(u)du}{\overline{G}_1(t)\overline{G}_2(t)\overline{G}_3(t)}.$$

We now present two examples.

Example 7.1: Suppose U_i has a linear failure rate function given by $h_i(t) = a_i + b_i t (i = 1, 2)$ and U_3 has an exponential distribution with failure rate θ_0 Then,

$$\mu_{T_1}(t) = \frac{1 - \Phi(t\sqrt{b_1 + b_2} + (\theta_0 + a_1 + a_2)/\sqrt{b_1 + b_2})}{\sqrt{b_1 + b_2}\phi(t\sqrt{b_1 + b_2} + (\theta_0 + a_1 + a_2)/\sqrt{b_1 + b_2})}. \tag{7.2}$$

Example 7.2: Suppose U_i has a generalized exponential distribution with parameter $\theta_i (i = 1, 2)$ and U_3 has an exponential distribution with failure rate θ_0. Then,

$$\begin{aligned}\mu_{T_1}(t) &= \frac{\int_t^\infty \{1 - (1 - e^{-u})^{\theta_1}\}\{1 - (1 - e^{-u})^{\theta_2}\}e^{-\theta_0 u}du}{\{1 - (1 - e^{-t})^{\theta_1}\}\{1 - (1 - e^{-t})^{\theta_2}\}e^{-\theta_0 t}} \\ &= \frac{(1/\theta_0)e^{-\theta_0 t} - \beta_{e^{-t}}(\theta_0, \theta_1 + 1) - \beta_{e^{-t}}(\theta_0, \theta_2 + 1) + \beta_{e^{-t}}(\theta_0, \theta_1 + \theta_2 + 1)}{\{1 - (1 - e^{-t})^{\theta_1}\}\{1 - (1 - e^{-t})^{\theta_2}\}e^{-\theta_0 t}}.\end{aligned} \tag{7.3}$$

7.2. Parallel System

It can be verified that the density and the reliability functions of T_2 are given by

$$f_{T_2}(t) = f_{X_1}(t) + f_{X_2}(t) - f_{T_1}(t)$$

and

$$\overline{F}_{T_2}(t) = \overline{F}_{X_1}(t) + \overline{F}_{X_2}(t) - \overline{F}_{T_1}(t),$$

respectively. Thus, for the model considered, we have the reliability function of T_2 as

$$\begin{aligned}\overline{F}_{T_2}(t) &= \overline{G}_1(t)\overline{G}_3(t) + \overline{G}_2(t)\overline{G}_3(t) - \overline{G}_1(t)\overline{G}_2(t)\overline{G}_3(t) \\ &= \overline{G}_3(t)\{\overline{G}_1(t) + \overline{G}_2(t) - \overline{G}_1(t)\overline{G}_2(t)\}.\end{aligned} \tag{7.4}$$

Note that Eq. (3.10) of Franco and Vivo [7] can be deduced from (7.4). Also, the failure rate of T_2 is given by

$$h_{T_2}(t) = -\frac{d}{dt}\ln \overline{F}_{T_2}(t) = h_V(t) + h_{U_3}(t),$$

where V is the length of life of a parallel system consisting of two independent components U_1 and U_2 and $h_V(t)$ is its failure rate.

Suppose U_1 and U_2 have exponential distributions with failure rates λ_1 and λ_2, respectively, and U_3 has an exponential distribution with failure rate θ_0. Then, in this case, we

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have

$$h_{T_2}(t) = h_V(t) + \theta_0.$$

Moreover,

$$P(V > t) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t},$$

which gives the failure rate of V as

$$h_V(t) = \frac{\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}}{e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}}.$$

Thus, $h'_{T_2}(t) = 0$ gives the equation

$$\lambda_1^2 e^{-\lambda_1 t} + \lambda_2^2 e^{-\lambda_2 t} - (\lambda_1 - \lambda_2)^2 = 0. \tag{7.5}$$

Since $\lambda_1^2 e^{-\lambda_1 t} + \lambda_2^2 e^{-\lambda_2 t}$ is a decreasing function of t, (7.5) has a unique solution, and it can also be verified that the change point is indeed a point of maxima. So, the failure rate of T_2 is of type U.

Remark 7.1: When $\lambda_1 = \lambda_2$, V has an increasing failure rate and consequently T_2 has an increasing failure rate as well.

Suppose U_1 and U_2 have generalized exponential distributions with parameters θ_1 and θ_2, respectively, and U has the exponential distribution with failure rate θ_0. Further, suppose $\theta_1 = \theta_2 = \theta$. In this special case, we have

$$h_{T_1}(t) = 2h(t) + \theta_0,$$

where $h(t)$ is the failure rate of the generalized exponential distribution with parameter θ. Then, if $\theta > 1$, $h(t)$ is increasing and so $h_{T_1}(t)$ is also increasing. Using a result of Takahasi [29], we have $h_{T_2}(t)$ also to be increasing and so T_2 to have an increasing failure rate.

Remark 7.2: When $\theta_1 \neq \theta_2$, and $\theta_1 > 1, \theta_2 > 1$, we have $h_{T_1}(t)$ to be increasing, but we can not make any statement about the monotonicity of $h_{T_2}(t)$. Similarly, if $\theta_1 < 1, \theta_2 < 1$, we have $h_{T_1}(t)$ to be decreasing, but we cannot make any statement about the monotonicity of $h_{T_2}(t)$.

8. CONCLUDING REMARKS

In this paper, we have considered a general class of bivariate distributions by following the construction of the Marshall–Olkin bivariate exponential distribution. In this GMO class, the dependence between X_1 and X_2 is created by taking $X_1 = \min(U_1, U_3)$ and $X_2 = \min(U_2, U_3)$, where U_1, U_2, and U_3 are independent random variables have some suitable non-negative continuous distributions.

For the proposed GMO class, we have studied the failure rate and the MRLF for the marginals and the conditionals as well as for the series and parallel systems formed with X_1 and X_2 as components. The monotonicity of the failure rates and the MRLFs have been discussed and the association measure due to Clayton [5] has been studied. In addition to developing general results for the GMO model, we have provided results for two special cases: (i) the distribution of Sarhan and Balakrishnan [25] and (ii) the distribution in which U_1 and U_2 have linear failure rates and U_3 has constant failure rate.

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