Ergod. Th. & Dynam. Sys. (1998), **18**, 153–170
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Closing lemma for holomorphic functions in $\mathbb{C}$

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(*Received* 26 *February* 1996 *and accepted in revised form* 16 *September* 1996)

Abstract. Let $F_\lambda(z) = \lambda z + \mathcal{O}(z^2)$ be a one parameter holomorphic family of holomorphic maps defined in a neighborhood of $\lambda_0 * \overline{\mathrm{D}}$. Assume that F_{λ_0} has a Siegel disc Δ_0, then the orbit of a point in the interior of the Siegel disc can be followed very closely by periodic orbits of nearby maps. The same technique is applied to Herman rings and Cremer points.

1. *Introduction*

Let f be an endomorphism of a smooth manifold M. Let d be a distance compatible with the topology. A point $x \in M$ is recurrent if

$$\underline{\lim}_{n \mapsto \infty} d(f^n(x), x) = 0.$$

A remarkable result, a closing lemma, of Pugh [**P**], see also Pugh and Robinson [**PR**], states that for any diffeomorphism f of a compact manifold, any recurrent point x and any $\mathcal{C}^1$ neighborhood V of f, there is a diffeomorphism $g \in V$ such that x is periodic for g.

The question is open for the $\mathcal{C}^k$ topology for $k \geq 2$. A similar question can be asked for symplectic diffeomorphisms, flows, etc.

Herman [**H**], see also the survey by Yoccoz [**Y**], has recently constructed a counterexample in the $\mathcal{C}^k$ topology for k large enough. More precisely, he has a symplectic form ω_A on the $2p$-dimensional torus T^{2p}, $p \geq 2$, and a Hamiltonian H_0 such that for H close enough to H_0 (in the $\mathcal{C}^k$ topology, k large enough) and c in an open interval, the orbits of the Hamiltonian flow on $H^{-1}(c)$ are dense. In particular, there are no periodic points.

In this paper we will discuss versions of Pugh's theorem for holomorphic maps in complex dimension 1. For the theory of iteration in one complex variable we refer to Carleson and Gamelin [**CG**] or Milnor [**M**].

Let us consider first the problem of approximation of recurrent orbits by periodic orbits for the same map.

§ First author supported by an NSF grant.

https://doi.org/10.1017/S0143385798097533 Published online by Cambridge University Press

Let $f : \mathbb{P}^1 \hookleftarrow$ be a rational map of degree $d \geq 2$. Let $\mathcal{J}$ be the Julia set of f. It is well known that repelling periodic points are dense in $\mathcal{J}$. The following question is quite natural.

Let X be a compact set in $\mathcal{J}$ such that $f(X) \subset X$. Note that X must contain a recurrent point. Let $\{X_\alpha\}$ be the collection of all nonempty compact subsets of X which are forward invariant. Then $\{X_\alpha\}$ is partially ordered by inclusion. By Zorn's lemma there is a minimal element X_0. The cluster set of the orbit of any point in X_0 must be all of X_0 so all points in X_0 are recurrent. Given $\epsilon > 0$, is there a periodic orbit in $X_\epsilon := \{z; \mathrm{d}(z, X) \leq \epsilon\}$? For example, when X is the boundary of a Siegel disc is there a periodic orbit entirely contained in X_ϵ? This seems unlikely in general.

Suppose $\mathcal{F}$ is a topological space of holomorphic self-maps of a domain $\Omega \subset \mathbb{C}$. For example, $\mathcal{F}$ could be the set of all entire functions with u.c.c. topology or all quadratic polynomials $z^2 + c$ on $\Omega = \mathbb{C}$.

We will formulate four versions of the closing lemma [**P**], depending on the amount of control one wants on the approximating periodic orbit. Suppose $f \in \mathcal{F}$ and $p \in \Omega$ is recurrent, i.e. $f^{n_i}(p) \mapsto p$ for some (n_i). (One might substitute this hypothesis by the assumption that p is non-wandering.) Let K denote the closure of the orbit $(f^n(p))_{n\geq 1}$. Also, let V denote a neighborhood of f and U denote a neighborhood of p.

(i) We say that the *weak closing lemma* holds if there is a $g \in V$ and a $q \in U$ so that q is periodic for g.

(ii) We say that the *closing lemma* holds if given f, p such that K is compact, then one can find some compact set $L, K \subset L \subset \Omega$, such that for all U, V there is a $g \in V, q \in U$ whose orbit is periodic and contained in L.

(iii) We say that the *strong closing lemma* holds if whenever W is a neighborhood of K, there is a $g \in V$ and a $q \in U$ such that q is periodic and its whole orbit is contained in W.

(iv) We say that the *very strong closing lemma* holds for f at p if whenever $\epsilon > 0$, there is a $g \in V$, a $q \in U$ and an $n \geq 1$ so that $g^n(q) = q$ and $|g^k(q) - f^k(p)| < \epsilon$ for all $0 \leq k \leq n$.

The very strong closing lemma follows from an even stronger property. We say that the *orbit control lemma* holds if whenever $\mathcal{L} \subset \mathcal{F}$ and $K \subset \Omega$ are compact sets and $\epsilon > 0$, there exists a constant $\delta(\epsilon) > 0$ so that whenever $f \in \mathcal{L}$ and $p \in K$, $f(p), \ldots, f^n(p) \in K$ and $|z - f^n(p)| < \delta(\epsilon)$, there is a $g \in \mathcal{F}, |g - f| < \epsilon$ so that $|g^j(p) - f^j(p)| < \epsilon \ \forall j = 1, \ldots, n, g^n(p) = z$.

A priori, given two families $\mathcal{F}, \mathcal{F}'$ containing f one closing lemma might hold for $\mathcal{F}$ and another type might hold for $\mathcal{F}'$.

The weak closing lemma holds when $\mathcal{F}$ is the class of entire maps in $\mathbb{C}$ (the case of $\mathbb{C}^k$ is considered in [**FS**]). The result was generalized to meromorphic maps in $\mathbb{C}$ by Kriete [**K**]. Kriete also proved the very strong closing lemma for classes of rational functions on $\mathbb{P}^1$. The classes considered were defined by restrictions on the global dynamics or by conditions reminiscent of hyperbolicity.

PROPOSITION 1.1. *Let f be a rational map of degree $d \geq 2$ in $\mathbb{P}^1$, with Julia set $\mathcal{J}$. There is a dense G_δ set G in $\mathcal{J}$ such that for every $p \in G, \forall \epsilon > 0, \exists q \in \mathcal{J}, f^n(q) = q$, for some $n \geq 1$ and $|f^k(p) - f^k(q)| < \epsilon, 0 \leq k \leq n$.*

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Proof. Define

$$G_\epsilon = \{p \in \mathcal{J}; \exists q \in \mathcal{J}, \exists n \geq 1,\ f^n(q) = q, |f^k(q) - f^k(p)| < \epsilon, 0 \leq k \leq n\}.$$

Clearly, G_ϵ is open. It is dense because periodic points are dense in $\mathcal{J}$. So $G = \cap_n G_{1/n}$ satisfies the requirement of the proposition. □

Let $\mathrm{Rec}(f)$ denote the set of recurrent points for $f \in \mathcal{J}$. Clearly, $G \subset \mathrm{Rec}(f)$, where G is the set from the previous proposition. Define $G' = \{p; \forall \epsilon > 0, \exists n \geq 1$ and a periodic orbit Q such that the Hausdorff distance $\delta[Q, \{p, \ldots, f^n(p)\}] < \epsilon\}$. Note that all points in G' are necessarily contained in $\mathrm{Rec}(f)$.

PROPOSITION 1.2. *Let f be a rational function of degree $d \geq 2$ in $\mathbb{P}^1$. Then* $\mathrm{Rec}(f) \cap f^{-1}(G') \subset G'$.

Proof. Suppose that $p_- \in \mathrm{Rec}(f) \cap f^{-1}(G')$. Set $p = f(p_-)$. Fix $\epsilon > 0$. Since p_- is recurrent there exists positive integers l such that $|f^l(p) - p_-| = |f^{l+1}(p_-) - p_-| < \epsilon$. Since $p \in G'$ there is $s \geq 0$ and a periodic orbit Q such that $\delta[Q, \{p, \ldots, f^{l+s}(p)\}]$ is arbitrarily small. Hence the orbit can be chosen to come closer to p_- than ϵ. Hence $p_- \in G'$. □

The reason for introducing G' is that we do not know how to prove that $f^{-1}(G) \cap \mathrm{Rec}(f) \subset G$.

Remark 1.3. Indeed, Proposition 1.1 and Proposition 1.2 are valid in the following context. Let f be a continuous surjective map $f : X \to X$, where X is a compact metric space. Assume periodic points are dense in X. Then G is a dense G_δ set for f and $\mathrm{Rec}(f) \cap f^{-1}(G') \subset G'$.

Perez Marco [**PM**] has shown that there is an injective holomorphic map $f : \Delta \to \mathbb{C}$, defined on the unit disc Δ, $f(z) = e^{i\theta}z + \mathcal{O}(z^2)$, such that 0 is an accumulation point for every orbit $(f^n(z))_{n\geq 0}$ which stays in D. In particular, if $z \neq 0$ is recurrent with $(f^n(z))_{n\geq 0}$ in D (and there are many such points) then z cannot be approximated by periodic points in the above sense. So, in general, one needs to perturb the original map before finding periodic orbits.

2. *The closing lemma in* $\mathbb{C}$

In this section we prove the very strong closing lemma for the Siegel discs for holomorphic families of maps. This follows from a slightly weakened version of an orbit control lemma. The orbit control lemma itself fails, as we also show. The orbit control lemma demands control on any orbit, while the weakened version which we prove gives control on any long enough orbit.

THEOREM 2.1. *Let $\{f_\lambda(z)\}_{\lambda\in\omega(\lambda_0)}$ be a holomorphic family of meromorphic (rational) maps, where $\lambda_0 = e^{i\theta}$, $\omega(\lambda_0)$ is a neighborhood of λ_0 and 0 is a fixed point with multiplier λ for each f_λ. Suppose that f_{λ_0} has a Siegel disc $U \subset \mathbb{C}$. For every $z_0 \in U$ and every $\epsilon > 0$ there exist $\lambda \in \omega(\lambda_0)$ and $N \in \mathbb{N}$ such that z_0 is periodic of order N for f_λ and $|f_\lambda^k(z_0) - f_{\lambda_0}^k(z_0)| < \epsilon$ for each $0 \leq k \leq N$.*

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Proof. Let $\lambda = \mu e^{i\theta}$. Define $F_\lambda(z) = \Phi \circ f_\lambda \circ \Phi^{-1}$, where Φ is the map conjugating f_{λ_0} to a rotation. After scaling we can assume $\Phi(z_0) = 1$ and that the maps F_λ are defined in a neighborhood of the closed unit disc $\overline{\Delta}$. Hence the theorem follows as a special case from the following more general result. □

THEOREM 2.2. *Let $r > 1, r' > 0$ denote two constants and suppose that*

$$F_\lambda(z) = F(\lambda, z)$$

is a holomorphic function on $\Delta(e^{i\theta}, r') * \Delta(0, r)$ such that $F_\lambda(z) = \lambda z + \mathcal{O}(z^2)$, $F_{e^{i\theta}}(z) = e^{i\theta} z$. Then given any $\epsilon > 0$ there exists an N and a $\lambda \in \Delta(e^{i\theta}, \epsilon)$ such that the orbit $\{F_\lambda^n(1)\}_{n=0}^{n=N}$ is closed and follows the orbit $\{F_{e^{i\theta}}^n(1)\}_{n=0}^N$ closer than ϵ.

In fact, we would have liked to prove the following stronger result, an orbit control lemma.

STATEMENT 2.3. *Let $r > 1, r' > 0$ denote two constants and suppose that $F_\lambda(z) = F(\lambda, z)$ is a holomorphic function on $\Delta(e^{i\theta}, r') \times \Delta(0, r)$ such that $F_\lambda(z) = \lambda z + \mathcal{O}(z^2)$, $F_{e^{i\theta}}(z) = e^{i\theta} z$. Then given any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that whenever $|z - F_{e^{i\theta}}^n(1)| < \delta(\epsilon)$ for some $n \geq 1$ then there exists $\lambda \in \Delta(e^{i\theta}, \epsilon)$ such that $F_\lambda^n(1) = z$ and $|F_\lambda^k(1) - F_{e^{i\theta}}^k(1)| < \epsilon$ for all $k = 1, \ldots, n$. The function $\epsilon \to \delta(\epsilon)$ can be chosen to depend only on* $\sup |F(\lambda, z)|$, *r and r'.*

Unfortunately, as the following example shows, this statement is false.

Example 2.1. Let $F_\lambda(z) = \lambda z + a(\lambda) z^2$. Let $e^{i\theta} \neq -1$ be on the unit circle. There is an analytic function $a(\lambda)$ defined in a neighborhood of $e^{i\theta}$, $a(e^{i\theta}) = 0$ so that $F_\lambda^2(1) \equiv e^{2i\theta}$. Consider the holomorphic function in $\mathbb{C}^2$:

$$G(\lambda, w) := \lambda(\lambda + w) + w(\lambda + w)^2 - e^{2i\theta}.$$

Let $a(\lambda)$ be the function obtained by the implicit function theorem, such that $G(\lambda, a(\lambda)) = 0, a(e^{i\theta}) = 0$. Clearly, $F_\lambda^2(1) = \lambda[\lambda + a(\lambda)] + a(\lambda)[\lambda + a(\lambda)]^2 \equiv e^{2i\theta}$.

Because of this example we will settle for a weaker version of an orbit control lemma, which still suffices for the proof of the very strong closing lemma in Theorem 2.2. The weaker version states that orbits can only be controlled if they are sufficiently long. How long depends also on the diophantine properties of θ.

THEOREM 2.4. *Let $r > 1, r' > 0$ denote two constants and fix $e^{i\theta}$, not a root of unity. Suppose that $F_\lambda(z) = F(\lambda, z)$ is a holomorphic function on $\Delta(e^{i\theta}, r') \times \Delta(0, r)$ such that $F_\lambda(z) = \lambda z + \mathcal{O}(z^2)$, $F_{e^{i\theta}}(z) = e^{i\theta} z$. There exists an integer $N \geq 1$ so that given any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that whenever $|z - F_{e^{i\theta}}^n(1)| < \delta(\epsilon)$ for some $n \geq N$ then there exists $\lambda \in \Delta(e^{i\theta}, \epsilon)$ such that $F_\lambda^n(1) = z$ and $|F_\lambda^k(1) - F_{e^{i\theta}}^k(1)| < \epsilon$ for all $k = 1, \ldots, n$. The function $\epsilon \to \delta(\epsilon)$ and the number N can be chosen to depend only on* $\sup |F(\lambda, z)|$, *r, r' and (the diophantine properties of) θ.*

Let $e^{i\theta}$ denote any number on the unit circle. For integers $m \geq 1$ and $\epsilon > 0$, let $D_m^\epsilon := \{|\lambda - e^{i\theta}| < \epsilon/m\}$. For each positive integer $k \geq 2$, let

$$K_{k,m}^\epsilon := \sup_{\lambda \in D_m^\epsilon} \left| \phi_{k,m}(\lambda) := \frac{\lambda^{(k-1)m} - 1}{\lambda^{k-1} - 1} \right|$$

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Note that $\phi_{k,m}(\lambda) = 1 + \lambda^{k-1} + \cdots + (\lambda^{k-1})^{m-1}$.

LEMMA 2.5. *One has that* $K^{\epsilon}_{k,m} \leq m(1+\epsilon/m)^{(m-1)(k-1)}$. *If* $e^{i\theta}$ *is not a root of unity, there exist constants* $C^{\epsilon}_{k,\theta} < \infty$ *so that* $K^{\epsilon}_{k,m} \leq C^{\epsilon}_{k,\theta}$ $\forall m$.

Proof of Lemma 2.5. The first part of the lemma is obvious from the second formula for $\phi_{k,m}$. To prove the second part, we only have to consider large values of m. Note that for a given k we know that $(e^{i\theta})^{k-1} \neq 1$. Hence there is a $\delta_{k,\theta} > 0$ such that if m is large enough, then $|\lambda^{k-1} - 1| > \delta_{k,\theta}$ on D^1_m. Hence it follows from the first formula for $\phi_{k,m}$ that $K^{\epsilon}_{k,m} \leq (((1+\epsilon/m)^m)^{k-1} + 1)/\delta_{k,\theta}$, which is clearly bounded uniformly in m. □

Proof of Theorem 2.4. We first shrink r, r' a little. Then we can assume that we have expansions

$$F_\lambda(z) = \lambda z + \sum_{n\geq 2} a_n(\lambda) z^n.$$

One has, by applying the Cauchy integral and the Schwarz lemma, the estimates

$$|a_n(\lambda)| \leq C_1 |\lambda - e^{i\theta}|/r^n$$

for a constant C_1 independent of n. Iterating the functions F_λ one observes that the domain of the iterates might shrink with increasing m. Let $V_{m,\lambda}$ denote the domain of F^m_λ, i.e. those points in $\Delta(0, r)$ for which the iterates of order strictly less than m remain in this disc.

Fix $1 < r^* < \tilde{r} < r$.

LEMMA 2.6. *Suppose that*

$$|\lambda - e^{i\theta}| < \sigma_m := \min\left\{r', \frac{\ln(\tilde{r}/r^*)}{m(1 + C_1/(r-\tilde{r}))}\right\}.$$

Then for every $|z| < r^*$, *the iterates* $|F^k_\lambda(z)| < \tilde{r}$ *for all* $k = 1, \ldots, m$.

Proof of Lemma 2.6. Suppose that $|z| < \tilde{r}$ and that $|\lambda - e^{i\theta}| < \sigma_m$. Then

$$\begin{aligned} |F_\lambda(z)| &\leq (1+\sigma_m)|z| + \sum_{n\geq 2} C_1 \sigma_m |z|^n / r^n \\ &\leq (1+\sigma_m)|z| + C_1 \sigma_m |z|/(r - |z|) \\ &\leq |z| e^{\sigma_m + C_1\sigma_m/(r-\tilde{r})}. \end{aligned}$$

Hence it suffices to show that

$$r^* e^{m\sigma_m + mC_1\sigma_m/(r-\tilde{r})} < \tilde{r}.$$

But this follows from the definition of σ_m. □

To continue the proof of Theorem 2.4, we compute an inductive formula for the iterates of F_λ. We write

$$F^m_\lambda(1) = \lambda^m + \sum_{k\geq 2} a_k(\lambda)\lambda^{m-1}\phi_{k,m}(\lambda) + C'_m|\lambda - e^{i\theta}|^2,$$

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where this equation serves to define the function $C'_m(\lambda)$. Pick $\epsilon_0 < 1/10$ so that $1 + 2\epsilon_0 =: \rho < r$. Then by Lemma 2.5 the infinite sum converges for $\lambda \in D_m^{\epsilon_0}$. We pick $\epsilon' \leq \epsilon_0$ so that $\epsilon'/m \leq \sigma_m \ \forall m$. Define $c'_m := \epsilon'/m$.

Let $D'_m := \{|\lambda - e^{i\theta}| \leq c'_m\}$. We denote by $L_{k,m} := K_{k,m}^{\epsilon'} = \sup_{\lambda \in D'_{m+1}} |\phi_{k,m}(\lambda)|$.

We will next estimate inductively C'_m on discs D''_m. The radii, c''_m, of the discs D''_m are determined in the following way. The radii c''_m are of the form $\hat{\epsilon}/m$, where $\hat{\epsilon} > 0$ is chosen as follows. Define

$$C_2 := 1 + \frac{2C_1}{r(1-\rho/r)^2}\left[2C_1\frac{\rho/r}{1-\rho/r} + 2\right].$$

Let $\hat{\epsilon} := \min\{\epsilon', 1/C_2, \delta_0\}$, where $\delta_0 > 0$ is a solution to the quadratic (in δ) equation

$$1 + 2\epsilon_0 = (1+2\delta)\left(1 + 2\delta\left(1 + C_1\sum_{n\geq 2}\left(\frac{\rho}{r}\right)^n\right)\right).$$

Let $\|C'_m\| := \sup_{\lambda \in D''_m} |C'_m(\lambda)|$. We want to prove that for every m, $\|C'_m\| \leq C_2 m^2$. We have $C'_1 = 0$ and set $c''_1 = c'_1$ to start the induction. Assume that $m \geq 1$ and that we have the estimate $\|C'_m\| \leq C_2 m^2$. Define $c''_{m+1} = \hat{\epsilon}/m + 1$.

We get, if $|\lambda - e^{i\theta}| < c''_{m+1}$, that

$$\begin{aligned} F_\lambda^{m+1}(1) &= \lambda\left[\lambda^m + \sum_{j\geq 2} a_j \lambda^{m-1}\phi_{j,m} + C'_m|\lambda - e^{i\theta}|^2\right] \\ &\quad + \sum_{k\geq 2} a_k(\lambda)\left[\lambda^m + \sum_{n\geq 2} a_n(\lambda)\lambda^{m-1}\phi_{n,m} + C'_m|\lambda - e^{i\theta}|^2\right]^k \\ &= \lambda^{m+1} + \sum_{k\geq 2} a_k \lambda^m \phi_{k,m+1} + C'_{m+1}|\lambda - e^{i\theta}|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} C'_{m+1}|\lambda - e^{i\theta}|^2 &= \sum_{j\geq 2} a_j\lambda^m\phi_{j,m} + \lambda C'_m|\lambda - e^{i\theta}|^2 + \sum_{k\geq 2} a_k(\lambda)\lambda^{mk} \\ &\quad + \sum_{k\geq 2} a_k(\lambda)\lambda^{mk}\left\{\left[1 + \sum_{n\geq 2} a_n(\lambda)\frac{\phi_{n,m}}{\lambda} + C'_m\frac{|\lambda - e^{i\theta}|^2}{\lambda^m}\right]^k - 1\right\} \\ &\quad - \sum_{k\geq 2} a_k\lambda^m\phi_{k,m+1}. \end{aligned}$$

Hence we get

$$\begin{aligned} |C'_{m+1}| &\leq |\lambda||C'_m| + \sum_{k\geq 2}\frac{|a_k(\lambda)|}{|\lambda - e^{i\theta}|^2}|\lambda^{mk}| \\ &\quad * \left\{\left[1 + \sum_{n\geq 2}|a_n(\lambda)||\phi_{n,m}|/|\lambda| + |C'_m||\lambda - e^{i\theta}|^2/|\lambda^m|\right]^k - 1\right\}. \end{aligned}$$

So,

$$|C'_{m+1}| \leq (1 + c''_{m+1})|C'_m| + \frac{C_1}{|\lambda - e^{i\theta}|}$$

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$$* \sum_{k\geq 2} \frac{(1+c''_{m+1})^{mk}}{r^k}$$
$$* \left\{ \left[1 + \sum_{n\geq 2} |\lambda - e^{i\theta}| \frac{C_1}{r^n} * |\phi_{n,m}|/|\lambda| + |C'_m| \frac{|\lambda - e^{i\theta}|^2}{|\lambda^m|} \right]^k - 1 \right\}.$$

Hence,

$$\begin{aligned} |C'_{m+1}| &\leq |C'_m| + c''_{m+1}|C'_m| + \frac{C_1}{|\lambda - e^{i\theta}|} * \sum_{k\geq 2} \frac{(1+c''_{m+1})^{mk}}{r^k} \\ &\quad * \left\{ \left[1 + \sum_{n\geq 2} |\lambda - e^{i\theta}| \frac{C_1}{r^n} * L_{n,m} \frac{1}{|\lambda|} + |C'_m| \frac{|\lambda - e^{i\theta}|^2}{|\lambda^m|} \right]^k - 1 \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} |C'_{m+1}| &\leq C_2 m^2 + \hat{\epsilon} C_2 m + \frac{C_1}{|\lambda - e^{i\theta}|} \\ &\quad * \sum_{k\geq 1} \left\{ \frac{(1+c''_{m+1})^m}{r} \left[1 + \sum_{n\geq 2} |\lambda - e^{i\theta}| \frac{C_1}{r^n} L_{n,m} \frac{1}{|\lambda|} + |C'_m| \frac{|\lambda - e^{i\theta}|^2}{|\lambda^m|} \right] \right\}^k \\ &\quad - \frac{C_1}{|\lambda - e^{i\theta}|} \sum_{k\geq 1} \left(\frac{(1+c''_{m+1})^m}{r} \right)^k. \end{aligned}$$

We have, using Lemma 2.5 and the induction hypothesis on C'_m, the estimate

$$\begin{aligned} &\frac{(1+c''_{m+1})^m}{r} \left[1 + \sum_{n\geq 2} |\lambda - e^{i\theta}| \frac{C_1}{r^n} L_{n,m} \frac{1}{|\lambda|} + |C'_m| \frac{|\lambda - e^{i\theta}|^2}{|\lambda^m|} \right] \\ &\leq \frac{\left(1 + \frac{\hat{\epsilon}}{m+1}\right)^m}{r} \\ &\quad * \left[1 + \sum_{n\geq 2} \frac{\hat{\epsilon}}{m+1} \frac{C_1}{r^n} m \left(1 + \frac{\epsilon'}{m+1} \right)^{(m-1)(n-1)} \frac{1}{1 - \frac{\hat{\epsilon}}{m+1}} + \frac{C_2 \hat{\epsilon}^2}{\left(1 - \frac{\hat{\epsilon}}{m+1}\right)^m} \right] \\ &\leq \frac{1+2\hat{\epsilon}}{r} \left[1 + 2C_1 \hat{\epsilon} \sum_{n\geq 2} \left(\frac{\rho}{r} \right)^n + C_2 \hat{\epsilon}^2 (1 + 4\hat{\epsilon}) \right] \\ &< \frac{\rho}{r} \\ &< 1 \end{aligned}$$

since $\hat{\epsilon} \leq \delta_0$. Hence the geometric series are convergent.

We obtain

$$\begin{aligned} &|C'_{m+1}| \\ &\leq C_2 m^2 + \hat{\epsilon} C_2 m + \frac{C_1}{|\lambda - e^{i\theta}|} \\ &\quad * \left[\frac{\frac{(1+c''_{m+1})^m}{r} \left[1 + \sum_{n\geq 2} |\lambda - e^{i\theta}| \frac{C_1}{r^n} L_{n,m} \frac{1}{|\lambda|} + |C'_m| \frac{|\lambda - e^{i\theta}|^2}{|\lambda^m|} \right]}{1 - \frac{(1+c''_{m+1})^m}{r} \left[1 + \sum_{n\geq 2} |\lambda - e^{i\theta}| \frac{C_1}{r^n} L_{n,m} \frac{1}{|\lambda|} + |C'_m| \frac{|\lambda - e^{i\theta}|^2}{|\lambda^m|} \right]} - \frac{\frac{(1+c''_{m+1})^m}{r}}{1 - \frac{(1+c''_{m+1})^m}{r}} \right]. \end{aligned}$$

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So performing the subtraction,

$$\begin{aligned}|C'_{m+1}| &\leq C_2m^2+\hat{\epsilon}C_2m+\frac{C_1}{|\lambda-e^{i\theta}|}\frac{(1+c''_{m+1})^m}{r}\\ &\quad *\left\{\frac{\left(1+\sum_{n\geq 2}|\lambda-e^{i\theta}|\frac{C_1}{r^n}L_{n,m}\frac{1}{|\lambda|}+|C'_m|\frac{|\lambda-e^{i\theta}|^2}{|\lambda^m|}\right)\left(1-\frac{(1+c''_{m+1})^m}{r}\right)}{(1-\rho/r)^2}\right\}\\ &\quad -\frac{C_1}{|\lambda-e^{i\theta}|}\frac{(1+c''_{m+1})^m}{r}\\ &\quad *\left\{\frac{\left[1-\frac{(1+c''_{m+1})^m}{r}*\left(1+\sum_{n\geq 2}|\lambda-e^{i\theta}|\frac{C_1}{r^n}L_{n,m}\frac{1}{|\lambda|}+|C'_m|\frac{|\lambda-e^{i\theta}|^2}{|\lambda^m|}\right)\right]}{(1-\rho/r)^2}\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}|C'_{m+1}| &\leq C_2m^2+\hat{\epsilon}C_2m+\frac{C_1}{|\lambda-e^{i\theta}|}\frac{(1+2\hat{\epsilon})}{r(1-\rho/r)^2}\\ &\quad *\left[\sum_{n\geq 2}|\lambda-e^{i\theta}|\frac{C_1}{r^n}L_{n,m}\frac{1}{|\lambda|}+|C'_m|\frac{|\lambda-e^{i\theta}|^2}{|\lambda^m|}\right]\\ &\leq C_2m^2+\hat{\epsilon}C_2m+m\frac{C_1(1+2\hat{\epsilon})}{r(1-\rho/r)^2}\left[2C_1\sum_{n\geq 2}\left(\frac{\rho}{r}\right)^n+2C_2\hat{\epsilon}\right]\\ &\leq C_2m^2+\left(\hat{\epsilon}C_2+\frac{2C_1}{r(1-\rho/r)^2}\left[2C_1\frac{\rho/r}{1-\rho/r}+2C_2\hat{\epsilon}\right]\right)m\\ &\leq C_2m^2+C_2m \qquad (*)\end{aligned}$$

since $\hat{\epsilon}C_2\leq 1$, and $\hat{\epsilon}\leq\delta_0$. But then $|C'_{m+1}|\leq C_2(m+1)^2$. Hence the inductive estimate follows.

Hence we have shown that if $|\lambda-e^{i\theta}|\leq\hat{\epsilon}/m$, then

$$F^m_\lambda(1)=\lambda^m+\sum_{k\geq 2}a_k(\lambda)\lambda^{m-1}\phi_{k,m}(\lambda)+C'_m(\lambda)|\lambda-e^{i\theta}|^2,$$

where $|C'_m(\lambda)|\leq C_2m^2$.

Pick any $\epsilon>0,\epsilon<\hat{\epsilon}$. We estimate $F^m_\lambda(1)$ for $|\lambda-e^{i\theta}|\leq\epsilon/m$. First of all

$$|\lambda^m-e^{im\theta}-me^{i(m-1)\theta}(\lambda-e^{i\theta})|\leq\frac{3\epsilon^2}{2} \qquad (**)$$

as we see from calculating that the left-hand side vanishes to second order at $\lambda=e^{i\theta}$ and the second derivative is at most $3m^2$ on the disc. Next,

$$\left|\sum_{k\geq 2}a_k(\lambda)\lambda^{m-1}\phi_{k,m}(\lambda)\right|\leq\sum_{k\geq 2}\frac{C_1|\lambda-e^{i\theta}|}{r^k}\left(1+\frac{\epsilon}{m}\right)^{m-1}m\left(\left(1+\frac{\epsilon_0}{m}\right)^{m-1}\right)^{k-1}.$$

The sum on the left-hand side is bounded by an expression of the form $\tilde{C}(m)m|\lambda-e^{i\theta}|$, where $\tilde{C}(m)\to 0, m\to\infty$ because of dominated convergence. Here we use the fact

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that $e^{i\theta}$ is not a root of unity and Lemma 2.5, hence for k fixed $\lim_{m\to\infty}(1/m)\sup_{D_m}|\phi_{k,m}| = 0$.

This estimate is the key of the proof. It shows the usefulness of the splitting in the formula for $F_\lambda^m(1)$ and why one needs large m before one can control the orbit.

We summarize that

$$|F_\lambda^m(1) - \lambda^m| \le \tilde{C}(m)m|\lambda - e^{i\theta}| + C_2 m^2|\lambda - e^{i\theta}|^2 \qquad (***)$$

with $\tilde{C}(m) \to 0$ as $m \to \infty$. This is valid if $|\lambda - e^{i\theta}| \le \epsilon/m$.

Hence,

$$\left|\sum_{k\ge 2} a_k(\lambda)\lambda^{m-1}\phi_{k,m}(\lambda)\right| \le \tilde{C}(m)\epsilon.$$

The third part $|C'_m||\lambda - e^{i\theta}|^2 \le C_2\epsilon^2$. Then for $\lambda \in D_m^\epsilon$,

$$|F_\lambda^m(1) - \lambda^m| \le \tilde{C}(m)\epsilon + C_2\epsilon^2.$$

If $\tilde{C}(m) \le \frac{1}{10}$ and $\epsilon < C_2^{-1}/10$, then $|F_\lambda^m(1) - \lambda^m| \le \epsilon/5$.

Let $\delta(\epsilon) = \epsilon^2$. Assume $|e^{im\theta} - z| < \delta(\epsilon)$. Then for $|\lambda - e^{i\theta}| = \epsilon/m$, $|\lambda^m - z| \ge m|\lambda - e^{i\theta}| - 3/2\epsilon^2 - \delta(\epsilon) \ge \epsilon/2$. Hence for $|\lambda - e^{i\theta}| = \epsilon/m$, $|F_\lambda^m(1) - z - (\lambda^m - z)| \le \epsilon/5 < \epsilon/2 \le |\lambda^m - z|$. So using Rouché's theorem we get the existence of (a unique) $\lambda \in D_m^\epsilon$ such that $F_\lambda^m(1) = z$.

Let A be an upper bound for $\tilde{C}_m$. ($\tilde{C}_m \to 0$.) Using the estimate $(***)$ we also get that for $\lambda \in D_m^{\epsilon/A}$, $|F_\lambda^k(1) - \lambda^k| < \epsilon/5$ for $0 \le k \le m$. □

COROLLARY 2.7. *Let $f_\lambda(z)$ be a holomorphic family of meromorphic maps. Assume $f_\lambda(z) = \lambda z + O(z^2)$. Assume that for λ_0 there is a Siegel disc U centered at the origin for f_{λ_0}. Let γ be an invariant 'circle' in U for f_{λ_0}. There is $\lambda_m \mapsto \lambda_0$ such that f_{λ_m} has a repelling or parabolic periodic orbit $\{z_1^{\lambda_m}, \dots, z_m^{\lambda_m}\} = X_m$ and γ is the limit of X_m in the Hausdorff metric.*

Proof. We can assume we are in the situation of Theorem 2.4. For each ζ, $|\zeta| \le 1$, we define

$$H_\lambda(z) := \lambda z + \sum_{k\ge 2} a_k(\lambda)\zeta^{k-1}z^k.$$

Observe that $H_\lambda(z) = (1/\zeta)F_\lambda(\zeta z)$. So if $H_\lambda^m(1) = 1$ as in Theorem 2.4 we have $F_\lambda^m(\zeta) = \zeta$.

For every $\zeta \in \Delta^*$ we find a unique $\lambda(\zeta) \in D_m^\epsilon$ such that $F_{\lambda(\zeta)}^m(\zeta) = \zeta$. The holomorphic function $\zeta \mapsto \lambda(\zeta)$ extends to 0 and $\lambda(0) = e^{2i\pi p_m/q_m}$. Define

$$G(\zeta) := (F_{\lambda(\zeta)}^m)'(z)|_{z=\zeta}.$$

The derivative is with respect to z.

Either G is constant, in which case all the approximating cycles are parabolic, or on every circle $|\zeta| = \rho$ there is ζ_ρ, $|G(\zeta_\rho)| > 1$, the corresponding cycle is repelling and is at distance $< 1/m$ from $|\zeta| = \rho$. □

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3. *Cremer points*

Let $g(z) = \lambda_0 z + \sum_{k>1} b_k z^k$ be a holomorphic function defined on $D(0,\rho), \rho > 1$. Assume $\lambda_0 = e^{i\theta}$ is not a root of unity, and that g is not linearizable at 0, i.e. is a Cremer point. We want to study the existence of small cycles near 0.

Recall that 0 is approximated by small cycles if for every neighborhood N of the origin there is a periodic orbit $(z_k)_{k=0}^{n-1}$ which is completely contained in $N \setminus (0)$.

Yoccoz [**Y**] proved that if 0 is a Cremer point for $\lambda z + z^2$, then it is approximated by small cycles. The question is open for polynomials of higher degree. As mentioned above, Perez Marco [**PM**] has shown the existence of a map g as above, say injective in the unit disc D but such that every orbit contained in D clusters at 0; in particular there are no cycles in D except 0.

Suppose that g as above belongs to a one-parameter family

$$g_\lambda(z) := \lambda z + \sum_{k>1} b_k(\lambda) z^k,$$

where $b_k(\lambda_0) = b_k$. (Assume g_λ converges uniformly in $D(0,\rho)$ for λ in $\Delta \supset \overline{\Delta(0,1)}$.) Following a version of Cremer's theorem [**M**] there exists a dense G_δ set G on the unit circle such that for $\lambda \in G$ the fixed point 0 is approximable by small cycles of nearby maps. We want to study approximation by small cycles near a Cremer point, using the method of Theorem 2.4. Let $\{p_m/q_m\}$ denote the mth continued fraction for $\theta/2\pi$. Recall [**L**] that

$$\left| \frac{\theta}{2\pi} - \frac{p_m}{q_m} \right| \leq \frac{1}{q_m q_{m+1}}.$$

THEOREM 3.1. *For every integer m large enough, and $0 < \epsilon \ll 1$, there exists $\lambda_1 \in D_m^\epsilon := \{|\lambda - \lambda_0| < \epsilon/q_m\}$ so that g_{λ_1} has a cycle of order q_m contained in $\Delta_m := \{z \mid |z| < 2\epsilon/q_m\} \setminus (0)$. Moreover, if $\{w_1, \ldots, w_{q_m}\}$ is the cycle, then $|w_{k+1} - \lambda_1^k w_1| \leq \epsilon^2/q_m$.*

Proof. Define

$$F_\lambda(z) = \lambda z + \sum_{k>1} b_k(\lambda)(\lambda - \lambda_0)^{k-1} z^k =: \lambda z + \sum_{k>1} a_k(\lambda) z^k.$$

If $\lambda \neq \lambda_0$ then

$$(\lambda - \lambda_0)^{-1} g_\lambda((\lambda - \lambda_0) z) = F_\lambda(z).$$

Observe that $|a_n(\lambda)| \leq C_1 |\lambda - \lambda_0|/r^n$, for some $r > 1$, λ close enough to λ_0, with C_1 independent of n.

Since the coefficients a_n satisfy the above estimate, if $|\lambda - e^{i\theta}| \leq c/q_m, c \ll 1$, we can apply Lemma 2.6 which shows that $F_\lambda^k(1)$ is well defined for $k = 1, \ldots, q_m$. The hypotheses of Theorem 2.4 are satisfied by $F_\lambda(z)$. Hence it follows from $(***)$ in §2 that

$$|F_\lambda^m(1) - \lambda^m| \leq \tilde{C}(m) m |\lambda - e^{i\theta}| + C_2 m^2 |\lambda - e^{i\theta}|^2$$

if $|\lambda - e^{i\theta}| < \epsilon'/m$ for some $\epsilon' > 0$ with $\tilde{C}(m) \to 0$ as $m \to \infty$. In particular, let $D_m^\epsilon := \{\lambda; |\lambda - \lambda_0| \leq \epsilon/q_m\}, 0 < \epsilon \ll 1$. For $\lambda \in D_m^\epsilon$,

$$|F_\lambda^{q_m}(1) - \lambda^{q_m}| \leq \tilde{C}(q_m)\epsilon + C_2 \epsilon^2 \leq \frac{\epsilon}{5}$$

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for m large enough and ϵ small enough. On the other hand, $|e^{2\pi i q_m \theta} - 1| \leq 2\pi/q_{m+1} \leq \epsilon^2$ for m large.

Since as we have already observed in $(**)$ in the proof of Theorem 2.4 that

$$|\lambda^m - \lambda_0^m - m\lambda_0^{m-1}(\lambda - \lambda_0)| \leq \frac{3\epsilon^2}{2}$$

if $|\lambda - \lambda_0| \leq \epsilon/m$, we get that for $|\lambda - \lambda_0| = \epsilon/q_m$

$$|\lambda^{q_m} - 1| \geq q_m|\lambda - e^{i\theta}| - \frac{3}{2}\epsilon^2 - \epsilon^2 \geq \frac{\epsilon}{2}.$$

So on the boundary of D_m^ϵ,

$$|F_\lambda^{q_m}(1) - 1 - (\lambda^{q_m} - 1)| \leq \frac{\epsilon}{5} < \frac{\epsilon}{2} \leq |\lambda^{q_m} - 1|.$$

But $e^{2\pi i p_m/q_m}$ is a zero for $\lambda^{q_m} - 1$, and

$$|e^{2\pi i p_m/q_m} - e^{i\theta}| \leq \left|\theta - \frac{2\pi p_m}{q_m}\right| \leq \frac{2\pi}{q_m q_{m+1}} < \frac{\epsilon}{q_m}.$$

Rouché's theorem implies that $F_\lambda^{q_m}(1) - 1$ has a zero for some $\lambda_1 \in D_m^\epsilon$. Observe that $\lambda_1 \neq \lambda_0$ since λ_0 is not a root of unity. Set $w_1 := \lambda_1 - \lambda_0$, $w_1 \neq 0$. We have $g_{\lambda_1}^{q_m}(w_1) = w_1 F_{\lambda_1}^{q_m}(1) = w_1$. So w_1 is contained in a cycle for g_{λ_1}. Since for all $k \leq q_m$,

$$|g_{\lambda_1}^k(\lambda_1 - \lambda_0) - \lambda_1^k(\lambda_1 - \lambda_0)| < |\lambda_1 - \lambda_0|\epsilon \leq \frac{\epsilon^2}{q_m}$$

we get that

$$|g_{\lambda_1}^k(w_1) - \lambda_1^k w_1| \leq \frac{\epsilon^2}{q_m}.$$

From Lemma 2.6 the orbit of $F_{\lambda_1}^k(1)$ is contained in $D(0, \tilde{r})$ with $0 < \tilde{r} < \rho$; if ϵ is small enough we can assume $\tilde{r} < 2$. So $g_{\lambda_1}^k(w_1) = w_1 F_{\lambda_1}^k(1)$ is contained in $\Delta = \{z; |z| < 2\epsilon/q_m\} \setminus \{0\}$. □

4. *Herman rings*

We want to prove a closing lemma for recurrent points in a Herman ring of a rational function. We prove first a general result.

For $0 < r_1 < r_2$, let $C(r_1, r_2) := \{z \in \mathbb{C}; r_1 < |z| < r_2\}$.

THEOREM 4.1. *Let r_1, r be two constants, $r_1 < 1 < r$. Suppose $F_\lambda(z) = F(\lambda, z)$ is a holomorphic function on $\Delta(e^{i\theta}, \rho) * C(r_1, r)$ such that $F_{\lambda_0}(z) = \lambda_0 z$, where $\lambda_0 = e^{i\theta}$ is not a root of unity. Assume*

$$F_\lambda(z) = g(\lambda)z + \sum_{k \neq 1} b_k(\lambda) z^k$$

with $\partial g/\partial\lambda(\lambda_0) \neq 0$. Then given any $\epsilon > 0$ there is an integer N and $\lambda \in \Delta(e^{i\theta}, \epsilon)$ such that the orbit $\{F_\lambda^n(1)\}_{n=0}^N$ is closed and follows the orbit $\{F_{e^{i\theta}}^n(1)\}_{n=0}^N$ closer than ϵ.

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Proof. Without loss of generality we can take as a new parameter λ the function $g(\lambda)$. Hence

$$F_\lambda(z) = \lambda z + \sum_{k\neq 1} b_k(\lambda) z^k,$$

with $b_k(\lambda_0) = 0$. One can make $b_0(\lambda)$ vanish by the following change of coordinates:

$$z = w + c(\lambda).$$

Then $F_\lambda(z)$ takes the form

$$G_\lambda(w) = \lambda w + (\lambda - 1)c(\lambda) + \sum_{n\neq 1} b_n(\lambda)(w + c(\lambda))^n = \tilde{\lambda} w + \sum_{n\neq 1} a_n(\lambda) w^n.$$

Then

$$a_0(\lambda) = (\lambda - 1)c(\lambda) + b_0(\lambda) + \sum_{n\geq 2} b_n(\lambda) c^n(\lambda).$$

Solve the equation $a_0(\lambda) = 0$ to get $c(\lambda) = O(|\lambda - e^{i\theta}|)$.

Next check

$$\tilde{\lambda} = \lambda + \sum_{n\geq 2} b_n(\lambda) n c^{n-1}(\lambda) = \lambda + O(|\lambda - e^{i\theta}|^2)$$

so we can use $\tilde{\lambda}$ as new parameter λ. So we can assume that

$$F_\lambda(z) = \lambda z + \sum_{k\neq 0,1} b_k(\lambda) z^k,$$

and $F_{\lambda_0}(z) = \lambda_0 z$.

Fix $r_1 < \tilde{r}_1 < r_1^* < 1 < r^* < \tilde{r} < r$. We can assume that $r_1 = r^{-1}, r_1^* = r^{*-1}$. Shrinking r and ρ, there is a constant C_1 such that $|b_n(\lambda)| \leq C_1 |\lambda - e^{i\theta}|/r^{|n|}$.

LEMMA 4.2. *There is a constant $c > 0$ such that if $|\lambda - e^{i\theta}| < \sigma_m := c/m$, then for every $r_1^* < |z| < r^*$, the iterates $F_\lambda^k(z)$ satisfy $\tilde{r}_1 < |F_\lambda^k(z)| < \tilde{r}$ for $k = 1, \ldots, m$.*

Proof of Lemma 4.2. Suppose $\tilde{r}_1 < |z| < \tilde{r}$ and $|\lambda - e^{i\theta}| < \sigma_m$. Then

$$\begin{aligned} |F_\lambda(z)| &\leq (1 + \sigma_m)|z| + C_1 \sigma_m \left(\frac{\tilde{r}}{r - \tilde{r}} + \frac{r_1}{\tilde{r}_1 - r_1} \right) \\ &\leq |z|(1 + \sigma_m(1 + A)) \end{aligned}$$

and

$$|F_\lambda(z)| \geq |z|(1 - \sigma_m(1 + A)).$$

If $(1 + \sigma_m(1 + A))^m \leq \tilde{r}/r^*$ and $(1 - \sigma_m(1 + A))^m > \tilde{r}_1/r_1^*$ for every m, then for $r_1^* < |z| < r^*$, we will have

$$\tilde{r}_1 < |F_\lambda^m(z)| < \tilde{r}.$$

Clearly, there is a $c > 0$ such that if $\sigma_m := c/m$, the required inequalities are valid. □

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Continuation of the proof of the Theorem 4.1. Let

$$\Phi_{k,m} := 1 + \lambda^{k-1} + \cdots + (\lambda^{k-1})^{m-1}, \quad k \in \mathbb{Z}, m \in \mathbb{N}.$$

Then

$$\begin{aligned} F_\lambda(z) &= \lambda z + \sum_{k\neq 0,1} b_k(\lambda) z^k \\ F_\lambda^m(1) &= \lambda^m + \sum_{k\neq 0,1} b_k(\lambda)\lambda^{m-1}\Phi_{k,m} + C'_m|\lambda - e^{i\theta}|^2 \end{aligned}$$

which defines C'_m. We get

$$\begin{aligned} F_\lambda^{m+1}(1) &= \lambda^{m+1} + \sum_{k\neq 0,1} b_k(\lambda)\lambda^m \Phi_{k,m+1} \\ &\quad + \sum_{k\neq 0,1} b_k(\lambda)\lambda^{mk}\left\{\left[1 + \sum_{n\neq 0,1} \frac{b_n(\lambda)}{\lambda}\Phi_{n,m}(\lambda) + \frac{C'_m}{\lambda^m}|\lambda - e^{i\theta}|^2\right]^k - 1\right\} \\ &\quad + C'_m\lambda|\lambda - e^{i\theta}|^2. \end{aligned}$$

As in the proof of Theorem 2.4 we want to prove by induction that $|C'_m| \leq C_3 m^2$, where

$$C_3 = 1 + 3\frac{C_1}{r(1-\rho/r)^2} * \left[9C_1\frac{\rho/r}{1-\rho/r} + 3\right].$$

Observe first that

$$|\Phi_{-n,m}| \leq 3m\left(1 + \frac{2\epsilon}{m}\right)^{(m-1)(n-1)}$$

for $|\lambda - e^{i\theta}| < \epsilon/m$, where ϵ is a small number as in the statement of the theorem. As in the proof of Theorem 2.4 let $c'' = \hat{\epsilon}/(m+1)$, where $\hat{\epsilon}$ is to be determined. We want to prove the existence of an $\hat{\epsilon} > 0$ such that for every m, $|C'_m| \leq C_3 m^2$ if $|\lambda - e^{i\theta}| < \hat{\epsilon}/m$.

Notice at first that if $|\lambda - e^{i\theta}| < c''_{m+1}$ then

$$\begin{aligned} &\sum_{n\geq 1} C_1\frac{|\lambda - e^{i\theta}|}{r^n}\left(\frac{|\Phi_{n,m}|}{|\lambda|} + \frac{|\Phi_{-n,m}|}{|\lambda|}\right) + \frac{|C'_m|}{|\lambda^m|}|\lambda - e^{i\theta}|^2 \\ &\leq \sum_{n\geq 1}\frac{\hat{\epsilon}}{m+1}\frac{C_1}{r^n}4m\left(1 + \frac{\epsilon}{m+1}\right)^{(m-1)(n-1)}\frac{1}{1-\frac{\hat{\epsilon}}{m+1}} + \frac{C_3\hat{\epsilon}^2}{\left(1-\frac{\hat{\epsilon}}{m+1}\right)^m}. \end{aligned}$$

If $\hat{\epsilon} \leq \delta_0$ is fixed, then the above sum is $\ll 1$. So, in particular,

$$1 - \sum_{n\geq 1} C_1\frac{|\lambda - e^{i\theta}|}{r^n}\left(\frac{|\Phi_{n,m}|}{|\lambda|} + \frac{|\Phi_{-n,m}|}{|\lambda|}\right) - \frac{|C'_m|}{|\lambda^m|}|\lambda - e^{i\theta}|^2 > 0.$$

Hence we have for $|\lambda - e^{i\theta}| < c''_{m+1}$,

$$\begin{aligned} |C'_{m+1}| &\leq |\lambda||C'_m| + \sum_{k\geq 2}\frac{|b_k(\lambda)|}{|\lambda - e^{i\theta}|^2}|\lambda^{mk}| \\ &\quad \times\left\{\left[1 + \sum_{n\geq 1} C_1\frac{|\lambda - e^{i\theta}|}{r^n}\left(\frac{|\Phi_{n,m}|}{|\lambda|} + \frac{|\Phi_{-n,m}|}{|\lambda|}\right) + \frac{|C'_m|}{|\lambda^m|}|\lambda - e^{i\theta}|^2\right]^k - 1\right\} \end{aligned}$$

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$$+\sum_{k\geq 1}\frac{|b_{-k}(\lambda)|}{|\lambda-e^{i\theta}|^2}|\lambda^{-mk}|$$
$$*\left\{\left[1-\sum_{n\geq 1}C_1\frac{|\lambda-e^{i\theta}|}{r^n}\left(\frac{|\Phi_{n,m}|}{|\lambda|}+\frac{|\Phi_{-n,m}|}{|\lambda|}\right)-\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right]^{-k}-1\right\}.$$

Here we use the inequality

$$|(1+\eta)^{-k}-1|\leq(1-|\eta|)^{-k}-1,\quad |\eta|<1.$$

As in the proof of Theorem 2.4 we get

$$\begin{aligned}|C'_{m+1}| &\leq (1+c''_{m+1})|C'_m|+\frac{C_1}{|\lambda-e^{i\theta}|}\sum_{k\geq 2}\frac{1}{r^k}(1+c''_{m+1})^{mk}\\ &*\left\{\left[1+\sum_{n\geq 1}|\lambda-e^{i\theta}|\frac{C_1}{r^n}\left(\frac{|\Phi_{n,m}|}{|\lambda|}+\frac{|\Phi_{-n,m}|}{|\lambda|}\right)+\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right]^{k}-1\right\}\\ &+\frac{C_1}{|\lambda-e^{i\theta}|}\sum_{k\geq 1}\frac{1}{r^k}(1+2c''_{m+1})^{mk}\\ &*\left\{\left[1-\sum_{n\geq 1}C_1\frac{|\lambda-e^{i\theta}|}{r^n}\left(\frac{|\Phi_{n,m}|}{|\lambda|}+\frac{|\Phi_{-n,m}|}{|\lambda|}\right)-\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right]^{-k}-1\right\}.\end{aligned}$$

We also have

$$\begin{aligned}&\left(1-\sum_{n\geq 1}C_1\frac{|\lambda-e^{i\theta}|}{r^n}\left(\frac{|\Phi_{n,m}|}{|\lambda|}+\frac{|\Phi_{-n,m}|}{|\lambda|}\right)-\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right)^{-k}\\ &\leq \left(1+2\sum_{n\geq 1}C_1\frac{|\lambda-e^{i\theta}|}{r^n}\left(\frac{|\Phi_{n,m}|}{|\lambda|}+\frac{|\Phi_{-n,m}|}{|\lambda|}\right)-\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right)^{k}\end{aligned}$$

since the sum is small as we noticed first. We get

$$\begin{aligned}&|C'_{m+1}|\\ &\leq (1+c''_{m+1})|C'_m|+\frac{C_1}{|\lambda-e^{i\theta}|}\sum_{k\geq 2}\frac{1}{r^k}(1+c''_{m+1})^{mk}\\ &*\left\{\left[1+\sum_{n\geq 1}|\lambda-e^{i\theta}|\frac{C_1}{r^n}\frac{4m\left(1+\frac{2\epsilon}{m}\right)^{(n-1)(m-1)}}{|\lambda|}+\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right]^{k}-1\right\}\\ &+\frac{C_1}{|\lambda-e^{i\theta}|}\sum_{k\geq 1}\frac{1}{r^k}(1+c''_{m+1})^{mk}\\ &*\left\{\left[1+2\sum_{n\geq 1}C_1\frac{|\lambda-e^{i\theta}|}{r^n}4m\left(1+\frac{2\epsilon}{m}\right)^{(n-1)(m-1)}+2\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right]^{k}-1\right\}\\ &\leq (1+c''_{m+1})|C'_m|+\frac{C_1}{|\lambda-e^{i\theta}|}\sum_{k\geq 1}\frac{1}{r^k}(1+c''_{m+1})^{mk}\\ &*2\left\{\left[1+8\sum_{n\geq 1}|\lambda-e^{i\theta}|\frac{C_1}{r^n}\frac{m\left(1+\frac{\epsilon}{m}\right)^{(n-1)(m-1)}}{|\lambda|}+2\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right]^{k}-1\right\}.\end{aligned}$$

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Therefore

$$\begin{aligned}|C'_{m+1}| &\leq C_3m^2+\hat{\epsilon}C_3m+2\frac{C_1}{|\lambda-e^{i\theta}|}*\sum_{k\geq 1}\frac{1}{r^k}(1+c''_{m+1})^{mk}\\ &\quad *\left\{\left[1+8\sum_{n\geq 1}|\lambda-e^{i\theta}|\frac{C_1}{r^n}\frac{m\left(1+\frac{2\epsilon}{m}\right)^{(n-1)(m-1)}}{|\lambda|}+2\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right]^k\right\}\\ &\quad -2\frac{C_1}{|\lambda-e^{i\theta}|}\sum_{k\geq 1}\frac{1}{r^k}(1+c''_{m+1})^{mk}.\end{aligned}$$

Hence, making δ_0 smaller if necessary,

$$\begin{aligned}&\frac{(1+c''_{m+1})^m}{r}\left[1+8\sum_{n\geq 1}|\lambda-e^{i\theta}|\frac{C_1}{r^n}\frac{m\left(1+\frac{2\epsilon}{m}\right)^{(n-1)(m-1)}}{|\lambda|}+2\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right]\\ &\leq \frac{\left(1+\frac{\hat{\epsilon}}{m+1}\right)^m}{r}\\ &\quad *\left[1+8\sum_{n\geq 1}\frac{\hat{\epsilon}}{m+1}\frac{C_1}{r^n}\frac{m\left(1+\frac{2\epsilon}{m}\right)^{(n-1)(m-1)}}{1/\left(1-\frac{\hat{\epsilon}}{m+1}\right)}+2\frac{C_3\hat{\epsilon}^2}{\left(1-\frac{\hat{\epsilon}}{m+1}\right)^m}\right]\\ &\leq \frac{1+2\hat{\epsilon}}{r}\left[1+9C_1\hat{\epsilon}\sum_{n\geq 1}\left(\frac{\rho}{r}\right)^n+2C_3\hat{\epsilon}^2(1+4\hat{\epsilon})\right].\end{aligned}$$

If we choose $\hat{\epsilon}$ small enough we get that the above sum is $< \rho/r < 1$. Therefore the geometric series are convergent.

We obtain

$$\begin{aligned}&|C'_{m+1}|\\ &\leq C_3m^2+\hat{\epsilon}C_3m+2\frac{C_1}{|\lambda-e^{i\theta}|}\\ &\quad *\left[\frac{\frac{1}{r}(1+c''_{m+1})^m\left[1+8\sum_{n\geq 1}|\lambda-e^{i\theta}|\frac{C_1}{r^n}\frac{m\left(1+\frac{2\epsilon}{m}\right)^{(n-1)(m-1)}}{|\lambda|}+2\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right]}{1-\frac{1}{r}(1+c''_{m+1})^m\left[1+8\sum_{n\geq 1}|\lambda-e^{i\theta}|\frac{C_1}{r^n}\frac{m\left(1+\frac{2\epsilon}{m}\right)^{(n-1)(m-1)}}{|\lambda|}+2\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right]}\right.\\ &\quad \left.-\frac{(1+c''_{m+1})^m/r}{1-(1+c''_{m+1})^m/r}\right].\end{aligned}$$

So performing the subtraction,

$$\begin{aligned}&|C'_{m+1}|\\ &\leq C_3m^2+\hat{\epsilon}C_3m+2\frac{C_1}{|\lambda-e^{i\theta}|}\frac{(1+c''_{m+1})^m}{r}\\ &\quad *\frac{\left(1+8\sum_{n\geq 1}|\lambda-e^{i\theta}|\frac{C_1}{r^n}\frac{m\left(1+\frac{2\epsilon}{m}\right)^{(n-1)(m-1)}}{|\lambda|}+2\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right)\left(1-\frac{(1+c''_{m+1})^m}{r}\right)}{(1-\rho/r)^2}\end{aligned}$$

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$$-2\frac{C_1}{|\lambda-e^{i\theta}|}\frac{(1+c''_{m+1})^m}{r}$$
$$*\frac{1-\frac{1}{r}(1+c''_{m+1})^m\left[1+8\sum_{n\geq 1}|\lambda-e^{i\theta}|\frac{C_1}{r^n}\frac{m\left(1+\frac{2\epsilon}{m}\right)^{(n-1)(m-1)}}{|\lambda|}+2\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right]}{(1-\rho/r)^2}.$$

Hence, by the definition of C_3,

$$\begin{aligned}
|C'_{m+1}| &\leq C_3m^2+\hat{\epsilon}C_3m+2\frac{C_1}{|\lambda-e^{i\theta}|}\frac{(1+2\hat{\epsilon})}{r(1-\rho/r)^2}\\
&\quad *\left[8\sum_{n\geq 1}|\lambda-e^{i\theta}|\frac{C_1}{r^n}\frac{m(1+\frac{2\epsilon}{m})^{(n-1)(m-1)}}{|\lambda|}+2\frac{|C'_m|}{|\lambda^m|}|\lambda-e^{i\theta}|^2\right]\\
&\leq C_3m^2+\hat{\epsilon}C_3m+2m\frac{C_1(1+2\hat{\epsilon})}{r(1-\rho/r)^2}*\left[9C_1\sum_{n\geq 1}\left(\frac{\rho}{r}\right)^n+3C_2\hat{\epsilon}\right]\\
&\leq C_3m^2+\left(\hat{\epsilon}C_3+3\frac{C_1}{r(1-\rho/r)^2}*\left[9C_1\frac{\rho/r}{1-\rho/r}+3C_3\hat{\epsilon}\right]\right)m\\
&\leq C_3m^2+C_3m
\end{aligned}$$

if $\hat{\epsilon}C_3<1$. But then $|C'_m|\leq C_3(m+1)^2$ and the inductive estimate follows. We have found only finitely many conditions on $\hat{\epsilon}$. Hence we have shown that if $|\lambda-e^{i\theta}|\leq\hat{\epsilon}/m$, then

$$F^m_\lambda(1)=\lambda^m+\sum_{k\neq 0,1}b_k(\lambda)\lambda^{m-1}\Phi_{k,m}(\lambda)+C'_m(\lambda)|\lambda-e^{i\theta}|^2,$$

where $|C'_m(\lambda)|\leq C_3m^2$.

Pick any $\epsilon>0,\epsilon<\hat{\epsilon}$. We estimate $F^m_\lambda(1)$ for $|\lambda-e^{i\theta}|\leq\epsilon/m$. First of all we get from $(**)$ in the proof of Theorem 2.4 that

$$|\lambda^m-e^{im\theta}-me^{i(m-1)\theta}(\lambda-e^{i\theta})|\leq\frac{3\epsilon^2}{2}.$$

Next

$$\begin{aligned}
\left|\sum_{k\neq 0,1}b_k(\lambda)\lambda^{m-1}\Phi_{k,m}\right| &\leq \sum_{k\geq 2}\frac{C_1|\lambda-e^{i\theta}|}{r^k}\left(1+\frac{\epsilon}{m}\right)^{m-1}m\left(\left(1+\frac{\epsilon_0}{m}\right)^{m-1}\right)^{k-1}\\
&\quad+\sum_{k\geq 1}\frac{C_1|\lambda-e^{i\theta}|}{r^k}\left(1+\frac{\epsilon}{m}\right)^{m-1}m\left(\left(1+\frac{2\epsilon_0}{m}\right)^{m-1}\right)^{k-1}.
\end{aligned}$$

Both sums are bounded by an expression of the form $\tilde{C}(m)m|\lambda-e^{i\theta}|$, where $\tilde{C}(m)\to 0$ as $m\to\infty$. Here we have used the fact that $e^{i\theta}$ is not a root of unity and Lemma 2.5, hence for fixed k,

$$\lim_{m\to\infty}\frac{1}{m}\sup_{D_m}|\Phi_{k,m}|=0.$$

So

$$F^m_\lambda(1)=\lambda^m+\tilde{C}(m)m|\lambda-e^{i\theta}|+C_3m^2|\lambda-e^{i\theta}|^2.$$

We can finish the proof as in Theorem 2.4 by applying Rouché's Theorem. □

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COROLLARY 4.3. *Let R_0 be a rational function of degree $d \geq 2$. Assume that R_0 has a Herman ring U with $R_0(U) = U$. For every $z_0 \in U$, there exists $\delta_0 > 0$ and a holomorphic family $R_\mu(z)$ of rational functions of degree $\leq d_1$, defined for $|\mu - 1| < \delta_0$, with $R_1(z) \equiv z$ such that for every $\epsilon > 0$ and $\delta > 0$ there exists μ_1 with $|\mu_1 - 1| < \delta$ and there is an N such that z_0 is periodic of order N for $R := R_{\mu_1} \circ R_0$ with $|R^k(z_0) - R_0^k(z_0)| < \epsilon$ for every $0 \leq k \leq N$.*

Proof. Choose $C(r) := C(1/r, r), r > 1$ such that there is $\Phi : C(r) \to U$, a parametrization satisfying $(\Phi^{-1} R_0 \Phi)(z) = \lambda_0 z$, where $z \to \lambda_0 z$ is an irrational rotation. Moreover, we can set $\Phi(1) = z_0$. Fix $1 < r' < r$ and set $V := \Phi(C(r'))$. Fix $\delta_0 > 0$ such that $S_\mu := \Phi(\mu \Phi^{-1})$ is well defined in a neighborhood of $\overline{V} \times \{\mu; |\mu - 1| \leq \delta_0\} =: Y$.

If f is defined on U and maps $\Phi(S^1)$ to U we define

$$\Lambda(f) = \frac{1}{2\pi} \int_0^{2\pi} \Phi^{-1}(f R_0(\Phi))(e^{i\theta}) e^{-i\theta}\, d\theta.$$

Then $\Lambda(f)$ is the coefficient of the linear term of the Laurent expansion of $\Phi^{-1}(f R_0(\Phi))$ around the origin. In order to apply Theorem 4.1 to a family $\lambda \to \Phi^{-1}(f_\lambda R_0(\Phi))$ we have to check that the derivative $\partial \Lambda(f_\lambda)/\partial \lambda$ does not vanish.

Clearly, $\Lambda(S_\mu) = \mu \lambda_0$. Let $\tilde{R}_\mu$ be an approximation of S_μ on Y by rational functions. We can assume that $\tilde{R}_\mu$ depends holomorphically in the parameter μ. We can also assume that $\tilde{R}_\mu$ is close enough to S_μ so that $\partial \Lambda / \partial \mu(\tilde{R}_\mu) \neq 0$. Clearly, the degree of R_μ is bounded. Then R_1 is close to the identity on $V \subset\subset U$.

Define $R_\mu := \tilde{R}_\mu - \tilde{R}_1 + \mathrm{Id}$. Then $R_1 = \mathrm{Id}$. Define $F_\lambda(z) := \Phi^{-1}(R_\lambda \circ R_0(\Phi))(z)$. The function F_λ satisfies the hypothesis of Theorem 4.1. The result follows. □

Remark 4.4. Assume now that U is periodic of period l, $R_0^l(U) = (U)$, $R_0^j(U) \neq U$, $j < l$. We have that $\Phi^{-1} R_0^l \Phi(z) = \lambda_0 z$. As above we consider $\Phi^{-1} R^l \Phi := F(\lambda, z)$. Given any ϵ, from Corollary 4.4, we get a value λ_1 arbitrarily close to 1 such that $(R_{\lambda_1} \circ R_0)^\ell(1)$ has a periodic point of order m, whose orbit is ϵ close to $(R_0)^{\ell k}(1)$, $k \leq m$. It follows easily that the orbits $(R_{\lambda_1} \circ R_0)^n(1)$ and $(R_0)^n(1)$ are also close for $n \leq m\ell$.

Acknowledgement. The authors would like to thank the referee for helpful observations that have been incorporated in this paper.

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