10

Positive Functionals

10.1 Motivation and Setup

Many physical quantities have a definite sign (e.g., positive mass, positive energy, negative charge, etc.). With this in mind, it is an important question whether the structure of a causal variational principle gives rise to positive functionals. We now briefly explain the structural results known at present. These were obtained with two different methods. One method is to make us of the fact that, given a minimizer of a variational principle, second variations are always nonnegative. This method was worked out in [49], and we will give an outline in Sections 10.2 and 10.3. The second method is to use that the action of a given minimizing measure ρ is smaller than the action of any other test measure $\tilde{\rho}$. By a suitable choice of $\tilde{\rho}$, one gets surface layer integral with a definite sign. This second method is explored in detail in [59], and applications are worked out. Here, we only explain the basic idea in Section 10.4.

For technical simplicity, we restrict attention to causal variational principles in the *non-compact smooth setting* (see (6.10) and Section 6.3).

10.2 Positivity of the Hessian of ℓ

Let ρ be a minimizer of the causal action. According to the EL equations (7.4), the function ℓ is minimal on M. This clearly implies that its Hessian (as computed in any chart) is positive semi-definite, that is,

$$D^2\ell(x) \geq 0 \qquad \text{for all } x \in M := \operatorname{supp}\rho\,. \tag{10.1}$$

This is the first nonnegative quantity obtained from the fact that ρ is a minimizer. In view of the restricted EL equations (7.13), the zero and first-order derivatives of ℓ vanish for all $x \in M$. Adding such lower derivative terms, we can write (10.1) with jet derivatives as

$$\nabla^2\ell|_x(\mathfrak{u},\mathfrak{u}) \geq 0 \qquad \text{for all } x \in M\,, \tag{10.2}$$

where, following our conventions (i) and (ii) introduced after (8.12) in Section 8.1,

$$\nabla^2\ell|_x(\mathfrak{u},\mathfrak{u}) := a(x)^2\,\ell(x) + 2\,a(x)\,D_u\ell(x) + D^2\ell|x(u,u)\,. \tag{10.3}$$

Integrating over M gives the following result:

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Proposition 10.2.1 *Let ρ be a minimizer of the causal action. Then*

$$\int_M \nabla^2 \ell|_x(\mathfrak{u},\mathfrak{u})\, \mathrm{d}\rho(x) \geq 0 \qquad \text{for all } \mathfrak{u} \in \mathfrak{J}_0\,. \tag{10.4}$$

10.3 Positivity of Second Variations Generated by Jets

We now analyze second variations for a special class of variations of the measure ρ to obtain another positive functional on jets. Similar to [62, Section 3], we consider measures of the form

$$\tilde{\rho}_\tau = (F_\tau)_*\big(f_\tau\, \rho\big) \qquad \text{for } \tau \in (-\tau_{\max}, \tau_{\max}), \tag{10.5}$$

with smooth mappings

$$\begin{aligned} &f \in C^\infty((-\tau_{\max}, \tau_{\max}) \times M, \mathbb{R}^+) \quad \text{and} \\ &F \in C^\infty((-\tau_{\max}, \tau_{\max}) \times M, \mathcal{F})\,, \end{aligned} \tag{10.6}$$

where the star denotes the push-forward measure defined by $((F_\tau)_*\mu)(\Omega) = \mu(F_\tau^{-1}(\Omega))$ (for details, see the preliminaries in Section 2.3 or, e.g., [15, Section 3.6]). We assume that for $\tau = 0$, the variation is trivial (8.5). Moreover, for technical simplicity, we assume that F_τ and f_τ are *trivial outside a compact set* $K \subset M$, meaning that

$$f_\tau|_{M\setminus K} \equiv 1 \qquad \text{and} \qquad F_\tau|_{M\setminus K} \equiv \mathbb{1}\,. \tag{10.7}$$

Finally, in order to satisfy the volume constraint on the right-hand side of (6.13), we assume that

$$\int_K f_\tau(x)\, \mathrm{d}\rho(x) = \rho(K) \qquad \text{for all } \tau \in (-\tau_{\max}, \tau_{\max})\,. \tag{10.8}$$

Then, the transformation (10.5) is described infinitesimally by the smooth and compactly supported jet

$$\mathfrak{u} = (a, u) := \big(\dot{f}_0, \dot{F}_0\big) \in \mathfrak{J}_0\,, \tag{10.9}$$

where the dot denotes the τ-derivative. Moreover, we differentiate the volume constraint (10.8) to obtain

$$\int_K a(x)\, \mathrm{d}\rho(x) = 0\,. \tag{10.10}$$

We now compute the first and second variations of the action. Combining (6.14) with the definition of the push-forward measure, we obtain

$$\begin{aligned} \mathcal{S}\big(\tilde{\rho}_\tau\big) - \mathcal{S}(\rho) = 2 &\int_K \mathrm{d}\rho(x) \int_{M\setminus K} \mathrm{d}\rho(y)\, \Big(f_\tau(x)\, \mathcal{L}\big(F_\tau(x), y\big) - \mathcal{L}(x,y)\Big) \\ &+ \int_K \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \Big(f_\tau(x)\, f_\tau(y)\, \mathcal{L}\big(F_\tau(x), F_\tau(y)\big) - \mathcal{L}(x,y)\Big)\,. \end{aligned} \tag{10.11}$$

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The first variation vanishes because

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{S}(\tilde{\rho}_\tau)\Big|_{\tau=0} = 2\int_K \mathrm{d}\rho(x)\int_M \mathrm{d}\rho(y)\,\nabla_{1,\mathfrak{u}}\mathcal{L}(x,y)$$
$$= 2\int_K \nabla_{\mathfrak{u}}\Big(\ell(x)+\mathfrak{s}\Big)\,\mathrm{d}\rho(x) = 0\,, \tag{10.12}$$

where in the last step we used (7.13) and (10.10) (and ∇_1 denotes the partial derivative acting on the first argument of the Lagrangian). Differentiating (10.11) twice, the second variation is computed to be

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathcal{S}(\tilde{\rho}_\tau)\Big|_{\tau=0} = 2\int_K \mathrm{d}\rho(x)\int_K \mathrm{d}\rho(y)\,\nabla_{1,\mathfrak{u}}\nabla_{2,\mathfrak{u}}\mathcal{L}(x,y)$$
$$+2\int_K \mathrm{d}\rho(x)\int_M \mathrm{d}\rho(y)\,\Big(a(x)\,D_{1,u}\mathcal{L}(x,y)$$
$$+D_{1,u}D_{1,u}\mathcal{L}(x,y)+\big(\ddot{f}_0(x)+D_{1,\ddot{F}_0}\big)\mathcal{L}(x,y)\Big)\,. \tag{10.13}$$

In the last line, we can carry out the y-integration using (7.3). Applying the EL equations (7.13), we obtain

$$\int_K \mathrm{d}\rho(x)\int_M a(x)\,D_{1,u}\mathcal{L}(x,y)\;\mathrm{d}\rho(y) = 0, \tag{10.14}$$
$$\int_K \mathrm{d}\rho(x)\int_M D_{1,u}D_{1,u}\mathcal{L}(x,y)\;\mathrm{d}\rho(y) = D^2\ell|_x(u,u) = \nabla^2\ell|_x(\mathfrak{u},\mathfrak{u}), \tag{10.15}$$
$$\int_K \mathrm{d}\rho(x)\int_M \big(\ddot{f}_0(x)+D_{1,\ddot{F}_0}\big)\mathcal{L}(x,y)\;\mathrm{d}\rho(y) = \int_K \ddot{f}_0(x)\,\mathfrak{s}\;\mathrm{d}\rho(x) \overset{(10.8)}{=} 0\,. \tag{10.16}$$

We thus obtain the simple formula

$$\frac{1}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathcal{S}(\tilde{\rho}_\tau)\Big|_{\tau=0} = \int_K \mathrm{d}\rho(x)\int_K \mathrm{d}\rho(y)\,\nabla_{1,\mathfrak{u}}\nabla_{2,\mathfrak{u}}\mathcal{L}(x,y)$$
$$+\int_K \nabla^2\ell|_x(\mathfrak{u},\mathfrak{u})\;\mathrm{d}\rho(x)\,. \tag{10.17}$$

Since ρ is a minimizer and the first variation vanishes, the second variation is necessarily nonnegative, giving rise to the inequality

$$\int_M \mathrm{d}\rho(x)\int_M \mathrm{d}\rho(y)\,\nabla_{1,\mathfrak{u}}\nabla_{2,\mathfrak{u}}\mathcal{L}(x,y)+\int_M \nabla^2\ell|_x(\mathfrak{u},\mathfrak{u})\;\mathrm{d}\rho(x)\geq 0\,, \tag{10.18}$$

subject to the condition that the scalar component of the jet $\mathfrak{u}$ must satisfy the volume constraint (10.10). In the next proposition, we remove this condition with a limiting procedure:

Proposition 10.3.1 *Let ρ be a minimizer of the causal action. Then, the inequality* (10.18) *holds for all $\mathfrak{u}\in\mathfrak{J}_0$.*

Proof Let $\mathfrak{u}=(a,u)\in\mathfrak{J}_0$ be a jet that violates the volume constraint (10.10). Then, choosing a compact set $\Omega\subset M$ with $\rho(\Omega)>0$, the jet $\hat{\mathfrak{u}}:=(\hat{a},u)$ with

$$\hat{a}(x)=a(x)-c(\Omega)\,\chi_\Omega(x) \qquad \text{and} \qquad c(\Omega):=\frac{1}{\rho(\Omega)}\int_\Omega a(x)\;\mathrm{d}\rho(x), \tag{10.19}$$

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(where χ_Ω is the characteristic function) does satisfy (10.10). Choosing the scalar variation $f_\tau = (1-\tau) + \tau\hat{a}$ and a family of diffeomorphisms F_τ with $\dot{F}_0 = u$, the resulting variation satisfies the volume constraint (10.8) (note that $\ddot{f} = 0$). Clearly, due to the characteristic function, the jet $\hat{\mathfrak{u}}$ is no longer smooth, but it has again compact support, and an approximation argument using Lebesgue's dominated convergence theorem shows that the inequality (10.18) also holds for $\hat{\mathfrak{u}}$. Expanding in powers of c, we thus obtain the inequality

$$\begin{aligned} 0 \leq \int_K \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{u}} \mathcal{L}(x,y) + \int_K \nabla^2 \ell|_x(\mathfrak{u},\mathfrak{u})\; \mathrm{d}\rho(x) \\ - 2c \int_M \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \chi_\Omega(x) \nabla_{2,\mathfrak{u}} \mathcal{L}(x,y) \\ + c^2 \int_M \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \chi_\Omega(x)\, \chi_\Omega(y)\, \mathcal{L}(x,y) \\ + \int_M \Big(- 2c\, \chi_\Omega(x)\, \nabla_{\mathfrak{u}} \ell(x) + c^2\, \chi_\Omega(x)^2\, \ell(x) \Big)\; \mathrm{d}\rho(x), \end{aligned} \tag{10.20}$$

(the integrand in the last line arises from the contributions to $\nabla^2\ell|x(\mathfrak{u},\mathfrak{u})$ involving the scalar components of the jets). The last line vanishes due to the restricted EL equations (7.13). Hence

$$\begin{aligned} \int_K \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{u}} \mathcal{L}(x,y) + \int_K \nabla^2 \ell|_x(\mathfrak{u},\mathfrak{u})\; \mathrm{d}\rho(x) \\ \geq 2c \int_K \mathrm{d}\rho(x) \int_\Omega \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \mathcal{L}(x,y) \\ - c^2 \int_K \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \mathcal{L}(x,y) =: A(\Omega)\,. \end{aligned} \tag{10.21}$$

We now let $(\Omega_n)_{n\in\mathbb{N}}$ be an exhaustion of M by compact sets. We distinguish the two cases when $\rho(M)$ is finite and infinite and treat these cases separately. If the total volume $\rho(M)$ is finite, one can take the limit $n \to \infty$ with Lebesgue's dominated convergence theorem to obtain

$$\begin{aligned} \lim_{n\to\infty} \int_K \mathrm{d}\rho(x) \int_{\Omega_n} \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \mathcal{L}(x,y) = \int_K \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \mathcal{L}(x,y) \\ = \int_K \nabla_{\mathfrak{u}} \Big(\ell(x) + \mathfrak{s} \Big)\; \mathrm{d}\rho(x) = \mathfrak{s} \int_K a(x)\; \mathrm{d}\rho(x) \end{aligned} \tag{10.22}$$

$$\begin{aligned} \lim_{n\to\infty} A(\Omega_n) &= 2\, c(M)\, \mathfrak{s} \int_K a(x)\; \mathrm{d}\rho(x) - c(M)^2\, \rho(M)\, \mathfrak{s} \\ &= \frac{\nu}{2\rho(M)} \left(\int_K a(x)\; \mathrm{d}\rho(x) \right)^2 \geq 0\,, \end{aligned} \tag{10.23}$$

where in the last line, we substituted the value of $c(M)$ in (10.19).

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In the remaining case that the volume $\rho(M)$ is infinite, we estimate the terms as follows,

$$\begin{aligned} &c(\Omega_n)^2 \int_K \mathrm{d}\rho(x) \int_K \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\ &\quad \leq c(\Omega_n)^2 \int_K \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \mathcal{L}(x,y) = c(\Omega_n)^2\, \mathfrak{s}\, \rho(K) \to 0 \qquad (10.24) \\ &\int_K \mathrm{d}\rho(x) \int_{\Omega_n} \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \mathcal{L}(x,y) \to \int_K \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \mathcal{L}(x,y) \\ &\quad = \int_K \nabla_{\mathfrak{u}} \Big(\ell(x) + \mathfrak{s} \Big)\, \mathrm{d}\rho(x) = \mathfrak{s} \int_K a(x)\, \mathrm{d}\rho(x)\,. \qquad (10.25) \end{aligned}$$

As a consequence, $A(\Omega_n)$ converges to zero as $n \to \infty$. This concludes the proof. □

We note that, restricting attention to scalar jets, that is, $\mathfrak{u} = (a, 0)$ with a a real-valued function on M, the inequality in Proposition 10.3.1 reduces to

$$\int_M \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, a(x)\, \mathcal{L}(x,y)\, a(y) \geq 0 \qquad \text{for all } a \in C_0^\infty(M)\,. \qquad (10.26)$$

This inequality was first derived in [74, Lemma 3.5] and used for the analysis of minimizing measures. For more details, see also Exercise 10.3.

10.3.1 Application: Hilbert Spaces of Jets

As an application, we now explain how our positive functionals can be used to endow spaces of jets in spacetime with Hilbert space structures. These Hilbert space structures should be very useful because they make functional analytic tools applicable to the analysis of the jet spaces and the causal action principle.

We introduce the following bilinear forms on $\mathfrak{J}_0$,

$$\begin{aligned} \langle \mathfrak{u}, \mathfrak{v} \rangle &:= \int_M \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{u}} \nabla_{2,\mathfrak{v}} \mathcal{L}(x,y) \\ &\quad + \int_M \nabla^2 \ell|_x(\mathfrak{u}, \mathfrak{v})\, \mathrm{d}\rho(x) \qquad (10.27) \\ \big\langle\!\big\langle \mathfrak{u}, \mathfrak{v} \big\rangle\!\big\rangle &:= \langle \mathfrak{u}, \mathfrak{v} \rangle + \int_M \nabla^2 \ell|_x(\mathfrak{u}, \mathfrak{v})\, \mathrm{d}\rho(x)\,. \qquad (10.28) \end{aligned}$$

By Propositions 10.2.1 and 10.3.1, both bilinear forms are positive semi-definite. The second bilinear form has the advantage that it is bounded from below by the bilinear form introduced in Proposition 10.2.1. Dividing out the null space and forming the completion gives real Hilbert spaces of jets denoted by $\mathcal{H}^{\langle \cdot,\cdot \rangle}$ and $\mathcal{H}^{\langle\!\langle \cdot,\cdot \rangle\!\rangle}$, respectively. Obviously,

$$\langle \mathfrak{u}, \mathfrak{u} \rangle \leq \big\langle\!\big\langle \mathfrak{u}, \mathfrak{u} \big\rangle\!\big\rangle\,, \qquad (10.29)$$

giving rise to a norm-decreasing mapping $\mathcal{H}^{\langle\!\langle \cdot,\cdot \rangle\!\rangle} \to \mathcal{H}^{\langle \cdot,\cdot \rangle}$.

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For the scalar components of the jets, the two scalar products (10.27) and (10.28) obviously agree. But they are quite different for the vector components. In order to understand this difference, it is instructive to consider a jet $\mathfrak{u} = (0, u)$, which describes a *symmetry* of the Lagrangian, that is (for details, see [61, Section 3.1])

$$\big(D_{1,u} + D_{2,u}\big)\mathcal{L}(x,y) = 0 \qquad \text{for all } x, y \in M\,. \tag{10.30}$$

For this jet, a direct computation shows that

$$\langle \mathfrak{u}, \mathfrak{u} \rangle = 0\,. \tag{10.31}$$

Hence, symmetry transformations lie in the kernel of the bilinear form $\langle ., . \rangle$ and thus correspond to the zero vector in $\mathcal{H}^{\langle .,. \rangle}$. Generally speaking, the scalar product $\langle ., . \rangle$ makes it possible to disregard symmetry transformations of the causal Lagrangian. However, jets describing symmetry transformations do, in general, correspond to nonzero vectors of the Hilbert space $\mathcal{H}^{\langle\langle .,. \rangle\rangle}$.

10.3.2 Application: A Positive Surface Layer Integral

We now derive a surface layer integral that is not necessarily conserved but that has a definite sign. As explained at the beginning of Section 10.3.1, this can be used to endow the jet space with a Hilbert structure. But in contrast to the scalar products in Section 10.3.1, where the jets were integrated over spacetime, here the scalar product is given as a surface layer integral. This should be useful for analyzing the dynamics of jets in spacetime.

Proposition 10.3.2 *Let $\mathfrak{v}$ be a solution of the linearized field equations* (8.15). *Then, for any compact $\Omega \subset M$, the following surface layer integral is positive,*

$$-\int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{v}} \nabla_{2,\mathfrak{v}} \mathcal{L}(x,y) \geq 0\,. \tag{10.32}$$

Proof Denoting the components of $\mathfrak{v}$ by $\mathfrak{v} = (b, v)$, we evaluate (8.15) for $\mathfrak{u} = \mathfrak{v}$ and integrate over Ω. The resulting integrals can be rewritten as follows,

$$\begin{aligned}
0 &= \int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{v}} \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big) \mathcal{L}(x,y) - \mathfrak{s} \int_\Omega b(x)^2\, \mathrm{d}\rho(x) \\
&= \int_\Omega \nabla^2 \ell|_x(\mathfrak{v},\mathfrak{v})\; \mathrm{d}\rho(x) + \int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{v}} \nabla_{2,\mathfrak{v}} \mathcal{L}(x,y) \\
&= \int_\Omega \nabla^2 \ell|_x(\mathfrak{v},\mathfrak{v})\; \mathrm{d}\rho(x) + \int_\Omega \mathrm{d}\rho(x) \int_\Omega \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{v}} \nabla_{2,\mathfrak{v}} \mathcal{L}(x,y) \qquad (10.33) \\
&\quad + \int_\Omega \mathrm{d}\rho(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\, \nabla_{1,\mathfrak{v}} \nabla_{2,\mathfrak{v}} \mathcal{L}(x,y)\,. \qquad (10.34)
\end{aligned}$$

Using characteristic functions, the expression (10.33) can be written as

$$\int_M \nabla^2 \ell|_x(\chi_\Omega \mathfrak{v}, \chi_\Omega \mathfrak{v})\; \mathrm{d}\rho(x) + \int_M \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \nabla_{1,\chi_\Omega \mathfrak{v}} \nabla_{2,\chi_\Omega \mathfrak{v}} \mathcal{L}(x,y)\,. \tag{10.35}$$

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Approximating the jet $\chi_\Omega \mathfrak{v}$ by smooth jets with compact support, one finds that the integrals in (10.33) are nonnegative by Proposition 10.3.1. Therefore, the last summand (10.34) must be nonpositive. This gives the result. □

We finally remark that in [50, Section 6], the surface layer integral in the last proposition is computed in Minkowski space.

10.4 A Positive Nonlinear Surface Layer Integral

In this section, we briefly mention another method for obtaining a positive surface layer integral. This method and the corresponding positivity results will not be used later in this book. We refer the reader interested in more explanations and applications of this method to [59].

As in Section 9.6, we again consider two measures: A measure ρ that describes the vacuum spacetime, and another measure $\tilde{\rho}$ that typically describes an interacting spacetime. We assume that the vacuum measure is a *minimizer* of the causal action principle as defined in Section 6.3 (see (6.15) and (6.14)). We choose subsets $\Omega \subset M$ and $\tilde{\Omega} \subset \tilde{M}$ having the same finite volume,

$$\rho(\Omega) = \tilde{\rho}(\tilde{\Omega}) < \infty \,. \tag{10.36}$$

In order to construct an admissible test measure $\hat{\rho}$, we "cut out" Ω from ρ and "glue in" the set $\tilde{\Omega}$, that is,

$$\hat{\rho} := \chi_{\tilde{\Omega}}\, \tilde{\rho} + \chi_{M \setminus \Omega}\, \rho \,. \tag{10.37}$$

The measure $\hat{\rho}$ differs from ρ only on a set of finite volume and preserves the volume constraint (see (6.13)). Therefore, we obtain from (6.15) and (6.14) (with $\tilde{\rho}$ replaced by $\hat{\rho}$) that

$$\begin{aligned}
0 &\leq \big(\mathcal{S}(\hat{\rho}) - \mathcal{S}(\rho)\big) \\
&= 2 \int_{\mathcal{F}} \mathrm{d}(\hat{\rho}-\rho)(x) \int_M \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\
&\quad + \int_{\mathcal{F}} \mathrm{d}(\hat{\rho}-\rho)(x) \int_M \mathrm{d}(\hat{\rho}-\rho)(y)\, \mathcal{L}(x,y) \\
&= 2 \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_M \mathrm{d}\rho(y)\, \mathcal{L}(x,y) - 2 \int_\Omega \mathrm{d}\rho(x) \int_M \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\
&\quad + \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(y)\, \mathcal{L}(x,y) - 2 \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_\Omega \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\
&\quad + \int_\Omega \mathrm{d}\rho(x) \int_\Omega \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\
&= 2 \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) - 2 \int_\Omega \mathrm{d}\rho(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\
&\quad + \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(y)\, \mathcal{L}(x,y) - \int_\Omega \mathrm{d}\rho(x) \int_\Omega \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \,.
\end{aligned} \tag{10.38}$$

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We thus obtain the inequality

$$\begin{aligned} 2 \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) &\leq 2 \int_{\Omega} \mathrm{d}\rho(x) \int_{M \setminus \Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y) \\ &\quad - \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(y)\, \mathcal{L}(x,y) - \int_{\Omega} \mathrm{d}\rho(x) \int_{\Omega} \mathrm{d}\rho(y)\, \mathcal{L}(x,y)\,. \end{aligned} \tag{10.39}$$

The left-hand side of this inequality coincides with the first summand in the nonlinear surface layer integral as introduced in (9.62). However, the second summand in (9.62) is now missing. We can regard the left-hand side of (10.39) again as a nonlinear surface layer integral, but with a somewhat different mathematical structure. It is not conserved, but it satisfies instead an *in*equality. The first summand on the right-hand side of (10.39) can be interpreted as the surface area of $\partial\Omega$. The two other summands in (10.39), on the other hand, can be regarded as volume integrals over $\tilde{\Omega}$ and Ω, respectively.

This method can be generalized and adapted in various ways, also to cases when $\tilde{\Omega}$ and Ω do not have the same volume. Moreover, the resulting inequality can be written in a particularly useful form if the measure $\tilde{\rho}$ also satisfies the EL equations. We finally remark that, assuming that $\tilde{\rho}$ is again of the form (10.5) and expanding in powers of τ, one gets inequalities for surface layer integrals involving jet derivatives.

10.5 Exercises

Exercise 10.1 *(Positive functionals for the causal variational principle on $\mathbb{R}$)* We return to the causal variational principles on $\mathbb{R}$ corresponding to the Lagrangian $\mathcal{L}_2$ introduced in Exercise 6.4. Let $\rho = \delta$ be the unique minimizer.

(a) Compute the function $\ell(x)$ and verify that its Hessian is positive (10.1).
(b) Compute the functional in Proposition 10.3.1 for the jets $\mathfrak{u} = (0, \partial_x)$ and $\mathfrak{u} = (1, 0)$.
(c) What are the resulting scalar products (10.27) and (10.28)?

Exercise 10.2 *(Positive functionals for the causal variational principle on S^1)* We return to the causal variational principle on $\mathbb{R}$ introduced in Exercise 6.5. Let ρ be a minimizing measure (6.47) for $0 < \tau < 1$. We choose $\mathfrak{J}_0^{\text{test}}$ as the four-dimensional vector space generated by the scalar jet $(1, 0)$ and the vector jet $(0, \partial_\varphi)$ at the two points.

(a) Compute the function $\ell(x)$ and verify that its Hessian.
(b) Compute the bilinear form in Proposition 10.3.1.
(c) What are the resulting scalar products (10.27) and (10.28)? What are the resulting Hilbert spaces of jets $\mathcal{H}^{\langle \cdot,\cdot \rangle}$ and $\mathcal{H}^{\langle\langle \cdot,\cdot \rangle\rangle}$? Which dimensions do they have? How can this result be understood in view of the space of linearized solutions as computed in Exercise 8.6?

Exercise 10.3 *(A positive operator on scalar jets)* In this exercise, we specialize the statement of Proposition 10.3.1 to scalar jets and work out a few consequences.

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(a) Show that for jets of the form $\mathfrak{u} = (a, 0)$, the statement of Proposition 10.3.1 reduces to the inequality (10.26).

(b) Let ρ be a minimizing measure and $x_0, \ldots, x_N \in M$ be a finite number of spacetime points. Show that the *Gram matrix* L defined by

$$L = \Big(\mathcal{L}(x_i, x_j)\Big)_{i,j=0,\ldots,N}, \tag{10.40}$$

is symmetric and positive semi-definite.

(c) Show that the operator $\mathcal{L}_\rho$ defined by

$$\mathcal{L}_\rho \,:\, C_0^\infty(M) \subset L^2(M, \mathrm{d}\rho) \to L^2(M, \mathrm{d}\rho)\,, \tag{10.41}$$

$$(\mathcal{L}_\rho \psi)(x) := \int_{\mathcal{F}} \mathcal{L}(x, y)\, \psi(y)\, \mathrm{d}\rho(y), \tag{10.42}$$

is a symmetric, densely defined operator on the Hilbert space $L^2(M, \mathrm{d}\rho)$. Prove that this operator is positive semi-definite.

Exercise 10.4 The goal of this exercise is to explore the positive nonlinear surface layer integral of Section 10.4 in the limiting case when the measures of the sets Ω and $\tilde{\Omega}$ tend to zero. For technical simplicity, let us assume that for given $x \in M$ and $y \in \tilde{M}$, there are sequences of open neighborhoods Ω_k of x and $\tilde{\Omega}_k$ of y with $\rho(\Omega_k) = \tilde{\rho}(\tilde{\Omega}_k)$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \rho(\Omega_k) = 0$. Show that, in the limit $k \to \infty$, the inequality (10.39) reduces to the inequality

$$\ell(y) \geq \ell(x)\,. \tag{10.43}$$

Thus, we get back the EL equation (7.9).

In view of this limiting case, the positive nonlinear surface layer integral in Section 10.4 can be regarded as a refined nonlinear version of the EL equations.

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