Math. Proc. Camb. Phil. Soc. (1977), **81**, 97
MPCPS 81–11
Printed in Great Britain

The wave front set of the solution of a simple initial-boundary value problem with glancing rays. II

By F. G. FRIEDLANDER
Department of Applied Mathematics and Theoretical Physics, Cambridge
and R. B. MELROSE
St John's College, Cambridge†

(*Received* 2 *July* 1976)

Introduction. This paper is a sequel to an earlier paper in these Proceedings by one of us ((5); this will be referred to as [I]). The question considered there was that of determining the wave front set of the solution of the boundary value problem

$$(*) \qquad Pu = (1+x)\frac{\partial^2 u}{\partial y_n^2} - \frac{\partial^2 u}{\partial y_1^2} - \ldots - \frac{\partial^2 u}{\partial y_{n-1}^2} - \frac{\partial^2 u}{\partial x^2} = 0,$$

$$u|_{x=0} = f(y) \in \mathscr{E}'(\mathbb{R}^n), \quad u = 0 \quad \text{if} \quad y_n < \inf\{z_n; z \in \operatorname{supp} f\},$$

where $x \in \mathbb{R}^+$, $y \in \mathbb{R}^n$, and $n > 1$; the precise meaning of the boundary condition at $x = 0$ is explained in section 1 below. The principal result of [I] can be expressed concisely by saying that singularities do not propagate along the boundary; a detailed statement is given in Theorem 1·9 of the present paper.

This result has now been extended to second order differential operators with real principal part, subject to appropriate hypotheses that ensure 'diffractive' behaviour at the boundary (10). Similar results have been obtained independently by M. E. Taylor in (16), for second-order hyperbolic equations in the exterior of a domain that is convex with respect to the bicharacteristics of the equation.

There are, nevertheless, some further aspects of the example (*) which are of interest, and these are discussed in the present paper. First, we have a simple argument by which the results of [I] can be extended to the inhomogeneous equation $Pu = v$; to do so we use the global uniqueness theorem for hyperbolic equations, and a technique of extending the operator P to $x < 0$. This is set out in section 1; it would carry over, as it stands, to other hyperbolic equations.

We then return to the homogeneous equation and examine the analytic wave front set of the solution to (*), and the wave-front set calculated with respect to Gevrey classes of functions; such wave front sets have been introduced by Hörmander (6). Our reason for doing this is perhaps best explained for the case of $f = \delta(y)$, the Dirac measure with support at the origin. Then the solution of (*) is a distribution k which is in effect a Green function for the boundary value problem. If one thinks of y_n as the time and of $x, y_1, \ldots, y_{n-1}$ as space variables, then k represents a disturbance generated, from rest, by a source at the origin of 'space–time'. The boundary of the support of k

† Current address: Department of Mathematics, M.I.T., Cambridge, Mass.

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

is generated by curves which can be divided into two classes: direct curves and indirect curves. The direct curves consist of all the forward-directed bicharacteristics, in $x \geqslant 0$, starting at the origin. The indirect curves arise from propagation in the boundary, $x = 0$, and each consists of a part in the boundary (boundary bicharacteristic or gliding ray) followed by a piece of bicharacteristic initially tangent to the boundary. The two classes have in common the bicharacteristics starting from the origin and initially tangent to the boundary. The corresponding subsets of the boundary of the support of k are here called the direct front and the diffracted front respectively.

Now, theorem 4·1 of [I] shows that k is smooth (C^∞) except at the direct front; in particular it is smooth at the diffracted front. It is therefore of interest to look at the wave front set of the solution of (*) in terms of analytic and Gevrey functions, to see at what stage of regularity the transition at the diffracted front becomes 'visible'. In section 2 the same bound is obtained for $WF_{G(s)}(u)$, the wave-front set calculated with respect to Gevrey classes of order $s \geqslant 3$, as was earlier found for $WF(u)$; the value 3 is probably the best possible, but we have not attempted to prove this here. It is very likely that this conclusion extends to the general diffraction problems, with analytic coefficients, considered in (10). On the other hand, we prove in section 3 that the analytic wave front set is contained in the union of the direct and diffracted fronts and contains at least part of the diffracted front. In other words, analytic singularities do propagate along the 'gliding rays' in the boundary. Following the recent counter-example to the appropriate problem in contact geometry ('Sato's conjecture', which is true in the C^∞ case (11)) obtained by Oshima (12) it is by no means clear that such a simple geometric description of the analytic singularities of general diffraction problems can be expected.

Our results are obtained by introducing the *boundary operator*

$$B: u_x\rceil(x = 0) \mapsto \partial u_x|\partial x\rceil(x = 0)$$

where $u_x \in C^\infty(\mathbb{R}^+; \mathscr{D}'(\mathbb{R}^n))$ is the distribution-valued function associated with the solution u of (*), as explained in section 1. The boundary operator is a convolution operator whose kernel has a Fourier transform which can be derived from the analysis in [I]. Both the Gevrey wave front set and the analytic wave front set of u can be bounded by finding bounds on the corresponding wave front sets of B. These bounds in turn depend on a detailed analysis of the asymptotic behaviour of the Airy function, carried out in the appendix. It is a remarkable fact, typical of diffraction, that B is a pseudo-differential operator. One can use this to give a more succinct proof of the principal results of [I], and we also present this, for $WF_{G(3)}$, in section 2.

1. *The inhomogeneous equation.* We begin with some remarks about the relation between distributions and distribution-valued functions. A distribution

$$u(x, y) \in \mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$$

may be said to be *regular in* x if, for all $\phi(x) \in C_0^\infty(\mathbb{R})$ and $\psi(y) \in C_0^\infty(\mathbb{R}^n)$, there is a C^∞ function $x \mapsto u.\psi$ such that

$$\langle u(x, y), \phi(x) \otimes \psi(y) \rangle = \int (u.\psi(x))\, \phi(x)\, dx.$$

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

(In the terminology of Schwartz ((14), p. 138), u is semi-regular to the right when viewed as a distribution kernel.) Let $\mathscr{K}(\mathbb{R}\times\mathbb{R}^n)$ denote the subspace of $\mathscr{D}'(\mathbb{R}\times\mathbb{R}^n)$ consisting of such distributions. When $u\in\mathscr{K}(\mathbb{R}\times\mathbb{R}^n)$, then $\psi\mapsto u.\psi$ is a distribution in $\mathscr{D}'(\mathbb{R}^n)$ for fixed x; denote this by u_x. As it follows from the closed graph theorem that $\psi\mapsto u.\psi$ is a continuous map $C_0^\infty(\mathbb{R}^n)\to C^\infty(\mathbb{R})$, it is clear that u_x is a smooth distribution-valued function. So one has a map

$$\tau\colon\mathscr{K}(\mathbb{R}\times\mathbb{R}^n)\to C^\infty(\mathbb{R};\mathscr{D}'(\mathbb{R}^n))$$

with $\tau u(x)=u_x$ defined by

$$\langle u(x,y),\phi(x)\otimes\psi(y)\rangle=\int\langle u_x,\psi\rangle\phi(x)\,dx\quad(\phi\in C_0^\infty(\mathbb{R}),\quad\psi\in C_0^\infty(\mathbb{R}^n)).\tag{1·1}$$

Conversely, suppose that a function $u_x\in C^\infty(\mathbb{R};\mathscr{D}'(\mathbb{R}^n))$ is given. Then one can set

$$\langle u(x,y),\chi(x,y)\rangle=\int\langle u_x,\chi(x,\,.\,)\rangle\,dx\quad(\chi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)).\tag{1·2}$$

It follows from elementary properties of bounded sets of test functions and of distributions that u is a distribution; it is evidently regular in x, and such that $\tau u=u_x$. Hence τ is a bijection.

The subspace $\mathscr{K}(\mathbb{R}^+\times\mathbb{R}^n)$ of $\mathscr{D}'(\mathbb{R}^+\times\mathbb{R}^n)$ is defined in the same way, and one again has a bijection $\tau\colon\mathscr{K}(\mathbb{R}^+\times\mathbb{R}^n)\to\mathscr{D}'(\mathbb{R}^+\times\mathbb{R}^n)$ which is given by (1·1) and (1·2), with $\phi(x)\in C_0^\infty(\mathbb{R}^+)$ and $\chi(x,y)\in C_0^\infty(\mathbb{R}^+\times\mathbb{R}^n)$. Such distributions may not be well behaved at the boundary. One must therefore also introduce the class of 'extendable' distributions which consists of elements of $\mathscr{D}'(\mathbb{R}^+\times\mathbb{R}^n)$ that are restrictions to $\mathbb{R}^+\times\mathbb{R}^n$ of distributions in $\mathscr{K}(\mathbb{R}\times\mathbb{R}^n)$; this will be denoted by $\mathscr{K}(\overline{\mathbb{R}}^+\times\mathbb{R}^n)$. (In (10), such distributions are called 'smooth up to the boundary'.) There is a simple relation between this and the smooth functions on the closed half-line with values in $\mathscr{D}'(\mathbb{R}^n)$.

Proposition 1·3. *The map*

$$\tau\colon\mathscr{K}(\overline{\mathbb{R}}^+\times\mathbb{R}^n)\to C^\infty(\overline{\mathbb{R}^+};\mathscr{D}'(\mathbb{R}^n))\tag{1·3}$$

defined by (1·1) *and* (1·2) *for* $\phi\in C_0^\infty(\mathbb{R}^+)$, $\psi\in C_0^\infty(\mathbb{R}^n)$, $\chi\in C_0^\infty(\mathbb{R}^+\times\mathbb{R}^n)$ *and* $\tau u=u_x$ *is a bijection.*

Proof. If $u\in\mathscr{K}(\overline{\mathbb{R}}^+\times\mathbb{R}^n)$ then there is a $\tilde{u}\in\mathscr{K}(\mathbb{R}\times\mathbb{R}^n)$ that is equal to u on $\mathbb{R}^+\times\mathbb{R}^n$. Then we have a function $\tilde{u}_x=\tau\tilde{u}\in C^\infty(\mathbb{R};\mathscr{D}'(\mathbb{R}^n))$ and $\tau u=\tau\tilde{u}|\mathbb{R}^+$ is given by (1·1) with $\phi\in C_0^\infty(\mathbb{R})$ and $\psi\in C_0^\infty(\mathbb{R}^n)$. By continuity, the derivatives $(d/dx)^j\tau u(0)$ are uniquely determined by τu, and so one obtains $u_x=\tau u\in C^\infty(\overline{\mathbb{R}}^+;\mathscr{D}'(\mathbb{R}^n))$ by cutting $\tilde{u}_x$ off at $x=0$.

Conversely, given $u_x\in C^\infty(\overline{\mathbb{R}}^+;\mathscr{D}'(\mathbb{R}^n))$, it is easily shown that this can be extended by the method of Seeley (15) to a function $\tilde{u}_x\in C^\infty(\mathbb{R};\mathscr{D}'(\mathbb{R}^n))$. One then defines $\tilde{u}=\tau^{-1}\tilde{u}_x$ by (1·2), and sets $\tau^{-1}u=\tau^{-1}\tilde{u}|\mathbb{R}^+\times\mathbb{R}^n$. It is obvious that $\tau^{-1}u$ is just (1·2) with $\chi\in C_0^\infty(\mathbb{R}^+\times\mathbb{R}^n)$, and that it is uniquely determined by u_x.

From now on, we shall write

$$\Omega=\mathbb{R}^+\times\mathbb{R}^n,\quad\overline{\Omega}=\overline{\mathbb{R}}^+\times\mathbb{R}^n,\quad\partial\Omega=\{0\}\times\mathbb{R}^n.\tag{1·4}$$

Also, $T^*\overline{\Omega}|_\Omega$ and $T^*\overline{\Omega}|_{\partial\Omega}$ will denote the restrictions of $T^*\overline{\Omega}$ to Ω and to $\partial\Omega$, respectively. We also write $\mathscr{K}(\overline{\Omega})$ for $\mathscr{K}(\overline{\mathbb{R}}^+\times\mathbb{R}^n)$.

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

The equation considered in [I] was

$$Pu_x = (1+x)\frac{\partial^2 u_x}{\partial y_n^2} - \frac{\partial^2 u_x}{\partial y_1^2} - \ldots - \frac{\partial^2 u_x}{\partial y_{n-1}^2} - \frac{\partial^2 u_x}{\partial x^2} = 0, \tag{1·5}$$

where $u_x \in C^\infty(\overline{\mathbb{R}}^+; \mathscr{D}'(\mathbb{R}^n))$, and $u_0 = f(y) \in \mathscr{E}'(\mathbb{R}^n)$ is given. This problem has a unique solution, provided one also imposes the condition that u is to vanish for

$$y_n < \inf\{z_n; z \in \operatorname{supp} f\}.$$

Furthermore, and this was the principal result of [I], the wave-front set of $u = \tau^{-1}u_x$ can be related to the wave front set of f. A point $(y, \eta) \in T^*\mathbb{R}^n \backslash \{0\}$ is called elliptic (with respect to our operator P) if $\eta_n^2 < |\eta'|^2$, hyperbolic if $\eta_n^2 > |\eta'|^2$, and glancing if $\eta_n^2 = |\eta'|^2$. All glancing points here are diffractive in the sense of (10). Let us denote by A the set of points in $T^*\mathbb{R}^n$ that are hyperbolic or glancing,

$$A = \{(y, \eta) \in T^*\mathbb{R}^n \backslash \{0\}; \eta_n^2 \geqslant |\eta'|^2\}, \tag{1·6}$$

and define a map $\sigma\colon A \to T^*\overline{\Omega}|_{\partial\Omega}$ by setting

$$\sigma(y, \eta) = (0, y, -(\eta_n^2 - |\eta'|^2)^{\frac{1}{2}} \operatorname{sign} \eta_n, \eta). \tag{1·7}$$

Then it follows from (2·10) and (2·11) of [I] that the bicharacteristic in $T^*\overline{\Omega}$ starting at $\sigma(y, \eta)$ is a forward one, that is to say that y_n is an increasing function of x along it. Theorem 7·1 of [I] then asserts that

$$WF(u) \subset \gamma(\sigma(WF(f) \cap A)), \tag{1·8}$$

where we use the notation $\gamma(B)$ for the forward bicharacteristic flow-out of a set $B \subset T^*\overline{\Omega}|_{\partial\Omega}$.

Now $u_x = k_x * f$, where k_x is defined by

$$Pk_x = 0 \quad \text{if} \quad x > 0, \quad k_0 = \delta(y), \quad k_x = 0 \quad \text{if} \quad y_n < 0.$$

The support of $k = \tau^{-1}k_x$ is contained in the cone

$$\{(x, y); \quad x > 0, \quad y_n \geqslant (x^2 + |y'|^2)^{\frac{1}{2}}\}.$$

This follows from standard hyperbolic estimates for mixed boundary value problems, and the fact that the coefficient of $(\partial/\partial y_n)^2$ in P is $1 + x \geqslant 1$. The results of [I] therefore still hold for $f \in \mathscr{D}'^+(\mathbb{R}^n)$, where $\mathscr{D}'^+(\mathbb{R}^n)$ consists of the distributions in $\mathscr{D}'(\mathbb{R}^n)$ whose supports are contained in some half-space $\{y; y_n \geqslant c > -\infty\}$. One then also has $u = \tau^{-1}u_x = 0$ when $y_n < c$. One can thus introduce the class $\mathscr{K}^+(\overline{\Omega})$ of elements of $\mathscr{K}(\overline{\Omega})$ for which there is a c such that $u = 0$ in $y_n < c$, and state the following variant of Theorem 7·1 of [I].

THEOREM 1·9. *There is a unique map $K\colon \mathscr{D}'^+(\mathbb{R}^n) \to \mathscr{K}^+(\overline{\Omega})$ such that, for $f \in \mathscr{D}'^+(\mathbb{R}^n)$ and $u = Kf$,*

$$P(\tau u) = 0, \quad \tau u(0) = f, \quad \tau u = 0 \quad \text{if} \quad y_n < \inf\{z_n; z \in \operatorname{supp} f\}. \tag{1·10}$$

Moreover, $WF(u)$ and $WF(f)$ are related by (1·8).

It is now easy to deduce an existence and uniqueness theorem for the inhomogeneous equation.

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

PROPOSITION 1·11. *Let $v \in \mathscr{K}'^{+}(\bar{\Omega})$ and $f \in \mathscr{D}'^{+}(\mathbb{R}^n)$. Then there is one and only one $u \in \mathscr{K}'^{+}(\bar{\Omega})$ such that $Pu = v$ in Ω, and $u(0) = f$.*

Proof. Without loss of generality, one may assume that v is the restriction to Ω of some $\tilde{v} \in \mathscr{K}'(\mathbb{R} \times \mathbb{R}^n)$ that vanishes for $y_n < c$, where c is a real number. For one can always multiply an extension of v by a suitable cut-off function that depends only on y_n.

Now, let $g(x) \in C^\infty(\mathbb{R})$ be such that

$$g'(x) > 0, \quad \lim_{x \to -\infty} g(x) = 1 - \delta, \quad g(x) = 1 + x \quad \text{if} \quad x \geqslant 0, \tag{1·12}$$

where $0 < \delta < 1$. (The condition that g is to be increasing will be needed later.) Extend the differential operator P to $\mathbb{R} \times \mathbb{R}^n$ by setting

$$\tilde{P} = g(x)\left(\frac{\partial}{\partial y_n}\right)^2 - \left(\frac{\partial}{\partial y_1}\right)^2 - \ldots - \left(\frac{\partial}{\partial y_{n-1}}\right)^2 - \left(\frac{\partial}{\partial x}\right)^2. \tag{1·13}$$

The hyperbolic equation $\tilde{P}\tilde{w} = \tilde{v}$ has a unique solution provided that one requires $\tilde{w}$ to vanish when $y_n < \inf\{z_n;\ (x,z) \in \operatorname{supp} \tilde{v}\}$, as all the planes $y_n = \text{const}$ are space-like. Again, it follows from well known results on partial hypoellipticity that $w = \tilde{w}/\Omega$ is in $\mathscr{K}'^{+}(\bar{\Omega})$; this can be proved by means of a partition of unity and theorem 4·3·1 in (7), applied to half-spaces $x \geqslant x_0 = \text{const}$. Hence $\tau w(0)$ is well defined, and one can put

$$u = w - K(\tau w(0)) + Kf. \tag{1·14}$$

Obviously, $Pu = v$ in Ω. Also, $\tau K \tau w(0) = \tau w(0)$ by Theorem 1·9; hence $\tau u(0) = f$. Finally, the uniqueness of the solution is a consequence of Proposition 1·1 of [I] and of Theorem 1·9 of the present paper.

Remark. It would clearly have been sufficient to assume that there exist an $\epsilon > 0$ and a $v_1 \in \mathscr{K}'^{+}(\bar{\Omega})$ such that $v = v_1$ for $0 < x < \epsilon$.

We must now discuss the bicharacteristics of P. Write

$$\tilde{p}(x,\xi,\eta) = \tilde{q}(x,\eta) - \xi^2, \quad \tilde{q}(x,\eta) = g(x)\eta_n^2 - |\eta'|^2. \tag{1·15}$$

The equations of the bicharacteristics of P are

$$dx/dt = -2\xi, \quad dy_j/dt = -2\eta_j \quad (j = 1, \ldots, n-1), \quad dy_n/dt = 2g(x)\eta_n,$$
$$d\xi/dt = -g'(x)\eta_n^2, \quad d\eta_j/dt = 0 \quad (j = 1, \ldots, n).$$

So η is constant on a bicharacteristic; also, if $\bar{x}$ and $\bar{\xi}$ are the initial values of x and ξ respectively, then

$$\tilde{p}(x,\xi,\eta) = \tilde{p}(\bar{x},\bar{\xi},\eta) = g(\bar{x})\eta_n^2 - |\eta'|^2 - \xi^2 = 0.$$

Hence $\bar{x}$ and $\bar{\xi}$ must satisfy the inequality

$$\tilde{q}(\bar{x},\eta) = g(\bar{x})\eta_n^2 - |\eta'|^2 \geqslant 0.$$

It follows that $\eta_n \neq 0$, as one would otherwise have $\bar{\xi} = \eta = 0$, and the bicharacteristic would degenerate to a point in the zero section of $T^*(\mathbb{R} \times \mathbb{R}^n)$.

If $\bar{Q} = (\bar{x},\bar{\xi},\bar{y},\eta) \in T^*(\mathbb{R} \times \mathbb{R}^n) \setminus \{0\}$ and $\tilde{p}(\bar{x},\bar{\xi},\eta) = 0$, then we denote by $\tilde{\gamma}(\bar{Q})$ the forward bicharacteristic from $\bar{Q}$, that is to say the bicharacteristic arc with end point

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

$\bar{Q}$ on which $y_n \geqslant \bar{y}_n$. On $\tilde{\gamma}(\bar{Q})$, one can use y_n as a parameter. The relevant information on the forward bicharacteristics is summarized in the following lemma.

LEMMA 1·16. (i) *If* $\bar{\xi}/\eta_n \leqslant 0$, *then* x *is an increasing function of* y_n *on* $\tilde{\gamma}(\bar{Q})$, *and* $x \to \infty$ *as* $y_n \to \infty$.

(ii) *If* $\bar{\xi}/\eta_n > 0$ *and*

$$\tilde{q}(-\infty, \eta) = (1-\delta)\eta_n^2 - |\eta'|^2 \geqslant 0,$$

then x *is a decreasing function of* y_n *on* $\tilde{\gamma}(\bar{Q})$, *and* $x \to -\infty$ *as* $y_n \to \infty$.

(iii) *If* $\bar{\xi}/\eta_n > 0$ *and* $\tilde{q}(-\infty, \eta) < 0$, *then there is a* $y_n^* > \bar{y}_n$ *such that* x *is decreasing on* $\tilde{\gamma}(\bar{Q})$ *for* $\bar{y}_n \leqslant y_n \leqslant y_n^*$, *increasing for* $y_n \geqslant y_n^*$, *and* $x \to \infty$ *as* $y_n \to \infty$; *the minimum* x^* *of* x, *attained when* $y_n = y_n^*$, *is the unique root of* $\tilde{q}(x, \eta) = 0$.

Proof. This is elementary geometrical optics, and is left to the reader. It follows from the equations

$$\frac{dx}{dy_n} = -\frac{\xi}{g(x)\eta_n}, \quad \frac{d\xi}{dy_n} = -\frac{g'(x)\eta_n}{2g(x)}, \quad \tilde{q}(x, \eta) = \xi^2,$$

and the hypotheses on $g(x)$.

Next, we introduce the broken bicharacteristic flow γ_b in $T^*\bar{\Omega}$, which includes reflexion at the boundary. If $\bar{Q} \in (T^*\bar{\Omega}\backslash\{0\}) \cap p^{-1}(0)$, then there are two possibilities. Either (*a*) $\tilde{\gamma}(\bar{Q}) \subset T^*\bar{\Omega}$, so that $\tilde{\gamma}(\bar{Q}) \subset T^*\Omega$, or $\tilde{\gamma}(\bar{Q})$ is tangent to the boundary $T^*\bar{\Omega}|_{\partial\Omega}$ and then turns back into $T^*\Omega$. In this case, $\gamma_b(\bar{Q}) = \tilde{\gamma}(\bar{Q}) = \gamma(\bar{Q})$. Or (*b*) $\tilde{\gamma}(\bar{Q})$ meets $T^*\bar{\Omega}|_{\partial\Omega}$ transversally; let Q_0 denote the point nearest to $\bar{Q}$ on $\tilde{\gamma}(\bar{Q})$ which is on $T^*\bar{\Omega}/_{\partial\Omega}$. Then

$$Q_0 = (0, (\eta_2^2 - |\eta'|^2)^{\frac{1}{2}} \operatorname{sign} \eta_n, y_0, \eta)$$

for some y_0. Then $\gamma_b(\bar{Q})$ consists of the arc $\bar{Q}Q_0$ and its reflexion at the boundary, which is $\gamma(Q_1)$, where

$$Q_1 = (0, -(\eta_n^2 - |\eta'|^2)^{\frac{1}{2}} \operatorname{sign} \eta_n, y_0, \eta).$$

Now the inclusion map $\iota: \partial\Omega \to \bar{\Omega}$ induces a pull-back $\iota^* T^*\bar{\Omega}|_{\partial\Omega} \to T^*\partial\Omega$; identifying $T^*\partial\Omega$ with $T^*\mathbb{R}^n$, one just has $\iota^*(0, \xi, y, \eta) = (y, \eta)$. Thus $Q_1 = \sigma \circ \iota^* \circ Q_0$, where σ is the map (1·7). Thus

$$\left.\begin{aligned} &\gamma_b(\bar{Q}) = \gamma(\bar{Q}) \quad \text{in case } (a), \\ &\gamma_b(\bar{Q}) = \gamma(\bar{Q}, Q_0) \cup \gamma(\sigma . \iota^* Q_0) \quad \text{in case } (b), \end{aligned}\right\} \tag{1·17}$$

where $\gamma(\bar{Q}, Q_0)$ denotes the component of $\tilde{\gamma}(\bar{Q}) \cap T^*\bar{\Omega}$ that contains $\bar{Q}$. It is easy to check that one has case (*a*) if $-(1+\bar{x})^{\frac{1}{2}} \leqslant \bar{\xi}/\eta_n \leqslant \bar{x}$, and case (*b*) if $\bar{x} < \xi/\eta_n \leqslant (1+\bar{x})^{\frac{1}{2}}$. If (*b*) holds and $\bar{x} = 0$, then $Q_0 = \bar{Q}$.

The distribution v of Proposition 1·11 has a wave front set $WF(v) \subset T^*\Omega\backslash\{0\}$. But it is an extendable distribution, by hypothesis, and so one can also consider the set

$$WF_{\bar{\Omega}}(v) = \bigcap WF(\tilde{v}), \tag{1·18}$$

where the intersection is over all $\tilde{v} \in \mathscr{D}'(\mathbb{R} \times \mathbb{R}^n)$ such that $\tilde{v}|_{\Omega} = v$. Note that $WF_{\bar{\Omega}}(v)$ is a subset of $T^*(\mathbb{R} \times \mathbb{R}^n)\backslash\{0\}$, and that its restriction to $T^*\Omega$ is obviously just $WF(v)$. If $\tilde{v}\restriction\Omega = v$ and $\rho(x) \in C^\infty(\mathbb{R})$, where $\rho = 1$ on a neighbourhood of $\mathbb{R}^+$, then also $v = \rho\tilde{v}|_{\Omega}$,

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

and from this it follows at once that $WF_{\bar{\Omega}}(v) \subset T^*\bar{\Omega}\backslash\{0\}$. It is not difficult to construct examples for which $WF_{\bar{\Omega}}(v)\backslash WF(v)$ is not empty. We therefore put

$$WF_{\partial\Omega}(v) = WF_{\bar{\Omega}}(v)\backslash WF(v) \subset T^*\bar{\Omega}|_{\partial\Omega}\backslash\{0\}. \tag{1·19}$$

We can now state the principal result of this section.

THEOREM 1·20. *Let u and v be as in Proposition* 1·11 *and suppose that $\tau u(0) = 0$. Then*

$$WF(u)\backslash WF(v) \subset \gamma_b(WF(v)) \cup \gamma_b(\sigma \circ \iota^* WF_{\partial\Omega}(v)), \tag{1·21}$$

where $WF(u)$ and $WF(v)$ are viewed as subsets of $T^*\Omega$.

Notes. (i) In the second member of (1·21), σ stands for the composition of σ with the restriction map to the set A defined by (1·7). (ii) If $\tau u(0) = f \not\equiv 0$, then one must combine (1·21) with (1·8).

Proof. Since $\tau u(0) = 0$, (1·14) reduces to

$$u = \tilde{w}|_\Omega - k \circ \tau\tilde{w}(0), \tag{1·22}$$

where $\tilde{w}$ is the (unique) solution of $\tilde{P}\tilde{w} = \tilde{v}$ that is in $\mathscr{D}'^+(\mathbb{R} \times \mathbb{R}^n)$, $\tilde{v}$ and $\tilde{P}$ being extensions of v and of P, respectively. From Hörmander's theorem on the invariance of $WF(\tilde{w})\backslash WF(\tilde{P}\tilde{w})$ under the bicharacteristic flow associated with $\tilde{P}$ (8), and the fact that $\tilde{w} \in \mathscr{D}'^+(\mathbb{R} \times \mathbb{R}^n)$, it follows easily that

$$WF(\tilde{w}) \subset \tilde{\gamma}(WF(\tilde{v})) \cup WF(\tilde{v}). \tag{1·23}$$

Again, Theorem 2·5·11 of (9) gives

$$WF(\tau\tilde{w}(0)) \subset \iota^* WF(\tilde{w})|_{\partial\Omega}$$

and so it follows from Theorem 1·9 that

$$WF(k \circ \tau\tilde{w}(0)) \subset \gamma(\sigma \circ \iota^* WF(\tau\tilde{w}(0))), \tag{1·24}$$

with the convention on σ made in Note (i) above. We now split up $WF(v)$ as

$$WF(\tilde{v}) = WF(\tilde{v})|_{\Omega^-} \cup WF(\tilde{v})|_\Omega \cup WF(\tilde{v})|_{\partial\Omega}, \tag{1·25}$$

where $\Omega^- = (\mathbb{R} \times \mathbb{R}^n)\backslash\Omega$, and consider the contributions of each summand to the second members of (1·23) and (1·24).

The crucial point is that u is uniquely determined by v (Proposition 1·11), while both $\tilde{v}$ and $\tilde{P}$ in (1·17) can be any admissible extensions of v and P, respectively. Note that $\tilde{w}$ is determined by $\tilde{P}$. Hence a point of $T^*\Omega$ can only be in $WF(u)$ if it is in the intersection of the second members of either (1·23) or (1·24) when both $\tilde{v}$ and $\tilde{P}$ run over all admissible extensions.

As an immediate consequence of this remark, we can ignore all (forward) bicharacteristics originating at points of $WF(v)|_{\Omega^-}$; such bicharacteristics can be removed by multiplying $\tilde{v}$ by a suitable $\rho(x) \in C^\infty(R)$ such that $\rho = 1$ on a neighbourhood of $\bar{\mathbb{R}}^+$.

Next, consider the contribution made by $WF(\tilde{v})|_\Omega = WF(v)$. Let Q be a point in $WF(v)|_\Omega$. Three cases must be distinguished. First, one can have $\tilde{\gamma}(Q) \subset T^*\Omega$, so that $\tilde{\gamma}(Q) = \gamma(Q) = \gamma_b(Q)$ does not meet the boundary. In this case, $\gamma_b(Q)$ figures in the

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

second member of (1·23), but not in that of (1·24). Again, $\tilde{\gamma}(Q) = \gamma(Q)$ may be a glancing bicharacteristic, touching $T^*\Omega|_{\partial\Omega}$ at Q_0, say. Then again $\tilde{\gamma}(Q) = \gamma(Q) = \gamma_b(Q)$ will be in the second member of (1·23). Also, as the fibre coordinate ξ of Q_0 vanishes, one has $\sigma \circ \iota^* Q_0 = Q_0$ and so the contribution to $WF(k \circ \tau w(0))$ is $\gamma(Q_0) \subset \gamma_b(Q)$. Finally, $\tilde{\gamma}(Q)$ may meet $T^*\Omega^-$. Then the component of $\tilde{\gamma}(Q) \cap T^*\overline{\Omega}$ is an arc QQ_0 with $Q_0 \in T^*\overline{\Omega}|_{\partial\Omega}$. So, in addition to QQ_0 which enters into the second member of (1·23), one has the contribution $\gamma(\sigma \circ \iota^* Q_0)$ to (1·24). We are clearly in case (b) of the definition of the broken geodesic flow, and so this last contribution is the reflected portion of $\gamma_b(Q)$, by (1·17). By (1·22), we have now accounted for all of $\gamma_b(WF(v))$ in (1·21).

However, in this third case it may happen that $\tilde{\gamma}(Q)$ returns from the 'otherworld' $T^*\Omega^-$, and crosses back into $T^*\Omega$ at a point $Q_1 \in T^*\overline{\Omega}|_{\partial\Omega}$, thus giving rise to a spurious contribution to both (1·23) and (1·24). But we are still free to chose the extension $\tilde{P}$. Let the coordinates of Q be (x, ξ, y, η). We are in case (iii) of Lemma 1·16 with x^* (the x-coordinate of the point at which the bicharacteristic turns) negative. So this can only happen if

$$\tilde{q}(-\infty, \eta) = (1-\delta)\eta_n^2 - |\eta'|^2 < 0 \quad (\eta_n^2 - |\eta'|^2 > 0).$$

But one can replace $\tilde{P}$ by an extension $\tilde{P}_1$ for which

$$\lim_{x \to -\infty} g_1(x) = 1 - \delta_1 \quad (\delta_1 < (\eta_n^2 - |\eta'|^2)/\eta_n^2).$$

With respect to $\tilde{P}_1$, Q is of type (ii) of Lemma 1·16 and so $\tilde{\gamma}(Q)$ escapes to $x = -\infty$. We can therefore conclude that returned otherworld bicharacteristics cannot appear in that part of $WF(u)$ which is due to $WF(v)$.

There remains the term $WF(\tilde{v})|_{\partial\Omega} = WF_{\partial\Omega}(v)$ in the second member of (1·25). There are two possibilities. If $Q \in WF_{\partial\Omega}(v) \cap p^{-1}(0)$ and $\tilde{\gamma}(Q) \subset T^*\overline{\Omega}$, then $\sigma \circ \iota^* Q = Q$, and so $\tilde{\gamma}(Q) = \gamma(Q) = \gamma_b(Q)$ enters into both $WF(\tilde{w})$ and $WF(k \circ \tau\tilde{w}(0))$. However, if $\tilde{\gamma}(Q) \backslash \{Q\} \subset T^*\Omega^-$, then in addition to a contribution $\gamma(\sigma \circ \iota^* Q) = \gamma_b(Q)$ to the second member of (1·23), there may also be one to $WF(\tilde{w})$, due to the fact that $\tilde{\gamma}(Q)$ returns from $T^*\Omega^-$ and crosses back into $T^*\Omega$. But one can argue as above that such a bicharacteristic cannot be in the intersection over all admissible extensions, by chosing a suitable $\tilde{P}$. So all the terms in (1·21) are accounted for, and the theorem is proved.

2. *The boundary operator and Gevrey regularity.* With K the solution map as in Theorem 1·9 we define the *boundary map* for the initial-boundary value problem (*) by

$$B: \mathscr{D}'^+(\mathbb{R}^n) \ni f \mapsto \partial_x[\tau K f](0) \in \mathscr{D}'^+(\mathbb{R}^n). \tag{2·1}$$

Thus, if u is the solution of (*) then $Bf = \partial_x u\rceil(x = 0)$. From the Fourier–Laplace representation of the kernel of K ([I], equation (3·7)) it is clear that B is a convolution operator with kernel $b \in \mathscr{D}'^+(\mathbb{R}^n)$ having as Fourier–Laplace transform

$$\hat{b}(\eta) = -\eta_n^{\frac{4}{3}} \operatorname{Ai}'(\zeta_0)/\operatorname{Ai}(\zeta_0). \tag{2·2}$$

Here we are using the notation of [I]: $\eta = (\eta', \eta_n) = (\eta_1, \ldots, \eta_{n-1}, \eta_n)$, $\eta' \in \mathbb{R}^{n-1}$, $\operatorname{Im}\eta_n < 0$, $\zeta_0(\eta) = \eta_n^{-\frac{4}{3}}(|\eta'|^2 - \eta_n^2)$ with $\eta_n^{-\frac{4}{3}} = |\eta_n|^{-\frac{4}{3}} \exp(-4i\theta/3)$ where

$$\eta_n = |\eta_n| \exp(i\theta),$$

$\pi < \theta < 2\pi$ and Ai is the standard Airy function.

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

The fact that $\hat{b}$, defined by (2·2), is the Fourier–Laplace transform of a distribution $b \in \mathscr{D}'^{+}(\mathbb{R}^n)$ follows from the existence and uniqueness in Theorem 1·9. Indeed using the results of [I] we can conclude that b is C^∞ outside the origin, i.e.

$$WF(b) \subset T_0^*\mathbb{R}^n \backslash \{0\}. \tag{2·3}$$

The information in the appendix on the asymptotic behaviour of Ai allows us to discuss the limit $\operatorname{Im} \eta_n \uparrow 0$, that is, we can examine the Fourier transform $\hat{b} \restriction \mathbb{R}^n$ of b directly. In this way we obtain an improvement of (2·3) and as a consequence a more refined version of Theorem 1·9, and incidentally a more direct proof of (2·3) and the propagation of singularities part, (1·8), of Theorem 1·9.

Recall that if $\Omega \subset \mathbb{R}^n$ is an open set then $f \in C^\infty(\Omega)$ is said to be of Gevrey class s ($s \in \mathbb{R}$) if to each compact set $K \Subset \Omega$ there corresponds a constant C_K such that for every multi-index α

$$\sup_{x \in K} |\partial_x^\alpha f(x)| \leqslant C_K^{1+|\alpha|} |\alpha|^{s|\alpha|}.$$

The space of such functions, $G_s(\Omega)$, is a ring closed under differentiation. Hörmander has considered (6) wavefront sets calculated with respect to these, and more general, spaces of functions and we shall use his results below. If $s > 1$, $G_s(\Omega)$ is not quasi-analytic (that is, contains non-trivial functions of compact support) and Hörmander has shown that if $f \in \mathscr{D}'(\mathbb{R}^n)$ and $(x, \eta) \in T^*\mathbb{R}^n \backslash \{0\}$ then $(x, \eta) \notin WF_{G(s)}(f)$ ($s > 1$) if and only if

$$\left.\begin{aligned} &\exists \rho \in G_s(\mathbb{R}^n) \text{ with } \rho(x) \not\equiv 0 \text{ and compact support such that in some conic} \\ &\text{neighbourhood } \gamma \text{ of } \eta \in \mathbb{R}^n \backslash \{0\} \text{ and for some } C, \\ &\qquad |\widehat{\rho f}(\eta)| \leqslant C^{1+N} N^{sN} |\eta|^{-N} \quad (N = 1, 2, \ldots). \end{aligned}\right\} \tag{2·4}$$

The improvement on (2·3) we will obtain is

Proposition 2·5. $WF(b) = WF_{G(3)}(b) = T_0^*\mathbb{R}^n \backslash \{0\}$.

To prove this we shall show that $T_0^*\mathbb{R}^n \backslash \{0\} \subset WF(b)$ and that $WF_{G(3)}(b) \subset T_0^*\mathbb{R}^n$, which suffices since $WF(b) \subset WF_{G(3)}(b)$ is always true. The main part of the proof is the second of these inclusions. We shall start with a lemma.

LEMMA 2·6. *For each $k \in \mathbb{N}$, $\hat{b}$ can be split*

$$\hat{b} = u_k + v_k,$$

where for some constant C independent of k,

$$|u_k(\eta)| \leqslant C^{1+p} k^{3(p+1)} |\eta|^{-p} \quad (\forall \eta \in \mathbb{R}^n, p \in \mathbb{N}) \tag{2·7}$$

$$|\partial_\eta^\alpha v_k(\eta)| \leqslant C^{1+|\alpha|} k^{|\alpha|} |\eta|^{1-\frac{1}{3}|\alpha|} \quad (\forall |\alpha| \leqslant k, \eta \in \mathbb{R}^n) \tag{2·8}$$

and $\operatorname{supp}(v_k) \subset \{\eta \in \mathbb{R}^n;\ |\eta| \geqslant k^3\}$.

Proof. The estimates obtained in the appendix, namely (A 23) and (A 24), show that for some C and for every n-multi-index α,

$$|\partial_\eta^\alpha \hat{b}(\eta)| \leqslant C^{1+|\alpha|} \{\alpha! |\eta|^{1-\frac{1}{3}|\alpha|} + |\alpha|^{2|\alpha|} |\eta|^{1-|\alpha|}\} \quad (\forall |\eta| \geqslant 1),$$

whilst $\hat{b}$ is bounded in $|\eta| \leqslant 1$. So, with a larger C it follows that in $|\eta| \geqslant k^3$

$$|\partial_\eta^\alpha \hat{b}(\eta)| \leqslant C^{1+|\alpha|} k^{|\alpha|} |\eta|^{1-\frac{1}{3}|\alpha|} \quad (\forall |\alpha| \leqslant k). \tag{2·9}$$

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

Choose $\rho \in C_0^\infty(\mathbb{R}^n)$ a function of Gevrey class 3, such that $\rho = 0$ in $|\eta| \leqslant 1$, $\rho = 1$ in $|\eta| \geqslant 2$ and put

$$u_k(\eta) = \hat{b}(\eta)(1 - \rho(\eta/k^3)) \quad (k \in \mathbb{N}).$$

Thus, supp $u_k \subset \{\eta \in \mathbb{R}^n;\ |\eta| \leqslant 2k^3\}$ and, as $\hat{b}$ is bounded by a multiple of $(1+|\eta|)$, for some C

$$|u_k(\eta)| \leqslant Ck^3(2k^3|\eta|)^p \quad (\forall p \in \mathbb{N}).$$

So (2·7) holds. It remains to show that

$$v_k(\eta) = \hat{b}(\eta)\,\rho(\eta/k^3)$$

satisfies (2·8). Now, by Leibniz's theorem

$$\partial_\eta^\alpha v_k(\eta) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial_\eta^{\alpha-\beta} \hat{b}(\eta)\, k^{-3|\beta|} [\partial_{\eta/k^3}^\beta \rho(\eta/k^3)].$$

By assumption ρ is of Gevrey class 3 and constant in $|\eta| \geqslant 2$, so for some C

$$|\partial_\eta^\gamma \rho(\eta)| \leqslant C^{1+|\gamma|}|\gamma|^{3|\gamma|} \quad (\forall \gamma,\ \forall \eta \in \mathbb{R}^n).$$

Writing H for the Heaviside step function and using (2·9), we find that there is a constant C such that for all $|\alpha| \leqslant k$

$$|\partial_\eta^\alpha v_k(\eta)| \leqslant C^{1+|\alpha|}k^{|\alpha|}|\eta|^{1-\frac{1}{3}|\alpha|}\left\{1 + H(|\eta| - k^3)\, H(2k^3 - |\eta|) \sum_{0<\beta\leqslant\alpha} \binom{\alpha}{\beta} k^{-|\beta|}\, |\eta|^{-\frac{1}{3}|\beta|}\right\}$$

which clearly implies (2·8). Thus we have proved the lemma.

Proof of Proposition 2·5. We wish to show first that $WF_{G(3)}(b) \subset T_0^*\mathbb{R}^n$, i.e. that b is of Gevrey class 3 outside the origin. Thus we must show that any point $x' \neq 0$ has a compact neighbourhood K', such that, for some constant C',

$$|\partial_x^\alpha b(x)| \leqslant (C')^{1+|\alpha|}|\alpha|^{3|\alpha|} \quad (\forall x \in K'). \tag{2·10}$$

From (2·7) and (2·8) it follows easily that b is (outside the origin) a symbol in the class $S^1_{\frac{1}{3},0}$ so (2·3) holds (since B is therefore a pseudodifferential operator, hence pseudolocal). Moreover from (2·7) it follows that if $\hat{f}_k = u_k$ then $f_k \in C^\infty(\mathbb{R}^n)$ for all k and

$$\begin{aligned}|\partial^\alpha f_k| &\leqslant (2\pi)^{-n} \int |\eta^\alpha \hat{f}_k|\, d\eta \\ &\leqslant (2\pi)^{-n}\left\{\int_{|\eta|\leqslant 1} |u_k| d\eta + \int_{|\eta|\geqslant 1} C^{1+p}\, k^{3(p+1)}\, |\eta|^{-p+|\alpha|}\, d\eta\right\}.\end{aligned}$$

So, choosing $p = |\alpha| + N + 2$ we find that

$$|\partial_x^\alpha f_k(x)| \leqslant C^{1+|\alpha|}|\alpha|^{3|\alpha|} \quad (k \leqslant 3(|\alpha| + N + 2)). \tag{2·11}$$

Similarly, put $\hat{g}_k = v_k \in S'(\mathbb{R}^n)$. Then, for any $p \in \mathbb{N}$ such that the right hand side is integrable and for any $j \leqslant n$,

$$|x_j^p \partial_x^\alpha g_k(x)| \leqslant (2\pi)^{-n} \int |\partial_{\eta_j}^p(\eta^\alpha v_k(\eta))|\, d\eta. \tag{2·12}$$

From (2·8), provided $p \leqslant k$

$$\begin{aligned}|\partial_{\eta_j}^p \eta^\alpha v_k(\eta)| &\leqslant \sum_{0\leqslant r\leqslant p} \binom{p}{r} |\eta|^{|\alpha|-r}\, |\partial_{\eta_j}^{p-r} v_k(\eta)| \alpha_j!/(\alpha_j - r)! \\ &\leqslant \sum_{0\leqslant r\leqslant p} \binom{p}{r} |\alpha|^r\, C^{1+p-r}\, k^{p-r}\, |\eta|^{1+|\alpha|-\frac{1}{3}p-\frac{2}{3}r}.\end{aligned} \tag{2·13}$$

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

Put $p = 3(|\alpha| + n + 2)$, insert (2·13) into (2·12) and integrate over the $(n-1)$-sphere to deduce that, for C sufficiently large,

$$|x_j^p \partial_x^\alpha g_k(x)| \leqslant \sum_{0 \leqslant r \leqslant p} \binom{p}{r} C^{1+p} k^{p-r} |\alpha|^r \int_{k^3}^\infty R^{-2-\frac{1}{3}r}\, dR$$

$$\leqslant k^p C^{1+p} \sum_{0 \leqslant r \leqslant p} \binom{p}{r} (|\alpha|/k^3)^r.$$

Choosing $k = p = 3(|\alpha| + n + 2)$ we find that

$$|x_j^{3(|\alpha|+n+2)} \partial_x^\alpha g_{3(|\alpha|+n+2)}| \leqslant C^{1+|\alpha|} |\alpha|^{3|\alpha|}$$

and this estimate, combined with (2·11), shows that (2·10) holds.

Thus we have proved the first inclusion, $WF_{G(3)}(b) \subset T_0^*\mathbb{R}^n$ and to complete the proof of Proposition 2·5 we need only show that $WF(b) \supset T_0^*\mathbb{R}^n \backslash \{b\}$. This is immediate from (A 18) and (A 19) which show that, in a conic neighbourhood of any point η with $\zeta_0(\eta) \neq 0$ and $\eta_n \neq 0$, $\hat{b}$ is a classical (i.e. of class $S^1_{1,0}$) and elliptic symbol. It therefore follows that all the points $(0, \eta)$ for these η are in $WF(b)$, which is closed. This concludes the proof of Proposition 2·5.

We shall now show how Proposition 2·5 can be used to prove (1·8). We need a result on the Gevrey wave front set of a convolution.

THEOREM 2·14. *If* $b \in \mathscr{D}'(\mathbb{R}^n)$, $f \in \mathscr{E}'(\mathbb{R}^n)$ *and* $s \geqslant 1$ *then*

$$WF_{G(s)}(b*f) \subset \{(x, \eta) \in T^*\mathbb{R}^n \backslash \{0\};$$

$$\exists (y, \eta) \in WF_{G(s)}(f) \quad \textit{such that} \quad (x-y, \eta) \in WF_{G(s)}(b)\}.$$

Proof. We shall only prove this in the non-quasianalytic case, $s > 1$. For $s = 1$, $WF_{G(s)} = WF_A$, and an analogous proof can be formulated using sequences of partitions of unity (cf. (6)); in fact the result holds for Hörmander's more general WF_L.

So suppose $(x, \eta) \in T^*\mathbb{R}^n \backslash \{0\}$ is such that

$$(y, \eta) \in WF_{G(s)}(f) \Rightarrow (x-y, \eta) \notin WF_{G(s)}(b), \tag{2·15}$$

we then wish to show that $(x, \eta) \notin WF_{G(s)}(b*f)$. If $\rho \in G_s(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n)$ is identically equal to 1 in a neighbourhood of x-supp f then $(\rho b)*f$ is equal to $b*f$ near x and $WF_{G(s)}(\rho b) \subset WF_{G(s)}(b)$ so (2·15) holds with b replaced by ρb.

Moreover, if $\{\rho_j\}$, $\rho_j \in G_s(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n)$ is a sufficiently fine finite partition of unity in a neighbourhood of supp $(\rho b) \cup$ supp f then, because of (2·15), there is a conic neighbourhood γ of $\eta \in \mathbb{R}^n$ such that

$$x \in \operatorname{supp} \rho_j + \operatorname{supp} \rho_k$$

$$\Rightarrow \text{either } (\rho_j \rho b)^\wedge \text{ or } \widehat{\rho_k f} \text{ is rapidly decreasing in } \gamma \text{ in the sense of (2·4).}$$

Putting $S = \{(j, k); x \in \operatorname{supp} \rho_j + \operatorname{supp} \rho_k\}$ it is clear that

$$v = \sum_{(j,k) \in S} (\rho_j \rho b)*\rho_k f \in \mathscr{E}'(\mathbb{R}^n)$$

is equal to $b*f$ near x. However,

$$\hat{v} = \sum_{(j,k) \in S} (\rho_j \rho b)^\wedge (\rho_k f^\wedge)$$

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

is clearly rapidly decreasing in γ in the sense of (2·4), so we can conclude that

$$(x,\eta)\notin WF_{G(s)}(b*f),$$

as desired, thereby proving the theorem.

Combining Proposition 2·5 and Theorem 2·14, applied to B, gives

$$WF_{G(s)}(Bf)\subset WF_{G(s)}(f)\quad(\forall f\in\mathscr{E}'(\mathbb{R}^n),s\geqslant 3). \tag{2·16}$$

Now, if u is the solution of (*) we know that u can be considered as a distribution $\tau u\in C^\infty(\overline{\mathbb{R}}^+;\mathscr{D}'(\mathbb{R}^n))$. So we can 'cut u off' at the boundary $x=0$ (see (10)) to give $u^c\in\mathscr{D}'(\mathbb{R}^{n+1})$ defined by

$$\langle u^c,\phi\rangle=\int_0^\infty\langle u(x),\phi(x,\,.\,)\rangle_{\mathbb{R}^n}dx\quad(\phi\in C_0^\infty(\mathbb{R}^{n+1})). \tag{2·17}$$

Let us now calculate Pu^c where P acts on $\mathscr{D}'(\mathbb{R}^{n+1})$ in the usual way as the dual of the formal dual of ${}^tP=P$ acting on $C_0^\infty(\mathbb{R}^{n+1})$. From (2·17) we find, by integration by parts,

$$\begin{aligned}\langle Pu^c,\phi\rangle&=\langle u^c,P_\phi\rangle\\&=\langle(Pu)^c+\delta'(x)\otimes f+\delta(x)\otimes Bf,\phi\rangle,\end{aligned} \tag{2·18}$$

where $\delta(x)$ and $\delta'(x)$ are the one-dimensional Dirac measure and its first derivative. Of course, $Pu=0$ in $x<0$ so (2·18) implies

$$Pu^c=\delta(x)\otimes Bf+\delta'(x)\otimes f;\quad u^c=0\quad\text{in}\quad y_n\leqslant 0\quad\text{and in}\quad x<0. \tag{2·19}$$

In fact u^c is the unique solution of (2·19) for given $f\in\mathscr{E}'(\mathbb{R}^n)$, as follows from the hyperbolicity of P in $x>-1$. Using (2·4) we easily see that for any $s\geqslant 1$ and $g\in\mathscr{E}'(\mathbb{R}^n)$

$$\begin{aligned}WF_{G(s)}(\delta(x)\otimes g)\subset\{(x,y,\xi,\eta)\in T^*\mathbb{R}^{n+1};\quad x=0,\quad\xi\neq 0,\quad\eta=0\\ \text{and}\quad y\in\operatorname{supp}g\quad\text{or}\quad x=0,\quad\xi\in\mathbb{R}\quad\text{and}\quad(y,\eta)\in WF_{G(s)}(g)\},\end{aligned}$$

and similarly for $\delta'(x)\otimes g$. Thus, using (2·14) and (2·19)

$$\begin{aligned}WF_{G(s)}(Pu^c)\subset\{(0,y,\xi,\eta)\in T^*\mathbb{R}^{n+1};\quad\text{either}\quad\xi\neq 0,\eta=0,\,y\in\mathbb{R}^n\\ \text{or}\quad\xi\in\mathbb{R},(y,\eta)\in WF_{G(s)}(f)\}\quad(\forall s\geqslant 3).\end{aligned} \tag{2·20}$$

Let us now recall the following fundamental theorem on the propagation of singularities from (6).

THEOREM 2·21 (*Hörmander*). *If P, a partial differential operator on $\Omega\subset\mathbb{R}^n$, has real principal symbol, p, and analytic coefficients then for all $s\geqslant 1$ and all $u\in\mathscr{D}'(\Omega)$,*

$$WF_{G(s)}(u)\subset WF_{G(s)}(Pu)\cup\{z\in T^*\Omega\backslash\{0\};p(z)=0\}$$

and any (connected) null bicharacteristic of p lying completely in $(T^*\Omega\backslash\{0\})$ $WF_{G(s)}(Pu)$ is either disjoint from or contained in $WF_{G(s)}(u)$.

Using this result we can now easily prove (1·8) with WF replaced by $WF_{G(s)}$, $s\geqslant 3$. Indeed we must show that if $z\in T^*((0,\infty)\times\mathbb{R}^n)\backslash\{0\}$, $z=(x,y,\xi,\eta)$ then $z\in WF_{G(s)}(u)$ only if $p(z)=0$ and the null bicharacteristic through z meets the boundary $x=0$ at a point $z'=(x',y',\xi',\eta')$ with $y_n'<y_n$ and $(y',\eta')\in WF_{G(s)}(f)$. First it is immediate from (2·20) and Theorem 2·21 that $z\in WF_{G(s)}$ implies $p(z)=0$.

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

The geometry of the null bicharacteristics of P was considered in detail in [I] and from that analysis it follows that if $z = (x, y, \xi, \eta)$, $p(z) = 0$ then the null bicharacteristic in the direction of decreasing y_n, $(-\infty, 0] \ni \sigma \mapsto z(\sigma)$, $z(0) = z$, through z falls into at least one of three classes:

(1) $z(\sigma)$ enters $y_n < \inf\{y_n;\ (x, y) \in \operatorname{supp} u\}$ without meeting $x = 0$,

(2) $z(\sigma)$ enters $y_n < \inf\{y_n; (x, y) \in \operatorname{supp} u\}$ after one point $(0, y^*, 0, \eta^*)$ of simple tangency and no other intersection with $x = 0$,

(3) $z(\sigma)$ meets $x = 0$ transversally first at $(0, y^*, \xi^*, \eta^*)$ and enters $x < 0$.

In case (1) we conclude immediately that $z \notin WF_{G(s)}(u)$ since u is zero, hence Gevrey 3 at one point on the bicharacteristic. In case (2) we similarly conclude that $z \notin WF_{G(s)}(u)$ provided $(0, y^*, 0, \eta^*) \notin WF_{G(s)}(Pu^c)$, and from (2·20) this follows if $(y^*, \eta^*) \notin WF_{G(s)}(f)$. Similarly again in case (3) we find $z \notin WF_{G(s)}(u)$ provided $(y^*, \eta^*) \notin WF_{G(s)}$ since $u^c = 0$ in $x < 0$. This concludes the proof of (1·8) with WF replaced by $WF_{G(s)}$, $s \geqslant 3$ and hence the desired refinement of Theorem 1·9.

3. *Propagation of analyticity.* The method used above can be applied again in the discussion of analyticity and allows us to deduce the desired result of $WF_A(u)$, where u satisfies (*), from an appropriate bound on $WF_A(b)$. Let

$$C_0^+ = \{(y, \eta) \in T^*\mathbb{R}^n \setminus \{0\};\quad y_n \geqslant 0, \eta_n^2 = |\eta'|^2 \quad \text{and} \quad \exists s \in \mathbb{R} \text{ s.t. } (y', y_n) = s(\eta', -\eta_n)\}.$$

Proposition 3·1. *The kernel b of B satisfies*

$$WF_A(b) \subset (T_0^*\mathbb{R}^n \setminus \{0\}) \cup C_0^+. \tag{3·2}$$

If u is replaced by E, the forward fundamental solution of the wave operator $P = \partial_y^2 - \Delta_{y'}$, (3·2) remains true. So, one might try to prove the proposition using the techniques developed for constant coefficient differential operators (see Andersson (1), Atiyah, Bott and Garding (3)). Although most of (3·2) can be proved in this way it turns out that there is an obstruction to this approach, which reduces to the use of Cauchy's (Stokes's) theorem to move contours in the Fourier–Laplace integral representation of the kernel, namely $\hat{b}(\eta)$ is not analytic in a neighbourhood of the hypersurface $\eta_n = 0$.

So, the proof of Proposition 3·1 which we shall give is based instead on the characterization of WF_A given by Andersson (2); we also use the idea of an analytic symbol introduced by Boutet de Monvel and Krée (4) (see also Hörmander (6), Sato, Kashiwara and Kawai (13)).

First we shall recall the relevant parts of (2) specializing somewhat to suit our purpose. Denote, as usual, by $S^m(F)$ the space of symbols supported by a given closed cone with compact base $F \subset \mathbb{R}^n \times \mathbb{R}^n$. Thus, $a \in S^m(F)$ if and only if $a \in C^\infty(\mathbb{R}^{2n})$, $\operatorname{supp} a \subset F$ and for each pair (α, β) of n-multi-indices there exists a constant $C_{\alpha\beta}$ such that

$$|D_y^\alpha D_\eta^\beta a(y, \eta)| \leqslant C_{\alpha\beta}(1 + |\eta|)^{m-|\beta|} \quad \text{in} \quad \mathbb{R}^{2n}.$$

By an A-sequence $\{a_k\}$ we shall mean a sequence $\mathbb{N} \to S^m(F)$ for some m and F, such that there are constants C, $\delta > 0$ for which

$$|D_y^\alpha D_\eta^\beta a_k(y, \eta)| \leqslant C^k k^{!\alpha|}(1 + |\eta|k)^{m-|\beta|} \quad (\forall k \geqslant \delta^{-2}, |\alpha + \beta| \leqslant \delta k). \tag{3·3}$$

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

Such a sequence is said to be non-characteristic at $(y_0, \eta_0) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ if there exists a conic neighbourhood F' of (y_0, η_0) and a constant $C > 0$ such that

$$a_k = a_k' + a_k''$$

with a_k'' an A-sequence of order $m-1$ and

$$|a_k'(y,\eta)| \geqslant C|\eta|^m \quad \text{in} \quad \{(y,\eta) \in F'; |\eta| \geqslant Ck\}.$$

Since $a_k \in S^m(F)$ for each k an A-sequence defines a sequence of pseudodifferential operators

$$a_k(y,D): \mathscr{S}'(\mathbb{R}^n) \ni f \to (2\pi)^{-n} \int e^{iy\eta} a_k(y,\eta) \hat{f}(\eta)\, d\eta \in \mathscr{E}'(\mathbb{R}^n),$$

where, as usual for pseudodifferential operators, this integral can be interpreted by introducing a sufficiently high power of $(1-\Delta_y)$ through

$$(1+|\eta|^2)^{-1}(1-\Delta_y)\, e^{iy\eta} = e^{iy\eta}.$$

LEMMA 3·4 (*Andersson*). *If $\{a_k\}$ is an A-sequence in $S^m(F)$, where F has compact base and $f \in \mathscr{S}'(\mathbb{R}^n)$ has $WF_a(f) \cap F = \varnothing$, then $a_k(y,D)f \in C_0^\infty(\mathbb{R}^n)$ and there exist constants $C, \delta > 0$ such that*

$$|D_y^\alpha a_k(y,D)f| \leqslant C^k k^{|\alpha|} \quad (\forall k \geqslant \delta^{-2}, |\alpha| \leqslant \delta k). \tag{3·5}$$

Conversely, $(y_0, \eta_0) \notin WF_a(f)$ if $\exists$ an A-sequence $\{a_k\}$ which is non-characteristic at (y_0, η_0) and for which (3·5) holds.

Note that suitable non-characteristic A-sequences do exist (Andersson (2), example (2·2) referring back to (6)). Indeed, if F_1 and F_2 are two closed cones with compact bases and

$$F_1 \backslash \{0\} \subset \operatorname{int}(F_2)$$

then there exists an A-sequence $\{a_k\}$ of order zero satisfying, in addition to (3·3) for $m = 0$,

$$a_k(y,\eta) = \begin{cases} 1 & \text{if} \quad (y,\eta) \in F_1, \quad |\eta| \geqslant 2Ck \\ 0 & \text{if} \quad |\eta| \leqslant Ck \quad \text{or} \quad (y,\eta) \notin F_2. \end{cases}$$

We shall denote the set of such sequences by $\tau^0(F_1, F_2)$. For later reference we note that a homogeneous real-analytic transformation preserves the τ^0 class. That is, if Γ_1 is an open conic neighbourhood of $F_2 \backslash \{0\}$ and g is a real-analytic diffeomorphism onto another open cone $g: \Gamma_1 \to \Gamma_2$ which is positive-homogeneous in the sense that if $(y,\eta) \in \Gamma_1$, $s > 0$ and $g(y,\eta) = (Y,H)$ then $g(y,s\eta) = (Y,sH)$, then $\{a_k \circ g^{-1}\}$ is an A-sequence in $\tau^0(g(F_1), g(F_2))$. To see this, first note that, as $a_k = 0$ in $|\eta| \leqslant Ck$, (3·5) is equivalent to the existence of $C > 0$ such that

$$|D_y^\alpha D_\eta^\beta a_k(y,\eta)| \leqslant C^k k^{|\alpha|+|\beta|} |\eta|^{-|\beta|} \quad (\forall |\alpha| + |\beta| \leqslant \delta k, k \geqslant \delta^{-2}). \tag{3·6}$$

To demonstrate (3·6) for $a_k \circ g^{-1}$ given that it holds for a_k itself, fix attention on a particular (but arbitrary) point (y_0, η_0) and replace a_k by $\tilde{a}_k$ its truncated Taylor series to order the integral part of δk at $g^{-1}(y_0, \eta_0)$. Clearly $D_y^\alpha D_\eta^\beta a_k \circ g^{-1} = D_y^\alpha D_\eta^\beta \tilde{a}_k \circ g^{-1}$ at (y_0, η_0) provided $|\alpha + \beta| \leqslant \delta k$. The estimates (3·6) then follow from the Cauchy inequalities for g and $\tilde{a}_k$ and are easily made uniform in (y_0, η_0) (compare Hörmander (6)).

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

Next, recall the notion of an analytic symbol as introduced by Boutet de Monvel and Krée (4). If Γ is an open cone in $\mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ such a symbol is a series (in general divergent)

$$\sum_{j \leqslant 0} p_j(y, \eta)$$

of functions $p_j \in C^\infty(\Gamma)$, each p_j being homogeneous of degree $m+j$ and such that for some $C > 0$ corresponding to each closed cone with compact base $K \Subset \Gamma$

$$|D_y^\alpha D_\eta^\beta p_j(y, \eta)| \leqslant C^{|j|+|\alpha|+|\beta|+1} |j|! \, |\alpha|! \, |\beta|! |\eta|^{j-|\beta|+m} \quad \text{in} \quad K. \tag{3·7}$$

In particular each p_j is real-analytic. Following Andersson we note that if Σp_j is an analytic symbol and $\{a_k\} \in \tau^0(F_1, F_2)$ with $F_2 \backslash \{0\} \subset \Gamma$ then the sequence

$$b_k = \sum_{j \geqslant -k} p_j(y, \eta) \, a_k(y, \eta)$$

is an A-sequence of order m.

Now, we are in a position to prove Proposition 3·1. First we shall prove the following:

$$(\bar{y}, \bar{\eta}) \notin WF_a(b) \quad \text{if} \quad \bar{y} \neq 0 \quad \text{and} \quad \bar{\eta}_n^2 \neq |\bar{\eta}'|^2. \tag{3·8}$$

To show this we apply an A-sequence $\{a_k\} \in \tau^0(F_1, F_2)$ where F_1 is a closed conic neighbourhood of $(\bar{y}, \bar{\eta})$, $F_1 \Subset F_2$ and F_2 is chosen so small (and with compact base) that

$$\inf \{|y|, (\eta_n^2 - |\eta'|^2)/|\eta|^2; \quad (y, \eta) \in F_2\} > 0.$$

Put $f_k = a_k(y, D) \, b$. This is well defined since $b \in \mathscr{S}'$ (Lemma 2·6). As a consequence of Lemma 3·4, (3·8) follows if we can show that $\exists C, \delta > 0$ such that

$$|D_y^\alpha f_k| \leqslant C^k k^{|\alpha|} \quad (|\alpha| \leqslant \delta k, |\alpha| \geqslant \delta^{-2}). \tag{3·9}$$

In the appendix the asymptotics of Ai are discussed and it is shown that there is an analytic symbol Σb_j depending only on η in the cone

$$\Gamma = \mathbb{R}^n \times \{\eta \in \mathbb{R}^n \backslash \{0\}; \eta_n^2 \neq |\eta'|^2\}$$

such that for any closed cone $K \Subset \Gamma \; \exists C > 0$ and $R > 0$ with

$$|\hat{b} - \sum_{j \geqslant -p} b_j| \leqslant C^{p+1} p! |\eta|^{-p} \quad (\forall p \in \mathbb{N}, |\eta| \geqslant R). \tag{3·10}$$

So, putting $T_p = \hat{b} - \sum_{j \geqslant -p} b_j$ for each $p \in \mathbb{N}$ it follows from the remark above that

$$c_k = a_k(\hat{b} - T_k)$$

defines an A-sequence in F_2 and hence that

$$g_k = f_k - (2\pi)^{-n} \int e^{iy\eta} a_k(y, \eta) \, T_k(\eta) \, d\eta = c_k(y, D)(\delta)$$

(δ the Dirac measure) satisfies (3·9) in place of f_k, as $WF_a(\delta) = T_0^* \mathbb{R}^n \backslash \{0\}$. So, (3·9) will hold for f_k if it holds for

$$h_k = (2\pi)^{-n} \int e^{iy\eta} a_k(y, \eta) T_k(\eta) \, d\eta.$$

We shall show this directly. Applying D_y^α to h_k gives, with appropriate interpretation of the integrals,

$$D_y^\alpha h_k = (2\pi)^{-n} \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} \int e^{iy\eta} \eta^\gamma D_y^{\alpha-\gamma} a_k \, T_k \, d\eta. \tag{3·11}$$

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

We can suppose that a_k is supported in $|\eta| \geqslant C'k$, with C' so large that (3·10) gives

$$|T_k| \leqslant (C')^{k+1} k^k |\eta|^{-k} \quad (\delta^{-2} \leqslant k \quad \text{in} \quad |\eta| \geqslant R).$$

Each integral in the sum for $D_y^\alpha h_k$ can then be estimated by

$$D \int_{c'k}^{\infty} r^{|\gamma|+N-1-k} (C')^k k^k C^k k^{|\alpha-\gamma|}\, dr \leqslant D(C')^{|\gamma|+n} C^k k^{|\alpha|+n} \leqslant (C'')^k k^{|\alpha|},$$

where D is a constant independent of k, α and γ, provided we assume that $|\gamma| \leqslant |\alpha| \leqslant k-n-1$ (which we can by decreasing δ in $|\alpha| \leqslant \delta k$, $k \geqslant \delta^{-2}$). Inserting these estimates in (3·11) and using the binomial theorem to sum we find that the h_k satisfy (3·9). This completes the proof of (3·8).

So, to prove (3·2) it now remains to show that

$$(\bar{y}, \bar{\eta}) \notin WF_a(b) \quad \text{when} \quad \bar{\eta}_n^2 = |\bar{\eta}'|^2 \quad \text{but} \quad (\bar{y}', \bar{y}_n) \neq s(\bar{\eta}', -\bar{\eta}_n) \quad \text{for any } s. \tag{3·12}$$

We shall prove this using the same basic technique. That is, we shall apply an A-sequence $\{a_k\}$ to b, with $\{a_k\} \in \tau^0(F_1, F_2)$, F_2 some conic neighbourhood of $(\bar{y}, \bar{\eta})$ chosen conveniently small.

It is a direct consequence of the assumption on $(\bar{y}, \bar{\eta})$ in (3·12) that in some such F_2 there is a (real) real-analytic vector field

$$\mathscr{L} = \sum_{j=1}^{n} \omega_j \partial/\partial\eta_j$$

with the ω_j homogeneous of degree zero and

$$\mathscr{L} e^{iy\eta} = i e^{iy\eta}, \quad \mathscr{L}\zeta(\eta) = 0 \quad \text{in} \quad F_2. \tag{3·13}$$

Indeed, (3·12) shows that there exists $\mu \in \mathbb{R}^n$ with $\mu.(\bar{\eta}', -\bar{\eta}_n) = 0$ and $\mu.\bar{y} \neq 0$. Since $|\bar{\eta}'|^2 = \bar{\eta}_n^2$, at $\bar{\eta}$ ζ_η' is a multiple of $(\bar{\eta}', -\bar{\eta}_n)$ so μ can be extended to a real-analytic vector field in some F_2, homogeneous of degree zero with $\mu(\eta).\zeta' = 0$. Then (3·13) holds for $\omega = \mu/\eta.\mu$ which is well defined if F_2 is small enough. Now, if necessary by shrinking F_2 even further (but still requiring it to be a conic neighbourhood of $(\bar{y}, \bar{\eta})$ of course), we can find a real-analytic map

$$\tilde{H}: F_2 \ni (y, \eta) \mapsto (y, H(y, \eta)) \in \mathbb{R}^{2n}$$

with the H_j homogeneous of degree one, mapping F_2 diffeomorphically onto some cone F_2' in such a way that

$$\tilde{H}_* \mathscr{L} = \partial/\partial H_1.$$

Then, with $\{a_k\} \in \tau^0(F_1, F_2)$ and any $s \in \mathbb{N}$

$$\begin{aligned} a_k(y, D)\, b &= (2\pi)^{-n} \int a_k(y, \eta)\, \hat{b}(\eta)\, e^{iy\eta}\, d\eta \\ &= (2\pi)^{-n} i^s \int - a_k(y, \eta)\, \eta_n^{\frac{s}{3}} \mathscr{L}^s (e^{iy\eta} \operatorname{Ai}'(\zeta_0)/\operatorname{Ai}(\zeta_0))\, d\eta \end{aligned}$$

since $\hat{b} = -\eta_n^{\frac{s}{3}} \operatorname{Ai}'(\zeta_0)/\operatorname{Ai}(\zeta_0)$ and $\mathscr{L}\zeta_0 = 0$. Making the substitution of variables $\eta \mapsto H$ we find

$$a_k(y, D)\, b = (2\pi)^{-n} i^s (-1)^{s+1} \int \frac{\partial^s}{\partial H_1^s} (a(y, \eta)\, \eta_n^{\frac{s}{3}} |J|)\, e^{iy\eta} \frac{\operatorname{Ai}'(\zeta_0)}{\operatorname{Ai}(\zeta_0)}\, dH.$$

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

Here J, the Jacobian, is real-analytic and homogeneous of degree zero (and non-zero). Noting the earlier remark on the transformation properties of $\tau^0(F_1, F_2)$ under analytic diffeomorphisms it is clear that

$$|D_y^\alpha D_{H_1}^s (a_k(y,\eta)\,\eta_n^{\frac{2}{3}}|J|)| \leqslant C^k k^{|\alpha|+s} |H|^{\frac{2}{3}-s} \quad (|\alpha|+s \leqslant \delta k, k \geqslant \delta^{-2}).$$

since $|\mathrm{Ai}'(\zeta_0)/\mathrm{Ai}(\zeta_0)| \leqslant C(1+|H|)^{\frac{1}{3}}$ this shows that

$$\begin{aligned} |D_y^\alpha a_k(y,D)\,b| &\leqslant (2\pi)^{-n} \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} \int |D^{\alpha-\gamma} D_{H_1}^s (a_k \eta_n^{\frac{2}{3}}|J|)\,|H|^{|\gamma|+\frac{1}{3}} C\,dH \\ &\leqslant C' \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} \int_{ck}^{\infty} C^k k^{|\alpha|-|\gamma|+s} r^{1+|\gamma|-s+n-1}\,dr \\ &\leqslant (C'')^k k^{|\alpha|} \end{aligned}$$

by choosing $C'' \gg C', C$, $s = |\alpha|+n+2 \leqslant \delta' k$, $0 < \delta' \ll \delta$. This concludes the proof of Proposition 3·1.

We can easily demonstrate the following 'inner bound' on $WF_A(b)$.

$$\text{analytic sing. supp. } b = \{y \in \mathbb{R}^n; y_n \geqslant 0, y_n^2 = |y'|^2\} = \mathscr{C}. \tag{3·14}$$

Indeed, it follows from (3·2) that b is zero outside $\mathscr{C}$ and analytic inside $\mathscr{C}$. So, if some point $\bar{y} \in \mathscr{C}$ is not in the analytic singular support of b, i.e. b is analytic near $\bar{y}$, then b must vanish within $\mathscr{C}$ as well; but b is C^∞ outside the origin, so this would imply that b has support the origin. A theorem of Schwartz shows that a distribution with support the origin is a finite sum of derivatives of the Dirac measure, so has Fourier transform a polynomial. Clearly $\hat{b}$ is not a polynomial so (3·14) holds.

It is relatively easy to demonstrate the following improvement on (3·14):

$$T_0^*\mathbb{R}^n \backslash \{0\} \subset WF_a(b) \text{ and either the interior or exterior normal field on } \mathscr{C} \text{ is contained in } WF_a(b). \tag{3·15}$$

The final task in this section is to deduce from Proposition 3·1 the results on the propagation of analytic singularities for the boundary value problem (*) mentioned in the introduction. We shall leave this to the reader since the argument follows almost verbatim the corresponding one in section two, and only state the result obtained.

PROPOSITION 3·16. *Let u and f be as in Theorem 1·9. Then $WF_A(u)$ is contained in the forward bicharacteristic flow-out of the set $F \subset T^*\bar{\Omega}|_{\partial\Omega}$ that is made up as follows*: (i) *If $(y,\eta) \in WF_A(f)$ is a hyperbolic point, then $\sigma(y,\eta) \in F$.* (ii) *If $(\bar{y},\bar{\eta}) \in WF_A(f)$ is a glancing point, then F contains*

$$\{(x,y,\xi,\eta);\quad x=\xi=0,\quad y'=\bar{y}'-s\bar{\eta}'\,\mathrm{sign}\,\bar{\eta}_n,\quad y_n=\bar{y}_n+s|\bar{\eta}'|,\quad \eta=\bar{\eta},\quad s\geqslant 0\}.$$

Elliptic points of $WF_A(f)$ do not contribute to $WF_A(u)$.

Appendix: the symbol of the boundary operator. The symbol of the boundary operator is

$$\hat{b}(\eta) = -|\eta_n|^{\frac{2}{3}} \frac{\mathrm{Ai}'(\zeta_0)}{\mathrm{Ai}(\zeta_0)} \exp(-\tfrac{2}{3}i\pi\,\mathrm{sign}\,\eta_n), \tag{A 1}$$

where

$$\zeta_0 = \frac{\eta_n^2 - |\eta'|^2}{|\eta_n|^{\frac{4}{3}}} \exp(\tfrac{1}{3}i\pi\,\mathrm{sign}\,\eta_n) \quad (\eta_n \neq 0). \tag{A 2}$$

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

We propose to show that $\hat{b}$, which is only defined for $\eta_n \neq 0$ by (A 1), extends by continuity to a C^∞ function on $\mathbb{R}^n\backslash\{0\}$, which we also denote by $\hat{b}(\eta)$; we shall then derive the estimates for the derivatives of $\hat{b}$ which are required in the text. For both purposes, one needs estimates for $\mathrm{Ai}'(\zeta)/\mathrm{Ai}(\zeta)$ and its derivatives when $|\zeta|$ is large and $\arg\zeta = \pm\frac{1}{3}i\pi$ or $\arg\zeta = \pm\frac{2}{3}i\pi$. Although the asymptotic expansion of the Airy function is well known, the estimates which are relevant here do not appear to be available in the literature, and must therefore be derived first.

Throughout, ζ denotes a complex number such that $|\arg\zeta| < \pi$, and if $\mu \in \mathbb{R}$ then ζ^μ denotes $|\zeta|^\mu \exp(i\mu\arg\zeta)$.

It follows from (A 1) and (A 2) that, if $\lambda \in \mathbb{R}^+$, then

$$\hat{b}(\eta', -\lambda) = \overline{\hat{b}(\eta', \lambda)}. \tag{A 3}$$

It is therefore sufficient to discuss $\hat{b}$ when $\eta_n \geqslant 0$; then $\arg\zeta_0 = \frac{1}{3}\pi$ if $\eta_n > |\eta'|$ and $\arg\zeta_0 = -\frac{2}{3}\pi$ if $0 < \eta_n < |\eta'|$. These cases are covered by the following lemma.

LEMMA A 4. *There exist functions $\phi(z), \psi(z)$, both in $C^\infty([0,\infty))$, such that*

$$\left.\begin{aligned} \zeta^{-\frac{1}{2}}\mathrm{Ai}'(\zeta)/\mathrm{Ai}(\zeta) &= \phi(z^{-\frac{3}{2}}) \quad \text{if} \quad \zeta = z\,e^{\frac{1}{3}i\pi} \quad (z > 0) \\ &= \psi(z^{-\frac{3}{2}}) \quad \text{if} \quad \zeta = z\,e^{-\frac{2}{3}i\pi} \quad (z > 0). \end{aligned}\right\} \tag{A 5}$$

Also, for any a, $0 < a < \infty$, there is a constant C such that

$$|\phi^{(j)}(z)| \leqslant C^{1+j}(j!)^2, \quad |\psi^{(j)}(z)| \leqslant C^{1+j}(j!)^2 \tag{A 6}$$

for $0 \leqslant z \leqslant a$ and $j = 0, 1, \ldots$.

Proof. By definition,

$$\mathrm{Ai}(\zeta) = \frac{1}{2\pi i}\int_{\infty\exp(-i\alpha)}^{\infty\exp(-i\beta)} \exp(t\zeta - \tfrac{1}{3}t^3)\,dt,$$

where $\frac{1}{2}\pi < \alpha < \frac{5}{6}\pi$ and $\frac{1}{2}\pi < \beta < \frac{5}{6}\pi$. We shall replace this by another integral representation, from which the lemma will follow easily. Suppose for the moment that $|\arg\zeta| < \frac{1}{3}\pi$. Then one can take $\alpha = \frac{2}{3}\pi + \frac{1}{2}\arg\zeta$, $\beta = \frac{2}{3}\pi - \frac{1}{2}\arg\zeta$, replace t by $t\zeta^{\frac{1}{2}}$, and obtain

$$\mathrm{Ai}(\zeta) = \frac{\zeta^{\frac{1}{2}}}{2\pi i}\int_{\infty\exp(-\frac{2}{3}i\pi)}^{\infty\exp(\frac{2}{3}i\pi)} \exp(\zeta^{\frac{3}{2}}(t - \tfrac{1}{3}t^3))\,dt. \tag{A 7}$$

Now consider the equation

$$\tfrac{1}{3}t^3 - t - \tfrac{2}{3} = s \quad (|\arg s| < \pi). \tag{A 8}$$

When $s = 0$, this has $t = -1$ as a double root, and it is easy to show that for small s there are two roots with Puiseux expansions

$$t = T(s^{\frac{1}{2}}), \quad t = T(-s^{\frac{1}{2}}),$$

where $s^{\frac{1}{2}}$ denotes the principal branch of the square root, $-s^{\frac{1}{2}}$ the other branch, and T is holomorphic in a neighbourhood of the origin. In fact, T can be obtained explicitly by solving the cubic equation (A 8). One finds that

$$T(s^{\frac{1}{2}}) = t_1(s^{\frac{1}{2}})\,e^{\frac{2}{3}i\pi} + t_2(s^{\frac{1}{2}})\,e^{-\frac{2}{3}i\pi}, \tag{A 9a}$$

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

where
$$\left.\begin{aligned} t_1^3 &= 1+\tfrac{3}{2}s+\tfrac{3}{2}s^{\frac{1}{2}}(s+\tfrac{4}{3})^{\frac{1}{2}}, \\ t_2^3 &= 1+\tfrac{3}{2}s-\tfrac{3}{2}s^{\frac{1}{2}}(s+\tfrac{4}{3})^{\frac{1}{2}}. \end{aligned}\right\} \tag{A 9b}$$

Here,
$$|\arg s^{\frac{1}{2}}| < \tfrac{1}{2}\pi, \quad |\arg (s+\tfrac{4}{3})^{\frac{1}{2}}| < \tfrac{1}{2}\pi,$$

and t_1 and t_2 are to be real and positive when s is real and positive. These equations clearly give $T(s^{\frac{1}{2}})$ and $T(-s^{\frac{1}{2}})$ on $\{s; s \neq 0, |\arg s| < \pi\}$.

It follows from (A 9a, b) that the path in the t-plane that is the union of
$$(\infty, 0] \ni s \mapsto T(-s^{\frac{1}{2}}) \quad \text{and} \quad [0, \infty) \ni s \mapsto T(s^{\frac{1}{2}})$$
runs from $\infty \exp(-\frac{2}{3}i\pi)$ to $\infty \exp(\frac{2}{3}i\pi)$, and so can be used as the path of integration in (A 7). Because of (A 8), this gives

$$\left.\begin{aligned} \operatorname{Ai}(\zeta) &= \zeta^{\frac{1}{2}} \exp(-\tfrac{2}{3}\zeta^{\frac{3}{2}}) \int_0^\infty s^{-\frac{1}{2}} \mu(s) \exp(-s\zeta^{\frac{3}{2}})\, ds, \\ \mu(s) &= \frac{1}{4\pi i}(T'(s^{\frac{1}{2}}) + T'(-s^{\frac{1}{2}})). \end{aligned}\right\} \tag{A 10}$$

As the two branches of $s^{\frac{1}{2}}$ are interchanged when s encircles the origin once, it follows from (A 9a, b) that $\mu(s)$ is analytic on the s-plane cut along $(-\infty, -\frac{4}{3})$. We also note that
$$\mu = 3^{-\frac{1}{2}} \frac{\cosh(\frac{2}{3}\sigma)}{\cosh \sigma}, \tag{A 11}$$
where $s = \frac{4}{3}\sinh^2 \sigma$, $\operatorname{Re} \sigma \geqslant 0$, $|\operatorname{Im} \sigma| < \frac{1}{2}\pi$.

The integral in (A 10) only converges if $|\arg \zeta| < \frac{1}{3}\pi$. But, as $\operatorname{Ai}(\zeta)$ is an entire function, (A 10) can be extended to larger values of $|\arg \zeta|$ by the usual device of rotating the path of integration. If one also makes the substitution $s \mapsto s\zeta^{-\frac{3}{2}}$, one obtains
$$\operatorname{Ai}(\zeta) = \zeta^{-\frac{1}{4}} \exp(-\tfrac{2}{3}\zeta^{\frac{3}{2}}) \int_0^{\infty \exp(i\gamma)} e^{-s} \mu(s\zeta^{-\frac{3}{2}})\, s^{-\frac{1}{2}}\, ds, \tag{A 12}$$
where γ is a real number such that
$$|\gamma| < \tfrac{1}{2}\pi, \quad |\gamma - \tfrac{3}{2}\arg \zeta| < \pi;$$
clearly, such a γ can be chosen provided that $|\arg \zeta| < \pi$.

Now $\zeta \mapsto Z = \zeta^{-\frac{3}{2}}$ maps the ζ-plane, cut along the negative real axis, on a certain Riemann surface M which can be described, somewhat loosely, as a 'sector'
$$\{Z; Z \neq 0, |\arg Z| < \tfrac{2}{3}\pi\}.$$
By (A 12), we have thus
$$\operatorname{Ai}(\zeta) = \zeta^{-\frac{1}{4}} \Phi(z) \exp(-\tfrac{2}{3}\zeta^{\frac{3}{2}}) \quad (z = \zeta^{-\frac{3}{2}}), \tag{A 13}$$
where
$$\Phi(z) = \int_0^{\infty \exp(i\gamma)} e^{-s} \mu(sZ)\, s^{-\frac{1}{2}}\, ds \tag{A 14}$$
is analytic on M and C^∞ on $M \cup \{0\}$. Also,
$$\Phi^{(j)}(Z) = \int_0^{\infty \exp(i\gamma)} e^{-s} \mu^{(j)}(sZ)\, s^{j-\frac{1}{2}}\, ds \quad (j = 1, 2, \ldots).$$

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

It has already been noted that $\mu(s)$ is analytic on the s-plane cut along $(-\infty, -\frac{4}{3})$, and it follows from (A 11) that $\mu(s)$ is bounded if, in addition, $|s+\frac{4}{3}| \geqslant \delta > 0$. So, once γ has been chosen (for fixed $\arg\zeta$), one has Cauchy estimates

$$|\mu^{(j)}(sZ)| \leqslant Cj!\, r^{-j} \quad (j = 0, 1, \ldots),$$

that are uniformly valid for fixed $\arg s = \gamma$, with C and r depending on γ. Hence

$$\begin{aligned} |\Phi^{(j)}(Z)| &\leqslant Cr^{-j} j! \int_0^\infty s^{j-\frac{1}{2}} \exp(-s\cos\gamma)\, ds \\ &= Cr^{-j} j!\, r(j+\tfrac{1}{2})(\cos\gamma)^{-j-\frac{1}{2}}, \end{aligned}$$

whence

$$|\Phi^{(j)}(Z)| \leqslant C^{1+j}(j!)^2 \quad (j = 0, 1, \ldots),$$

with a new constant C. So Φ is of Gevrey class 2 at $Z = 0$.

Again, (A 13) gives

$$\zeta^{-\frac{1}{2}} \frac{\mathrm{Ai}'(\zeta)}{\mathrm{Ai}(\zeta)} = \Psi(Z) = -1 - \tfrac{1}{4}Z - \tfrac{3}{2}Z^2 \frac{\Phi'(Z)}{\Phi(Z)}. \tag{A 15}$$

By (A 13), and the fact that the zeros of the Airy function are all on the negative real axis, $|\Phi(z)|$ is bounded away from zero if $|\arg\zeta| \leqslant \pi - \delta$, $\delta > 0$. In view of this, it is easy to show that $\Psi(z)$ is also of Gevrey class 2 at $Z = 0$, and of course analytic on the Riemann surface M. So, if $0 \leqslant Z \leqslant a$, there is a constant $C = C(a, \gamma)$ such that

$$|\Psi^{(j)}(Z)| \leqslant (C(a,\gamma))^{1+j}(j!)^2 \quad (0 \leqslant Z \leqslant a, j = 0, 1, \ldots). \tag{A 16}$$

Now one can take $\gamma = 0$ when $\arg\zeta = \frac{1}{3}\pi$ and $\gamma = -\frac{1}{4}\pi$ when $\arg\zeta = -\frac{2}{3}\pi$. So the assertion of the Lemma follows, if one puts

$$\phi(z) = \Psi(z\, e^{-\frac{1}{2}i\pi}), \quad \psi(z) = \Psi(z\, e^{i\pi}),$$

with $C = \max(C(a, 0), C(a, -\frac{1}{4}\pi))$.

We can now discuss $\hat{b}(\eta)$. Suppose that $\eta_n > 0$; then, by (A 1) and (A 2),

$$\hat{b}(\eta) = -\eta_n^{\frac{2}{3}} \frac{\mathrm{Ai}'(\zeta_0)}{\mathrm{Ai}(\zeta_0)} e^{-\frac{2}{3}i\pi}, \quad \zeta_0 = \frac{\eta_n^2 - |\eta'|^2}{\eta_n^{\frac{4}{3}}} e^{\frac{1}{3}i\pi}. \tag{A 17}$$

Also, if $\zeta_0 \neq 0$, then (A 1), (A 2) and (A 5) give

$$\hat{b}(\eta) = i(\eta_n^2 - |\eta'|^2)^{\frac{1}{2}} \phi(\eta_n^2(\eta_n^2 - |\eta'|^2)^{-\frac{3}{2}}) \quad \text{if} \quad \eta_n > |\eta'|, \tag{A 18}$$

$$\hat{b}(\eta) = (|\eta'|^2 - \eta_n^2)^{\frac{1}{2}} \psi(\eta_n^2(|\eta'|^2 - \eta_n^2)^{-\frac{3}{2}}) \quad \text{if} \quad 0 < \eta_n < |\eta'|. \tag{A 19}$$

PROPOSITION A 20. (i) *The function $\hat{b}(\eta)$, which is defined for $\eta_n \neq 0$ by* (A 1), *extends by continuity to a function in $C^\infty(\mathbb{R}^n \backslash \{0\})$.*

(ii) *There exist functions $b_{-j}(\eta)$, $j = 0, 1, \ldots$, defined on $\Gamma = \{\eta; \eta_n^2 - |\eta'|^2 \neq 0\}$, such that Σb_{-j} is an analytic symbol and, for every closed cone $K \subset \Gamma$ and every non-negative integer p,*

$$|\hat{b} - \sum_{j \leqslant p} b_{-j}| \leqslant C^{1+p} p! |\eta|^{-p} \quad \text{if} \quad |\eta| \geqslant 1, \tag{A 21}$$

where C is a constant that depends on K.

Proof. (i) The only difficulty arises at $\eta_n = 0$, where ζ_0 is not defined. It is clear from (A 19) that the restriction of $\hat{b}$ to $\eta_n > 0$ extends, by continuity, to a function

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

that is C^∞ on $\{\eta; \eta_n \geqslant 0, \eta \neq 0\}$. It then follows from (A 3) that $\hat{b}$ is also defined by continuity as a C^∞ function on $\{\eta; \eta_n \leqslant 0, \eta \neq 0\}$. So it only remains to prove that $\hat{b}$ and its derivatives tend to the same values as $\eta_n \downarrow 0$ and $\eta_n \uparrow 0$, respectively, when $\eta' \neq 0$. Now for $0 < \lambda < |\eta'|$,

$$\hat{b}(\eta', \lambda) = -\lambda^{\frac{2}{3}} e^{-\frac{2}{3}i\pi} \frac{\mathrm{Ai}'(z_0 e^{-\frac{2}{3}i\pi})}{\mathrm{Ai}(z_0 e^{-\frac{2}{3}i\pi})}, \quad \hat{b}(\eta', -\lambda) = -\lambda^{\frac{2}{3}} e^{\frac{2}{3}i\pi} \frac{\mathrm{Ai}'(z_0 e^{\frac{2}{3}i\pi})}{\mathrm{Ai}(z_0 e^{\frac{2}{3}i\pi})},$$

where

$$z_0 = \frac{|\eta'|^2 - \lambda^2}{\lambda^{\frac{4}{3}}}.$$

It is well known that the Airy function satisfies the identity

$$\mathrm{Ai}(z_0) + e^{-\frac{2}{3}i\pi} \mathrm{Ai}(z_0 e^{-\frac{2}{3}i\pi}) + e^{\frac{2}{3}i\pi} \mathrm{Ai}(z_0 e^{\frac{2}{3}i\pi}) = 0,$$

whence

$$\mathrm{Ai}'(z_0) + e^{-\frac{4}{3}i\pi} \mathrm{Ai}'(z_0 e^{-\frac{2}{3}i\pi}) + e^{\frac{4}{3}i\pi} \mathrm{Ai}'(z_0 e^{\frac{2}{3}i\pi}) = 0.$$

So a simple computation gives

$$\hat{b}(\eta', \lambda) - \hat{b}(\eta', -\lambda) = \lambda^{\frac{2}{3}} e^{\frac{2}{3}i\pi} \frac{\mathrm{Ai}'(z_0)}{\mathrm{Ai}(z_0 e^{-\frac{2}{3}i\pi})} - \lambda^{\frac{2}{3}} e^{\frac{4}{3}i\pi} \frac{\mathrm{Ai}(z_0)\,\mathrm{Ai}'(z_0 e^{\frac{2}{3}i\pi})}{\mathrm{Ai}(z_0 e^{-\frac{2}{3}i\pi})\,\mathrm{Ai}(z_0 e^{\frac{2}{3}i\pi})},$$

and the asymptotic behaviour of the Airy function therefore implies that

$$\hat{b}(\eta', \lambda) - \hat{b}(\eta', -\lambda),$$

and its derivatives of all orders, tend to zero as $\lambda \downarrow 0$ for $\eta' \neq 0$.

(ii) Note that Γ has three components,

$$\Gamma_1 = \{\eta; \eta_n > |\eta'|\}, \quad \Gamma_2 = \{\eta; \eta_n < -|\eta'|\}, \quad \Gamma_3 = \{\eta; |\eta_n| < |\eta'|\},$$

so that the assertion will follow if it is proved for $K \subset \Gamma_m$, $m = 1, 2, 3$. If $K \subset \Gamma_1$, then $\hat{b}$ is given by (A 18), and one can replace ϕ by its formal Taylor series, whence

$$\hat{b} \sim i \sum_{j \geqslant 0} \phi^{(j)}(0)\, \eta_n^{2j} (\eta_n^2 - |\eta'|^2)^{-\frac{3}{2}j + \frac{1}{2}} / j! = \sum_{j \geqslant 0} b_{-j}(\eta).$$

There is an $\epsilon > 0$ such that

$$K \subset \{\eta; \eta_n \geqslant (1+\epsilon)^{\frac{1}{2}} (1-\epsilon)^{-\frac{1}{2}} |\eta'|\}.$$

Then

$$\eta_n^2 (\eta_n^2 - |\eta'|^2)^{-\frac{3}{2}} \leqslant \epsilon^{-\frac{3}{2}} |\eta|^{-1}.$$

As the b_{-j} are real analytic and homogeneous of degree $-1-j$, and the $\phi^{(j)}(0)$ can be estimated by (A 6) (or computed explicitly), it is obvious that the estimates (3·7), which characterize analytic symbols, hold. It also follows from (A 6) that (A 21) is valid.

Because of (A 3), this argument also settles the case of $K \subset \Gamma_2$. When $K \subset \Gamma_3$, a little care is needed. If $0 < \eta_n < |\eta'|$, then $\hat{b}$ is given by (A 19), and so

$$\hat{b} \sim \sum_{j \geqslant 0} \psi^{(j)}(0)\, \eta_n^{2j} (|\eta'|^2 - \eta_n^2)^{-\frac{3}{2}j + \frac{1}{2}} = \sum_{j \geqslant 0} b_{-j}(\eta).$$

When $-|\eta'| < \eta_n < 0$, one has to replace the second member of (A 19) by its complex conjugate. But is clear from (A 14) (where one must take, say, $\gamma = \frac{1}{4}\pi$ when $\arg \zeta = \frac{2}{3}\pi$) that ψ, like $\hat{b}$, is 'almost real' inasmuch as $\psi(z)$ and $\overline{\psi(z)}$ have the same Taylor series at $z = 0$. Hence the assertion again follows by an appeal to (A 14).

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

We now come to the estimates required for the determination of the Gevrey wave front set.

PROPOSITION A 22. (i) *If* $|\eta_n| \geqslant \epsilon|\eta|$, *where* $0 < \epsilon < 1$, *and* $|\eta| \geqslant 1$, *then*

$$|\partial^\alpha \hat{b}(\eta)| \leqslant C^{1+|\alpha|}\alpha!\,|\eta|^{1-\frac{1}{3}|\alpha|} \quad (\alpha \geqslant 0), \tag{A 23}$$

where C *is a constant which depends on* ϵ.

(ii) *If* $|\eta_n| \leqslant \delta|\eta|$, *where* $0 < \delta < 2^{-\frac{1}{2}}$, *and* $|\eta| \geqslant 1$, *then*

$$|\partial^\alpha \hat{b}(\eta)| \leqslant C^{1+|\alpha|}(\alpha!)^2\,|\eta|^{1-|\alpha|} \quad (\alpha \geqslant 0), \tag{A 24}$$

where C *is a constant which depends on* δ.

Proof. By (A 3), it is sufficient to prove this for $\eta_n \geqslant 0$. Let first $\eta_n \geqslant \epsilon|\eta| \geqslant \epsilon$. For $|\alpha| > 0$,

$$\partial_\eta^\alpha(\mathrm{Ai}'(\zeta_0)/\mathrm{Ai}(\zeta_0)) = \sum_{j=1}^{|\alpha|} \chi_j(\eta)\,(d/d\zeta_0)^j(\mathrm{Ai}'(\zeta_0)/\mathrm{Ai}(\zeta_0))/j!, \tag{A 25}$$

where

$$\chi_j(\eta) = \partial_\eta^\alpha(\zeta_0 - t)^j|_{t=\zeta_0} \quad (j = 1, 2, \ldots). \tag{A 26}$$

This identity is established by observing that

$$\mathrm{Ai}'(\zeta_0)/\mathrm{Ai}(\zeta_0) = \sum_{j=0}^{|\alpha|} (\zeta_0 - t)^j\,(d/dt)^j(\mathrm{Ai}'(t)/\mathrm{Ai}(t))/j! + (\zeta_0 - t)^{|\alpha|+1} r_\alpha(\eta, t),$$

where r_α is a smooth function. As $\eta \mapsto \zeta_0$ is homogeneous of degree $\frac{2}{3}$, and (A 26) becomes, when expanded,

$$\chi_j = \sum_{k=1}^{j} \frac{(-1)^k}{k!(j-k)!}\,\zeta_0^{j-k}\,\partial_\eta^\alpha \zeta_0^k,$$

it will be seen that χ_j is homogeneous of degree $\frac{2}{3}j - |\alpha|$. Hence, for $|\eta| \geqslant 1$,

$$|\chi_j(\eta)| \leqslant |\eta|^{\frac{2}{3}j-|\alpha|} \sup\{|\chi_j(\eta)|;\ |\eta| = 1,\ \eta_n \geqslant \epsilon\}.$$

Now $\eta \mapsto \zeta_0$ is real analytic when $\eta_n \neq 0$, so that, for $\eta_n \geqslant \epsilon$,

$$\chi_j = \partial_\eta^\alpha(\zeta_0 - t)^j|_{t=\zeta_0} = \frac{\alpha!}{(2\pi i)^n}\int_D \frac{(\zeta_0(s) - \zeta_0(\eta))^j}{(s_1 - \eta_1)^{\alpha_1+1} \ldots (s_n - \eta_n)^{\alpha_n+1}}\,ds,$$

where

$$D = \{s \in \mathbb{C}^n;\ s = (\eta_1 + \tfrac{1}{2}\epsilon\, e^{i\theta_1}, \ldots, \eta_n + \tfrac{1}{2}\epsilon\, e^{i\theta_n}),\ \theta \in \mathbb{T}^n\}.$$

Therefore

$$|\chi_j(\eta)| \leqslant M^j\alpha!(\tfrac{1}{2}\epsilon)^{-|\alpha|}|\eta|^{\frac{2}{3}j-|\alpha|},$$

where M is an upper bound for $|\zeta_0(s) - \zeta_0(\eta)|$ when $|\eta| = 1$, $\eta_n \geqslant \epsilon$, and $s \in D$. As $|\eta| \geqslant 1$, we therefore have from (A 25) that

$$|\partial_\eta^\alpha(\mathrm{Ai}'(\zeta_0)/\mathrm{Ai}(\zeta_0))| \leqslant \alpha!(\tfrac{1}{2}\epsilon)^{-|\alpha|}\,|\eta|^{-\frac{1}{3}|\alpha|} \sum_{j=1}^{|\alpha|} M^j|(d/d\zeta_0)^j(\mathrm{Ai}'(\zeta_0)/\mathrm{Ai}(\zeta_0))|/j!. \tag{A 27}$$

To bound the derivatives of $\mathrm{Ai}'(\zeta_0)/\mathrm{Ai}(\zeta_0)$, one can also use Cauchy estimates. It follows from (A 13) that, if Ω is a subset of $\mathbb{C}$ that is at a finite distance from the set of zeros of $\mathrm{Ai}(\zeta)$, then there is a constant $A = A(\Omega)$ such that $|\mathrm{Ai}'(\zeta)/\mathrm{Ai}(\zeta)| \leqslant A|\zeta^{\frac{1}{2}}|$

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

for $\zeta \in \Omega$. Now the set of zeros of $\mathrm{Ai}(\zeta)$ is of the form $\{\zeta \in \mathbb{C};\ \zeta = -\alpha_\nu,\ \nu = 1, 2, \ldots\}$, where the α_ν are real numbers, and $0 < \alpha_1 < \alpha_2 < \ldots$. Again, for $\eta_n > 0$, ζ_0 is on the straight line $\zeta_0 = s e^{i\pi/3}$, $s \in \mathbb{R}$, whose distance from the zero set of $\mathrm{Ai}(\zeta)$ is clearly $(\sqrt{\frac{3}{2}})\alpha_1$. Hence, if $0 < \delta < (\sqrt{\frac{3}{2}})\alpha_1$, then

$$|(d/d\zeta_0)^j (\mathrm{Ai}'(\zeta_0)/\mathrm{Ai}(\zeta_0))| \leqslant Aj!\,\delta^{-j}(|\zeta_0| + \delta)^{\frac{1}{2}} \leqslant Bj!\delta^{-j}|\eta|^{\frac{1}{3}}, \tag{A 28}$$

where B is another constant, and we have used the inequality $|\zeta_0| \leqslant \epsilon^{-\frac{2}{3}}|\eta|^{\frac{1}{3}}$, which holds because $\eta_n \geqslant \epsilon$. Substituting this in (A 27), one finds that

$$|\partial_\eta^\alpha(\mathrm{Ai}'(\zeta_0)/\mathrm{Ai}(\zeta_0))| \leqslant \alpha!\,(\tfrac{1}{2}\epsilon)^{-|\alpha|}\,|\eta|^{\frac{1}{3}-\frac{1}{3}|\alpha|} \sum_{j=1}^{|\alpha|} B(M/\delta)^j,$$

whence

$$|\partial_\eta^\alpha(\mathrm{Ai}'(\zeta_0)/\mathrm{Ai}(\zeta_0))| \leqslant C^{1+|\alpha|}\alpha!\,|\eta|^{\frac{1}{3}-\frac{1}{3}|\alpha|} \tag{A 29}$$

for $\alpha > 0$; by (A 28), with $j = 0$, (A 29) also holds when $\alpha = 0$. It now follows easily from (A 29) and (A 19) that the estimate (A 23) is valid, with a different C, for $\eta_n \geqslant \epsilon|\eta| \geqslant \epsilon$. As already noted, the same estimate then holds for $\eta_n \leqslant -\epsilon|\eta| \leqslant -\epsilon$, because of (A 3).

The proof of (A 24) is similar, and will only be indicated briefly. It is again sufficient to consider $\eta_n > 0$, and one then has (A 19) which can be written as follows:

$$\hat{b}(\eta) = (|\eta'|^2 - \eta_n^2)^{\frac{1}{2}}\,\psi(z), \quad z = \eta_n^2(|\eta'|_i^2 - \eta_n^2)^{-\frac{3}{2}}. \tag{A 30}$$

The derivatives of $\psi(z)$ can be estimated by means of the identity

$$\partial_\eta^\alpha \psi(z) = \sum_{j=1}^{|\alpha|} \psi^{(j)}(z)\,\partial_\eta^\alpha (z-t)^j|_{t=z}/j!.$$

As $z(\eta)$ is real analytic in $\{\eta; |\eta_n| < |\eta'|\}$ and homogeneous of degree -1, and as $|\eta'|^2 - \eta_n^2 \geqslant 1 - 2\delta^2 > 0$ when $|\eta| = 1$, one can use Cauchy estimates to obtain

$$|\partial_\eta^\alpha (z-t)^j|_{t=z}| \leqslant M^j \alpha!\, r^{-|\alpha|}\,|\eta|^{-1-|\alpha|} \quad (\alpha > 0), \tag{A 31}$$

where M and r are constants. To estimate the $\psi^{(j)}(z)$ one now has to use (A 6), noting that z is bounded when $|\eta_n| \leqslant \delta|\eta|$ with $0 < \delta < 2^{-\frac{1}{2}}$, and $|\eta| \geqslant 1$. The inequalities (A 24) then follow from Leibniz's theorem and (A 30); the factor $|\eta|^{1-|\alpha|}$, rather than $|\eta|^{-|\alpha|}$, is obtained because (A 31) only holds for $\alpha > 0$, and $\partial^\alpha(|\eta'|^2 - \eta_n^2)^{\frac{1}{2}}$ is homogeneous of degree $1 - |\alpha|$.

During the initial part of the preparation of this paper the second author received support from the Science Research Council.

Note added in proof. Professor Hörmander kindly pointed out to us, at the 1976 Durham Symposium on Partial Differential Equations, that the inclusion (3·2) can be strengthened to an equality by virtue of the principle of analytic continuation, expressed in terms of wave front sets. (See Theorem 4·1 in L. Hörmander, *The spectral analysis of singularities*. Proc. Fifth National Math. Conf., Pahlavi Univ., Shiraz, Iran (1974) 129–139.)

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press

REFERENCES

(1) ANDERSSON, K. G. Propagation of analyticity of solutions of partial differential equations with constant coefficients. *Ark. Mat.* **8** (1971), 277–302.
(2) ANDERSSON, K. G. Analytic wave front sets for solutions of linear differential equations of principal type. *Trans. Amer. Math. Soc.* **177** (1973), 1–27.
(3) ATIYAH, M. F., BOTT, R. and GÅRDING, L. Lacunas for hyperbolic differential operators with constant coefficients: I. *Acta Math.* **124** (1970), 109–189.
(4) BOUTET DE MONVEL, L. and KRÉE, P. Pseudo-differential operators and Gevrey classes. *Ann. Inst. Fourier (Grenoble)* **17** (1967), 295–323.
(5) FRIEDLANDER, F. G. The wave front set of the solution of a simple initial-boundary value problem with glancing rays. *Math. Proc. Cambridge Philos. Soc.* **79** (1976), 145–159.
(6) HÖRMANDER, L. Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients. *Comm. Pure Appl. Math.* **24** (1971), 671–703.
(7) HÖRMANDER, L. *Linear partial differential operators* (Springer, 1963).
(8) HÖRMANDER, L. On the existence and regularity of solutions of linear pseudo-differential equations. *Enseignement Math.* **17** (1971), 99–163.
(9) HÖRMANDER, L. Fourier integral operators: I. *Acta Math.* **127** (1971), 79–183.
(10) MELROSE, R. B. Microlocal parametrices for diffractive boundary value problems. *Duke Math. J.* **42** (1975), 605–635.
(11) MELROSE, R. B. Equivalence of glancing hypersurfaces. *Invent. Math.* (to appear).
(12) OSHIMA, T. *Sato's problem in contact geometry* (manuscript, 1976).
(13) SATO, M., KAWAI, T. and KASHIWARA, M. Microfunctions and pseudo-differential equations. *Lecture Notes in Math.* (Springer) **287** (1973), 265–529.
(14) SCHWARTZ, L. *Théorie des distributions* (Paris, Hermann, 1966).
(15) SEELEY, R. T. Extension of C^∞ functions defined in a half space, *Proc. Amer. Math. Soc.* **15** (1964), 625–626.
(16) TAYLOR, M. E. Grazing rays and reflexion of singularities of solutions to wave equations. *Comm. Pure Appl. Math.* **29** (1976), 1–38.

https://doi.org/10.1017/S030500410000030X Published online by Cambridge University Press