as may easily be seen by noticing that the four latter complexes are of the same degrees in the coefficients of the quadrics a_1, a_2 as four complexes in Turnbull's list and that no invariant combination of the cubic complexes other than these has the same degrees, while the remaining complexes given by Turnbull are the duals of these. The syzygies expressing the squares of these complexes in terms of the quadratic complexes can be obtained at once from (31), (32) and (50).

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POINT-CURVE CORRESPONDENCES

I. THE THEORY OF THE BASE

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Received 16 *February* 1945

Attempts have been made to extend to higher dimensional varieties the theory of correspondences developed for algebraic curves. So far efforts have been concentrated on 'point-point' correspondences (i.e. between two varieties of the same dimension such that a generic point of one corresponds to a 0-dimensional variety of the other), and even in the case of surfaces important problems, such as the base number for the correspondences, are still unsolved (cf. Hodge(3)). The purpose of these papers is to draw attention to, and study in the simpler cases, another class of correspondences, the 'point-primal' type, in which to a generic point of one variety corresponds a primal of the other. These correspondences provide at least as natural a generalization of the theory for curves as point-point correspondences, but have so far only been touched on by Severi(7, 8) in two simple cases. In this paper we give a number of general results for point-primal correspondences (mostly immediate generalizations of Severi's results), and embark on a detailed discussion of such correspondences, which we naturally call point-curve correspondences, between two surfaces, the chief result being the determination of the base in that case. Further results on the transformation of the cycles of a surface by a point-curve correspondence, the correspondences induced between curves of the two surfaces and, in the case where the surfaces coincide, the 'united curve' of the correspondence, as well as the question of correspondences of non-zero valency, will be dealt with in a later paper, where we shall also consider the case of correspondences between a curve and a surface discussed in Severi's paper(7).

This work depends very largely on the results of an earlier paper (Scott(6)), referred to in future as S), and we adopt as far as possible the same notational conventions. Matrices will be denoted by symbols in clarendon type, and the elements of a matrix $\mathbf{a}$ by a_{ij}. We shall denote the transposed matrix of $\mathbf{a}$ by $\mathbf{a}'$, and the elements of the unit

https://doi.org/10.1017/S0305004100022490 Published online by Cambridge University Press

matrix **1** by δ_{ij}. Bars will be used as superscripts to denote complex conjugates. We use the same summation convention as in S: any expression, whether involving numbers, cycles, or the elements of matrices, is to be summed over the permissible values of any repeated *lower* suffixes occurring in it. A compound suffix such as i_k is to be regarded as a single indivisible entity, and summation is intended only if the whole suffix recurs, no matter how many times k may be repeated. We are concerned only with homologies with division, and do not rule out the use of fractional cycles or 'fractionally algebraic' cycles (i.e. the statement that a cycle is algebraic is merely to be interpreted as meaning that there is some multiple of it which is homologous to an effective or virtual algebraic curve).

1. GENERALITIES

We consider an irreducible correspondence T between two algebraic varieties U and V of respective dimensions r and s. We suppose that T is not degenerate, i.e. that to every point of U or V corresponds at least one point of the other variety. (This condition is not really restrictive, because a degenerate correspondence is always a non-degenerate, though possibly trivial, correspondence between two subvarieties of U and V.) We shall also assume that T is not the trivial correspondence in which any point of either variety corresponds to the whole of the other; this implies that T is represented by a proper subvariety W on the product $U \times V$. If x and y are generic points of U and V respectively, and t is the dimension of W, the dimensions of $T(x)$ and $T^{-1}(y)$ are respectively $t-r$ and $t-s$ (van der Waerden (9), p. 141). The conditions of non-degeneracy and non-triviality thus require that

$$\max(r,s) \leqslant t \leqslant r+s-1. \tag{1·1}$$

Let Γ_i^p $(i = 1, \ldots, \mathrm{R}_p)$ and Δ_j^q $(j = 1, \ldots, \mathrm{R}'_q)$ be bases respectively for the p-cycles of U and the q-cycles of V. Since W is algebraic, and hence its intersections with $x \times V$ and $U \times y$ must also be algebraic, we can represent it as a $2t$-cycle on $U \times V$ by a homology of the form

$$W \approx U \times Y + \sum_{k=1}^{2(r+s-t)-1} \xi^{(k)}_{i_k j_k} \Gamma^{2r-k}_{i_k} \times \Delta^{2t-2k+k}_{j_k} + X \times V$$
$$(i_k = 1, \ldots, \mathrm{R}_{2r-k};\ j_k = 1, \ldots, \mathrm{R}'_{2t-2r+k}), \tag{1·2}$$

where X and Y are algebraic subvarieties, of dimensions $t-s$ and $t-r$ respectively, on U and V. The correspondence T is thus determined, from the topological point of view, by the varieties X and Y and the matrices $\xi^{(k)}$.

Since a point of U is transformed into a $2(t-r)$-cycle of V homologous to Y, we should expect a p-cycle of U to be transformed into a $(p+2(t-r))$-cycle of V, thus getting $2(r+s-t)-1$ sets of homologies‡ of the form

$$T(\Gamma_i^p) \approx \epsilon^{(p)}_{ij} \Delta_j^{2(t-r)+p} \quad \text{on } V$$
$$(i = 1, \ldots, \mathrm{R}_p;\ j = 1, \ldots, \mathrm{R}'_{2(t-r)+p} : p = 1, \ldots, 2(r+s-t)-1). \tag{1·3}$$

‡ Since naturally we are only interested in cycles whose transforms are of dimension $\leqslant 2s-1$.

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and similarly another $2(r+s-t)-1$ sets of homologies of the form

$$T^{-1}(\Delta_k^q) \approx \eta_{kl}^{(q)} \Gamma_l^{2(t-s)+q} \quad \text{on } U$$
$$(k = 1, \ldots, \mathrm{R}_q'; \; l = 1, \ldots, \mathrm{R}_{2(t-s)+q}: \; q = 1, \ldots, 2(r+s-t)-1). \quad (1{\cdot}4)$$

The basic problems to be solved fall into three classes. First, to determine necessary and sufficient conditions to be satisfied by the matries $\boldsymbol{\xi}^{(k)}$ in order that W be an algebraic cycle; secondly, to express the matrices $\boldsymbol{\epsilon}^{(p)}$ and $\boldsymbol{\eta}^{(q)}$ in terms of the $\boldsymbol{\xi}^{(k)}$; and thirdly, to discuss the relationship between the topological and geometrical properties of the correspondence, usually with reference to a favoured class of correspondences, the correspondences of valency zero.

We cannot expect a general solution of the first problem in the present state of the theory, because sufficient conditions that a $2t$-cycle of an n-dimensional variety be algebraic are known only in the cases $t = 1$ or $n-1$. In the first case (1·1) requires that $r = s = 1$, and we have the classical case of correspondences between curves. In the second alternative (of which the first is a special case) we have a point-primal correspondence, so that we see at once that this is a specially favourable type. In this case (1·2) is greatly simplified and we are concerned only with a single matrix $\boldsymbol{\xi}^{(1)}$.

The second problem is one that is completely solved for point-point correspondences, but is also not particularly difficult in the point-primal case. In the first place, a point-primal correspondence can have on U no fundamental locus ‡ of dimension $r-1$. For, if Z were such a locus, the transform of any point of it would be the whole irreducible variety V, and W would thus contain the $(r+s-1)$-dimensional variety $Z \times V$, contradicting our hypothesis that W is irreducible. We can thus easily replace any 1-cycle of U by a convenient homologous cycle not meeting the fundamental locus, so that its transform will be a $(2s-1)$-cycle of V. Our equations (1·2), (1·3) and (1·4) are reduced to the forms

$$W \approx U \times Y + \xi_{ij} \Gamma_i^{2r-1} \times \Delta_j^{2s-1} + X \times U \quad \text{on } U \times V, \quad (1{\cdot}2')$$
$$T(\Gamma_i^1) \approx \epsilon_{ij} \Delta_j^{2s-1} \quad \text{on } V, \quad (1{\cdot}3')$$

and

$$T^{-1}(\Delta_j^1) \approx \eta_{ji} \Gamma_i^{2r-1} \quad \text{on } U, \quad (1{\cdot}4')$$

where, in each formula, $i = 1, \ldots, \mathrm{R}_1$; $j = 1, \ldots, \mathrm{R}_1'$.

If Γ^1 and Δ^1 are arbitrary 1-cycles of U and V respectively, we have, as a natural generalization of the corresponding result for curves given by Lefschetz ((4), p. 348), the formula

$$(W . \Gamma^1 \times \Delta^1) = -(T(\Gamma^1) . \Delta^1),$$

where the first Kronecker index is calculated on $U \times V$ and the second on V. Applying this formula to the cycles Γ_k^1 and Δ_l^1, we obtain

$$\xi_{ij} (\Gamma_i^{2r-1} . \Gamma_k^1) (\Delta_j^{2s-1} . \Delta_l^1) = \epsilon_{kj} (\Delta_j^{2s-1} . \Delta_l^1) \quad (i, k = 1, \ldots, \mathrm{R}_1; \; j, l = 1, \ldots, \mathrm{R}_1'). \quad (1{\cdot}5)$$

Introducing the matrices $\mathbf{g}$ and $\mathbf{h}$ defined by

$$g_{ik} = (\Gamma_i^1 . \Gamma_k^{2r-1}) = -(\Gamma_k^{2r-1} . \Gamma_i^1), \quad h_{jl} = (\Delta_j^1 . \Delta_l^{2s-1}) = -(\Delta_l^{2s-1} . \Delta_j^1) \quad (i, kj, l \text{ as in } (1{\cdot}5)),$$

(1·5) reduces to

$$g_{ki} \xi_{ij} h_{lj} = -\epsilon_{kj} h_{lj},$$

and, since $\mathbf{h}$ is non-singular, we deduce

$$\boldsymbol{\epsilon} = -\mathbf{g}\boldsymbol{\xi}, \quad (1{\cdot}6)$$

and similarly

$$\boldsymbol{\eta} = \mathbf{h}\boldsymbol{\xi}'. \quad (1{\cdot}7)$$

‡ A fundamental locus of T is a variety consisting of points x, such that the dimension of $T(x)$ is greater than $t-r$.

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These results are very similar to those for correspondences between curves, to which they reduce if r and s are equal to one. If, however, r and s are greater than one, (1·6) and (1·7) are capable of further simplification, since, by a suitable choice of bases for the various cycles, we may arrange that $\mathbf{g}$ and $\mathbf{h}$ are the unit matrices of orders R_1 and R'_1 respectively. We thus obtain the simplified formulae

$$\boldsymbol{\epsilon} = -\boldsymbol{\xi}, \quad \boldsymbol{\eta} = \boldsymbol{\xi}'. \tag{1·8}$$

An alternative proof of these results for point-curve correspondences will be given in a later paper.

The third of our basic problems is also much simplified for point-primal correspondences. In the case of point-point correspondences between surfaces two different definitions of zero valency have been given, by Albanese and Severi, which have different geometrical properties. Albanese has defined correspondences of valency zero by the property that every 1-cycle is transformed into a bounding cycle; this class of correspondence includes Severi's correspondences of valency zero, defined by the property that the transforms of the points of one surface belong to a series of equivalence on the other. With point-primal correspondences no such difficulties arise, for, instead of systems of equivalence, we can base ourselves on linear systems of primals. In fact, if we define a correspondence of valency zero by the property that the transforms of the points of U are primals of a single linear system on V, the following theorems hold:

I. *If a correspondence T is of valency zero, so is T^{-1}.*

II. *A correspondence of valency zero is represented by a single equation connecting the points of the two varieties.*

III. *If T is of valency zero, it is represented by a variety W of $U \times V$ satisfying the linear equivalence*

$$W \equiv U \times Y + X \times V \quad \text{on } U \times V, \tag{1·9}$$

where $X \equiv T^{-1}(y)$, $Y \equiv T(x)$, and the products refer to the geometrical product of the two varieties. Conversely, a variety W of the form (1·9) *represents a correspondence of valency zero.*

IV. *A necessary and sufficient condition that the correspondence represented by* (1·2′) *be of valency zero is that $\boldsymbol{\xi} = \mathbf{0}$.*

V. *Between two different varieties of general moduli all correspondences are of valency zero.*

VI. *A necessary and sufficient condition that a correspondence be of valency zero is that every 1-cycle of U is transformed into a bounding cycle of V.* (This may clearly be replaced by the alternative condition obtained by interchanging U and V. Either condition implies the other by I.)

VII. *If the varieties U and V coincide, the correspondence of valency zero defined by* (1·9) *has a 'united primal' (locus of self-corresponding points) linearly equivalent to $X + Y$.*

I–III, V and VII are immediate generalizations of Severi's results for point-curve correspondences (Severi(8)), and the proofs are trivial extensions of those given by

https://doi.org/10.1017/S0305004100022490 Published online by Cambridge University Press

him. A theorem completely equivalent to VI is given by Albanese ((1), p. 176); this result, together with (1·8), clearly implies IV, which is, however, not difficult to prove directly. These results are exactly those which hold for correspondences between curves.

The problem of correspondences of non-zero valency, in the case where U and V coincide, is not so simple for point-primal as it is for point-point correspondences. In the latter case the identical correspondence is a member of the class under consideration, and with its aid we can introduce correspondences of arbitrary valency as soon as we have defined zero valency. We shall consider the problem in some detail for point-curve correspondences in a later paper, where we shall show that, if the period matrix of the simple integrals of the first kind is sufficiently simple, all correspondences are valency correspondences, but that there may be surfaces which have no correspondences of non-zero valency.

2. Riemann matrices

The period matrix of the simple integrals of the first kind of an algebraic variety is a Riemann matrix. In this section we give some lemmas on such matrices which we shall use later. We first recall some elementary properties of Riemann matrices; references on this subject will be found in Lefschetz's report(5).

If $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are two Riemann matrices of genera p_1 and p_2 respectively, a rational $(2p_1, 2p_2)$ matrix $\mathbf{c}$ such that $\boldsymbol{\omega}_1 \mathbf{c} \boldsymbol{\omega}_2' = \mathbf{0}$ is called a *mutual multiplier* of $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$, and if a non-zero matrix of this type exists $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are said to be *related*. The number, λ, of linearly independent mutual multipliers, is called the simultaneity index of $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. If $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = \boldsymbol{\omega}$, $\mathbf{c}$ is called a *multiplier* of $\boldsymbol{\omega}$, and the number of linearly independent multipliers is called the *multiplication index* of $\boldsymbol{\omega}$. A skew-symmetric multiplier is called an *alternate matrix* of $\boldsymbol{\omega}$. A Riemann matrix is said to be *pure* if it is not possible, by a change of bases for the integrals and for the cycles, to reduce it to the form

$$\boldsymbol{\omega} = \begin{pmatrix} \boldsymbol{\sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_2 \end{pmatrix}.$$

The multipliers of a pure Riemann matrix are either zero or non-singular.

It is known that the matrices $\begin{pmatrix} \boldsymbol{\omega}_1 \\ \bar{\boldsymbol{\omega}}_1 \end{pmatrix}$ and $\begin{pmatrix} \boldsymbol{\omega}_2 \\ \bar{\boldsymbol{\omega}}_2 \end{pmatrix}$ are non-singular, and that $\mathbf{c}$ is a mutual multiplier of $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ if, and only if,

$$\begin{pmatrix} \boldsymbol{\omega}_1 \\ \bar{\boldsymbol{\omega}}_1 \end{pmatrix} \mathbf{c} \begin{pmatrix} \boldsymbol{\omega}_2 \\ \bar{\boldsymbol{\omega}}_2 \end{pmatrix}' = \begin{pmatrix} \mathbf{0} & \mathbf{P} \\ \overline{\mathbf{P}} & \mathbf{0} \end{pmatrix},$$

and further, if $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$ and $\mathbf{c}$ is non-singular so is $\mathbf{P}$. This property enables us to prove that if $\mathbf{a}_1$ and $\mathbf{a}_2$ are any non-singular‡ multipliers of $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ respectively, $\mathbf{b}_1$ and $\mathbf{b}_2$ are any multipliers of $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$, and $\mathbf{c}$ is a mutual multiplier, then $\mathbf{b}_1 \mathbf{a}_1^{-1} \mathbf{c} \mathbf{a}_2^{-1} \mathbf{b}_2$ is a mutual multiplier of $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. For convenience of reference we state as lemmas the two following simple corollaries of this result:

Lemma 2·1. *If the matrices* $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b}_1$, $\mathbf{b}_2$ *are as above, and* $\mathbf{x}$ *is a matrix such that* $\mathbf{a}_1 \mathbf{x} \mathbf{a}_2$ *is a mutual multiplier of* $\boldsymbol{\omega}_1$ *and* $\boldsymbol{\omega}_2$, *then so is* $\mathbf{b}_1 \mathbf{x} \mathbf{b}_2$.

‡ A Riemann matrix has always at least one non-singular multiplier, its principal matrix.

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LEMMA 2·2. *If* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *are any non-zero multipliers of a pure matrix* $\boldsymbol{\omega}$, *then* $\mathbf{ac^{-1}b}$ *is also a multiplier.*

With the aid of this latter result we can now prove the less trivial

LEMMA 2·3.‡ *If* $\boldsymbol{\omega}$ *is a pure Riemann matrix, and* $\mathfrak{M}$ *the additive group formed by the multipliers of* $\boldsymbol{\omega}$, *the set* $\mathfrak{M}^{-1}$, *consisting of the inverses of the non-zero elements of* $\mathfrak{M}$ *together with the zero-matrix, is an additive group isomorphic with* $\mathfrak{M}$. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are any non-zero elements of* $\mathfrak{M}$ (*not necessarily distinct*), *an isomorphism is defined by*

$$\mathfrak{M}^{-1} = \mathbf{a}^{-1}\mathfrak{M}\mathbf{b}^{-1}.$$

Let $\mathbf{c}$ be an arbitrary element of $\mathfrak{M}$ and let $\phi(\mathbf{c}) = \mathbf{a}^{-1}\mathbf{c}\mathbf{b}^{-1}$. Then $\phi(\mathbf{0}) = \mathbf{0}$. Let us now suppose $\mathbf{c} \neq \mathbf{0}$, and let $\mathbf{d} = \phi(\mathbf{c})$. Then $\mathbf{d}^{-1} = \mathbf{b}\mathbf{c}^{-1}\mathbf{a}$ belongs to $\mathfrak{M}$, by Lemma 2·2. Thus $\mathbf{d}$ belongs to $\mathfrak{M}^{-1}$, and because ϕ clearly preserves addition, it defines a homeomorphic mapping of $\mathfrak{M}$ into $\mathfrak{M}^{-1}$. But ϕ is obviously reversible, so that the mapping is in fact an isomorphism, and our lemma is established.

If we have two varieties U and V it is possible, by a suitable choice of bases for the 1-cycles and simple integrals of the first kind of the two varieties, to reduce the period matrices simultaneously to the forms

$$\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\pi}^1 & & & \mathbf{0} \\ & \boldsymbol{\pi}^2 & & \\ & & \ddots & \\ & & & \boldsymbol{\pi}^t \\ \mathbf{0} & & & \boldsymbol{\omega} \end{pmatrix}, \qquad \boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\rho}^1 & & & \mathbf{0} \\ & \boldsymbol{\rho}^2 & & \\ & & \ddots & \\ & & & \boldsymbol{\rho}^t \\ \mathbf{0} & & & \boldsymbol{\psi} \end{pmatrix}, \tag{2·1}$$

where $\boldsymbol{\pi}^\alpha$, defined by the equation

$$\boldsymbol{\pi}^\alpha = \begin{pmatrix} \boldsymbol{\sigma}^\alpha & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \boldsymbol{\sigma}^\alpha \end{pmatrix} \quad (\alpha = 1, \ldots, t),$$

is a reducible Riemann matrix with m_α pure submatrices, of genus p_α, each equal to $\boldsymbol{\sigma}^\alpha$, and $\boldsymbol{\rho}^\alpha$ is the similar matrix with n_α submatrices each equal to $\boldsymbol{\sigma}^\alpha$. Further, $\boldsymbol{\sigma}^\alpha$ is not related to $\boldsymbol{\sigma}^\beta$ unless $\alpha = \beta$, and $\boldsymbol{\omega}$ and $\boldsymbol{\psi}$ are not related to one another or to any $\boldsymbol{\sigma}^\alpha$. This is a consequence of the normal decomposition of a Riemann matrix into pure submatrices, and the theorem that two Riemann matrices are related only if they have equivalent submatrices. If $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ are not related, then clearly $t = 0$.

It is also convenient to write the matrices in the alternative form

$$\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\omega}^1 & & & \mathbf{0} \\ & \boldsymbol{\omega}^2 & & \\ & & \ddots & \\ \mathbf{0} & & & \boldsymbol{\omega}^{u+1} \end{pmatrix}, \qquad \boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\psi}^1 & & & \mathbf{0} \\ & \boldsymbol{\psi}^2 & & \\ & & \ddots & \\ \mathbf{0} & & & \boldsymbol{\psi}^{v+1} \end{pmatrix}, \tag{2·2}$$

‡ This result and the preceding are simple consequences of the relation between the multipliers and the complex multiplications of a Riemann matrix; they are equivalent to the theorem that if the matrix is pure the complex multiplications give rise to a division algebra. The present forms of these results are, however, the handiest for our applications.

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where $\boldsymbol{\omega}^{u+1} = \boldsymbol{\omega}$, $\boldsymbol{\psi}^{v+1} = \boldsymbol{\psi}$, and $\boldsymbol{\omega}^1, \ldots, \boldsymbol{\omega}^u$ and $\boldsymbol{\psi}^1, \ldots, \boldsymbol{\psi}^v$ are renumberings of the remaining pure submatrices down the diagonals of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ in (2·1). Clearly

$$u = m_1 + m_2 + \ldots + m_t \quad \text{and} \quad v = n_1 + n_2 + \ldots + n_t.$$

If $\mathbf{c}$ is a mutual multiplier of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ it can be divided into blocks concordantly with the division of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ in (2·1). We denote the (α, β) block in this subdivision by $\mathbf{c}^{\alpha\beta}$. Then, if neither α nor β is greater than t, the (α, β) block of the zero product $\boldsymbol{\Omega}\mathbf{c}\boldsymbol{\Psi}'$ is $\boldsymbol{\pi}^\alpha . \mathbf{c}^{\alpha\beta}(\boldsymbol{\rho}^\beta)'$. But $\boldsymbol{\pi}^\alpha$ and $\boldsymbol{\rho}^\beta$ are only related if $\alpha = \beta$, and hence $\mathbf{c}^{\alpha\beta}$ is zero in all other cases. Similarly, since $\boldsymbol{\omega}$ and $\boldsymbol{\psi}$ are not related to one another or to any $\boldsymbol{\pi}^\alpha$ or $\boldsymbol{\rho}^\beta$, $\mathbf{c}^{\alpha\beta}$ must be zero if either α or β is equal to $t+1$. We can therefore take $\mathbf{c}$ in the form

$$\mathbf{c} = \begin{pmatrix} \mathbf{c}^1 & & & \mathbf{0} \\ & \mathbf{c}^2 & & \\ & & \ddots & \\ & & & \mathbf{c}^t \\ \mathbf{0} & & & \mathbf{0} \end{pmatrix}, \tag{2·3}$$

where $\mathbf{c}^\alpha$ is a mutual multiplier of $\boldsymbol{\pi}^\alpha$ and $\boldsymbol{\rho}^\alpha$.

Let us now further subdivide $\mathbf{c}^\alpha$ into blocks concordantly with the subdivision of $\boldsymbol{\pi}^\alpha$ and $\boldsymbol{\rho}^\alpha$ into pure submatrices, and denote by $\mathbf{c}^\alpha_{\beta\gamma}$ the (β,γ) block in this subdivision. The (β,γ) block in the zero product $\boldsymbol{\pi}^\alpha\mathbf{c}^\alpha(\boldsymbol{\rho}^\alpha)'$ is $\boldsymbol{\sigma}^\alpha\mathbf{c}^\alpha_{\beta\gamma}(\boldsymbol{\sigma}^\alpha)'$, so that $\mathbf{c}^\alpha_{\beta\gamma}$ is a multiplier (possibly zero) of $\boldsymbol{\sigma}^\alpha$. We thus have a complete reduction of $\mathbf{c}$ to the following form:

$$\mathbf{c} = \left(\begin{array}{c|c|c|c|c} \begin{matrix} \mathbf{c}^1_{11} & \ldots & \mathbf{c}^1_{1n_1} \\ \vdots & & \\ \mathbf{c}^1_{m_1 1} & & \mathbf{c}^1_{m_1 n_1} \end{matrix} & & & & \mathbf{0} \\ \hline & \begin{matrix} \mathbf{c}^2_{11} & \ldots & \mathbf{c}^2_{1n_2} \\ \vdots & & \\ \mathbf{c}^2_{m_2 1} & & \mathbf{c}^2_{m_2 n_2} \end{matrix} & & & \\ \hline & & \ddots & & \\ \hline & & & \begin{matrix} \mathbf{c}^t_{11} & \ldots & \mathbf{c}^t_{1n_t} \\ \vdots & & \\ \mathbf{c}^t_{m_t 1} & & \mathbf{c}^t_{m_t n_t} \end{matrix} & \\ \hline \mathbf{0} & & & & \mathbf{0} \end{array}\right), \tag{2·4}$$

where the principal blocks are divided in accordance with the subdivision of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ in (2·1), the subblocks in accordance with the subdivision in (2·2), and $\mathbf{c}^\alpha_{\beta\gamma}$ belongs to the additive group $\mathfrak{M}_\alpha$ of multipliers of $\boldsymbol{\sigma}^\alpha$.

A matrix of the form (2·4) in which, instead of taking the $\mathbf{c}^\alpha_{\beta\gamma}$ from $\mathfrak{M}_\alpha$, we take them from $\mathfrak{M}_\alpha^{-1}$, will be called a *mutual anti-multiplier* of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$. We can now generalize Lemma 2·3 to give

LEMMA 2·4. *The mutual anti-multipliers of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ form an additive group $\mathfrak{G}^{-1}$ isomorphic with the group $\mathfrak{G}$ of mutual multipliers. If $\mathbf{a}$ and $\boldsymbol{\alpha}$ are any non-singular multipliers of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ respectively, they define an isomorphism $\mathfrak{G} = \mathbf{a}\mathfrak{G}^{-1}\boldsymbol{\alpha}$.*

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Let $\mathbf{d}^\alpha$ be any non-zero multiplier of $\boldsymbol{\sigma}^\alpha$, and consider the non-singular multipliers $\mathbf{e}$ and $\mathbf{f}$ of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ respectively, defined by

$$\mathbf{e} = \begin{pmatrix} \mathbf{e}^1 & & & \mathbf{0} \\ & \mathbf{e}^2 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{e}^{t+1} \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} \mathbf{f}^1 & & & \mathbf{0} \\ & \mathbf{f}^2 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{f}^{t+1} \end{pmatrix},$$

where $\mathbf{e}^\alpha$, defined by the equation

$$\mathbf{e}^\alpha = \begin{pmatrix} \mathbf{d}^\alpha & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{d}^\alpha \end{pmatrix} \quad (\alpha = 1, \ldots, t),$$

has m_α submatrices each equal to $\mathbf{d}^\alpha$, $\mathbf{f}^\alpha$ is the similar matrix with n_α submatrices each equal to $\mathbf{d}^\alpha$, and $\mathbf{e}^{t+1}$ and $\mathbf{f}^{t+1}$ are convenient non-singular multipliers of $\boldsymbol{\omega}$ and $\boldsymbol{\psi}$ respectively. Then, for any mutual multiplier $\mathbf{c}$ of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$, the product $\mathbf{e}^{-1}\mathbf{c}\mathbf{f}^{-1}$ is a matrix of the form (2·4) in which every $\mathbf{c}^\alpha_{\beta\gamma}$ is replaced by $(\mathbf{d}^\alpha)^{-1}\mathbf{c}^\alpha_{\beta\gamma}(\mathbf{d}^\alpha)^{-1}$, which, by Lemma 2·3, is an element of $\mathfrak{M}_\alpha^{-1}$. Thus $\mathbf{e}^{-1}\mathfrak{G}\mathbf{f}^{-1}$ is a homomorphism of $\mathfrak{G}$ into $\mathfrak{G}^{-1}$. It is easy to see that this transformation is reversible, and hence it is an isomorphism. If $\boldsymbol{\xi}$ is any element of $\mathfrak{G}^{-1}$, $\mathbf{e}\boldsymbol{\xi}\mathbf{f}$ belongs to $\mathfrak{G}$, and hence, by Lemma 2·1, so does $\mathbf{a}\boldsymbol{\xi}\boldsymbol{\alpha}$. Thus $\mathbf{a}\mathfrak{G}^{-1}\boldsymbol{\alpha}$ gives an isomorphism of $\mathfrak{G}^{-1}$ into $\mathfrak{G}$, since, as $\mathbf{a}$ and $\boldsymbol{\alpha}$ are non-singular, no non-zero matrix of $\mathfrak{G}^{-1}$ can be mapped on to the zero matrix of $\mathfrak{G}$.

Two obvious consequences of this last lemma will be useful later. The first is that the number of linearly independent anti-multipliers of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ is equal to the simultaneity index λ. The second is that a set $\mathfrak{X}$ of matrices defined by the property $\mathbf{a}\mathfrak{X}\boldsymbol{\alpha} = \mathfrak{G}$ is the set $\mathfrak{G}^{-1}$. It is also worth noting that, in view of the isomorphisms between $\mathfrak{G}$ and $\mathfrak{G}^{-1}$, the idea of an anti-multiplier is not dependent, as might appear from the way in which it was introduced, on our having chosen a canonical form for $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$.

3. The base for point-curve correspondences

Given a pair of algebraic surfaces F and G, we can choose bases $\gamma_1, \ldots, \gamma_{2p}$ and $\delta_1, \ldots, \delta_{2q}$ for the 1-cycles of F and G respectively, together with bases $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$ for the simple integrals of the first kind of the surfaces, so that the period matrices $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ of these integrals take the form (2·1). We can also choose bases $\Gamma_1, \ldots, \Gamma_{2p}$ and $\Delta_1, \ldots, \Delta_{2q}$ for the 3-cycles such that

$$(\gamma_i . \Gamma_k) = \delta_{ik}, \quad (\delta_j . \Delta_l) = \delta_{jl},$$

where $i, k = 1, \ldots, 2p$ and $j, l = 1, \ldots, 2q$. It will be assumed henceforth that the indices i, k, j, l always have these ranges. We may suppose that the models of F and G with which we are concerned are non-singular surfaces in some suitable spaces with generic prime sections P and Q respectively. Then $|P|$ and $|Q|$ are both linear systems of freedom at least three. If C^* and D^* are proper curves (S, p. 415) of F and G we denote their respective a-matrices by $\mathbf{a}^*$ and $\boldsymbol{\alpha}^*$. These matrices, as we have shown in S, are non-singular alternate matrices of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ respectively, defined by the properties

$$\Gamma_i . C^* \approx a^*_{ik}\gamma_k \text{ on } F, \quad \Delta_j . D^* \approx \alpha^*_{jl}\delta_l \text{ on } G. \tag{3·1}$$

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A point-curve correspondence between F and G is represented on $F \times G$ (cf. (1·2′)) by a homology

$$W \approx F \times Y + \xi_{ij} \Gamma_i \times \Delta_j + X \times G, \tag{3·2}$$

where X and Y are algebraic curves of F and G respectively, such that, if x and y are generic points of F and G, $\quad X = T^{-1}(y), \quad Y = T(x)$.

We shall establish the following

THEOREM 3·1. *A necessary and sufficient condition that a 6-cycle W of the form* (3·2) *be algebraic is that its intersection with a cycle of the form $C^* \times D^*$ is algebraic.*

The proof is in three stages. First we prove the simpler

THEOREM 3·2. *If W is a cycle of the form* (3·2), *and if $W \,.\, C^* \times D^*$ is algebraic, then so also is $W \,.\, C^\dagger \times D^\dagger$, where $C^\dagger$ and $D^\dagger$ are any other proper curves of F and G respectively.*

For
$$W \,.\, C^* \times D^* \approx (F \times Y + \xi_{ij} \Gamma_i \times \Delta_j + X \times G) \,.\, C^* \times D^*$$
$$\approx (Y \,.\, D^*)\, C^* \times y + (X \,.\, C^*)\, x \times D^* - a^*_{ki} \xi_{ij} \alpha^*_{jl} \gamma_k \times \delta_l, \tag{3·3}$$

as is seen at once from (3·1), taking account of the skew-symmetry of $\mathbf{a}^*$. But $W \,.\, C^* \times D^*$ is algebraic if and only if the double integrals of $F \times G$ all have zero periods over it (Hodge (2), p. 24). But the double integrals of $F \times G$ are the double integrals of F and G, all of which have zero period over (3·3), and the integrals of the form

$$u_r \times v_s \quad (r = 1, \ldots, p;\ s = 1, \ldots, q).$$

The period of $u_r \times v_s$ over the first two terms of (3·3) is zero and over the third it is $-a^*_{ki} \xi_{ij} \alpha^*_{jl} \Omega_{rk} \Psi_{sl}$. If this is to vanish for all permissible r and s we must have

$$\boldsymbol{\Omega} \mathbf{a}^* \boldsymbol{\xi} \boldsymbol{\alpha}^* \boldsymbol{\Psi}' = \mathbf{0},$$

i.e. $\mathbf{a}^* \boldsymbol{\xi} \boldsymbol{\alpha}^*$ is a mutual multiplier of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$. Similarly, the condition for $W \,.\, C^\dagger \times D^\dagger$ to be algebraic is that $\mathbf{a}^\dagger \boldsymbol{\xi} \boldsymbol{\alpha}^\dagger$ is a mutual multiplier of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$. Hence, by Lemma 2·1, $W \,.\, C^\dagger \times D^\dagger$ is algebraic if $W \,.\, C^* \times D^*$ is, and conversely. In particular we may replace $C^\dagger$ and $D^\dagger$ by the prime sections P and Q.

The second stage is to show that $|K| \equiv |F \times Q + P \times G|$ is a system of 'prime sections' of $F \times G$. By this we mean that there is a birational transformation of $F \times G$ into a variety with the transform of $|K|$ as the system of prime sections, and that this birational transformation does not affect our topology. It is sufficient for this to prove that $|K|$ is ∞^5 at least, without base points, and is simple and has no exceptional loci, i.e. that there are no varieties of $F \times G$ which this transformation sends into varieties of lower dimension.

First of all $|K|$ contains the two linear systems $F \times |Q| + P_1 \times G$ and $F \times Q_1 + |P| \times G$, each of which is ∞^3 at least and which have in common only the variety $F \times Q_1 + P_1 \times G$ (P_1 and Q_1 being any convenient prime sections of F and G respectively). $|K|$ is therefore at least ∞^6. Also, since $|P|$ and $|Q|$ are cut off (at least in part) by prime sections of F and G, it is easy to show that it is possible to pick out six particular members of $|K|$ which have no common point, thus showing that $|K|$ has no base points. Similarly, given any point (and not merely a generic point) $x \times y$ of $F \times G$, we can find five members of $|K|$ containing $x \times y$ and having no further intersection. This shows that $|K|$ is simple and also free from exceptional elements. The proofs of these statements are perfectly straightforward and need not detain us here.

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The third stage is to apply the theorem that a primal of a V_m is algebraic if and only if its intersection with $(m-2)$ generic prime sections of V_m is algebraic (Hodge(2)). In our case the problem reduces to finding the condition that $W.K^2$ is algebraic, where K^2 is the cycle of intersection of two generic K's. But

$$K^2 \approx (P \times G + F \times Q).(P \times G + F \times Q)$$
$$\approx (P.P)\, x \times G + (Q.Q)\, F \times y + 2P \times Q.$$

Hence

$$W.K^2 \approx (F \times Y + \xi_{ij}\Gamma_i \times \Delta_j + X \times G).((P.P)\, x \times G + (Q.Q)\, F \times y + 2P \times Q)$$
$$\approx (P.P)\, x \times Y + (Q.Q)\, X \times y + 2W.P \times Q.$$

Thus, since $x \times Y$ and $X \times y$ are algebraic, W is algebraic if and only if $W.P \times Q$ is so, or, by Theorem 3·2, if and only if $W.C^* \times D^*$ is algebraic. We use this to prove

THEOREM 3·3. *The point-curve correspondence given by* (3·2) *is algebraic if and only if the matrix* $\boldsymbol{\xi}$ *is a mutual anti-multiplier of* $\boldsymbol{\Omega}$ *and* $\boldsymbol{\Psi}$. *The number of independent*‡ *algebraic point-curve correspondences between two surfaces is* $\rho + \sigma + \lambda$, ρ *and* σ *being the respective base numbers for the algebraic curves of the two surfaces, and* λ *the simultaneity index of the period matrices of the simple integrals of the first kind.*

We have shown above that $W.C^* \times D^*$ is algebraic if, and only if, $\mathbf{a}^*\boldsymbol{\xi}\boldsymbol{\alpha}^*$ is a mutual multiplier of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$. But, as we have shown in § 2, this is simply the condition that $\boldsymbol{\xi}$ is a mutual anti-multiplier of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$. Further, in (3·2) we have ρ choices for X and σ for Y, and, as we have pointed out earlier, there are exactly λ independent matrices $\boldsymbol{\xi}$, so that both parts of the theorem are established.

The explicit expression of allowable matrices $\boldsymbol{\xi}$ is given by replacing every $\mathbf{c}^{\alpha}_{\beta\gamma}$ in (2·4) by an element of $\mathfrak{M}_{\alpha}^{-1}$. A correspondence $T^{\alpha}_{\beta\gamma}$ corresponding to a matrix in which every element is zero, except for the (β, γ) sub-block of the αth block, will be called an elementary correspondence of type $(\alpha; \beta, \gamma)$. These correspondences provide a natural base for the non-valency correspondences between F and G (cf. the latter part of Severi's paper(8)). Corresponding to the division of $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ into the blocks of (2·1) we have a division of the 1- and 3-cycles of F and G into $t+1$ sets, every cycle Γ_i or Δ_j being placed in the same set as γ_i or δ_j. The further decomposition of these blocks in (2·2) implies a subdivision of the αth sets $(\alpha \leqslant t)$ of F and G respectively into m_{α} and n_{α} subsets, each subset containing $2p_{\alpha}$ 1-cycles and 3-cycles. It is clear from (1·8) that any correspondence transforms a 1-cycle of the αth set $(\alpha \leqslant t)$ of F (or G) into a linear combination of 3-cycles of the αth set of G (or F), and a 1-cycle of the $(t+1)$th set into a bounding cycle. A $T^{\alpha}_{\beta\gamma}$ transforms every 1-cycle of F into a bounding cycle, except for the cycles of the βth subset of the αth set, which are transformed into a linear combination of the 3-cycles of the γth subset of the αth set of G. The effect of the reverse correspondence is obtained by transposing F and G and β and γ in the last statement. If h_{α} is the multiplication index of $\boldsymbol{\sigma}^{\alpha}$, then h_{α} is the maximum number of correspondences of type $(\alpha; \beta, \gamma)$ of which no linear combination is a correspondence of valency zero.

In conclusion I would like to express my gratitude to Prof. Hodge for his kindly encouragement, and for some valuable suggestions.

‡ A set of correspondences being by definition independent if they are represented by cycles of $F \times G$ independent with respect to $\approx$.

https://doi.org/10.1017/S0305004100022490 Published online by Cambridge University Press

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ON THE SUMMATION OF CERTAIN TRIGONOMETRIC SERIES

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Communicated by W. V. D. HODGE

Received 15 November 1944

INTRODUCTION

In the present note, which is introductory to the following paper, closed expressions, suitable for computational purposes, are found for the sums of the series

$$\mathfrak{S}_n^t(\alpha) = \alpha^2 n^2 \sum_{m=1}^{\infty} \frac{\sin^2(\pi m/\alpha)}{m^t(m^2-\alpha^2 n^2)}, \quad \mathfrak{T}_n^t(\alpha) = \alpha^2 n^2 \sum_{m=1}^{\infty} \frac{\sin^2(\pi m/\alpha)}{m^{t-2}(m^2-\alpha^2 n^2)^2},$$

where $\alpha > 1$, $t = 1, 2, 3, \ldots$, and n is a positive integer. In each case a recurrent relation is found giving the values of $\mathfrak{S}_n^t(\alpha)$ and $\mathfrak{T}_n^t(\alpha)$ for $t > 2$ in terms of $\mathfrak{S}_n^1(\alpha)$, $\mathfrak{S}_n^2(\alpha)$, $\mathfrak{T}_n^1(\alpha)$, $\mathfrak{T}_n^2(\alpha)$ and the series $\Theta_\kappa(\alpha)$ ($\kappa = 1, 2, \ldots, t$), where

$$\Theta_\kappa(\alpha) = \sum_{m=1}^{\infty} \frac{1}{m^\kappa} \sin^2 \frac{\pi m}{\alpha}.$$

When κ is even the last series is expressed in closed form in terms of the Bernoullian polynomial $\phi_\kappa(1/\alpha)$ and, when κ is odd and α is *rational*, a closed form is found involving the polygamma function $\Psi^{(\kappa)}(z)$, where $\Psi^{(\kappa-1)}(z) = \dfrac{d^\kappa}{dz^\kappa} \log \Gamma(z+1)$. The general expressions for $\mathfrak{S}_n^1(\alpha)$ and $\mathfrak{T}_n^1(\alpha)$ involve $\Psi(z)$ and $\Psi'(z)$ when α is rational, but for special values of α they reduce to a form independent of the Ψ-function. $\mathfrak{S}_n^2(\alpha)$ and $\mathfrak{T}_n^2(\alpha)$ are independent of n and are expressible as simple rational functions of α.

https://doi.org/10.1017/S0305004100022490 Published online by Cambridge University Press