Mathematika 63 (2017) 1136–1151 © 2017 University College London
doi:10.1112/S0025579317000274

SIMULTANEOUS APPROXIMATION TO TWO REALS: BOUNDS FOR THE SECOND SUCCESSIVE MINIMUM

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In memory and appreciation of Klaus Roth

Abstract. Introduced in Schmidt and Summerer [Parametric geometry of numbers and applications. *Acta Arith.* **140** (2009), 67–91], approximation exponents $\underline{\varphi}_i, \overline{\varphi}_i$, $(i = 1, 2, 3)$, attached to points $\xi = (\xi_1, \xi_2)$ in $\mathbb{R}^2$, give information on Diophantine approximation properties of these points. Laurent [Exponents of Diophantine approximation in dimension two. *Canad. J. Math.* **61** (2009), 165–189] had described all possible quadruples $(\underline{\varphi}_1, \overline{\varphi}_1, \underline{\varphi}_3, \overline{\varphi}_3)$ arising in this way. Our emphasis here will be on $\underline{\varphi}_2, \overline{\varphi}_2$ and the construction of suitable "3-systems".

§1. *Introduction.* In his paper [1] Michel Laurent has proved that for any $(\xi_1, \xi_2) \in \mathbb{R}^2$ with $1, \xi_1, \xi_2$ linearly independent over $\mathbb{Q}$, the four classical approximation constants $\omega, \omega^*, \hat{\omega}, \hat{\omega}^*$ satisfy the relations

$$2 \leqslant \hat{\omega}^* \leqslant \omega^* \leqslant +\infty, \tag{1.1}$$

$$\hat{\omega} = \frac{\hat{\omega}^* - 1}{\hat{\omega}^*} \text{ (Jarnik's identity)}, \tag{1.2}$$

$$\frac{\omega^*(\hat{\omega}^* - 1)}{\omega^* + \hat{\omega}^*} \leqslant \omega \leqslant \frac{\omega^* - \hat{\omega}^* + 1}{\hat{\omega}^*}. \tag{1.3}$$

Conversely, he proved also that for each quadruple of numbers $\omega, \omega^*, \hat{\omega}, \hat{\omega}^*$ satisfying (1.1), (1.2) and (1.3), there exists some $(\xi_1, \xi_2) \in \mathbb{R}^2$ with $1, \xi_1, \xi_2$ linearly independent over $\mathbb{Q}$ whose approximation constants are those numbers.

In [4] an alternative approach to this and several other problems in simultaneous approximation was introduced that is based on the study of the successive minima of a suitable family of convex bodies with respect to a lattice defined in terms of the real numbers to be approximated.

In the context of approximation to two real numbers ξ_1, ξ_2, the authors considered the lattice $\Lambda(\xi_1, \xi_2)$ of points $(y, \xi_1 y - x_1, \xi_2 y - x_2)$ where (y, x_1, x_2) runs through $\mathbb{Z}^3$ and the box $K(q)$ of points (z_0, z_1, z_2) with

$$|z_0| \leqslant \mathrm{e}^{2q}, \qquad |z_1| \leqslant \mathrm{e}^{-q}, \qquad |z_2| \leqslant \mathrm{e}^{-q}.$$

If $\lambda_1(q), \lambda_2(q), \lambda_3(q)$ denote the successive minima of Λ with respect to $K(q)$, we set $L_i(q) := \log \lambda_i(q)$ for $i = 1, 2, 3$. The functions L_i are continuous,

Received 19 October 2016.
MSC (2010): 11H06, 11J13 (primary).

https://doi.org/10.1112/S0025579317000274 Published online by Cambridge University Press

piecewise linear functions with slopes 1 and -2 and

$$-\log 6 \leqslant L_1(q) + L_2(q) + L_3(q) \leqslant 0$$

by Minkowski's second theorem on convex bodies. For $i = 1, 2, 3$ we set $\varphi_i(q) = L_i(q)/q$ and

$$\underline{\varphi}_i := \liminf_{q\to\infty} \varphi_i(q), \qquad \overline{\varphi}_i := \limsup_{q\to\infty} \varphi_i(q). \tag{1.4}$$

For $i = 1, 2, 3$, the $\underline{\varphi}_i$ and $\overline{\varphi}_i$ will be called *geometric approximation constants*.

In [4] it was shown that there is a direct correspondence between $(\omega, \hat{\omega}, \omega^*, \hat{\omega}^*)$ and $(\underline{\varphi}_1, \overline{\varphi}_1, \underline{\varphi}_3, \overline{\varphi}_3)$ through the relations

$$(1 + \omega)(2 + \underline{\varphi}_1) = (1 + \hat{\omega})(2 + \overline{\varphi}_1) = 3$$

and

$$(1 + \omega^*)(1 - \overline{\varphi}_3) = (1 + \hat{\omega}^*)(1 - \underline{\varphi}_3) = 3.$$

As a consequence, (1.1), (1.2) and (1.3) may be reformulated as

$$0 \leqslant \underline{\varphi}_3 \leqslant \overline{\varphi}_3 \leqslant 1, \tag{1.5}$$

$$\underline{\varphi}_3 + \overline{\varphi}_1 + \underline{\varphi}_3\overline{\varphi}_1 = 0, \tag{1.6}$$

$$2\underline{\varphi}_1+\overline{\varphi}_3 \leqslant -\underline{\varphi}_3(3+\underline{\varphi}_1+2\overline{\varphi}_3) \quad \text{and} \quad \underline{\varphi}_1+2\overline{\varphi}_3 \geqslant -\overline{\varphi}_1(3+\overline{\varphi}_3+2\underline{\varphi}_1). \tag{1.7}$$

The reader will notice that the above inequalities do not involve $\underline{\varphi}_2$, $\overline{\varphi}_2$ and the aim of our paper will be to establish bounds for $\underline{\varphi}_2$ and $\overline{\varphi}_2$. Concretely we will prove the following.

THEOREM 1.1. *For any $(\xi_1, \xi_2) \in \mathbb{R}^2$ with $1, \xi_1, \xi_2$ linearly independent over $\mathbb{Q}$, the approximation constants $\underline{\varphi}_1, \overline{\varphi}_1, \underline{\varphi}_2, \overline{\varphi}_2, \underline{\varphi}_3, \overline{\varphi}_3$ satisfy the relations (1.5), (1.6), (1.7) and, if $\underline{\varphi}_3 < 1$, also*

$$\overline{\varphi}_2 \leqslant \overline{\Omega} := \frac{\overline{\varphi}_1 - \underline{\varphi}_1}{2 - \overline{\varphi}_1 - \overline{\varphi}_1\underline{\varphi}_1} \tag{1.8}$$

as well as the dual inequality

$$\underline{\varphi}_2 \geqslant \underline{\Omega} := \frac{\underline{\varphi}_3 - \overline{\varphi}_3}{2 - \underline{\varphi}_3 - \underline{\varphi}_3\overline{\varphi}_3}. \tag{1.9}$$

Moreover, these bounds are best possible in the sense that for given numbers $\underline{\varphi}_1, \overline{\varphi}_1, \underline{\varphi}_3, \overline{\varphi}_3$ with $0 \leqslant \underline{\varphi}_3 < 1$ and (1.6) and (1.7) as well as

$$\underline{\varphi}_2 = \underline{\Omega}, \qquad \overline{\varphi}_2 = \overline{\Omega}, \tag{1.10}$$

there are $(\xi_1, \xi_2) \in \mathbb{R}^2$ with $1, \xi_1, \xi_2$ linearly independent over $\mathbb{Q}$ whose geometric approximation constants are the given numbers.

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Remark. We had already pointed out at the end of [4] (in terms of $\psi_i = \varphi_i/2$) that, when $\underline{\varphi}_3 = 1$, then $\underline{\varphi}_1 = -2$, $\overline{\varphi}_1 = \underline{\varphi}_2 = -1/3$ and $\overline{\varphi}_2 = \underline{\varphi}_3 = \overline{\varphi}_3 = 1$.

Our theorem, together with the remark, contains Laurent's [1] results on $\underline{\varphi}_1, \overline{\varphi}_1, \underline{\varphi}_3, \overline{\varphi}_3$. However it does not give a full description of the spectrum of $\underline{\varphi}_1, \overline{\varphi}_1, \underline{\varphi}_2, \overline{\varphi}_2, \underline{\varphi}_3, \overline{\varphi}_3$, i.e. the set of all such six-tuples arising from Diophantine approximation exponents. Recently Roy [3], using a different approach, could give a detailed description of this spectrum as a semi-algebraic set.

§2. *The setting.* We will start by recalling some needed notions. In a more general situation such an exposition has already been given in [5]. We will thus content ourselves to present it here for the two-dimensional approximation problem. Let $\Delta \subseteq \mathbb{R}^3$ consist of the points $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 \leqslant \alpha_2 \leqslant \alpha_3$. Observe that $\mathcal{L}_\xi := (L_1, L_2, L_3)$ with L_1, L_2, L_3 as in §1 is a map $(0, \infty) \to \Delta$.

A 3-system will be a map $\mathcal{P} : (0, \infty) \to \Delta$, where $\mathcal{P} = (P_1, P_2, P_3)$ with each P_i continuous and piecewise linear with slopes among $1, -2$ with $P_1(q) + P_2(q) + P_3(q) = 0$, and with a further condition formulated below.

The *combined graph* of $\mathcal{P}$ is the union of the graphs of P_1, P_2, P_3. When for some point (q, y) in the combined graph of $\mathcal{P}$ the set of slopes of the graph emanating to the left of (q, y) is not the same as the set of slopes to the right, then q is called a *switch number*. There will be some P_u which changes slope at q, and since $P_1 + P_2 + P_3 = 0$ there will be some P_v, $v \neq u$, which changes slope the "other way". Say P_u changes from rising to falling, P_v from falling to rising. For a 3-system the extra condition alluded to above is in this case always $u > v$. For given $u > v$, the number q is named a *switch number* of type $\binom{u}{v}$. As a consequence, when P_1 has a local maximum at q, then also the graph of P_2 passes through $(q, P_1(q))$ and $P_1(q) = P_2(q)$; when P_3 has a local minimum at q, then $P_2(q) = P_3(q)$.

Call $\mathcal{P}$ *proper* if $q - P_3(q)$ tends to infinity with q. Call $\mathcal{P}$ *agreeable* if there is no q with $P_1(q) = P_2(q) = P_3(q)$ (hence $= 0$), and there is no interval of positive length where $P_1(q) = P_2(q)$ or $P_2(q) = P_3(q)$.

THEOREM 2.1.

(a) *For each* $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ *with* $1, \xi_1, \xi_2$ *linearly independent over* $\mathbb{Q}$, *there is an agreeable proper* 3*-system* $\mathcal{P} : (0, \infty) \to \Delta$ *with* $\mathcal{P} - \mathcal{L}_\xi$ *bounded on* $(0, \infty)$.

(b) *Given a proper* 3*-system* $\mathcal{P} : (0, \infty) \to \Delta$, *there exists some* $\xi = (\xi_1, \xi_2)$ *with* $1, \xi_1, \xi_2$ *linearly independent over* $\mathbb{Q}$ *having* $\mathcal{P} - \mathcal{L}_\xi$ *bounded on* $(0, \infty)$.

Proof. As shown in Theorem 3.1 of our paper [6], assertion (b) is an easy consequence of the fundamental work [2] of Roy.

As for (a), it was shown in [6] that for each $\xi \in \mathbb{R}^2$ there is a 3-system $\mathcal{P} : (\eta_0, \infty) \to \Delta$ for some $\eta_0 \geqslant 0$ such that $\mathcal{P} - \mathcal{L}_\xi$ is bounded on (η_0, ∞). But [2, Theorem 1.3] asserts the existence of a *rigid* system $\mathcal{P}$ of positive mesh

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with $\mathcal{P}-\mathcal{L}_\xi$ bounded, and such a system is certainly agreeable. By the argument in [6], such a system leads to an agreeable 3-system $\mathcal{P}$ in our context with $\mathcal{P}-\mathcal{L}_\xi$ bounded. Note that an agreeable 3-system $\mathcal{P}:(\eta_0,\infty)\to\Delta$, where $\eta_0>0$ can be extended to an agreeable 3-system $\mathcal{P}:(0,\infty)\to\Delta$, and then $\mathcal{P}-\mathcal{L}_\xi$ will be bounded on $(0,\eta_0]$, hence on $(0,\infty)$.

It had been shown in [4] that when $1,\xi_1,\xi_2$ are linearly independent over $\mathbb{Q}$, then there are arbitrarily large numbers r with $L_2(r)=L_3(r)$ as well as arbitrarily large r^* with $L_1(r^*)=L_2(r^*)$. Since $\mathcal{P}-\mathcal{L}_\xi$ is bounded, these relations give $P_2(r)=P_3(r)+O(1)$, $P_1(r^*)=P_2(r^*)+O(1)$ with the implied constants independent of r,r^*. The second relation gives $P_2(r^*)\ll 1$, hence $P_2(q)<q-r^*+O(1)$ for $q>r^*$. When $r>r^*$, the first relation gives $P_3(r)<P_2(r)+O(1)<r-r^*+O(1)$, and when $q>r$, $P_3(q)<q-r^*+O(1)$. Since r^* is arbitrarily large, $q-P_3(q)\to\infty$. □

The argument in the above proof shows that if $P_2(r)=P_3(r)$ and $P_1(r^*)=P_2(r^*)$ for certain arbitrarily large r,r^*, then a 3-system $\mathcal{P}$ is proper. Conversely, when $\mathcal{P}$ is proper, then there are intervals $[s,r]$ with large s where P_3 declines, and has a local minimum at r, so that $P_2(r)=P_3(r)$. Now $P_1(r)=P_2(r)=P_3(r)$, or P_2 declines to the right of r, hence P_1 inclines, and reaches a local maximum at some r^*, so that $P_1(r^*)=P_2(r^*)$.

We set $\varphi_i(q):=P_i(q)/q$ for $i=1,2,3$ and note that the quantities $\underline{\varphi}_i$ and $\overline{\varphi}_i$ defined by (1.4) in terms of $\mathcal{P}$ are the same as the quantities defined in terms of $\mathcal{L}_\xi$ when $\mathcal{P}-\mathcal{L}_\xi$ is bounded. By Theorem 2.1, to prove Theorem 1.1 up to the best possible assertion, we may deal with an agreeable, proper 3-system $\mathcal{P}$. For such a system, the numbers r with $P_2(r)=P_3(r)$ form a sequence tending to infinity, and so do the numbers r^* with $P_1(r^*)=P_2(r^*)$, and these sequences have no common element. In the mixed combined sequence of these r^* and r arranged in increasing order, blocks of r^* will alternate with blocks of r (possibly reduced to one single r^* respectively r). Starting with a block of r^*, in the jth block of r^* denote the first and last element by b_j^* and a_j^*. In the directly following block of r denote the first and last element by a_j and b_j. Note that if $b_j^*=a_j^*$ or $a_j=b_j$, then the corresponding block consists of only one element. Set $\mathcal{I}_j:=[b_{j-1},b_{j+1}^*]$, and note that $\mathcal{I}_j,\mathcal{I}_{j+1}$ overlap on $[b_j,b_{j+1}^*]$.

In (a_j^*,b_{j+1}^*) we have $P_1(q)\neq P_2(q)$, so that P_1 has no local maximum. P_1 decreases up to some p_j^*, then increases. Similarly, P_3 increases up to some p_j, then decreases up to a_j. In the case when P_2 increases throughout $[a_j^*,a_j]$, then $p_j=p_j^*$ and this number is a switch number of type $\binom{3}{1}$. Otherwise, P_2 increases in $[a_j^*,p_j^*]$, decreases in $[p_j^*,p_j]$ where $p_j^*<p_j$, and increases in $[p_j,a_j]$. In this case p_j^*, p_j are respectively switch numbers of type $\binom{2}{1}$ and $\binom{3}{2}$.

When $b_j^*<a_j^*$, then $\frac{1}{2}(P_1+P_2)$ will decrease in (b_j^*,a_j^*) with slope $-1/2$, and P_1, P_2 will zigzag around the dotted line of Figure 1. Any local minimum of P_1 in the interval will be assumed at a switch number of type $\binom{2}{1}$. Similarly, when $a_j<b_j$, then P_2, P_3 will zigzag around a line of slope $-1/2$ and a local maximum of P_3 in $[a_j,b_j]$ will be asumed at a switch number of type $\binom{3}{2}$.

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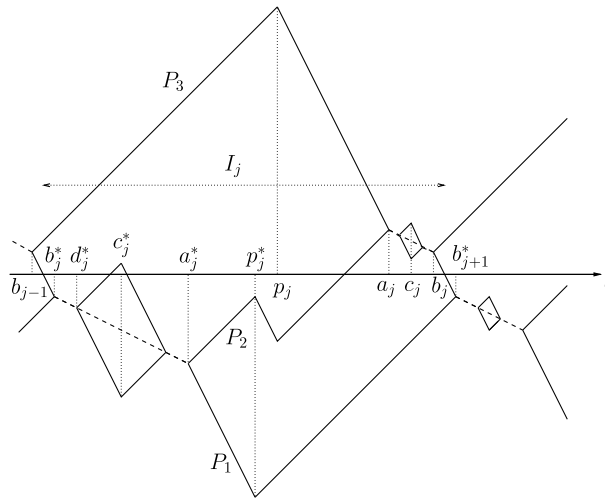


Figure 1.

For the convenience of the reader we illustrate the described situation within some interval $\mathcal{I}_j$ in Figure 1.

Now let $c_j^* \in \mathcal{I}_j$ be a number where P_1 assumes a local minimum. Then $b_j^* < c_j^* \leqslant p_j^*$. We will always have such a local minimum within $[a_j^*, a_j]$, namely at $c_j^* = p_j^*$ and when $b_j^* < a_j^*$ there will be at least one more such c_j^* within (b_j^*, a_j^*). For any $c_j^* \in (b_j^*, a_j^*)$ let $d_j^* < c_j^*$ be the largest number where $P_1(d_j^*) = P_2(d_j^*)$. For $c_j^* = p_j^*$ let $d_j^* = a_j^*$.

Let $c_j \in \mathcal{I}_j$ be a number where P_3 assumes a local maximum. Then $p_j \leqslant c_j < b_j$. Thus c_j may be p_j, and when $a_j < b_j$ there will be at least one such c_j within (a_j, b_j). For any $c_j \in (a_j, b_j)$ let $d_j > c_j$ be the least number where $P_3(d_j) = P_2(d_j)$. For $c_j = p_j$ let $d_j = a_j$.

Let $\overline{\varphi}_{ij}$ be the maximum of φ_i in $\mathcal{I}_j$ respectively $\underline{\varphi}_{ij}$ the minimum. Note that when $\overline{\phi}_i > \overline{\varphi}_i$, $\underline{\phi}_i < \underline{\varphi}_i$, then $\overline{\varphi}_{ij} < \overline{\phi}_i$, $\underline{\varphi}_{ij} > \underline{\phi}_i$ when j is large.

In what follows, we will assume

$$\overline{\varphi}_{ij} < \overline{\phi}_i, \qquad \underline{\varphi}_{ij} > \underline{\phi}_i. \tag{2.1}$$

§3. *Basic inequalities.* In this section we derive a few relations that result from the basic properties of proper 3-systems, in particular from $P_1+P_2+P_3 = 0$ and the knowledge of the possible slopes for these functions.

For any c_j^* as defined in Section 2 we have

$$\begin{aligned} P_3(c_j^*) &= P_3(b_j^*) + c_j^* - b_j^*, \\ P_1(c_j^*) &= P_1(b_j^*) - \tfrac{1}{2}(d_j^* - b_j^*) - 2(c_j^* - d_j^*) \\ &= P_1(b_j^*) - 2(c_j^* - b_j^*) + \tfrac{3}{2}(d_j^* - b_j^*), \end{aligned}$$

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and therefore

$$2P_3(c_j^*) + P_1(c_j^*) = 2P_3(b_j^*) + P_1(b_j^*) + \tfrac{3}{2}(d_j^* - b_j^*) = -3P_1(b_j^*) + \tfrac{3}{2}(d_j^* - b_j^*),$$

hence upon division by c_j^*

$$2\varphi_3(c_j^*) + \varphi_1(c_j^*) = -3\frac{b_j^*}{c_j^*}\varphi_1(b_j^*) + \frac{3}{2}\frac{d_j^* - b_j^*}{c_j^*}. \tag{3.1}$$

Also

$$P_3(c_j^*) + 2P_1(c_j^*) = P_3(b_j^*) + 2P_1(b_j^*) - 3(c_j^* - b_j^*) + 3(d_j^* - b_j^*),$$

and division by c_j^* yields

$$3 + \varphi_3(c_j^*) + 2\varphi_1(c_j^*) = 3\frac{b_j^*}{c_j^*} + 3\frac{d_j^* - b_j^*}{c_j^*}. \tag{3.2}$$

Combining (3.1) and (3.2) we obtain

$$\begin{aligned}
2\varphi_3(c_j^*) + \varphi_1(c_j^*) &= -\varphi_1(b_j^*)\left(3 + \varphi_3(c_j^*) + 2\varphi_1(c_j^*) - 3\frac{d_j^* - b_j^*}{c_j^*}\right) + \frac{3}{2}\frac{d_j^* - b_j^*}{c_j^*} \\
&= -\varphi_1(b_j^*)(3 + \varphi_3(c_j^*) + 2\varphi_1(c_j^*)) + 3\frac{d_j^* - b_j^*}{c_j^*}\left(\frac{1}{2} + \varphi_1(b_j^*)\right) \\
&\geqslant -\varphi_1(b_j^*)(3 + \varphi_3(c_j^*) + 2\varphi_1(c_j^*)),
\end{aligned}$$

since $\varphi_1(b_j^*) \geqslant -1/2$. Here

$$3 + \varphi_3(c_j^*) + 2\varphi_1(c_j^*) = 3 + \varphi_1(c_j^*) - \varphi_2(c_j^*) \geqslant 0$$

in view of $\varphi_1(c_j^*) \geqslant -2$ and $\varphi_2(c_j^*) \leqslant 1$. Moreover $\varphi_1(b_j^*) \leqslant \overline{\phi}_1$, so that

$$2\varphi_3(c_j^*) + \varphi_1(c_j^*) \geqslant -\overline{\phi}_1(3 + \varphi_3(c_j^*) + 2\varphi_1(c_j^*)), \tag{3.3}$$

which may also be written as

$$(1 + 2\overline{\phi}_1)\varphi_1(c_j^*) + (2 + \overline{\phi}_1)\varphi_3(c_j^*) \geqslant -3\overline{\phi}_1. \tag{3.4}$$

Thus $(\varphi_1(c_j^*), \varphi_3(c_j^*))$ lies on or above the line L^* with equation

$$(1 + 2\overline{\phi}_1)x + (2 + \overline{\phi}_1)y = -3\overline{\phi}_1.$$

Dually to (3.4),

$$(1 + 2\underline{\phi}_3)\varphi_3(c_j) + (2 + \underline{\phi}_3)\varphi_1(c_j) \leqslant -3\underline{\phi}_3, \tag{3.5}$$

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so that $(\varphi_1(c_j), \varphi_3(c_j))$ lies on or below the line L with equation

$$(2+\underline{\phi}_3)x + (1+2\underline{\phi}_3)y = -3\underline{\phi}_3.$$

Now choose c_j^* to be a number where φ_1 assumes its minimum in $\mathcal{I}_j$. Then $\varphi_1(c_j^*) = \underline{\varphi}_{1j}$ and $\varphi_3(c_j^*) < \overline{\phi}_3$, so that (3.4) yields

$$(1+2\overline{\phi}_1)\underline{\varphi}_{1j} + (2+\overline{\phi}_1)\overline{\phi}_3 \geqslant -3\overline{\phi}_1.$$

Since $\underline{\varphi}_1 = \liminf_j \underline{\varphi}_{1j}$, we obtain

$$(1+2\overline{\phi}_1)\underline{\varphi}_1 + (2+\overline{\phi}_1)\overline{\phi}_3 \geqslant -3\overline{\phi}_1$$

or, since $\overline{\phi}_i$ may be chosen arbitrarily close to $\overline{\varphi}_i$ for $i = 1, 3$,

$$\underline{\varphi}_1 + 2\overline{\varphi}_3 \geqslant -\overline{\varphi}_1(3 + 2\underline{\varphi}_1 + \overline{\varphi}_3). \tag{3.6}$$

Dually[1] to (3.6),

$$\overline{\varphi}_3 + 2\underline{\varphi}_1 \leqslant -\underline{\varphi}_3(3 + 2\overline{\varphi}_3 + \underline{\varphi}_1). \tag{3.7}$$

§4. *Bounds for $\overline{\varphi}_{2j}$ and $\underline{\varphi}_{2j}$.* φ_2 does have local maxima where P_2 does. In $\mathcal{I}_j$ these local maxima are at a_j, at p_j^* if $p_j^* < p_j$ and at numbers c_j^* in (b_j^*, a_j^*).

The point $(a_j, P_2(a_j))$ is the intersection of the line of slope 1 through $(p_j, P_2(p_j))$ and the line of slope -2 through $(p_j, P_3(p_j))$. Therefore

$$P_2(a_j) = P_2(p_j) + a_j - p_j = -P_1(p_j) - P_3(p_j) + a_j - p_j$$

and also

$$P_2(a_j) = P_3(p_j) - 2(a_j - p_j).$$

Twice the first of these equations added to the second gives

$$3P_2(a_j) = -2P_1(p_j) - P_3(p_j) = -(2\varphi_1(p_j) + \varphi_3(p_j))p_j. \tag{4.1}$$

Moreover

$$P_3(p_j) - P_2(p_j) = 3(a_j - p_j)$$

by Figure 1, so that

$$\begin{aligned} 3a_j &= 3p_j + 3(a_j - p_j) = 3p_j + P_3(p_j) - P_2(p_j) \\ &= 3p_j + P_1(p_j) + 2P_3(p_j) = (3 + \varphi_1(p_j) + 2\varphi_3(p_j))p_j, \end{aligned} \tag{4.2}$$

[1] *Remark*: In [**4**] we had proved (3.7), but the dual argument for (3.6) presented there does not work. The sentence. "Then equation (8.2) follows" on p. 89 does not hold water. We apologize and give the correct proof in the appendix using the notation from [**4**].

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so that in combination with (4.1)

$$\varphi_2(a_j) = P_2(a_j)/a_j = -\frac{2\varphi_1(p_j) + \varphi_3(p_j)}{3 + \varphi_1(p_j) + 2\varphi_3(p_j)} =: f(\varphi_1(p_j), \varphi_3(p_j)), \quad (4.3)$$

with

$$f(x, y) = -\frac{2x + y}{3 + x + 2y}.$$

Since $f(x, y)$ is decreasing in x as well as in y when $3 + x + 2y > 0$, and since $\varphi_1(p_j) \geqslant \varphi_1(p_j^*)$, $\varphi_3(p_j) \geqslant \varphi_3(p_j^*)$, we obtain

$$\varphi_2(a_j) \leqslant f(\varphi_1(p_j^*), \varphi_3(p_j^*)). \quad (4.4)$$

Given $c_j^* \in (b_j^*, a_j^*)$, let $(\tilde{a}_j, \tilde{P}_j)$ be the point of intersection of the line of slope 1 through $(c_j^*, P_2(c_j^*))$ and the line of slope -2 through $(c_j^*, P_3(c_j^*))$. By the argument above, with c_j^* in place of p_j we obtain, in analogy to (4.3)

$$\tilde{\varphi}_2(\tilde{a}_j) = f(\varphi_1(c_j^*), \varphi_3(c_j^*)),$$

where $\tilde{\varphi}_2(\tilde{a}_j) = \tilde{P}_j/\tilde{a}_j$. Since clearly $\varphi_2(c_j^*) \leqslant \tilde{\varphi}_2(\tilde{a}_j)$, we may conclude that $\varphi_2(c_j^*) \leqslant f(\varphi_1(c_j^*), \varphi_3(c_j^*))$, hence (as p_j^* is also some c_j^*)

$$\overline{\varphi}_{2j} \leqslant \max_{c_j^* \in I_j} (f(\varphi_1(c_j^*), \varphi_3(c_j^*))), \quad (4.5)$$

and therefore

$$\overline{\varphi}_{2j} \leqslant \max f(x, y) \quad (4.6)$$

with the maximum ranging over points (x, y) on or above L^* and with $x \geqslant \underline{\phi}_1$.

Dually to (4.3), (4.6)

$$\varphi_2(a_j^*) = f(\varphi_3(p_j^*), \varphi_1(p_j^*)) = f^*(\varphi_1(p_j^*), \varphi_3(p_j^*)) \quad (4.3^*)$$

with $f^*(x, y) = f(y, x) = -(2y + x)/(3 + y + 2x)$, and

$$\underline{\varphi}_{2j} \geqslant \min f^*(x, y) \quad (4.6^*)$$

with the minimum ranging over points on or below L, and with $y \leqslant \overline{\phi}_3$.

§5. *Proof of (1.8) and (1.9).* Since $f(x, y)$ increases as y decreases, $f(x, y)$ with (x, y) as in (4.6) is $\leqslant f(x, y(x))$ with $x \geqslant \underline{\phi}_1$, where L^* is given by

$$y(x) = -\frac{(1 + 2\overline{\phi}_1)x + 3\overline{\phi}_1}{2 + \overline{\phi}_1}.$$

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Hence we obtain

$$\begin{aligned} f(x, y(x)) &= -\frac{2x + y(x)}{3 + x + 2y(x)} \\ &= -\frac{2x - ((1 + 2\overline{\phi}_1)x + 3\overline{\phi}_1)/(2 + \overline{\phi}_1)}{3 + x - 2((1 + 2\overline{\phi}_1)x + 3\overline{\phi}_1)/(2 + \overline{\phi}_1)} \\ &= -\frac{2x(2 + \overline{\phi}_1) - x(1 + 2\overline{\phi}_1) - 3\overline{\phi}_1}{6 + 3\overline{\phi}_1 + x(2 + \overline{\phi}_1) - 2x(1 + 2\overline{\phi}_1) - 6\overline{\phi}_1} \end{aligned}$$

and simplification leads to

$$f(x, y(x)) = \frac{-x + \overline{\phi}_1}{2 - \overline{\phi}_1 - \overline{\phi}_1 x}. \tag{5.1}$$

Since $x \geqslant \underline{\phi}_1$, this is at most $(-\underline{\phi}_1 + \overline{\phi}_1)/(2 - \overline{\phi}_1 - \overline{\phi}_1\underline{\phi}_1)$, which is therefore an upper bound for φ_{2j}. This is also an upper bound for $\underline{\varphi}_2$, and since $\overline{\phi}_1, \underline{\phi}_1$ can be chosen arbitrarily close to $\overline{\varphi}_1, \underline{\varphi}_1$, inequality (1.8) follows.

Here $f^*(x, y)$ decreases as x increases, so that $f^*(x, y)$ with (x, y) as in (4.6*) is $\geqslant f^*(x(y), y)$ where L consists of points $(x(y), y)$. Dually to (5.1)

$$f^*(x(y), y) = \frac{-y + \underline{\phi}_3}{2 - \underline{\phi}_3 - \underline{\phi}_3 y}. \tag{5.1*}$$

Taking into account $y \leqslant \overline{\phi}_3$, inequality (1.9) easily follows.

§6. *Standard pieces.* Before turning to the "best possible" assertion of Theorem 1.1 we need to define standard functions on some intervals.

Suppose $\underline{\varphi}_1\overline{\varphi}_1, \underline{\varphi}_2, \overline{\varphi}_2, \underline{\varphi}_3, \overline{\varphi}_3$ are numbers with (1.5), ..., (1.9) and with $\underline{\varphi}_3 < 1$ (hence $\overline{\varphi}_1 > -1/2$) and $\underline{\varphi}_1 < 0 < \overline{\varphi}_3$. Further let $\underline{t}, \bar{t}$ be numbers, not both zero, with $\bar{t} < 1$,

$$2\underline{t} + \bar{t} \leqslant -\underline{\varphi}_3(3 + \underline{t} + 2\bar{t}), \qquad 2\bar{t} + \underline{t} \geqslant -\overline{\varphi}_1(3 + \bar{t} + 2\underline{t}), \tag{6.1}$$

so that (1.7) holds with $\underline{t}, \bar{t}$ in place of $\underline{\varphi}_1, \overline{\varphi}_3$. Thus $(\underline{t}, \bar{t})$ is on or below the line L given by $(2 + \underline{\varphi}_3)x + (1 + 2\underline{\varphi}_3)y = -3\underline{\varphi}_3$, and on or above the line L^* with equation $(1 + 2\overline{\varphi}_1)x + (2 + \overline{\varphi}_1)y = -3\overline{\varphi}_1$. (Note that L, L^* are analogues to the lines of §3.) So $(\underline{t}, \bar{t})$ is also on or below the line $y = 1$, hence is in the triangle T bounded by these three lines. A short computation shows that the vertices of T are $(-2, 1)$, $(\alpha, 1)$ with $\alpha := -(1 + 5\underline{\varphi}_3)/(2 + \underline{\varphi}_3) > -2$ and $(\underline{\beta}, \bar{\beta})$ with $\underline{\beta} := -3\underline{\varphi}_3/(1 - \overline{\varphi}_1\underline{\varphi}_3)$, $\bar{\beta} := -3\overline{\varphi}_1/(1 - \overline{\varphi}_1\underline{\varphi}_3)$, so that $\underline{\beta} \leqslant 0$, $\bar{\beta} \geqslant 0$ and

$$-2 \leqslant \underline{t} \leqslant 0 \quad \text{and} \quad 0 \leqslant \bar{t} < 1. \tag{6.2}$$

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By (6.1),

$$(2+\overline{\varphi}_1)\bar{t} \geqslant -3\overline{\varphi}_1 - (1+2\underline{\varphi}_1)\underline{t},$$
$$(2+\overline{\varphi}_1)\bar{t} + (1+\tfrac{1}{2}\overline{\varphi}_1)\underline{t} \geqslant -3\overline{\varphi}_1 - \tfrac{3}{2}\overline{\varphi}_1\underline{t} = -3\overline{\varphi}_1(1+\tfrac{1}{2}\underline{t}) \geqslant 0,$$

hence $2\bar{t} + \underline{t} \geqslant 0$. Dually, $2\underline{t} + \bar{t} \leqslant 0$, so that

$$2\bar{t} + \underline{t} \geqslant 0 \quad \text{and} \quad 2\underline{t} + \bar{t} \leqslant 0. \tag{6.3}$$

Not both relations can hold with equality, for then $\underline{t} = \bar{t} = 0$, which was ruled out. We may infer that $\underline{t} < 0 < \bar{t}$. The case where both relations in (6.3) are strict inequalities will be called the *generic case*, the other case is the *special case*. This case can only occur when $\overline{\varphi}_1 = 0 = \underline{\varphi}_3$.

After this tedious beginning we now come to the core of this section. For suitable numbers

$$c^* \leqslant b^* \leqslant a^* \leqslant 1 \leqslant a \leqslant b \leqslant c \tag{6.4}$$

with $c^* < c$ (which have no relation with the numbers c_j^*, c_j of §3) we will construct functions P_1, P_2, P_3 on $\mathcal{I}(\underline{t}, \bar{t}) := [c^*, c]$ with $P_1 + P_2 + P_3 = 0$, where each P_i is continuous and piecewise linear with slopes among $1, -2$. We will have

$$P_1(1) = \underline{t}, \qquad P_3(1) = \bar{t}. \tag{6.5}$$

P_1 will rise (with slope 1) in $[1, c]$. P_2 will rise, P_3 will fall (with slope -2) to the right of $q = 1$, and they will meet at $q = a$, so that $P_2(a) = P_3(a)$. Since

$$P_2(a) = P_2(1) + a - 1 = -\underline{t} - \bar{t} + a - 1$$

and also

$$P_2(a) = P_3(1) - 2(a-1) = \bar{t} - 2(a-1),$$

we obtain

$$a = 1 + \tfrac{1}{3}(2\bar{t} + \underline{t}) \geqslant 1, \tag{6.6}$$

in fact > 1 when $2\bar{t} + \underline{t} > 0$. It follows that

$$P_2(a) = \bar{t} - \tfrac{2}{3}(2\bar{t} + \underline{t}) = -\tfrac{1}{3}(\bar{t} + 2\underline{t}). \tag{6.7}$$

Moreover $P_2(b) = P_3(b) = \underline{\varphi}_3 b$, and when $a < b$, then P_2, P_3 will zigzag with alternating slopes $1, -2$ around a line of slope $-1/2$ (indicated by a dotted line segment in Figure 2), so that

$$\underline{\varphi}_3 b = P_2(b) = P_2(a) - \tfrac{1}{2}(b-a),$$

hence

$$(1+2\underline{\varphi}_3)b = a + 2P_2(a) = 1 + \tfrac{1}{3}(2\bar{t} + \underline{t} - 2\bar{t} - 4\underline{t}) = 1 - \underline{t},$$

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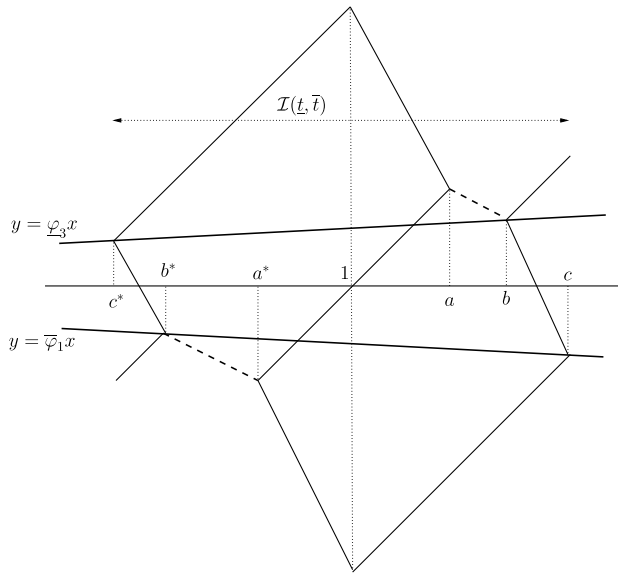


Figure 2.

so that

$$b=\frac{1-\underline{t}}{1+2\underline{\varphi}_3}. \tag{6.8}$$

Finally $P_1(c)=P_2(c)$, and when $b<c$, then P_1, P_3 will rise, P_2 will fall in $[b,c]$. Since $P_1(b)=-2P_2(b)=-2\underline{\varphi}_3 b$, we have $3(c-b)=P_3(b)-P_1(b)=-3\underline{\varphi}_3 b$, hence $c=(1+\underline{\varphi}_3)b$, $c>b$ if $\underline{\varphi}_3>0$. Moreover by (6.8), (1.6)

$$c=(1-\underline{t})\frac{1+\underline{\varphi}_3}{1+2\underline{\varphi}_3}=\frac{1-\underline{t}}{1-\overline{\varphi}_1}. \tag{6.9}$$

Also $P_2(c)=P_3(b)-2(c-b)=\underline{\varphi}_3 b-2\underline{\varphi}_3 b=-\underline{\varphi}_3 b$. So

$$P_1(c)=P_2(c)=-\underline{\varphi}_3\frac{1-\overline{\varphi}_1}{1+2\underline{\varphi}_3}c=\overline{\varphi}_1 c.$$

But $P_1(c)=\underline{t}+c-1$ since P_1 rises in $[1,c]$, and we obtain again (6.9). By (6.7), (6.8), (6.1),

$$\begin{aligned}3(1+2\underline{\varphi}_3)(b-a)&=3(1-\underline{t})-(1+2\underline{\varphi}_3)(3+2\overline{t}+\underline{t})\\&=-4\underline{t}-2\overline{t}-2\underline{\varphi}_3(3+2\overline{t}+\underline{t})\geqslant 0.\end{aligned} \tag{6.10}$$

We give a dual construction on the left of $q=1$. In our duality, we have c^*, b^*, a^* in place of c, b, a, and $\overline{\varphi}_1$, $\underline{\varphi}_3$, $\overline{t}$, $\underline{t}$, P_1, P_2, P_3 are replaced by $\underline{\varphi}_3$, $\overline{\varphi}_1$, $\underline{t}$, $\overline{t}$, P_3, P_2, P_1. Now P_3 rises in $[c^*,1]$, $P_1(a^*)=P_2(a^*)$ and P_2 rises, P_1 falls

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in $[a^*, 1]$. Hence

$$a^* = \tfrac{1}{3}(3 + 2\underline{t} + \overline{t}) \leqslant 1, \tag{6.6*}$$

$$P_2(a^*) = -\tfrac{1}{3}(\underline{t} + 2\overline{t}). \tag{6.7*}$$

Also

$$b^* = \frac{1 - \overline{t}}{1 + 2\overline{\varphi}_1}, \tag{6.8*}$$

$$c^* = \frac{1 - \overline{t}}{1 - \underline{\varphi}_3} \tag{6.9*}$$

and

$$a^* - b^* \geqslant 0. \tag{6.10*}$$

By (6.6), (6.8), (6.9), (6.6*), (6.8*), (6.9*) (and using (6.3)) we have in fact (6.4). Not both relations $a = 1$ and $a^* = 1$ can hold, so that $a^* < a$, hence $c^* < c$.

The triple of functions on the interval $\mathcal{I}(\underline{t}, \overline{t})$ will be called a *standard piece* and denoted $\mathcal{P}(\underline{t}, \overline{t})$. When $\overline{\varphi}_1 = 0 = \underline{\varphi}_3$, then $c^* = b^*$, $b = c$, and $P_1(q) = P_2(q) = P_3(q) = 0$ for $q = c^*$ and $q = c$. Moreover there will be an interval $[c^*, g^*]$ with $c^* < g^*$ where $P_1(q) = P_2(q)$, and an interval $[g, c]$ with $g < c$ where $P_2(q) = P_3(q)$. A degenerate version of Figure 2 will ensue.

Define $\varphi_i(q) = P_i(q)/q$ for $i = 1, 2, 3$ on $\mathcal{I}(\underline{t}, \overline{t})$, and let $\overline{\varphi}_i(\mathcal{P}(\underline{t}, \overline{t}))$ (respectively $\underline{\varphi}_i(\mathcal{P}(\underline{t}, \overline{t}))$) be the maximum (respectively minimum) of $\varphi_i(q)$ in $\mathcal{I}(\underline{t}, \overline{t})$. Clearly $\underline{\varphi}_3(\mathcal{P}(\underline{t}, \overline{t})) = \underline{\varphi}_3$ and $\overline{\varphi}_1(\mathcal{P}(\underline{t}, \overline{t})) = \overline{\varphi}_1$. But caution: $\mathcal{P}(\underline{t}, \overline{t})$ is not completely determined if $b^* < a^*$ or $a < b$. In the generic case, $a^* < 1 < a$, $P_1(a^*) > \underline{t}$, $P_3(a) < \overline{t}$, and we can pick alternating functions in $[b^*, a^*]$ and $[a, b]$ such that $\underline{\varphi}_1(\mathcal{P}(\underline{t}, \overline{t})) = \underline{t}$, $\overline{\varphi}_3(\mathcal{P}(\underline{t}, \overline{t})) = \overline{t}$. In the special case we can for each $\epsilon > 0$ pick a standard piece $\mathcal{P}_\epsilon(\underline{t}, \overline{t}))$ with

$$\underline{\varphi}_1(\mathcal{P}_\epsilon(\underline{t}, \overline{t})) > \underline{t} - \epsilon, \qquad \overline{\varphi}_3(\mathcal{P}_\epsilon(\underline{t}, \overline{t})) < \overline{t} + \epsilon.$$

We will now suppose $\overline{\varphi}_3 < 1$, hence $\underline{\varphi}_1 < 0 < \overline{\varphi}_3 < 1$. Three choices for $(\underline{t}, \overline{t})$ will be needed, denoted by $(\underline{s}_j, \overline{s}_j)$ for $j = 0, 1, 2$. First

$$(\underline{s}_0, \overline{s}_0) = (\underline{\varphi}_1, \overline{\varphi}_3), \tag{6.11}$$

which is allowed by (1.7). Here

$$\underline{\varphi}_1(\mathcal{P}(\underline{s}_0, \overline{s}_0)) = \underline{\varphi}_1 \quad \text{and} \quad \overline{\varphi}_3(\mathcal{P}(\underline{s}_0, \overline{s}_0)) = \overline{\varphi}_3 \tag{6.12}$$

in the generic case,

$$\underline{\varphi}_1(\mathcal{P}_\epsilon(\underline{s}_0, \overline{s}_0)) > \underline{\varphi}_1 - \epsilon \quad \text{and} \quad \overline{\varphi}_3(\mathcal{P}_\epsilon(\underline{s}_0, \overline{s}_0)) < \overline{\varphi}_3 + \epsilon \tag{6.13}$$

in the special case.

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Next,

$$(\underline{s}_1, \overline{s}_1) = (\underline{\varphi}_1, y(\underline{\varphi}_1)), \tag{6.14}$$

where $y = y(x)$ is the equation of the line L^*. Thus $(\underline{s}_1, \overline{s}_1)$ is the downward projection of $(\underline{\varphi}_1, \overline{\varphi}_3)$ on L^*, hence is in T, hence satisfies the condition on pairs $(\underline{t}, \overline{t})$. Moreover $y(\underline{\varphi}_1) \leqslant \overline{\varphi}_3$, so that

$$\underline{\varphi}_1(\mathcal{P}(\underline{s}_1, \overline{s}_1)) = \underline{\varphi}_1 \quad \text{and} \quad \overline{\varphi}_3(\mathcal{P}(\underline{s}_1, \overline{s}_1)) \leqslant \overline{\varphi}_3 < 1 \tag{6.15}$$

respectively

$$\underline{\varphi}_1(\mathcal{P}_\epsilon(\underline{s}_1, \overline{s}_1)) > \underline{\varphi}_1 - \epsilon \quad \text{and} \quad \overline{\varphi}_3(\mathcal{P}_\epsilon(\underline{s}_1, \overline{s}_1)) < \overline{\varphi}_3 + \epsilon < 1 \tag{6.16}$$

in the generic respectively in the special case for sufficiently small ϵ.

Moreover by (4.3) with $p_j = 1$, and by (6.5),

$$\overline{\varphi}_2(\mathcal{P}(\underline{s}_1, \overline{s}_1)) = \varphi_2(a) = f(\underline{s}_1, \overline{s}_1) = f(\underline{s}_1, y(\underline{s}_1)),$$

hence by (5.1)

$$\overline{\varphi}_2(\mathcal{P}(\underline{s}_1, \overline{s}_1)) = \bar{\Omega}, \tag{6.17}$$

with $\bar{\Omega}$ as in (1.8).

Finally,

$$(\underline{s}_2, \overline{s}_2) = (x(\overline{\varphi}_3), \overline{\varphi}_3), \tag{6.18}$$

where $x = x(y)$ is the equation of L. By an argument dual to the one for $(\underline{s}_1, \overline{s}_1)$,

$$\overline{\varphi}_3(\mathcal{P}(\underline{s}_2, \overline{s}_3)) = \overline{\varphi}_3 \quad \text{and} \quad \underline{\varphi}_1(\mathcal{P}(\underline{s}_2, \overline{s}_2)) \geqslant \underline{\varphi}_1, \tag{6.15*}$$

$$\overline{\varphi}_3(\mathcal{P}_\epsilon(\underline{s}_2, \overline{s}_2)) < \overline{\varphi}_3 + \epsilon \quad \text{and} \quad \underline{\varphi}_1(\mathcal{P}_\epsilon(\underline{s}_2, \overline{s}_2)) > \underline{\varphi}_1 - \epsilon \tag{6.16*}$$

in the general and the special case. By (4.3*) with $p_j^* = 1$,

$$\underline{\varphi}_2(\mathcal{P}(\underline{s}_2, \overline{s}_2)) = \varphi_2(a^*) = f^*(\underline{s}_2, \overline{s}_2) = f^*(x(\overline{s}_2), \overline{s}_2),$$

so that by (5.1*)

$$\underline{\varphi}_2(\mathcal{P}(\underline{s}_2, \overline{s}_2)) = \underline{\Omega}. \tag{6.17*}$$

$\mathcal{P}(\underline{s}_0, \overline{s}_0)$ is agreeable in an obvious sense unless $\underline{\varphi}_1 + 2\overline{\varphi}_3 = 0$ or $2\underline{\varphi}_1 + \overline{\varphi}_3 = 0$, and $\mathcal{P}(\underline{s}_1, \overline{s}_1)$, $\mathcal{P}(\underline{s}_2, \overline{s}_2)$ are agreeable when $\overline{\varphi}_1 < 0 < \underline{\varphi}_3$ but not otherwise, since $\overline{\varphi}_1 = 0 = \underline{\varphi}_3$ implies $y(\overline{\varphi}_1) = -\frac{1}{2}\overline{\varphi}_1$, $x(\overline{\varphi}_3) = -12\overline{\varphi}_3$.

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§7. *Best possible.* Given a 3-system $\mathcal{P}$ we had in §2 defined quantities $\underline{\varphi}_i$, $\overline{\varphi}_i$, $i = 1, 2, 3$, which will now be denoted $\underline{\varphi}_i(\mathcal{P}), \overline{\varphi}_i(\mathcal{P})$. To prove that the conditions (1.5), ..., (1.9) are best possible, we will for given numbers $\underline{\varphi}_i, \overline{\varphi}_i$ satisfying these five conditions, as well as $\underline{\varphi}_3 < 1$, construct a proper 3-system $\mathcal{P}$ with

$$\underline{\varphi}_i(\mathcal{P}) = \underline{\varphi}_i, \qquad \overline{\varphi}_i(\mathcal{P}) = \overline{\varphi}_i \quad \text{for } i = 1, 3 \tag{7.1}$$

and

$$\underline{\varphi}_2(\mathcal{P}) = \underline{\Omega}, \qquad \overline{\varphi}_2(\mathcal{P}) = \overline{\Omega}, \tag{7.2}$$

with $\underline{\Omega}, \overline{\Omega}$ as in (1.8), (1.9). In the generic case, when (6.3) holds with strict inequalities, $\mathcal{P}$ will be self-similar, i.e. will be invariant under dilation by a factor $\rho > 1$.

When $\mathcal{P}(\underline{t}, \overline{t})$ is as in the preceding section, let $\mathcal{P}^-(\underline{t}, \overline{t})$ (respectively $\mathcal{P}^+(\underline{t}, \overline{t})$) be its part with $q \in [c^*, b^*]$ (respectively $q \in [b, c]$). For any $(\underline{t}, \overline{t}), (\underline{t}', \overline{t}')$, the parts $\mathcal{P}^-(\underline{t}, \overline{t}), \mathcal{P}^+(\underline{t}, \overline{t}), \mathcal{P}^-(\underline{t}', \overline{t}'), \mathcal{P}^+(\underline{t}', \overline{t}')$ will all be similar, i.e. obtained from each other by multiplication by factors $\rho > 0$. In particular, $\mathcal{P}^+(\underline{t}, \overline{t})$ is obtained from $\mathcal{P}^-(\underline{t}, \overline{t})$ by a factor $\rho(\underline{t}, \overline{t}) > 1$. In fact, $\rho(\underline{t}, \overline{t}) = b/c^* = c/b^*$.

Let $(\underline{t}_j, \overline{t}_j)$, $j \in \mathbb{Z}$, again be pairs with $\overline{t}_j \leqslant 1$ and with (6.1). Then

$$\mathcal{P}^+(\underline{t}_j, \overline{t}_j) = \rho(\underline{t}_j, \overline{t}_j)\mathcal{P}^-(\underline{t}_{j+1}, \overline{t}_{j+1}),$$

so that

$$\mathcal{P}(\underline{t}_j, \overline{t}_j) \cup \rho(\underline{t}_j, \overline{t}_j)\mathcal{P}(\underline{t}_{j+1}, \overline{t}_{j+1})$$

defines functions on $\mathcal{I}(\underline{t}_j, \overline{t}_j) \cup \rho(\underline{t}_j, \overline{t}_j)\mathcal{I}(\underline{t}_{j+1}, \overline{t}_{j+1})$: we could say that the standard pieces $\mathcal{P}(\underline{t}_j, \overline{t}_j), \mathcal{P}(\underline{t}_{j+1}, \overline{t}_{j+1})$ are patched together. Define ω_j ($j \in \mathbb{Z}$) by $\omega_0 = 1$ and $\omega_{j+1}/\omega_j = \rho(\underline{t}_j, \overline{t}_j)$. If $\omega_j \to 0$ (respectively $\omega_j \to \infty$) as $j \to -\infty$ (respectively $j \to \infty$), then

$$\mathcal{P} = \bigcup_{j \in \mathbb{Z}} \omega_j \mathcal{P}(\underline{t}_j, \overline{t}_j) \tag{7.3}$$

defines a proper 3-system on $(0, \infty)$.

Again suppose $\underline{\varphi}_1 < 0 < \overline{\varphi}_3 < 1$. We now set $(\underline{t}_j, \overline{t}_j) = (\underline{s}_i, \overline{s}_i)$ for $i = 0, 1, 2$ when $j \equiv i \pmod 3$. In the case when $\overline{\varphi}_1 < 0 < \underline{\varphi}_3$, it follows from (6.12), (6.15), (6.17), (6.15*), (6.17*) and the bounds (1.8), (1.9) that (7.1), (7.2) hold. Our $\mathcal{P}$ is agreeable and self similar with factor $\rho := \rho(\underline{s}_0, \overline{s}_0)\rho(\underline{s}_1, \overline{s}_1)\rho(\underline{s}_2, \overline{s}_2)$. In the case when $\overline{\varphi}_1 = 0 = \underline{\varphi}_3$ we pick $\epsilon_j > 0$ with $\epsilon_j \to 0$ as $j \to \infty$, and in (7.3) replace $\mathcal{P}(\underline{t}_j, \overline{t}_j)$ by $\mathcal{P}_\epsilon(\underline{t}_j, \overline{t}_j)$. Again (7.1), (7.2) will hold. Thus Theorem 1.1 is true when $\underline{\varphi}_1 < 0 < \overline{\varphi}_3 < 1$.

Next suppose $\underline{\varphi}_1 < 0 \leqslant \underline{\varphi}_3 < \overline{\varphi}_3 = 1$. Pick a sequence of points $(\underline{s}_{0j}, \overline{s}_{0j}) \in T$ tending to $(\underline{\varphi}_1, \overline{\varphi}_3)$ as $j \to \infty$, with $\underline{s}_{0j} > \underline{\varphi}_1$, $\frac{1}{2}(\underline{\varphi}_3 + \overline{\varphi}_3) < \overline{s}_{0j} < \overline{\varphi}_3 = 1$. Also set

$$(\underline{s}_{1j}, \overline{s}_{1j}) := (\underline{s}_{0j}, y(\underline{s}_{0j})), \qquad (\underline{s}_{2j}, \overline{s}_{2j}) := (x(\overline{s}_{0j}), \overline{s}_{0j}),$$

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so that for $i = 0, 1, 2$,

$$(\underline{s}_{ij}, \overline{s}_{ij}) \to (\underline{s}_i, \overline{s}_i) \quad \text{as } j \to \infty,$$

where $(\underline{s}_i, \overline{s}_i)$ is given by (6.11), (6.14), (6.18). Set $(\underline{t}_j, \overline{t}_j) = (\underline{s}_{ij}, \overline{s}_{ij})$ for $i = 0$, 1, 2 when $j \equiv i \pmod 3$. When $j \equiv 0 \pmod 3$, then (6.9*) with $\overline{t}_j$ in place of $\overline{t}$ yields

$$c_j^* \leqslant \psi := \frac{1 - \frac{1}{2}(\underline{\varphi}_3 + \overline{\varphi}_3)}{1 - \underline{\varphi}_3} < 1,$$

so that $\rho(\underline{t}_j, \overline{t}_j) \geqslant 1 - c_j^* > 1 - \psi > 0$. So if again $\mathcal{P}$ is defined by (7.3), with $\mathcal{P}(\underline{t}_j, \overline{t}_j)$ or $\mathcal{P}_\epsilon(\underline{t}_j, \overline{t}_j)$, then again (7.1), (7.2) hold.

This takes care of the case when $\underline{\varphi}_1 < 0 < \overline{\varphi}_3$. Otherwise $\underline{\varphi}_1 = 0 = \overline{\varphi}_3$, and $\underline{\varphi}_i = 0 = \overline{\varphi}_i$ for $i = 1, 2, 3$. But as is well known, this is true for almost every $\xi \in \mathbb{R}^2$ in the sense of Lebesgue measure. With a little effort one could construct a proper 3-system $\mathcal{P}$ with each $\underline{\varphi}_i = \overline{\varphi}_i = 0$. The proof of Theorem 1.1 is thus complete.

As for the remark below the Theorem, it is best possible since all the quantities $\underline{\varphi}_i, \overline{\varphi}_i$ are completely described. But one really should prove that $\xi = (\xi_1, \xi_2)$ with $1, \xi_1, \xi_2$ linearly independent over $\mathbb{Q}$ with $\overline{\varphi}_3 = 1$ exist. We invite the reader to construct a proper 3-system $\mathcal{P}$ with this property.

A. *Appendix.* With the notation from [**4**], equation (8.2) can be proved as follows. We have

$$\begin{aligned}\tilde{L}_3(m_j^*) &= \tilde{L}_3(a_j^*) + \tfrac{1}{2}(m_j^* - a_j^*) + O(1),\\ \tilde{L}_1(m_j^*) &= \tilde{L}_1(a_j^*) - (m_j^* - a_j^*) + \tfrac{3}{4}(b_j^* - a_j^*) + O(1).\end{aligned}$$

It follows that on the one hand

$$2\tilde{L}_3(m_j^*) + \tilde{L}_1(m_j^*) = -3\tilde{L}_1(a_j^*) + \tfrac{3}{4}(b_j^* - a_j^*) + O(1)$$

and hence

$$2\tilde{\chi}_3(m_j^*) + \tilde{\chi}_1(m_j^*) = -3\tilde{\chi}_1(a_j^*)\frac{a_j^*}{m_j^*} + \frac{3}{4}\frac{b_j^* - a_j^*}{m_j^*} + O(1/m_j^*).$$

On the other hand we get

$$2\tilde{L}_3(m_j^*) + 4\tilde{L}_1(m_j^*) = -3(m_j^* - a_j^*) + 3(b_j^* - a_j^*) + O(1)$$

and hence

$$3 + 2\tilde{\chi}_3(m_j^*) + 4\tilde{\chi}_1(m_j^*) = 3\frac{a_j^*}{m_j^*} + 3\frac{b_j^* - a_j^*}{m_j^*} + O(1/m_j^*).$$

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The combination of these two estimates gives

$$\begin{aligned}2\tilde{\chi}_3(m_j^*)+\tilde{\chi}_1(m_j^*) &= -\tilde{\chi}_1(a_j^*)\left(3+2\tilde{\chi}_3(m_j^*)+4\tilde{\chi}_1(m_j^*)-3\frac{b_j^*-a_j^*}{m_j^*}\right)\\ &\quad +\frac{3}{4}\frac{b_j^*-a_j^*}{m_j^*}+O(1/m_j^*)\\ &= -\tilde{\chi}_1(a_j^*)(3+2\tilde{\chi}_3(m_j^*)+4\tilde{\chi}_1(m_j^*))\\ &\quad +3\frac{b_j^*-a_j^*}{m_j^*}\left(\frac{1}{4}+\tilde{\chi}_1(a_j^*)\right)+O(1/m_j^*).\end{aligned}$$

Taking into account (7.4) the assertion in equation (8.2) follows.

References

1. M. Laurent, Exponents of Diophantine approximation in dimension two. *Canad. J. Math.* **61** (2009), 165–189.

2. D. Roy, On Schmidt and Summerer parametric geometry of numbers. *Ann. of Math.* (2) **182** (2015), 739–786.

3. D. Roy, On the topology of Diophantine approximation spectra. *Compos. Math.* **153** (2017), 1512–1546.

4. W. M. Schmidt and L. Summerer, Parametric geometry of numbers and applications. *Acta Arith.* **140** (2009), 67–91.

5. W. M. Schmidt and L. Summerer, Diophantine approximation and parametric geometry of numbers. *Monatsh. Math.* **169** (2013), 51–104.

6. W. M. Schmidt and L. Summerer, The generalization of Jarnik's identity. *Acta Arith.* **175** (2016), 119–136.

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https://doi.org/10.1112/S0025579317000274 Published online by Cambridge University Press