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Colouring games

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There are various kinds of two-person games in which the players construct a colouring of a graph (or of a subgraph), and some additional games are closely related to graph colourings. Here we describe several variants, most of which deal with 'maker–breaker games'. We also consider the game version of list-colourings.

1. Introduction

The connection between graph colourings and game theory has its roots in the works of P. M. Grundy in the latter field as early as the 1930s. His research later motivated the introduction of some graph invariants, one of which is discussed in detail in Berge's monograph [9]. For undirected graphs the analogous parameter is the *Grundy number*, also called the *online chromatic number*, which we will consider in Section 9. But although Grundy colouring may be formulated in terms of games, it remains a solitaire game in which the player aims at finding a solution that is as bad as possible.

In this chapter we mainly consider non-cooperative two-person games that either construct colourings or are otherwise closely related to some kind of colouring concept, even when they do not colour the graph.

https://doi.org/10.1017/CBO9781139519793.017 Published online by Cambridge University Press

A good starting point is the following *maker–breaker game*, which was introduced by Brams [24] for planar graphs in 1981, and independently by Bodlaender [12] for general graphs in 1991. Given a graph G, two players alternately colour vertices of G from a set of colours, say $\{1, 2, \ldots, k\}$. The two players are Alice (the maker) and Bob (the breaker); each of them colours one uncoloured vertex at a time, under the condition that monochromatic edges must not occur, Alice moving first. Alice wins if a proper k-colouring of the entire graph G is eventually obtained, whereas Bob wins if a situation is reached where the neighbourhood of some uncoloured vertex contains all the k colours of the palette. Certainly, if the number of colours is sufficiently large (for example, if $k = |V(G)|$), then Alice surely wins the game.

This leads to the following definition. The *game-chromatic number* $\chi_g(G)$ of G is the smallest number k for which Alice has a winning strategy.

For every graph the inequalities

$$\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1 \tag{1}$$

hold for the chromatic number χ, the game-chromatic number χ_g and the maximum vertex-degree Δ: a proper colouring with $\chi_g(G)$ colours is constructed by the end of the game whenever Alice wins, and a free colour is available for any uncoloured vertex whenever the number of colours exceeds the number of neighbours. The path P_4 of length 3 has $\chi = 2$, but $\chi_g = 3$, because if Alice colours a vertex with colour 1 in her first move, then Bob can assign colour 2 to the vertex at distance 2 from the vertex coloured first. It follows that Brooks's theorem cannot be extended directly to the game-chromatic number, and that χ_g may exceed χ, even when G is a path.

For some trees T, $\chi_g(T)$ can be as large as $\chi(T) + 2$ – that is, $\chi_g(T) = 4$ may occur. For example, consider the colouring game played with colours α, β, γ on the caterpillar of diameter 5 on 14 vertices (see Fig. 1(a)). No matter which vertex v Alice colours first, Bob can assign the same colour α to some vertex w at distance 3 from v, giving the subtree in Fig. 1(b).

If Alice colours a vertex on the central path, its colour β must differ from α, and then Bob can assign the third colour γ to a leaf adjacent to the last (uncoloured) vertex of the central path, thereby winning the game. If Alice assigns colour β to a leaf, Bob wins by assigning γ to the other leaf adjacent to the uncoloured middle vertex. If Alice assigns colour α to a leaf, Bob assigns colour β to a leaf adjacent to

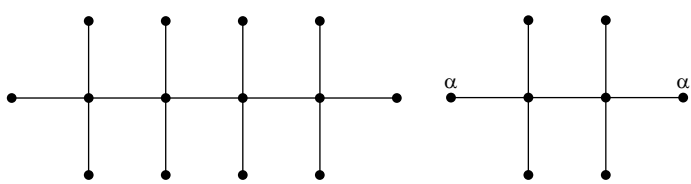


Fig. 1. A tree T with $\chi_g(T) = 4$

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the other vertex of the central path. This creates a double threat on the neighbour of colour β, because Bob has a chance to assign colour γ to any of its two uncoloured neighbours. The only way for Alice to eliminate this threat is to assign colour γ to the neighbour of colour β. Now the other vertex on the central path has colours α, γ in its neighbourhood, and still has an uncoloured leaf neighbour to which Bob can assign colour β on his next move, hence winning the game. Actually, the upper bound of 4 is tight on trees. We show this in a stronger form in the next section.

We note that a completely different game would be obtained if the first move were made by Bob. For example, the 'cocktail-party graph' $K_{k,k} - kK_2$, obtained by omitting a perfect matching from $K_{k,k}$, has $\chi_g = k$, since Bob can win for any fewer than k colours by assigning to the matching pair of the vertex just coloured the same colour that Alice used in her latest move. On the other hand, if Bob makes the first move, then Alice can assign a different colour to the matching pair of the coloured vertex, and hence win with just two colours.

Most studies on the game-chromatic number concentrate on upper bounds for classes of graphs. For a class $\mathcal{H}$ of graphs, let

$$\chi_g(\mathcal{H}) = \max\{\chi_g(G) : G \in \mathcal{H}\}.$$

Zhu [61] proved that if $\mathcal{H}$ is closed under the operations of taking disjoint unions and adding K_1 as a component, then $\chi_g(\mathcal{H})$ does not depend on who has the first move. Nevertheless, as the cocktail-party graph shows, for individual graphs it can make a difference who starts the game. In this chapter we concentrate on games started by Alice.

We observe further that $\chi_g(K_{k,k}) = 3$. Indeed, after Alice's first move Bob's strongest answer is to assign a second colour to a vertex that is not adjacent to the first one; but then Alice can assign a third colour in the other vertex-class and thus ensure that the colouring can be completed with three colours, no matter how Bob plays. In fact, if we re-insert just one edge of the omitted 1-factor into $K_{k,k} - kK_2$, we already obtain a graph with $\chi_g = 3$.

As a consequence of these examples, we deduce that the game-chromatic numbers of subgraphs of G are not bounded above by $\chi_g(G)$.

A general upper bound

The *acyclic chromatic number* $\chi_a(G)$ of a graph G is the smallest integer k for which G has a proper vertex k-colouring such that no cycle is 2-coloured. Dinski and Zhu [19] proved the following result.

Theorem 1.1 *For any graph G,*

$$\chi_g(G) \le \chi_a(G)(\chi_a(G) + 1). \tag{2}$$

This upper bound is usually not sharp, but still the result offers a general approach to proving that the members of some graph classes have bounded game-chromatic

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number; for example, it trivially implies that $\chi_g \leq 6$ holds for all trees. More interestingly, applying Borodin's result [13] that $\chi_a(G) \leq 5$ holds for every planar graph G, we deduce that the game-chromatic number of every planar graph is at most 30.

Algorithmic complexity

Although we do not devote a separate section to this issue, we should say something about it, as this was the main topic of Bodlaender's seminal paper [12]. However, Bodlaender concentrated on a somewhat different game on a graph, where a linear ordering $v_1, v_2, \ldots, v_n$ of the vertices is also given, together with the number of colours, and the vertices have to be coloured in this fixed order. He used the term 'sequential colouring construction game'.

Bodlaender proved that, if the number of colours is at least 3, then it is PSPACE-complete to decide which of the two players has a winning strategy in the sequential colouring construction game, even when the input is restricted to graphs with maximum degree at most 5. On the other hand, if just two colours are available, the winner can be determined in time $O(n + m)$.

We will return to complexity issues in Section 9.

2. Marking games

The *colouring number* $\mathrm{col}(G)$ of a graph G is defined as

$$\mathrm{col}(G) = \min \max_{1 \leq i \leq n} (d^-(v_i) + 1),$$

where the minimum is taken over all vertex orderings $v_1, v_2, \ldots, v_n$, and $d^-(v_i)$ denotes the number of neighbours preceding v_i in the vertex order under consideration. The colouring number is an upper bound for $\chi(G)$ – a rather bad one in some cases. The point is that, when we colour the graph in the order of increasing subscripts according to an order that attains $\mathrm{col}(G)$, each vertex has fewer than $\mathrm{col}(G)$ coloured neighbours at the time that it has to be coloured, and so there is always a free colour available.

For the game version, consider the game in which Alice and Bob alternately select vertices of G, constructing a vertex order in which the vertices with odd index are chosen by Alice and those with even index are chosen by Bob. Alice's goal is to minimize $\max_{1 \leq i \leq n} d^-(v_i) + 1$, while Bob's goal is to maximize it. This leads to define the *game-colouring number* of a graph G, denoted by $\mathrm{col}_g(G)$, to be the value of $\max_{1 \leq i \leq n} d^-(v_i) + 1$ when both players play optimally.

This is an interesting parameter in its own right. However, the main purpose of introducing it here is because it is an upper bound for the game-chromatic number.

Theorem 2.1 *For any graph G, $\chi_g(G) \leq \mathrm{col}_g(G)$.*

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In the play of a marking game we say that a vertex is *marked* if it has been chosen by a player. If $\mathrm{col}_g(G) = k$, Alice has a strategy for playing so that, at any moment of the game, any unmarked vertex has at most $k-1$ marked neighbours. In the maker–breaker game Alice uses her strategy for the marking game to choose vertices to be coloured, and she can then ensure that, at any moment of the game, any uncoloured vertex has at most $k-1$ coloured neighbours. Thus, with k colours available, all the vertices are eventually coloured. This argument proves Theorem 2.1.

As a significant difference between χ_g and col_g, it was proved by Wu and Zhu [56] that, if H is a subgraph of G, then $\mathrm{col}_g(H) \le \mathrm{col}_g(G)$. Perhaps this more natural behaviour – which is not satisfied by χ_g – is one reason why many upper bounds on χ_g are proved by showing that they are valid for col_g.

The game-colouring numbers for many classes of graphs were determined or estimated by Faigle, Kern, Kierstead and Trotter [23]. The first result of this kind was that $\mathrm{col}_g(F) \le 4$ for every forest F. We sketch Alice's winning strategy.

Alice must ensure that, after each of her moves, each unmarked subtree of F (that is, each component of the subgraph of F induced by the currently unmarked vertices) is joined by edges to at most two marked vertices; this property evidently holds after the first move, since just one vertex is marked. At any later stage Bob marks a vertex inside some unmarked component T of F. After Bob's move, the uncoloured vertices of T induce one or more components. In any case, at most one of the components has three coloured neighbours, while all the other components have at most two. If Bob leaves a component with three marked neighbours, then Alice marks the unique vertex separating them. This splits the component into at least three unmarked components, and three of the components have two marked neighbours each. If Bob leaves each component with at most two marked neighbours, then in some component Alice marks a vertex adjacent to a marked vertex, or marks an arbitrary vertex if the component has no marked neighbours. This strategy works because only one tree component of the unmarked subforest requires special attention – namely, the one for which Bob has created three marked neighbours.

Sidorowicz [53] showed that, if G contains cycles no two of which have a common edge (a cactus), then $\mathrm{col}_g(G) \le 5$ and this upper bound is tight also for $\chi_g(G)$. More generally, Junosza-Szaniawski and Rożej [32] showed that if each edge is contained in at most c cycles of G, then $\mathrm{col}_g(G) \le c+4$.

The activation strategy

A frequently used strategy in establishing upper bounds for col_g is the so-called 'activation strategy'. To use it for a graph G, we first need to fix a vertex order $v_1, v_2, \ldots, v_n$ of G. For a vertex v, let $N^<(v)$ be the set of neighbours preceding v in the vertex order and $N^>(v)$ be the set of neighbours occurring after v. During the play of the game Alice constructs a directed graph D with $V(D) = V(G)$; this digraph helps her to choose vertices in later moves. Initially, D has no arcs, and during play arcs are added to D.

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By 'processing' a vertex v, we mean the following self-calling procedure: If there are no unmarked vertices, then stop (without doing anything). Assume that there are unmarked vertices.

Step 1. If $N^<(v)$ contains no unmarked vertices and v is unmarked, then mark v and stop.
Step 2. If $N^<(v)$ contains no unmarked vertices and v is marked, then mark the unmarked vertex of smallest index and stop.
Step 3. Otherwise, let x be the unmarked vertex of smallest index in $N^<(v)$. If x has an in-neighbour in D, then add an arc from v to x, mark x and stop. If x has no in-neighbour in D, then add an arc from v to x and process x (that is, go to Step 1, with v replaced by x).

The activation strategy is as follows:
On her first move, Alice marks v_1. Later, when Bob marks a vertex v, Alice processes v.

Note that, when processing a vertex v, the procedure may repeatedly call itself to process a sequence of vertices. In this sequence the vertices have decreasing indices, so the procedure eventually ends. The procedure always ends by marking an unmarked vertex, provided that there are still such vertices.

Suppose that Alice has just completed a move. We say that a vertex v is *active* if v is incident to an arc of D, or if v and all vertices in $N^<(v)$ are marked.

The following observations follow easily from the processing procedure and the activation strategy.

(a) Each marked vertex is active.
(b) If v is active and unmarked, then v has out-degree 1 in D.
(c) If v is active and $w \in N^<(v)$ is unmarked, then v has out-degree 1 in D, and the out-neighbour of v precedes or equals w.
(d) If a vertex w has in-degree 2 in D, then w is marked. Moreover, once a vertex w is marked, no more arcs incident to w can be added to D. Hence each vertex of D has in-degree at most 2.

To derive an upper bound on $\mathrm{col}_g(G)$ by using the activation strategy, we need the following result:
Assume that Alice has just completed a move and that v is unmarked, and suppose that there are two sets $A(v), B(v)$ of vertices for which the following properties hold:

(i) $N^<(v) \subseteq A(v)$; and
(ii) if u precedes v and there is a vertex $w \notin B(v)$ for which $u, v \in N^<(w)$, then $u \in A(v)$.

Then v has at most $3|A(v)| + |B(v)|$ marked neighbours.

To prove this, we let $P(v)$ be the set of marked vertices in $N^<(v)$ and $Q(v)$ be the set of marked vertices in $N^>(v)$. By (i), $P(v) \subseteq A(v)$. By (ii) and the observation above,

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if $w \in Q(v) - B(v)$, then there is an arc wu in D such that either $u = v$ or $u \in A(v)$. If there is an arc wv in D, then v is active, and so there is an arc from v to $A(v)$; so there are at least $|Q(v)| - |B(v)|$ arcs from $(Q(v) - B(v)) \cup \{v\}$ to $A(v)$. Since each vertex $u \in A(v)$ has in-degree at most 2, we conclude that $|Q(v)| - |B(v)| \leq 2|A(v)|$. Thus the total number of marked neighbours of v is

$$|P(v)| + |Q(v)| \leq 3|A(v)| + |B(v)|.$$

By this result, if v is unmarked at any moment of the game, and if the sets $A(v)$ and $B(v)$ satisfy the conditions listed above, then v has at most $3|A(v)| + |B(v)| + 1$ marked neighbours (since if Bob has just completed a move he could have marked a further neighbour of v). Therefore, if $A(v), B(v)$ satisfy the conditions above for each vertex v of G, then

$$\mathrm{col}_g(G) \leq \max_{v \in V(G)} 3|A(v)| + |B(v)| + 2.$$

A good vertex order of $V(G)$ is crucial for good performance of the activation strategy. If G is a chordal graph, then it has a vertex order in which $N^<(v)$ induces a complete subgraph for each v. Under this vertex order, taking $A(v) = N^<(v)$ and $B(v) = \varnothing$, we can easily verify that (i) and (ii) are satisfied. Hence

$$\mathrm{col}_g(G) \leq \max_{v \in V(G)} 3|A(v)| + 2 = 3\omega(G) - 1.$$

In particular, if G is a k-tree, then $\mathrm{col}_g(G) \leq 3k + 2$. By the monotonicity of $\mathrm{col}_g(G)$ this implies that the game-colouring number of any partial k-tree is bounded above by $3k + 2$. This upper bound is sharp for all $k \geq 2$ (see [56]).

Planar and outerplanar graphs

When Brams invented the colouring game he was interested only in planar graphs. One can easily see that four colours are not enough for the octahedron graph and five colours are not enough for the icosahedron graph. Robert High then found a planar graph for which six colours are not enough. Whether there is a universal finite upper bound on the game-chromatic number of all planar graphs remained an open problem for some time. The existence of such a bound was conjectured by Bodlaender and proved by Kierstead and Trotter [35] with bound 33. This was subsequently improved in a sequence of papers, and the current record of 17 is due to Zhu [60].

A further tool handling the game-colouring number is based on edge decomposition [58].

Theorem 2.2 *If G_1 and G_2 are subgraphs of G with $E(G) = E(G_1) \cup E(G_2)$, then*

$$\mathrm{col}_g(G) \leq \mathrm{col}_g(G_1) + \Delta(G_2). \tag{3}$$

For the more restricted class of C_4-free planar graphs, Borodin *et al.* [14] proved that the edge-set can be decomposed into a forest and a subgraph with maximum

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degree at most 6; $\mathrm{col}_g \leq 10$ then follows by inequality (3). For the class of planar graphs with girth at least 8, it was proved by Montassier *et al.* [43], and independently by Wang [55], that the edge-set can be decomposed into a forest and a matching, so $\mathrm{col}_g \leq 5$.

From below, it is known that there exist planar graphs with $\mathrm{col}_g \geq 11$ (see Wu and Zhu [56]) and with $\chi_g \geq 8$ (see Kierstead and Trotter [35]).

As outerplanar graphs are all partial 2-trees, their game-colouring number is bounded by 8. By observing that the edge-set of an outerplanar graph can be decomposed into a forest and a graph of maximum degree 3, Guan and Zhu [26] improved this to $\mathrm{col}_g \leq 7$ (using inequality (3)), and this is sharp (see [38]). Since $\chi_g(G) \leq \mathrm{col}_g(G)$, the game-chromatic number of outerplanar graphs is at most 7. Kierstead and Trotter [35] showed that there are outerplanar graphs with game-chromatic number 6. It remains an open question as to whether there exists an outerplanar graph with game-chromatic number 7.

Cartesian product graphs

Recall that the Cartesian product $G \,\square\, H$ of two graphs G and H has vertex-set $V(G) \times V(H)$ and edge-set $\{(vw, v'w') : v = v' \in V(G) \text{ and } ww' \in E(H)\} \cup \{(vw, v'w') : vv' \in E(G) \text{ and } w = w' \in V(H)\}$. Various graph invariants have the property that their value on $G \,\square\, H$ is bounded above by some function of their values on G and H.

Bartnicki *et al.* [8] proved that this is not the case for colouring games: χ_g is unbounded on the Cartesian products of complete bipartite graphs $K_{q,q}$ (recall that $\chi_g(K_{q,q}) = 3$ for all $q \geq 2$) and col_g is unbounded on the Cartesian products of stars $K_{1,q}$. On the other hand, Zhu [61] has proved that if H has acyclic chromatic number k, and if joining to each vertex of G a set of $|V(H)|$ neighbours of degree 1 gives a graph with game-colouring number m, then $\chi_g(G \,\square\, H) \leq k(k + m - 1)$. As a consequence, the Cartesian product of two forests has game-chromatic number at most 10, and the Cartesian product of two planar graphs has game-chromatic number at most 105.

Game-perfect graphs

Recall from Chapter 7 that a perfect graph is defined in terms of the equality $\chi = \omega$, which is required to hold for every induced subgraph. In this vein, Andres [7] investigated the structure of graphs G for which $\chi_g(G') = \omega(G')$ or $\mathrm{col}_g(G') = \omega(G')$ is valid for every induced subgraph G' of G. We call these graphs (ω, χ_g)-*perfect* and (ω, col_g)-*perfect*, respectively. (The terminology of [7] is different, because other types of game-perfectness are also studied there.) It turns out that these graphs admit a concise characterization in terms of forbidden induced subgraphs.

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Theorem 2.3
A graph is (ω, χ_g)-perfect if and only if it contains none of the graphs P_4, C_4, $K_3 + 3K_1$, $K_2 + 2P_3$, $(K_1 + 2P_3) \cup (K_2 + 3K_1)$, $2(K_1 + 2P_3)$ or $2(K_2 + 3K_1)$ as an induced subgraph.
A graph is (ω, col_g)-perfect if and only if it contains none of the graphs P_4, C_4, $2(K_4 - e)$ or $K_1 + (K_4 - e)$ as an induced subgraph.

3. Greedy colouring games

In each round of the maker–breaker game each player chooses an uncoloured vertex and assigns a colour to it, whereas in the marking game each player just chooses an unmarked vertex and no colour is involved. The *greedy colouring game* is, in some sense, a game lying between these two. Here, each player chooses any uncoloured vertex, but the chosen vertex has to be coloured with the smallest positive integer not assigned to any of its neighbours. So the colours are positive integers and the chosen vertices are coloured greedily. Alice's goal is to minimize the largest used colour, while Bob's goal is to maximize it. The *game Grundy number* $\Gamma_g(G)$ of a graph G is the maximum colour used in playing the game when both players use optimal strategies.

When playing the greedy colouring game, the player does not choose the colour. However, the colour is a critical concern. It follows from the definition that $\Gamma_g(G) \leq \mathrm{col}_g(G)$, for every graph G. Therefore, $\Gamma_g(F) \leq 4$ for any forest F, and $\Gamma_g(G) \leq 8$ for any partial 2-tree G. Havet and Zhu [28] proved that $\Gamma_g(F) \leq 3$ for any forest F, and this bound is tight. For any partial 2-tree G, $\Gamma_g(G) \leq 7$. Since there are trees T for which $\chi_g(T) = 4$, there are graphs G for which $\Gamma_g(G)$ is strictly less than $\chi_g(G)$: indeed, Havet and Zhu also proved that, for any positive integer $k \geq 2$, there is a graph G for which $\Gamma_g(G) = k$ and $\chi_g(G) \geq 2^k - 1$. However, it is unknown whether $\Gamma_g(G)$ is always bounded by $\chi_g(G)$. It is also an open question as to whether $\chi_g(G)$ is bounded by a function of $\Gamma_g(G)$.

4. Playing on the edge-set

The edge-colouring version – equivalent to playing the original vertex version on line graphs – was investigated first by Cai and Zhu [15]. We adopt the commonly used 'prime' notation for the edge versions of parameters and denote by $\chi'_g(G)$ the smallest number of colours sufficient for Alice to win.

The particular case of (1) when applied to line graphs means that

$$\Delta \leq \chi'(G) \leq \chi'_g(G) \leq 2\Delta - 1,$$

since each edge can share a vertex with at most $2\Delta - 2$ other edges.

Vizing's classic theorem tells us that the upper bound for $\chi'(G)$ can be improved to $\Delta + 1$ for all graphs (see Chapter 5). Estimates of this strength are not known

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(and are not even valid) for the colouring game, although bounds of the form $\Delta + c$, for a constant c, have been established for several graph classes. One such result by Cai and Zhu [15] states that, if G is a d-degenerate graph, then

$$\mathrm{col}'_g(G) \leq \Delta(G) + 3d - 1.$$

In particular, this inequality yields the upper bounds $\Delta + 6k - 4$ for graphs of arboricity k, $\Delta + 3k - 1$ for partial k-trees, $\Delta + 14$ for planar graphs, $\Delta + 5$ for outerplanar graphs and $\Delta + 2$ for forests. It may also be the case that $\chi'_g(F) \leq \Delta(F) + 1$ holds for all forests F, and this has been proved except for $\Delta = 4$ (see Erdős *et al.* [22] and Andres [5]).

It is not true, however, that $\chi'_g \leq \Delta + c$ must hold for some suitably chosen constant c. Beveridge *et al.* [10] proved the existence of an infinite family of graphs G with any large maximum degree, for which $\chi'_g \geq 1.008\Delta(G)$. An important consequence of this result is that it is substantially harder to win the colouring game for some graphs than it is to list-colour them, since lists of size $\Delta + o(\Delta)$ suffice for the latter, as $\Delta \to \infty$. On the positive side, one can prove under some minimum-degree conditions that $\chi'_g \leq (2 - \varepsilon)\Delta$, for some $\varepsilon > 0$ (see [10] for details). It remains a conjecture that, for a suitably chosen $\varepsilon > 0$, the upper bound $\chi'_g(G) \leq (2 - \varepsilon)\Delta(G)$ is valid for all graphs G.

The *game Grundy index* $\Gamma'_g(G)$ of a graph G is the game Grundy number of its line graph. The *Grundy index* $\Gamma'(G)$ of G is the Grundy number of its line graph. Just as for the game-chromatic index, we have

$$\Delta \leq \Gamma'_g(G) \leq \Gamma'(G) \leq 2\Delta - 1.$$

Zhang and Zhu [57] proved that if T is a tree with $\Delta \geq 5$, then $\Gamma'_g(T) \leq \Delta + 1$;
if G is a partial 2-tree with $\Delta \geq 11$, then $\Gamma'_g(G) \leq \Delta + 4$;
and if G is an outerplanar graph with $\Delta \geq 14$, then $\Gamma'_g(G) \leq \Delta + 3$.

5. Oriented and directed graphs

An *oriented graph* is obtained from an undirected graph by assigning an orientation to each of its edges; there is thus at most one arc between any two vertices, whereas a *directed graph* may contain directed cycles of length 2. Below we consider two oriented kinds of colouring games. For an oriented or directed graph we use the notation $D = (V, A)$, where V is the vertex-set and A is the arc-set.

Oriented graphs

A colour assignment $\varphi : V \to \{1, 2, \ldots, k\}$ of an oriented graph $D = (V, A)$ is an *oriented colouring* if it is a proper vertex-colouring and all arcs between any two colour classes are in the same direction; that is, if $vw \in A$, then there does not exist $v'w' \in A$ with $\varphi(v') = \varphi(w)$ and $\varphi(w') = \varphi(v)$. Note that digraphs containing cycles of length 2 do not admit any such colouring.

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In accordance with this definition, in the oriented colouring game Alice and Bob alternately assign colours $\varphi(v)$ to vertices v of D under the following restrictions:

(i) If $vw \in A$, then $\varphi(w) \neq \varphi(v)$;
(ii) If $uw \in A$ and $wv \in A$, then $\varphi(u) \neq \varphi(v)$;
(iii) If $vw \in A$ and $v'w' \in A$, and if $\varphi(v) = \varphi(w')$, then $\varphi(v') \neq \varphi(w)$.

Clearly, (i) and (iii) together require that the partial colouring obtained at each stage should be an oriented colouring. The role of (ii) is to ensure that Bob cannot win the game in a trivial way.

The *oriented game-chromatic number* $\vec{\chi}_g(D)$ of D is the smallest number k for which Alice has a winning strategy when the game is played with k colours; this parameter was introduced by Nešetřil and Sopena [44]. Unlike χ_g for undirected graphs, the maximum value of $\vec{\chi}_g(T)$ for oriented trees is not known.

Motivated by the marking game we may consider, for any uncoloured vertex v, the set of coloured neighbours and the set of coloured vertices w at (directed) distance 2 from v (in either direction) for which the middle vertices of every length-2 path between w and v is uncoloured. Using the strategy that proves $\chi_g(T) \leq 4$ for trees, Alice can ensure that both of these sets have at most three elements. Combining this idea with the construction of tournaments satisfying a certain adjacency/orientation property, we can prove that $\vec{\chi}_g$ is bounded above (by at most 19) on the class of trees. Also, by using marking games, Kierstead and Tuza [37] proved the following theorem.

Theorem 5.1 *For each positive integer k, there is a constant $C(k)$ for which the oriented game-chromatic number of oriented partial k-trees is at most $C(k)$.*

Moreover, Kierstead and Trotter [36] proved the following theorem.

Theorem 5.2 *For any positive integer k, orientations of graphs containing no subdivision of K_k have bounded oriented game-chromatic number. In particular, the oriented game-chromatic number of oriented planar graphs is bounded by a constant.*

The problem of determining the exact value of $\vec{\chi}_g$ appears to be much more difficult than that of χ_g. To indicate the complications, we quote the following result of Nešetřil and Sopena [44]: every oriented path and cycle has $\vec{\chi}_g \leq 7$, and this bound is sharp since there exist oriented paths with $\vec{\chi}_g = 7$.

Directed graphs

The oriented colouring game cannot be played on digraphs that contain 2-cycles. For such digraphs, a natural game is given by the following rule: each vertex must be assigned a colour different from those of its coloured in-neighbours. This rule eliminates the danger originating from a monochromatic pair at distance 2, which arose in the oriented version above.

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This kind of colouring game was introduced by Andres [6]. It includes the undirected version as a particular case, by replacing each edge vw in an undirected graph by the two arcs vw and wv (a cycle of length 2), so we need not introduce any new terminology: on a digraph D the minimum number of colours sufficient for a winning strategy for Alice is called the *game-chromatic number* $\chi_g(D)$. Analogously, we introduce the *game-colouring number* $\mathrm{col}_g(D)$, based upon the directed version of the marking game. It is clear that, for all digraphs D and their underlying undirected graphs G_D, we have the inequalities

$$\chi_g(D) \le \chi_g(G_D) \qquad \text{and} \qquad \mathrm{col}_g(D) \le \mathrm{col}_g(G_D).$$

Here, strict inequality may hold when some arcs of D are not part of a symmetric pair. For example, as proved by Andres [6], Alice can win the marking game with three colours on any forest without 2-cycles – that is, $\chi_g(D) \le \mathrm{col}_g(D) \le 3$, whenever D is an oriented forest. Andres also observed that inequality (3) has its analogue for the directed game: if (D_1, D_2) is an arc decomposition of D, then

$$\mathrm{col}_g(D) \le \mathrm{col}_g(D_1) + \Delta^+(D_2),$$

where Δ^+ denotes the maximum out-degree.

6. Asymmetric games

In this variant, one or both players may colour (or mark) more than one vertex at a time – that is, in the game parametrized by a pair (a, b) of natural numbers, at each step Alice colours or marks a vertices and then Bob colours or marks b vertices.

The first systematic study of asymmetric colouring games was carried out by Kierstead [34], but its roots date back to the earlier paper of Faigle, Kern, Kierstead and Trotter [23]. The authors considered the case $(a, b) = (1, 3)$ on trees, and proved that Alice has a strategy in the marking game to keep the score below $c \log n$, for some constant c; this logarithmic upper bound is sharp, apart from the value of c. It was also shown in [23] how this upper bound can be extended for any fixed $b \ge 3$; these results were applied there to prove that $\mathrm{col}_g = O(\log n)$ in every class of graphs whose edge-sets decompose into a bounded number of forests.

Kierstead [34] considered (a, b)-game-colourings and marking of forests. Let $\chi_g(\mathcal{F}; a, b)$ and $\mathrm{col}_g(\mathcal{F}; a, b)$ be the supremum taken over all forests, needed for Alice to win in the colouring game and in the marking game. The characteristic ranges for all positive integers a and b can be summarized as follows; they are the same if one considers trees instead of forests.

Theorem 6.1 *Assume that a, b are positive integers.*
If $a < b$, then $\chi_g(\mathcal{F}; a, b) = \mathrm{col}_g(\mathcal{F}; a, b) = \infty$.
If $b \le a < 2b$, or $(a, b) = (2, 1)$, then $\chi_g(\mathcal{F}; a, b) = \mathrm{col}_g(\mathcal{F}; a, b) = b + 3$.
If $2b \le a < 3b$ and $b > 1$, then $\chi_g(\mathcal{F}; a, b) = b + 2$ and $\mathrm{col}_g(\mathcal{F}; a, b) = b + 3$.
If $3b \le a$, then $\chi_g(\mathcal{F}; a, b) = \mathrm{col}_g(\mathcal{F}; a, b) = b + 2$.

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Note that the range $2b \le a < 3b$ exhibits cases where $\chi_g < \mathrm{col}_g$.

For graphs in general, Kierstead and Yang [38] proved the following result.

Theorem 6.2 *If G has an orientation with maximum out-degree k, then* $\mathrm{col}_g(G; k, 1) \le 2k + 2$.

By letting $k = 1$, we see that the game-colouring number of any forest (as well as that of any graph with unicyclic components) is at most 4.

7. Relaxed games

For a positive integer d, Chou, Wang and Zhu [18] introduced the *d-relaxed game*. It is played by Alice and Bob with the same rules as the original game, except that the subgraphs induced by the monochromatic sets need not be independent but are allowed to have maximum degree at most d. Let $\chi_g^d(G)$ denote the minimum integer k for which Alice has a winning strategy on G when k colours are available.

Not surprisingly, the extended flexibility provided by the parameter d makes it easier for Alice to win. A typical example is the class of trees and forests for which $\chi_g^2 \le 2$, by a theorem of He, Wu and Zhu [29]. In fact, there is a smooth transition from the maximum game-chromatic number 4 for trees, which can be expressed in the following uniform way: in the range $0 \le d \le 2$ all forests have $\chi_g^d \le 4 - d$, and this upper bound is sharp for each d.

For partial k-trees G, Dunn and Kierstead [20] proved that $\chi_g^d(G) \le 6k + 1$, under the assumption $d \ge 4k - 1$; the particular case $k = 2$ implies that $\chi_g^7(G) \le 3$ for every outerplanar graph G. On the other hand, if $d \ge 2$ and G is outerplanar, then $\chi_g^d(G) \le 5$, by [29]. It is not known in general, however, whether the assumption $d \ge 7 - k$ implies that $\chi_g^d(G) \le k$ for all $2 \le k \le 7$ and all outerplanar graphs G. The sharpness of this upper bound on $\chi_g^d(G)$ (if valid) is not known either. It is known that the assumption $d = 4$ is not strong enough to ensure that $\chi_g^d(G) \le 2$.

8. Paintability

We next consider the list-colouring version of the colouring game; list-colouring and the choice number of a graph are discussed in Chapter 6. Let L be a list assignment for a graph G: without loss of generality, we may assume that the set of colours is $\bigcup_{v \in V(G)} L(v) = \{1, 2, \ldots, q\}$. For $i = 1, 2, \ldots, q$, let $L_i = \{v : i \in L(v)\}$. Then the sequence $(L_1, L_2, \ldots, L_q)$ is another way of specifying the list assignment. An L-colouring of G is equivalent to a sequence $(S_1, S_2, \ldots, S_q)$ of independent sets of vertices that form a partition of $V(G)$, and such that $S_i \subseteq L_i$ for all $i = 1, 2, \ldots, q$. This approach to list-colouring motivates the definition of the following game on a graph G, which was introduced by Schauz [50], [51].

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Let G be a graph and let $f : V(G) \to \mathbb{N}$ be a mapping, where $f(v)$ is the number of permissible colours for v. The *f-painting game on G* is played by two players: Lister and Painter.

1. In the ith round, Lister releases the set L'_i of uncoloured vertices that permit colour i (that is, the yet-uncoloured subset of L_i). Painter decides which of these vertices are coloured with colour i, and hence specifies S_i.

2. If, at the end of some round an uncoloured vertex runs out of permissible colours – that is, a vertex v appeared in $f(v)$ sets L'_i and remains uncoloured – then the game ends with Lister as the winner.

3. Otherwise, at the end of some round all vertices are coloured, the game ends, and Painter is the winner.

Note that once coloured, a vertex does not participate further in the game and can be considered as having been removed from the graph.

We say that G is *f-paintable* if Painter has a winning strategy in this game, and that G is *k-paintable* if G is f-paintable for the constant function with $f(v) = k$ for all v. The *paint number* $\chi_p(G)$ of G is the minimum k for which G is k-paintable.

The following result can be easily derived from the definition, and is convenient for inductive proofs.

A graph G is f-paintable if and only if $f(v) \geq 1$ for each $v \in V(G)$ and, for any subset U of $V(G)$, there is an independent set X of G contained in U for which $G - X$ is $(f - \delta_U)$-paintable, where $\delta_U(v) = 1$ if $v \in U$ and $\delta_U(v) = 0$ otherwise.

It follows from the definition that $\chi_p(G) \geq \chi_l(G)$ for any graph G, where χ_l is the list-chromatic number. There exist graphs G with $\chi_p(G) > \chi_l(G)$. For example, for any integer $k \geq 1$, the theta graph $\theta_{2,2,2k}$, which consists of three paths of lengths $2, 2, 2k$ joining two vertices, is 2-choosable but not 2-paintable (see [62]). It is not known whether, for a graph G, the difference $\chi_p(G) - \chi_l(G)$ can be arbitrarily large.

Upper bounds

Many currently known upper bounds for the choice numbers of classes of graphs are also upper bounds for their paint numbers. For example, the paint number of any planar graph is at most 5 (see Schauz [50]), and the paint number of planar graphs with girth at least 5 is at most 3 (see [50], [17]). Indeed, if an upper bound for $\chi_l(G)$ is proved by induction, then it is likely that the proof can be adapted to a strategy for Painter, establishing the same upper bound for $\chi_p(G)$.

One method to prove an upper bound for $\chi_l(G)$ is the *kernel method*:

if G has an orientation D for which each induced subdigraph has a kernel, and if $f(v) = d^+_D(v) + 1$ for every vertex v, then G is f-choosable (see Chapter 6).

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The proof of this result actually shows that, under the condition above, G is f-paintable: Painter simply selects a kernel S_i in the subdigraph induced by L'_i in each step. In particular, this implies that the line graph $L(G)$ of a bipartite graph G is $\Delta(G)$-paintable (see [50]).

Another method for proving an upper bound for $\chi_l(G)$ is to use the following result of Alon and Tarsi [2]:

if G has an orientation D for which the numbers of Eulerian subdigraphs with odd and even numbers of arcs are not equal, and if $f(v) = d_D^+(v) + 1$ for every vertex v, then G is f-choosable.

Schauz [52] proved that, under the same assumption, G is f-paintable, so any upper bound for $\chi_l(G)$ proved by using the Alon–Tarsi theorem is also an upper bound for $\chi_p(G)$. To mention one, Hladký, Král' and Schauz [30] gave a new proof of Brooks's theorem, based on the Alon–Tarsi theorem. This new proof yields an upper bound for the paint number of graphs: G is Δ-paintable if G is connected and is not a complete graph or an odd cycle.

Some upper bounds for $\chi_l(G)$ are proved by the probabilistic method. If only expectations are used in the proof, then the proof maybe de-randomized to give a winning strategy for Painter.

For complete bipartite graphs, Erdős, Rubin and Taylor [21] proved that $\chi_l(K_{n,n}) \leq \log_2 n + 2$. The same upper bound also holds for the paint number. Let G be any bipartite graph of order n. The following simple strategy for Painter to win the $(\log_2 n + 1)$-painting game on G was given by Zhu [62]. Let A and B be the two partite sets of G.

1. Initially, assign weight 1 to each vertex of G.

2. Each time Lister presents a set U of uncoloured vertices, Painter chooses the independent set $S = A \cap U$ or $S = B \cap U$, according to whichever has the larger total weight, and then doubles the weight of each vertex in $U \setminus S$. The vertices in S are then coloured.

Since the weight of $U \setminus S$ does not exceed that of S, the total weight of uncoloured vertices is never increased; in particular, each uncoloured vertex has weight at most $2n$. On the other hand, each time that a vertex is given a permissible colour but is not coloured, its weight doubles. If an uncoloured vertex has been given k permissible colours, then its weight is $2^k \leq n$, implying that $k \leq \log_2 n$. So if a vertex is given $\log_2 n + 1$ colours, then it is coloured. The above strategy is thus a winning strategy for Painter in the $(\log n + 1)$-painting game on G. Thus $\chi_p(G) \leq \log n + 1$, and in particular, $\chi_p(K_{n,n}) \leq \log n + 2$.

A similar argument shows that if G is a k-chromatic n-vertex graph, then $\chi_p(G) \leq k \log n + 1$.

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The online version of Ohba's conjecture

A graph G is *chromatic-choosable* if $\chi(G) = \chi_l(G)$, and is *chromatic-paintable* if $\chi_p(G) = \chi(G)$. Since $\chi_p(G) \geq \chi_l(G) \geq \chi(G)$, all chromatic-paintable graphs are chromatic-choosable, but the converse is not true. It was conjectured by Ohba [46], and proved by Noel, Reed and Wu [45], that graphs G with $n \leq 2\chi(G) + 1$ are chromatic-choosable.

Kim *et al.* [39] proved that the complete multipartite graph G, with k parts of size 2 and one part of size 3, has $\chi_p(G) = k + 2$. These are graphs G with $n = 2\chi(G) + 1$ and are not chromatic-paintable. Nevertheless, Huang *et al.* [31] proposed the online version of Ohba's conjecture, as follows.

Online version of Ohba's conjecture. Every graph G with $\chi(G) \geq \frac{1}{2}n$ is chromatic-paintable.

This conjecture is largely open. Caraher *et al.* [16] proved the following result.

Theorem 8.1 *If G is k-paintable and if $n \leq kt/(t-1)$, then the graph G' obtained from G by adding an independent set T of t vertices is $(k+1)$-paintable.*

We may assume that each vertex of T is adjacent to every vertex of G. Denote by R_i the set of uncoloured vertices after the ith move. For each vertex v, let $f_i(v)$ be the number of remaining permissible colours of v after the ith move, and let

$$\rho_i(v) = \max\{0, |R_i \cap N_G(v)| - f_i(v) + 1\}.$$

So $f_0(v) = k + 1$ for each vertex v, and $\rho_0(v) = \max\{0, d_G(v) - k\}$. Note that if $\rho_i(v) = 0$, then after the ith move the number of permissible colours of v exceeds the number of its uncoloured neighbours. In this case, the vertex v can be ignored throughout the rest of the game, since it can never run out of permissible colours.

We may assume that, for any i, the set of uncoloured vertices with i as a permissible colour is not an independent set, since otherwise Painter can simply colour all these vertices with colour i. The strategy for Painter to win the online $(G', k+1)$-list-colouring game is simple:

In the ith move, let L_i be the set of vertices with i as a permissible colour presented by Lister. Painter ignores the vertices of T and uses his winning strategy for the k-painting game on G to colour an independent set S_i of G (contained in L_i), with one exception: if $\rho_{i-1}(v) = 0$ for each vertex $v \in T \setminus L_i$, then Painter colours the vertices in $T \cap L_i$ with colour i.

Since the exceptional move is performed only once, and since each vertex of G has one more permissible colour than is required in the k-painting game, the vertices of G can never run out of permissible colours. Also, once the exceptional move is made, each vertex in T is either coloured or can be ignored. So to show that this is indeed a winning strategy for Painter, it suffices to show that before the exceptional move is made, no vertex in T can run out of permissible colours.

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At the ith round, if the exceptional move is not carried out then (by the description of the strategy), some vertex $v \in T$ is not contained in L_i and hence has $0 \le \rho_i(v) \le \rho_{i-1}(v) - 1$. Now suppose that a vertex $v \in T$ runs out of permissible colours before the exceptional move. Then there are moves $j_1, j_2, \ldots, j_{k+1}$ such that $v \in L_{j_s}$ for $s = 1, 2, \ldots, k+1$. For each such s, there is a vertex $w \in T \setminus \{v\}$ for which $0 \le \rho_{j_s}(w) \le \rho_{j_s-1}(w) - 1$. Therefore,

$$k + 1 \le \sum_{w \in T - \{v\}} \rho_0(w) = (t-1)(n-k).$$

This contradicts our assumption that $n \le kt/(t-1)$.

A consequence of Theorem 8.1 is that the online version of Ohba's conjecture holds for graphs with independence number 2. Kozik *et al.* [40] proved that the conjecture also holds for graphs with independence number 3.

The fractional paint number

A q-tuple colouring φ of G assigns to each vertex v a set $\varphi(v)$ of q colours so that adjacent vertices receive disjoint colour sets. The *fractional chromatic number* $\chi^*(G)$ of a graph G is the infimum of the ratios p/q for which there exists a q-tuple colouring that uses p colours in total.

A graph is called (a, b)-*choosable* if, for any list assignment L that assigns to each vertex a set of a permissible colours, there is a b-tuple colouring φ with $\varphi(v) \subseteq L(v)$ for each vertex v; in particular, '$(a, 1)$-choosable' means a-choosable. The *fractional choice number* $\chi_l{}^*(G)$ of G is the infimum of the ratios a/b for which G is (a, b)-choosable.

Given a graph G and two positive integers a, b, the *b-tuple a-painting game on G* is defined analogously to the f-painting game. In the ith round Lister reveals the information as to which vertices can be coloured with colour i, and Painter then decides which of these vertices should be so coloured. The difference is that each vertex needs to be assigned b colours. If, at any moment of the game, a vertex v has not been assigned b colours but runs out of permissible colours, then Lister wins the game. Otherwise, at some round, each vertex is assigned b colours and Painter wins the game.

The *fractional paint number* $\chi_p{}^*(G)$ of G is the infimum of the ratios a/b for which Painter has a winning strategy in the online $(G, (a, b))$-list-colouring game. It follows from the definition that, for any graph G,

$$\chi^*(G) \le \chi_l{}^*(G) \le \chi_p{}^*(G).$$

Alon, Tuza and Voigt [4] proved that the fractional choice number of a graph is always equal to its fractional chromatic number. Gutowski [25] extended this by proving the following result.

Theorem 8.2 *For any graph G, $\chi_p{}^*(G) = \chi_l{}^*(G)$.*

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Suppose that $\chi^*(G) = a/b$ and that φ is a b-tuple a-colouring of G. We shall prove that, for any $\varepsilon > 0$, there exist sufficiently large integers a', b' for which $a'/b' < a/b + \varepsilon$ and Painter has a winning strategy for the b'-tuple a-painting game on G. The strategy for Painter is simple. Suppose that in the ith round, Lister presents a set L_i, and that there are p sets among $L_1, L_2, \ldots, L_{i-1}$ that are identical to L_i, and let $p' \equiv p \pmod a$ with $1 \leq p' \leq a$. Then Painter's choice is $S_i = L_i \cap \varphi^{-1}(p')$. We may consider the permissible colour i as identified with the colour p' in the b-tuple a-colouring of G. A key observation is the following:

for any vertex v and for any two different colours p' and p'', the number of permissible colours of v that are identified with p' and the number of permissible colours of v that are identified with p'' may differ by at most 2^n.

Since the graph G is fixed, the number 2^n is constant. When a' and b' are sufficiently large, this difference may be ignored – that is, counted in the ε factor. So, roughly speaking, about b/a of the permissible colours of v are eventually assigned to v, and Painter wins the game.

This argument extends the identity $\chi^* = {\chi_l}^*$ to $\chi^* = {\chi_p}^*$. Nevertheless, ${\chi_l}^*$ and ${\chi_p}^*$ behave in a somewhat different way – namely, as proved in [4], for each graph G there exists a pair (a, b) of integers for which G is (a, b)-choosable with $b = a/\chi^*(G)$. Consequently, in the definition of ${\chi_l}^*$ we can take 'minimum' instead of 'infimum'. This is not valid for ${\chi_p}^*$, as we really need the infimum for some graphs, despite the corresponding proof for ${\chi_l}^*$ being constructive.

9. Achievement and avoidance games

Unlike the games already discussed, in achievement and avoidance games the goal of the two players is symmetrical. The players construct partial colourings together, playing on a given graph G within a given colour bound k: at most k colours may be used and monochromatic edges are forbidden. The players alternately colour one uncoloured vertex at a time, and the game is over when no further legal move is possible – either because the entire vertex-set is coloured, or because the neighbourhood of each uncoloured vertex contains all the k colours.

The *achievement game* is *won* by the player who makes the last legal move, and the *avoidance game* is *lost* by the player who makes the last legal move. These colouring games were introduced by Harary and Tuza [27], motivated by the algorithmic decision problem called Precolouring Extension (see [11] for the latter). An instance of the algorithmic problem PrExt is a graph $G = (V, E)$, a colour bound k, and a precoloured set $W \subset V$ – that is, a partial colouring $\varphi_W : W \to \mathbb{N}$ is given, which is supposed to be a proper vertex-colouring of the subgraph induced by W. The question to decide is whether G admits a proper k-colouring in which each precoloured vertex $v \in W$ has its pre-assigned colour $\varphi_W(v)$.

The most interesting particular cases of these games correspond to small values of k. If $k = 1$, then the game terminates with a maximal independent set, and

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the parity of its cardinality determines who is the winner. Schaefer [49] developed hardness results and studied various games concerning Boolean satisfiability, and also gave three examples of natural games, one of which (Node Kayles) is exactly our achievement game with $k = 1$. He proved that it is PSPACE-complete to decide whether the first player has a winning strategy. The same complexity was proved by Bodlaender [12] for $k = 2$ – that is, where the game terminates with a non-extendable bipartite subgraph of G.

Unlike the games discussed in previous sections, the case $k = 1$ already raises enormous difficulties, even for some very simple types of graph. Recall that the players in this game sequentially construct a maximal independent set. On a path of given length the game is equivalent to a game known as *Dawson's chess*, created by T. R. Dawson in 1935. Dawson's chess is played on a $3 \times n$ chessboard, with n white pawns in the first row and n black pawns in the third row. Pawns move in the standard way, except that capture is mandatory whenever possible.

The game on paths, with $k = 1$, is also known as Node Kayles. In the terminology of game theory, achievement corresponds to 'normal play' (last player wins) whereas avoidance corresponds to 'misère play' (last player loses). (Dawson proposed the latter.) The former game was solved after two decades by Guy and Smith (1956), but the latter game has remained open, even after three-quarters of a century, in spite of considerable efforts in game theory. The game with normal play can be analyzed by computing the Grundy function; it turns out that the second player has a winning strategy if and only if $n \equiv 4, 8, 20, 24$ or $28 \pmod{34}$ or $n = 14$ or $n = 34$ – that is, the score is ultimately periodic modulo 34.

As a rather vague intuition concerning reasons why the achievement game is easier than the avoidance one, we mention a simple strategy for the former, based on symmetry, that can be applied under certain conditions. Suppose that G has a cut-vertex v with the property that $G-v$ is the union of two disjoint graphs G_1 and G_2 for which $G-G_1 \cong G-G_2$ and some isomorphism between these two subgraphs maps v onto itself. Then the first player can win the achievement game with any given colour bound k as follows: colour v first, fix an automorphism $f : G \to G$ with $f(v) = v$, $f(G_1) = G_2$ and $f(G_2) = G_1$, and then copy the moves of the second player: thus, if the second player assigns colour c to a vertex $w \in V(G_i)$, answer by assigning c to $f(w) \in V(G_{3-i})$. An analogous winning strategy works for the second player if G is the disjoint union of two isomorphic graphs, but there seems to be no similarly trivial strategy for the avoidance game.

For a deeper discussion about why misère play is much harder than normal play, and for substantial partial results concerning the former, see [48] and some of the papers cited there.

10. The acyclic orientation game

Finally, we discuss a game that involves no colours. Its original motivation was to enable us to check efficiently whether a 'hidden' colouring of a graph is proper or

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not. It is related to the fact that every proper colouring $\varphi : V \to \mathbb{N}$ defines an orientation without directed cycles in a natural way, from smaller colours to larger ones. Should a monochromatic edge occur, its orientation remains undetermined.

The first version of this game was introduced by Manber and Tompa [42]; here we consider the following variation, presented by Aigner, Triesch and Tuza [1]. The game is played on a simple undirected graph G by two players, Algy and Strategist. At each step, Algy selects an edge $vw \in E$ and Strategist orients it either from w to v or from v to w, as he chooses, but in such a way that no directed cycle occurs. The goal is to identify an acyclic orientation of G unambiguously, and the game is over when the actual partial orientation has a unique extension to an acyclic orientation of G. Algy's goal is to minimize the number of steps, while Strategist aims at maximizing it. We denote by $c(G)$ the number of steps when both players play optimally.

When played on a complete graph, the game ends with a linear ordering of the vertices, and is thus equivalent to comparison sorting, where $c(K_n) = (1 + o(1))n \log_2 n$. On the other hand, when played on a bipartite graph, Strategist may even promise at the beginning to orient all but one edge from the first vertex-class to the second; this leaves the possibility of orienting the last edge in either way, yielding $c(G) = |E|$. More generally, the same holds if G is the cover graph of a partially ordered set. (A graph G is a *cover graph* if it admits an orientation $\vec{G}$ for which there exists a partially ordered set $P = (V, <)$ over the vertex-set V of G such that vw is an arc in $\vec{G}$ if and only if v covers w in P – that is, $v > w$, and there is no $u \in V$ with $v > u > w$.)

As proved in [1], the following general bounds are valid for all graphs G of order n and independence number α :

$$n \log_2 n\alpha - n \log_2 e + 1 + \tfrac{1}{2} \log_2 n < c(G) < \alpha n \, (\log_2(n/\alpha) + 1). \qquad (4)$$

It follows in particular that, for any graph with n vertices and $\binom{n}{2} - o(n^2)$ edges, $c(G) = o(n^2)$ as $n \to \infty$. On the other hand, for each $\varepsilon < \frac{1}{2}$ there exists $\delta > 0$ such that, for every sufficiently large n, there exists a graph G with at least $(\frac{1}{2} - \varepsilon)n^2$ edges and with $c(G) \geq \delta n^2$.

A less explicit inequality is the information-theoretic lower bound $c(G) \geq \log_2 a(G)$, where $a(G)$ denotes the number of acyclic orientations of G. Interestingly, $a(G)$ can be expressed in terms of the chromatic polynomial $P(G, \lambda)$ of G, as $a(G) = |P(G, -1)|$, by a theorem of Stanley [54].

The above observation on bipartite graphs yields $c(K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}) = \lfloor \frac{1}{4} n^2 \rfloor$. Pikhurko [47] proved that no graph of order n can require substantially more steps – that is, $c(G) \leq \frac{1}{4} n^2 + o(n^2)$. But from below, it is not known whether or not $c(G) - \frac{1}{4} n^2$ can be arbitrarily large; so far, no example has been found where the difference exceeds 1.

The typical value of $c(G)$, however, grows much more slowly as a function of n : Alon and Tuza [3] proved that, for any constant $p \in (0, 1)$, the random graph $G_{n,p}$ of order n with edge probability p satisfies $c(G_{n,p}) = \Theta(n \log n)$, with probability $1 - o(1)$ as $n \to \infty$. A similar slowly growing (and slightly larger) upper bound holds for an arbitrary edge probability function $p = p(n)$ – namely, $c(G_{n,p}) = O(n \log^3 n)$

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holds almost surely. The exponent 3 can be improved; it remains open as to whether $O(n \log n)$ is a universal upper bound for all probability functions $p(n)$.

We say that $G = (V, E)$ is an *exhaustive graph* if $c(G) = |E|$ – that is, if Strategist can force Algy to probe all edges of G. Unlike the value of $c(G)$, it is known (see [1]) that for $n \geq 6$, the number of edges in an exhaustive graph of order n is at most $\lfloor \frac{1}{4}n^2 \rfloor$, and if $n \geq 7$ then $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ is the unique extremal graph.

The cover graphs mentioned earlier, which are necessarily triangle-free, show that exhaustiveness is not a local property. However, it is proved in [1] that the exclusion of induced cycles of length larger than 3 has a different effect: a chordal graph is exhaustive if and only if it contains no K_4 or $P_4 + K_1$ as an induced subgraph.

As regards time complexity, Algy can design in polynomial time a sequence of sufficient questions fewer than the upper bound in the inequality 4 (see [1]). On the other hand, Pikhurko [47] proved that $c(G)$ is hard to compute, and is even hard to approximate within arbitrary precision: $c(G)$ admits no polynomial-time approximation scheme unless P = NP. As an upper bound for the approximation ratio achievable in polynomial time, only a $O(n/\log n)$-approximation is known (see [47]), and hence a wide gap remains to be narrowed.

As with several other games, it may turn out to be PSPACE-complete to determine the winner for a generic input graph G and value k which is part of the input, but this seems not to have been proved so far; on the other hand, it is known that $a(G)$ is #P-hard to compute (see [41]). A further open problem is to determine the computational complexity of recognizing exhaustive graphs.

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