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DIFFUSION LIMITED AGGREGATION IN THE LAYERS MODEL

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Abstract

In the classical model of diffusion limited aggregation (DLA), introduced by Witten and Sander, the process begins with a single-particle cluster placed at the origin of a space. Then, one at a time, particles make a random walk from infinity until they halt by colliding with the existing cluster. We consider an analogous version of this process on large but finite graphs with a designated source and sink vertex. Initially the cluster of halted particles contains a single particle at the sink vertex. Starting one at a time from the source, each particle makes a random walk in the direction of the sink vertex. The particle halts at the last unoccupied vertex before the walk enters the cluster for the first time, thus increasing the size of the cluster. This continues until the source vertex becomes occupied, at which point the process ends. We study this DLA process on several classes of layered graphs, including Cayley trees of branching factor at least two with a sink vertex attached to the leaves. We determine the finish time of the process for the given classes of graphs and show that the subcomponent of the final cluster linking source to sink is essentially a unique path.

Keywords: DLA; Witten–Sander model

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1. Introduction

1.1. Diffusion limited aggregation

In the classical model of diffusion limited aggregation (DLA), introduced by Witten and Sander [13, 14], the process begins with a single-particle cluster placed at the origin of a space, and then, one at a time, particles make a random walk 'from infinity' until they collide with and stick to the existing cluster. The process is particularly natural in Euclidean space with particles making Brownian motion, or on the d-dimensional lattice $\mathbb{Z}^d$. Simulations of DLA in two dimensions show tree-like figures with long branches. For $\mathbb{Z}^d$, Kesten [8] proved that when the cluster size is N, the length of these arms is almost surely upper bounded by $CN^{2/3}$ when $d = 2$, and by $C_d N^{2/d}$ when $d \geq 3$.

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A distinct but related process, internal diffusion limited aggregation (IDLA), was introduced by Diaconis and Fulton [3], as a protocol for recursively building a random aggregate of particles. In IDLA particles are added to the source vertex of an infinite graph, and make a random walk (over occupied vertices) until they visit an unoccupied vertex at which point they halt. Thus, the first particle occupies the source, and subsequent particles stick to the outside of the component rooted at the source. Although the DLA and IDLA processes differ, the point in common is that they both describe the evolution of a unique cluster by adhesion to the cluster boundary. As with DLA, the focus in IDLA has been on the limiting shape of the component formed by the occupied vertices. The formative work by Lawler *et al.* [9], proved that, on d-dimensional lattices the limiting shape approaches a Euclidean ball; a result subsequently refined in [1, 7, 10], amongst others.

DLA has been proposed as a model of physical processes in systems as diverse as coagulated aerosols [13] and urban growth [2], a common factor being the dendritic shape of the cluster so obtained (see [5] for illustrative examples). However, the main and an original motivation for the DLA process was as a model of dendritic growth in dielectric breakdown and lightning formation. Niemeyer *et al.* [12], introduced a dielectric breakdown model which considers DLA in the presence of an electric field which biasses the particles to move in a given direction (e.g. downwards). In established models of lightning formation, paths of negatively charged particles (leaders) descend downwards from the base of the cloud layer, and paths of positively charged particles percolate upwards from the ground towards them, inducing a lightning strike on meeting. The DLA process was seen as a first approximation of this process, which is itself finite in extent with a source at the top (cloud layer) and a sink (the ground) at the bottom.

We consider a version of the DLA process on large but finite graphs with a designated source and sink vertex. DLA on finite graphs was studied previously for complete binary trees by Hastings and Halsey [6], and for the Boolean lattice (hypercube) by Frieze and Pegden [4]. The current paper continues this analysis of DLA on finite layered graphs, of which the binary tree and hypercube are typical examples.

1.2. The layers model

Let G be a finite graph $G=(V,E)$, whose vertices can be partitioned into sets $S_0, S_1, \ldots, S_k, S_{k+1}$ to form a *layered structure* in which all edges are between layer S_i and S_{i+1} $(i=0,1,\ldots,k)$. The sets $S_0,\ S_{k+1}$ are of size one with $S_0=\{v\}$ and $S_{k+1}=\{z\}$. The vertex v is the *source vertex* and the vertex z is the *sink vertex*. In certain cases the source and sink may be attached as extra (*artificial*) vertices to an existing graph G to complete the layered structure. This induces a natural orientation of the edges downwards from source to sink.

Examples of layered graphs with a symmetric structure include the following: the Boolean lattice (hypercube) $\mathcal{B}=\{0,1\}^n$, with $v=\mathbf{1}$, $z=\mathbf{0}$, see [4]; two-dimensional square grids with source the top left=hand corner and sink the bottom right-hand corner, thus inducing a diagonal orientation; finite Cayley trees with source the root and an artificial sink; and layered multipartite graphs with attached source and sink. A layers model on $\mathbb{Z}^2$ with diagonal downwards orientation was studied by Martineau [11]. The paper gives bounds on the height and width at a given step of the cluster rooted at the origin and within the positive quadrant.

Graphs in the layers model have a top (the source) and a bottom (the sink). Particles are constrained to move downwards. As such they can be seen as simple models of particles percolating downwards through some porous medium.

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1.3. DLA in the layers model

Informally, at a given step s, a particle labelled s starts from the source and percolates downwards through the graph, halting when it attempts to move to a vertex which is already occupied by a previous particle.

We next give a formal description of the DLA process, and the occupied component. Initially at step $t = 0$, only the sink vertex z is occupied, and the occupied component is $C_0 = \{z\}$. At a given step t, let ρ_t be a random walk on the underlying graph starting at the source v and moving forwards level by level to the sink z. A particle placed on the source vertex (assumed unoccupied) follows the walk ρ_t until it encounters an occupied vertex. Let the trajectory of the random walk from v to z be ρ_t be $v = x_0x_1x_2, \ldots, x_kx_{k+1} = z$, where v and z are the start and end vertices of the walk, x_i is the ith vertex on the walk, and $x_0 = v$.

Let x_i ($i \geq 1$) be the first occupied vertex encountered by particle t on the walk ρ_t. The particle halts at position x_{i-1} and permanently occupies that vertex. The component C_t is formed by adding the vertex x_{i-1} and directed edge $x_{i-1}x_i$ to the component C_{t-1} thus extending the directed tree formed by the occupied vertices and rooted at the sink z. As the sink is occupied from the start, any particle which reaches level k automatically halts there. The process ends at a step t_f, *the finish time*, when the source vertex v first becomes occupied by a halted particle. Thus, t_f is the final number of particles occupying the graph (not including the sink). Thus, the term *step s* refers to the introduction of the particle s to the network and encompasses all actions performed by the particle s from the source until it halts, the next step being the addition of particle $s + 1$. We do not count the individual transitions of the random walk undertaken by particle s.

1.4. The models of this paper

Let $N_i = |S_i|$ be the size of the set S_i, the ith layer (or level) of the graph G. We regard all edges as directed from the source towards the sink. We consider two models.

- The *Cayley tree* $G(d, k)$ with branching factor d and height k. For convenience we take the size of the last level to be $d^k = n$. Here $d \geq 2$ is fixed or tends to infinity with n sufficiently slowly, so that $k = \log n / \log d$ also tends to infinity with n. The source v is the root vertex at level zero of the tree and the sink z is an artificial vertex connected to all vertices in the final layer S_k of the tree. Excluding the sink, $G(d, k)$ is a $(d^{k+1} - 1)/(d - 1)$-vertex graph.

- The *multipartite layers model.* The sets S_0, S_{k+1} have size one, the layers S_i have size N_i. For $i = 0, \ldots, k$ there is a complete bipartite graph between S_i and S_{i+1}. Here $k \geq 1$ is fixed or tends to infinity with n.
  The *equal (multipartite) layers model* is a graph $M_E(n, k)$ in which the sets of S_i ($i = 1, \ldots, k$) are the same size $N_i = n/k$, where $N_i \to \infty$. Excluding the source and sink, the equal layers model G is an n-vertex graph. The parameter k can either be a fixed integer $k \geq 1$, or a function of n. In the extremes the graph has one layer, S_1, of n vertices ($k = 1$), or is a path ($k = n$).
  The *growing layers model* is a graph $M_G(d, k)$ in which the sets S_i grow geometrically in size with parameter d. As $|S_0| = 1$, then $|S_i| = N_i = d^i$, and we take $d^k = n$. Here $d, k \to \infty$ with n. Excluding the sink, the growing layers model G is a $(d^{k+1} - 1)/(d - 1)$-vertex graph.

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Analysis of the DLA process is in terms of n, which is, up to a constant multiple, the number of vertices in the graph. The model is probabilistic and corrections arising from the exceptional events are estimated as a function of n, even if this is not always made explicit. The parameter k determines the number of levels in the graph, and d the growth rate (if any) of the levels. The value of k or d can be constant in some models or it can tend to infinity as a function of n within some bounds. As $d^k = n$ in the growing layers model, they are related by $k = \log n/\log d$.

1.5. Notation

We say a sequence of events $\mathcal{E}_n$, $n \geq 0$ occurs with high probability (w.h.p.) if $\mathrm{P}(\mathcal{E}_n) = 1 - \varepsilon_n$ where $\varepsilon_n = o_n(1)$ for some function $o_n(1) \to 0$ with n, and thus $\lim_{n\to\infty} \mathrm{P}(\mathcal{E}_n) = 1$. Typically we write $o_n(1) = o(1)$ and similarly for other asymptotic notation such as $O(\cdot)$, $\Theta(\cdot)$, $\Omega(\cdot)$. We use $A_n \sim B_n$ to denote $A_n = (1 + o(1))B_n$ and, thus, $\lim_{n\to\infty} A_n/B_n = 1$. We use $\omega = \omega_n$ in two ways; either to denote any quantity ω_n which tends to infinity with n but suitably slowly as required in the given proof context, or as a fixed divergent quantity whose value is stated explicitly. The expression $f(n) \ll g(n)$ indicates $f(n)/g(n) = o_n(1)$. The notation $f \to \infty$ indicates that $f = f(n)$ grows unboundedly with increasing n. All results claimed are for sufficiently large n. The notation $u = x_0x_1 \cdots x_j = w$ indicates a path from u to w whose vertices have been labelled x_i, $i = 0, \ldots, j$, in the path order.

1.6. Results for the multipartite layers model

An arborescence is a rooted tree in which all edges are directed towards the root vertex. At any step t, the occupied component C_t is an arborescence, with edges directed downwards towards the sink z. At the finish time t_f the source becomes occupied, and the component C_{t_f} contains a directed *connecting path* from the source v to the sink z, all of whose vertices contain halted particles. In either layers model there is a value T_f, depending on the model under consideration, which serves to approximate the finish time t_f.

Theorem 1. *The following results hold w.h.p. in the multipartite layers model.*

(1) *Equal layers model, $M_E(n, k)$. Let $2 \leq k \leq \sqrt{\log n}/\log\log n$, and for $i = 1, \ldots, k$ let $N_i = n/k$. Let*

$$T_f = \left[(k+1)!(n/k)^k\right]^{1/(k+1)} = \gamma_k\, n^{k/(k+1)},$$

where $\gamma_k > 0$ constant, and where $\gamma_k \to 1/e$ as $k \to \infty$.
If k is constant then the finish time t_f satisfies $T_f/\omega' \leq t_f \leq \omega' T_f$ with probability $1 - O(1/\omega')$; where ω' tends to infinity arbitrarily slowly with n. Whereas if $k \to \infty$ then w.h.p. $t_f = \Theta(T_f)$.

(2) *Growing layers model, $M_G(d, k)$. For $i = 0, \ldots, k$, let $N_i = d^i$, where $d^k = n$ and, thus, $k = \log n/\log d$. Let*

$$T_f = \sqrt{k}\, d^{k+3/2-\sqrt{2k+2}}.$$

(a) *Provided $d \to \infty$ and $k \to \infty$, but $k \leq ((\log d)/(\log\log d))^2$, the finish time t_f satisfies $d^{-O(1/\sqrt{k})}\, T_f \leq t_f \leq d^{\,O(1/\sqrt{k})}\, T_f$.*

(b) *At t_f, the levels $i = 1, \ldots, k - \lceil\sqrt{2k+2} - 1\rceil$ contain a single occupied vertex, the vertex u_i of the connecting path.*

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The $d^{\pm O(1/\sqrt{k})}$ terms in the bounds on the finish time of the growing layers model come from possible rounding errors arising in the proof. Given the bounds on k, this error diverges, albeit slowly.

On deletion of the sink vertex, the digraph $D_t = C_t \backslash \{z\}$ consists of a forest of smaller arborescences whose components are rooted at the occupied vertices of level k. At the end of the process, let $v = u_0 u_1 \cdots u_k u_{k+1} = z$ be the connecting path from source $v = u_0$ to sink $z = u_{k+1}$. We prove that w.h.p. the arborescence rooted at vertex u_k of the connecting path is precisely the path $v = u_0 u_1 \cdots u_k$. Thus, this path grew back to the source without gaining any off-path branches due to other particles colliding with it.

Theorem 2. *Let $v = u_0 u_1 \cdots u_k z$ be the path connecting source v and sink z at t_f, and let $D_{t_f} = C_{t_f} \backslash \{z\}$ be the occupied component with the sink deleted.*

In either multipartite model, w.h.p., the vertices $\{u_1, \ldots, u_k\}$ have in-degree one in C_{t_f}. Thus the directed tree rooted at u_k in D_{t_f} is exactly this path.

1.7. Results for Cayley trees

For the Cayley tree $G = G(d, k)$, the connecting path is by definition the unique path from source to sink. We give values for the finish time both for d finite and for $d \to \infty$. By an indicative argument Hastings and Halsey [6] derived an estimate of $\sqrt{2k}\, 2^{k-\sqrt{2k}}$ for the expected finish time of DLA on the binary tree $G(2, k)$ of height k. We confirm their estimate is of the correct order of magnitude, and give results for $G(d, k)$ for any constant d.

Theorem 3. *With high probability the finish time t_f of DLA on a Cayley tree of branching factor d and height k is as follows.*

(1) Let $T_f = \sqrt{k}\, d^{k+3/2-\sqrt{2k+2}}$. Provided $d \to \infty$ and $k \to \infty$, but $k = o(d)$, the finish time t_f satisfies $d^{-O(1/\sqrt{k})}\, T_f \le t_f \le d^{\,O(1/\sqrt{k})}\, T_f$.

(2) Let $T_f = \sqrt{k}\, d^{k-\sqrt{2k}}$. If $d \ge 2$ is constant, then $t_f = \Theta(T_f)$.

1.8. Comparison of occupancy

How to compare the results of DLA on different models with each other? One possibility is to define the *packing ratio* $\rho = N/|V|$ of a process as the number of particles N at the finish divided by the number of vertices in the graph. Ignoring constants and lower order terms, $\rho = n^{-1/(k+1)}$ for the equal layers model and $\rho = n^{-\sqrt{2/k}}$ for the growing layers model.

How can the packing ratio be used to compare results for DLA on infinite graphs such as $\mathbb{Z}^d$? The sink in $\mathbb{Z}^d$ is the origin $\mathbf{0}$, as this is where the initial particle is located. At any fixed moment the longest arm of the figure formed by the occupied vertices defines a path back to the edge of the current bounding figure; which we take as the source (of particles entering from outside).

For the two-dimensional grid it is a result of Kesten [8], that, when N particles are added, the longest arm of the DLA figure is upper bounded by $C_2 N^{2/3}$, for some constant C_2. A circle of radius $N^{2/3}$ contains order $n = N^{4/3}$ vertices in $\mathbb{Z}^2$, so $N = \Theta(n^{3/4})$. Ignoring constants, this gives $\rho \le n^{-1/4}$ for $\mathbb{Z}^2$, at any point in the (infinite) process. For $d \ge 3$, the longest arm length in $\mathbb{Z}^d$ is $C_d N^{2/d}$. A figure of radius $R = N^{2/d}$ has order $n = R^d = N^2$ vertices, so $\rho = n^{-1/2}$.

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1.9. Road map of proofs

The following is an outline description and we often suppress minor details and qualifiers such as w.h.p.

The first step is to derive a solution to the recurrence (1)–(2) for the expected occupancy of the levels in the multipartite layers model. Under certain assumptions, the solution is asymptotic to (3), and the actual occupancy is concentrated around this value. This is particularly true in the equal layers model where the levels fill up in a more or less regular manner as given by (3), allowing us to estimate the finish time.

Things differ somewhat for the growing layers model. Although the higher levels fill in a regular manner, as given by (3), eventually we reach a level (with a well defined index $k - j^*$) where, in expectation, only a single occupancy occurs. The expected waiting time for further occupancy of this level is much longer than the expected waiting time for a unique path of halted particles to grow backwards to the origin, thus terminating the process. This is the content of Theorem 1.(2).(b). On the other hand Theorem 2 (which requires a separate proof) says something stronger. Namely that the connecting path grows back from level k to the source as a unique path, and without gaining any off-path neighbours due to other particles colliding with it.

Because of the similarity in which the layers grow, the results for the growing layers model tell us enough about the likely behaviour of the Cayley tree model to allow us to construct proofs for the finish time. The results for Cayley trees of branching factor d at least 2, are given in Theorem 3.

2. Bounds on occupancy in the layers model

The term step t (or time t) refers to the entire DLA process undertaken by particle t from introduction at the source until it halts and permanently occupies a vertex. The main task is to estimate the number of particles halted in each level (or layer) at the end of step t; that is, after the DLA process of tth particle has been completed.

Recall that N_i is the size of layer i for $i = 0, 1, \ldots, k$, where the value of N_i depends on the model in question. For $t \geq 0$, let $L_i(t)$ be the number of particles halted in level i at the end of step t. Thus $L_i(0) = 0$ for all $i \leq k$, and $t = \sum_{i=0}^{k} L_i(t)$. We refer to L_i as the occupancy of level i. Note that $L_0(t) = 0$ for $t < t_f$, $L_k(t) \leq t$, and generally $L_i(t) \leq \min(t, N_i)$.

2.1. General formulation of layers occupancy

Let $\mathcal{H}(t) = (L_0(t), L_1(t), \ldots, L_k(t))$ be the occupancy vector of the DLA process at step t. Then,

$$\mathrm{E}(L_i(t+1) \mid \mathcal{H}(t)) = L_i(t) + \frac{L_{i+1}(t)}{N_{i+1}} \prod_{j=0}^{i} \left(1 - \frac{L_j(t)}{N_j}\right), \quad i < k, \tag{1}$$

$$\mathrm{E}(L_k(t+1) \mid \mathcal{H}(t)) = L_k(t) + \prod_{j=0}^{k} \left(1 - \frac{L_j(t)}{N_j}\right). \tag{2}$$

Note that (2) follows from (1) on defining $L_{k+1}(t)/N_{k+1} = 1$ for all t, and that if $L_0(t) = 1$, the above recurrences give $L_i(t+1) = L_i(t)$.

The next proposition gives the solution to these recurrences under suitable conditions.

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Proposition 1. *For $t \geq 0$ let $\mu_k(t) = t$, and for $1 \leq j \leq k-1$, let*

$$\mu_{k-j}(t) = \frac{1}{N_k N_{k-1} \cdots N_{k-j+1}} \frac{t^{j+1}}{(j+1)!}. \tag{3}$$

For $j < k$ in the equal layers model, and for $j \leq \sqrt{2k+2} - 2$ in the growing layers model there are deterministic values $T_k < \cdots < T_{k-\ell} < \cdots < T_{k-j}$, such that w.h.p. for all $\ell \leq j$ and $T_{k-\ell} \leq t \leq t_f$ we have $L_{k-\ell}(t) \sim \mu_{k-\ell}(t)$, whereas for $\ell > j$, and $0 \leq t \leq T_{k-j}$, $L_{k-\ell}(t) = 0$.

This proposition summarizes a gap property that when level $k-j$ starts to fill enough for the occupancy to become concentrated, the levels with lower index $i = 0, 1, \ldots, k-(j+1)$ still remain empty. Most of the work is to prove that T_{k-j} can be chosen to be $t_{k-j}(\omega)$ as given by (6) and (7). The proof of Proposition 1 is inductive backwards from level k. The precise growth of the levels, and values of j which make it work are to be determined. The first steps are common to the equal and growing layers models, and we include them together in this section. For the equal layers model we complete the proof of Proposition 1 in Sections 3.1 and 3.2. The analogous proof for the growing layers model is given in Section 4.

2.2. Upper bound on occupancy at step *t*

The underlying random walk ρ_t from source v to sink z at step t defines a path given by $v = u_0 u_1 \cdots u_k u_{k+1} = z$, where u_i is a random vertex in level i. Particle t follows this walk until halting at a vertex u_i, where u_{i+1} is the first occupied vertex encountered by ρ_t.

Let $B_i(t)$ denote the occupied (blocked) vertices in level i in the DLA process at the end of step t. We define an *upper-blocked process* which we use to upper bound $L_i(t) = |B_i(t)|$. This new process gives rise to sets $\widehat{B}_i(t) \supseteq B_i(t)$ and random variables $\widehat{L}_i(t) = |\widehat{B}_i(t)| \geq L_i(t)$. For *every* vertex u_j, $0 \leq j \leq k$, on the walk ρ_{t+1}, if u_{j+1} is occupied ($u_{j+1} \in \widehat{B}_{j+1}(t)$) add a vertex to $\widehat{B}_j(t)$ as follows. If $u_j \notin \widehat{B}_j(t)$ add u_j to $\widehat{B}_j(t+1)$. If $u_j \in \widehat{B}_j(t)$ add some other $u'_j \in S_j \backslash \widehat{B}_j(t)$ to $\widehat{B}_j(t+1)$. If the layer S_j becomes full we continue with the layer S_{j-1}. As we prove, this contingency is unlikely to occur, as w.h.p. the first layer to become full is the source.

If particle $t+1$ halts at vertex u_i in the DLA process, then either u_i is added to both $B_i(t+1)$ and $\widehat{B}_i(t+1)$, or u_i is already a member of $\widehat{B}_i(t)$. In either case $B_i(t) \subseteq \widehat{B}_i(t)$ for all i and $t \geq 0$. It follows that $L_i(t) \leq \widehat{L}_i(t) \leq t$, as ρ_{t+1} can add at most one vertex to $\widehat{B}_i(t)$. Moreover, $\widehat{L}_k(t) = t$ deterministically (provided $t \leq N_k$). As a consequence the finish time $t_f(UB)$ of the upper-blocked process satisfies $t_f(UB) \leq t_f(DLA)$.

Let $\widehat{\mathcal{H}}(t) = (\widehat{L}_0(t), \widehat{L}_1(t), \ldots, \widehat{L}_k(t))$ be the occupancy vector of the upper-blocked process at step t. For $0 \leq i \leq k$, the expectation $\mathrm{E}\,\widehat{L}_i(t)$ satisfies the recurrence

$$\mathrm{E}\big(\widehat{L}_i(t+1) \mid \widehat{H}(t)\big) = \widehat{L}_i(t) + \frac{\widehat{L}_{i+1}(t)}{N_{i+1}} \mathbb{1}_{\{\mathcal{E}(t)\}}, \tag{4}$$

where $\mathcal{E}(t)$ is the event that $\widehat{L}_i(t) < N_i$ and that $\widehat{L}_{i+1}(t) < N_{i+1}$ for $i < k$. Setting $\mathbb{1}_{\{\mathcal{E}(t)\}} = 1$ always gives an upper bound on $\mathrm{E}\,\widehat{L}_i(t)$, and as we only propose to analyse the process as long as no level i ($1 \leq i \leq k$) is full, we assume $\mathbb{1}_{\{\mathcal{E}(t)\}} = 1$ forthwith. Equation (4) follows because the upper-blocked process increases the size of $\widehat{B}_j(t)$ (if possible) whenever the walk ρ_t contains a vertex of $\widehat{B}_{j+1}(t)$, this being true at all levels $j = 0, \ldots, k$.

The evolution of $\widehat{\mathcal{H}}(t) = (\widehat{L}_0(t), \widehat{L}_1(t), \ldots, \widehat{L}_k(t))$ is Markovian, and for $t \leq t_f$ we henceforth assume for $i \geq 1$ that $\widehat{L}_i(t) < N_i$ in our calculations. If so, referring to (1) and (2), we have

$$\mathrm{E}(L_i(t+1) \mid \mathcal{H}(t)) \leq L_i(t) + \frac{L_{i+1}(t)}{N_{i+1}} \leq \widehat{L}_i(t) + \frac{\widehat{L}_{i+1}(t)}{N_{i+1}} = \mathrm{E}\big(\widehat{L}_{i+1}(t+1) \mid \widehat{\mathcal{H}}(t)\big).$$

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When $\widehat{L}_0(t) = 1$ at $t = \widehat{t}_f$ the upper-blocked process stops, and $\widehat{t}_f$ gives a lower bound on the finish time t_f of the corresponding DLA process.

We next give w.h.p. bounds for $\widehat{L}_i(t)$ assuming none of the layers $i = 1, \dots, k$ are full for $t \le \widehat{t}_f$. The proofs of level occupancy are inductive backwards from level k. For a given level $k - j$ we identify two (not necessarily integer) times, $t_1(k-j)$ and $t_{k-j}(\omega)$, defined as follows. For level k, as $\mu_k(t) = t$, $t_1(k) = 1$ and for $j \ge 1$, let $t_1(k-j)$ is the solution to $\mu_{k-j}(t) = 1$ in (3). Thus,

$$t_1(k-j) = \left[(j+1)!N_kN_{k-1}\cdots N_{k-j+1}\right]^{1/(j+1)}. \tag{5}$$

The variable $t_1(k-j)$ is used as a reference point in many of our calculations.

Let $t_k(\omega) = 1$, and for $1 \le j \le k-1$, let $t_{k-j}(\omega) = (4\omega^3)^{1/(j+1)}t_1(k-j)$. Thus,

$$t_{k-j}(\omega) = \left((4\omega^3)\left[(j+1)!N_kN_{k-1}\cdots N_{k-j+1}\right]\right)^{1/(j+1)}. \tag{6}$$

Henceforth, we choose

$$\omega = 6\log n. \tag{7}$$

In Section 2.3 we show that with the value of ω given in (7), for $t \ge t_{k-j}(\omega)$, the value of $\widehat{L}_{k-j}(t)$ is concentrated around $\mu_{k-j}(t)$ w.h.p., thus establishing the first part of the claim in Proposition 1.

Note. For $j \ge 1$, the values of $t_1(k-j)$ and $t_{k-j}(\omega)$ are not necessarily integer. In this case the nearest step t to t_1 would be obtained by rounding to $t = t_1 + \varepsilon$ for some $-1/2 < \varepsilon < 1/2$, so that $\mu_{k-j}(t) = (1+\varepsilon/t_1)^{j+1}$. For $j \ge 1$, the values of k we consider satisfy $j/t_1(k-j) = o(1)$. Thus, $\mu_{k-j}(t) \sim 1$ and the effect of rounding is negligible. This is shown for the equal layers model in Lemma 4(3) and for the growing layers model in (45).

The next lemma gives the expected value of $\widehat{L}_i$ and conditions for its concentration at any step $t \le t_f$. Recall that $t_1(k-j)$ is given by (5), and $t_{k-j}(\omega)$ is given by (6).

Lemma 1. *Let $\mu_i(t)$ be given by (3), then for $i = 0, 1, \dots, k$ and any $t \ge 0$, it holds that $\mathrm{E}\,\widehat{L}_i(t) \le \mu_i(t)$. Moreover, provided $\widehat{L}_i(t) < N_i$ for $k - j \le i \le k$ then the following hold.*

(1) *For $j \ge 1$, $\mathrm{E}\,\widehat{L}_{k-j}(t) = \mu_{k-j}(t)(1 - O(j^2/t))$. Thus, if $j^2/t = o(1)$ then $\mathrm{E}\,\widehat{L}_{k-j}(t) \sim \mu_{k-j}(t)$.*

(2) *Suppose that $t_1(k-j+1) \ll t_1(k-j)$, and that $j^2/t_1(k-j) = o(1)$. Let $\omega = 6\log n$. If $t \ge t_{k-j}(\omega)$ then w.h.p. $\widehat{L}_{k-j}(t) = \mu_{k-j}(t)(1 + O(1/\omega))$.*

Proof. As the sink is occupied at $t = 0$, $N_{k+1} = 1$, and we define $\widehat{L}_{k+1}(t) = 1$. This implies that, provided $t \le N_k$, $\widehat{L}_k(s) = s = \mu_k(s)$ for $0 \le s \le t$. Moreover, as at most one vertex can be added to $\widehat{L}_i(t-1)$ at step t, this implies $\widehat{L}_i(t) \le t$.

Iterating (4) backwards for $0 \le s \le t$, and using $\widehat{L}_i(0) = 0$, gives

$$\mathrm{E}\,\widehat{L}_i(t) = \frac{1}{N_{i+1}}\sum_{s=0}^{t-1}\mathrm{E}\,\widehat{L}_{i+1}(s). \tag{8}$$

We claim for $j \ge 0$ that

$$\mathbb{1}_{\{t\ge j\}}\frac{(t-j)^{j+1}}{(j+1)!} \le \left(N_kN_{k-1}\cdots N_{k-j+1}\right)\mathrm{E}\,\widehat{L}_{k-j}(t) \le \frac{t^{j+1}}{(j+1)!}. \tag{9}$$

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For given t, the induction is backwards on $k-j$ from $j=0$. When $j=0$, (9) becomes $\mathrm{E}\,\widehat{L}_k(t)=t$ which is true, so the first non-trivial case is $j=1$. From (8) we see that

$$\mathrm{E}\,\widehat{L}_{k-1}(t)=\frac{1}{N_k}\sum_{s=0}^{t-1}s, \tag{10}$$

which illustrates how (9) arises from bounding this sum.

For the induction at step $i=k-(j+1)$, let

$$M_{j-1}=N_kN_{k-1}\cdots N_{k-j+1}. \tag{11}$$

Multiply (8) by M_{j-1}, and insert the bounds on $M_{j-1}\mathrm{E}\,\widehat{L}_{k-j}(s)$ from (9) (with $i+1=k-j$) into this, to give

$$\frac{1}{N_{k-j}}\sum_{s=j}^{t-1}\frac{(s-j)^{j+1}}{(j+1)!}\le M_{j-1}\mathrm{E}\,\widehat{L}_{k-(j+1)}(t)\le\frac{1}{N_{k-j}}\sum_{s=1}^{t-1}\frac{s^{j+1}}{(j+1)!} \tag{12}$$

By comparison of the sum with the related integral we have that

$$\frac{(t-1)^{m+1}}{m+1}\le 1^m+2^m+\cdots+(t-1)^m\le\frac{t^{m+1}}{m+1}. \tag{13}$$

Use (13) in (12) with $m=j+1$ to give

$$\frac{\mathbb{1}_{\{t\ge j+1\}}}{N_{k-j}}\frac{(t-(j+1))^{j+2}}{(j+2)!}\le M_j\mathrm{E}\,\widehat{L}_{k-(j+1)}(t)\le\frac{1}{N_{k-j}}\frac{t^{j+2}}{(j+2)!},$$

which completes the induction for (9).

Setting the indicator $\mathbb{1}_{\{\mathcal{E}(t)\}}=1$ in (4), provides an upper bound on $\mathrm{E}\,\widehat{L}_i(t)$, and as the upper bound in this induction is $\mu_i(t)$, this gives $\mathrm{E}\,\widehat{L}_i(t)\le\mu_i(t)$ as required.

Furthermore, provided $\widehat{L}_{k-j}(t)<N_{k-j}$, $j^2/t=o(1)$,

$$\mathrm{E}\,\widehat{L}_{k-j}(t)=\frac{1}{N_kN_{k-1}\cdots N_{k-j+1}}\frac{t^{j+1}}{(j+1)!}\big(1-O\big(j^2/t\big)\big)=\mu_{k-j}(t)\big(1-O\big(j^2/t\big)\big). \tag{14}$$

This completes the proof of Lemma 1(1). □

The next thing to check is that, for the values of k given in Theorem 1, for $t\ge t_1(k-j)$, $j^2/t=o(1)$, allowing us to use Lemma 1(1) in the proof of Lemma 1(2).

Lemma 2. *Let $t_1(k-j)$ be the value of t such that $\mu_{k-j}(t)=1$ as given by (5). The condition $j^2/t_1(k-j)=o(1)$ is satisfied in the equal layers model provided $k=o(n^{1/5})$ and in the growing layers model provided $d\to\infty$. This allows us to assume that for $t\ge t_{k-j}(\omega)$, $\mathrm{E}\,\widehat{L}_{k-j}(t)\sim\mu_{k-j}(t)$ in subsequent calculations.*

Proof. For $j\ge 1$, the value of $t_{k-j}(\omega)$ given in (6) satisfies

$$\begin{aligned}t_{k-j}(\omega)&=(4\omega^3)^{1/(j+1)}t_1(k-j)\ge t_1(k-j)=((j+1)!N_k\cdots N_{k-j+1})^{1/(j+1)}\\&=\big((j+1)!M_{j-1}\big)^{1/(j+1)},\end{aligned}$$

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where M_{j-1} is given in (11). The product M_{j-1} is model specific. For the equal layers model it has value $M_{j-1}(E) = (n/k)^j$, as $N_i = n/k$. For the growing layers model the value is $M_{j-1}(G) = d^{kj-j(j-1)/2}$, see (42).

In the first case $M_{j-1}(E)^{1/(j+1)} \geq (n/k)^{1/2}$, and in the second $M_{j-1}(G) \geq d^{k/2}$, this minimum being achieved at $j = 1$ or $j = k$. Checking $j^2/M_{j-1}^{1/(j+1)}$ we see that the condition $j^2/t_1(k-j) = o(1)$ is satisfied in the equal layers model provided $k = o(n^{1/5})$ and in the growing layers model provided $d \to \infty$. □

The other assumptions of Lemma 1(1) and (2) are dealt with as follows. Theorem 1 gives a w.h.p. finish time $t_f \approx T_f$. As $\mathrm{E}\,\widehat{L}_i(t) \leq \mu_i(t)$ follows from Lemma 1, checking that $\mu_i(T_f) \ll N_i$ and using the Markov inequality will imply that w.h.p. $\widehat{L}_i(T_f) \ll N_i$ holds as asserted. For the equal layers model, $\mu_i(T_f) \ll N_i$ is proved in Lemma 4(1), and for the growing layers model in (54) and (55). The condition that $t_1(k-(j-1)) \ll t_1(k-j)$ is checked in Lemma 4(2) for the equal layers model and in (53) of for the growing layers model.

With this in hand, we can now continue with the proof of Lemma 1(2).

2.3. Concentration of $\widehat{L}_i(t)$ for sufficiently large t

Lemma 3. *Let $\mu_{k-j}(t)$ as be given by (3), let k be as given in Theorem 1 and let $\omega = 6\log n$. Let $t_k(\omega) = 1$, and for $1 \leq j \leq k-1$, let $t_{k-j}(\omega)$ be as given in (6). Then with probability $1 - O(k^2\omega N_k \mathrm{e}^{-\omega})$ it holds that for all $j \leq k-1$ and all $t \geq t_{k-j}(\omega)$ that $\widehat{L}_{k-j}(t) \in [\mu_{k-j}(t)(1-1/\omega), \mu_{k-j}(t)(1+1/\omega)]$.*

Proof. The proof is inductive and for level $k-j$ it depends on establishing the result for levels $k, k-1, \ldots, k-j+1$. By definition, $\widehat{L}_k(t) = t = \mu_k(t)$. Let $t_k(\omega) = 1$, establishing Lemma 1(2) for $j = 0$. For $j \geq 0$, let $t_1(k-j)$ as given by (5) and note that $\mu_{k-j}(t_1(k-j)) = 1$. For $j \geq 1$, as $t_{k-j}(\omega) = (4\omega^3)^{1/(j+1)} t_1(k-j)$ then $\mu_{k-j}(t_{k-j}(\omega)) = 4\omega^3$.

The random variable $\widehat{L}_{k-j}(t)$ is obtained from $\widehat{L}_{k-j}(t-1)$ by choosing a random vertex $u \in S_{k-j}$ irrespective of the current occupancy of u, and a random neighbour $w \in S_{k-j+1}$. If w is occupied then $\widehat{L}_{k-j}(t+1) = \widehat{L}_{k-j}(t) + 1$. Thus, $\widehat{L}_{k-j}(t+1) = \widehat{L}_{k-j}(t) + Q_{k-j}(t)$, where $\mathrm{P}(Q_{k-j}(t) = 1) = (\widehat{L}_{k-j+1}(t)/N_{k-j+1})$ independently of any previous outcomes. As $t_1(k-j) \gg j^2$, Lemma 1(1) implies that $\mathrm{E}\,\widehat{L}_{k-j}(t) = \mu_{k-j}(t)(1+o(1))$

For $j \geq 1$, and $t \geq t_{k-j}(\omega)$ let $\mathcal{A}_j(t)$ denote the event that $\widehat{L}_{k-j}(t) \in [\mu_{k-j}(t)(1-1/\omega), \mu_{k-j}(t)(1+1/\omega)]$. As $\widehat{L}_{k-j}(t)$ is the sum of independent Bernoulli random variables, by Hoeffding's inequality,

$$\mathrm{P}(\neg\mathcal{A}_j(t)) \leq 2\exp\left\{-\frac{\mu_{k-j}(t)}{3\omega^2}(1-\delta)\right\} \leq 2\exp\left\{-\frac{\mu(t_{k-j}(\omega))}{4\omega^2}\right\} \leq 2\mathrm{e}^{-\omega}, \tag{15}$$

where $\delta = O(1/\omega + j^2/t)$, includes the correction from (14).

Let $\mathcal{E}_j(t)$ be the event that $\mathcal{A}_j(s)$ holds for all $s \in [t_{k-j}(\omega), t]$. For $j = 0$, $\mathrm{P}(\mathcal{E}_0(t)) = 1$, and for $j \geq 1$, given $\mathcal{E}_{j-1}(t)$, as an inductive assumption, we have that

$$\mathrm{P}\big(\neg\mathcal{E}_j(t) \mid \mathcal{E}_{j-1}(t)\big) \leq \sum_{s=t_{k-j}(\omega)}^{t} \mathrm{P}\big(\neg\mathcal{A}_j(s)\big). \tag{16}$$

Summing over $i \leq j$ we will be able to complete the induction, via

$$\mathrm{P}\big(\neg\mathcal{E}_j\big) \leq \mathrm{P}\big(\neg\mathcal{E}_j \mid \mathcal{E}_{j-1}\big) + \mathrm{P}\big(\neg\mathcal{E}_{j-1}\big) \leq \sum_{i\leq j}\sum_{t\geq t_{k-i}(\omega)} \mathrm{P}\big(\neg\mathcal{A}_i(t)\big) \leq cj\omega N_k \mathrm{e}^{-\omega}, \tag{17}$$

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provided we can establish the bound on the right-hand side of (17), which we now do. Here c is some absolute constant.

As $\mu_{k-j}(t)$ in (3) is monotone increasing in t and by (15) is bounded above by $2\mathrm{e}^{-\omega} = o(1)$, we use the Euler–MacLaurin theorem to replace the summation over t in (16) by an integral. Thus,

$$\sum_{t\geq t_{k-j}(\omega)} \mathrm{P}\big(\neg\mathcal{A}_j(t)\big) \leq 2 \sum_{t\geq t_{k-j}(\omega)} \exp\left\{-\frac{\mu(t_{k-j}(\omega))}{4\omega^2}\right\} \leq 3\int_{t\geq t_0} \exp\left\{-\frac{t^{j+1}}{C}\right\} \mathrm{d}t,$$

where, $t_0 = t_{k-j}(\omega)$, $C = [4\omega^2(j+1)!N_k\cdots N_{k-j+1}]$, and from (5) and (6), $t_{k-j}(\omega) = (4\omega^3 C)^{1/(j+1)}$.

Put $t^{j+1}/C = z^2/2$, so that $t = (Cz^2/2)^{1/(j+1)}$, and $dt/dz = (2/(j+1))(C/2)^{1/(j+1)} z^{2/(j+1)-1}$. Let $z_0 = z(t_0) = \sqrt{2\omega}$, and note that as $j \geq 1$, $z^{2/(j+1)-1} \leq 1$. Finally $(C/2)^{1/(j+1)} = O(\omega k N_k)$, as N_k is a largest level in both models. Thus,

$$\int_{t\geq t_0} \exp\left\{-\frac{t^{j+1}}{C}\right\} \mathrm{d}t \leq \frac{2}{j+1}\left(\frac{C}{2}\right)^{1/(j+1)} \int_{z\geq z_0} \mathrm{e}^{-z^2/2}\,\mathrm{d}z \leq O(\omega k N_k)\frac{1}{\sqrt{\omega}}\mathrm{e}^{-\omega}, \tag{18}$$

by using a standard bound on the tail of the normal distribution, $\mathrm{P}(Z \geq z) \leq 1/(\sqrt{2\pi}z)\mathrm{e}^{-z^2/2}$.

Lemma 3 follows for all $1 \leq j \leq k$ by adding the upper bound on the right-hand side of (17) given in (18) over $j \leq k$, and choosing $\omega = 6\log n$. □

2.4. Lower bound on occupancy at step t

So far we only have an upper bound on $\mathrm{E}\,L_j(t)$ given by $\mathrm{E}\,L_j(t) \leq \mathrm{E}\,\widehat{L}_j(t)$. We construct a lower bound $\mathrm{E}\,\widetilde{L}_j(t)$ and prove that for large enough t these bounds converge, thus giving the asymptotic value of $\mathrm{E}\,L_j(t)$.

For a given level j, a lower bound on $L_j(t)$ can be found as follows. Define a subprocess of DLA which requires that the particle avoids upper-blocked vertices. Thus, to reach level j, a particle must avoid choosing neighbours in $\widehat{B}_1, \ldots, \widehat{B}_j$ at steps $0, \ldots, j-1$ of its random walk.

Let $L_j^*(t)$ be an upper bound on $\widehat{L}_j(t)$ that holds w.h.p. This will, for example, be obtained from Lemma 3 if $t \geq t_j(\omega)$. Referring to (1) and (2), let $\widetilde{L}_i(t)$ be obtained by replacing L by L^* in the bracketed terms on the right-hand side. This defines a lower bound $\widetilde{L}_j(t)$, such that w.h.p. $\widetilde{L}_j(t) \leq L_j(t) \leq \widehat{L}_j(t)$. Let $\widetilde{\mathcal{H}}(t) = (\widetilde{L}_0(t), \widetilde{L}_1(t), \ldots, \widetilde{L}_k(t))$ be the occupancy vector obtained from this lower bound, then we have the following recurrences:

$$\mathrm{E}\big(\widetilde{L}_i(t+1) \mid \widetilde{\mathcal{H}}(t)\big) = \widetilde{L}_i(t) + \frac{\widetilde{L}_{i+1}(t)}{N_{i+1}} \prod_{j=0}^{i}\left(1 - \frac{L_j^*(t)}{N_j}\right), \tag{19}$$

$$\mathrm{E}\big(\widetilde{L}_k(t+1) \mid \widetilde{\mathcal{H}}(t)\big) = \widetilde{L}_k(t) + \prod_{j=0}^{k}\left(1 - \frac{L_j^*(t)}{N_j}\right). \tag{20}$$

These recurrences are solved in Section 3.2 for the equal layers model and in Section 4 for the growing layers model.

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2.5. The gap property

It is clear that $0 \le \widetilde{L}_{k-j}(t) \le L_{k-j}(t) \le \widehat{L}_{k-j}(t) \le t$, and that these values are monotone non-decreasing. We choose times

$$t^-(k-j) < t_1(k-j) < t^+(k-j) = t_{k-j}(\omega),$$

(where $\omega = 6 \log n$) such that

$$\mu_{k-j}(t^-) = 1/\omega^3, \quad \mu_{k-j}(t_1) = 1, \quad \mu_{k-j}(t^+) = 4\omega^3.$$

Below $t^-(k-j)$, $\widehat{L}_{k-j}(t) = 0$ w.h.p. for all $j = 1, \dots, k$. This follows from Lemma 1, and the Markov inequality, as $k = o(\omega)$ in either model. Above $t^+(k-j)$ we will see that $\mathrm{E}\,\widetilde{L}_{k-j}(t) \sim \mathrm{E}\,\widehat{L}_{k-j}(t)$ and thus $\mathrm{E}\,L_{k-j}(t) \sim \mu_{k-j}(t)$. The main content of Sections 3 and 4 is a proof of this via a *gap property* which implies that

$$t^+(k-j) = t_{k-j}(\omega) \ll t^-(k-(j+1)) \ll t_1(k-(j+1)),$$

so that w.h.p. concentration for L_{k-j} occurs while $L_{k-(j+1)}$ is still zero.

3. Analysis of DLA in the equal layers model

It is convenient at this point to obtain an asymptotic estimate for the finish time of DLA in the equal layers model. From (3), as $N_1 = \cdots = N_k = n/k$,

$$\mu_{k-j}(t) = \left(\frac{k}{n}\right)^j \frac{t^{j+1}}{(j+1)!}.$$

Let $t_1(k-j)$ be given by (5) and let $T_f = t_1(0)$ be the value of t for which $\mu_0(t) = 1$. Then

$$T_f = \left(\frac{(k+1)!}{k^k} n^k\right)^{1/(k+1)}. \tag{21}$$

At this time the source has expected occupancy asymptotic to one in the upper-blocked process. That T_f indeed approximates the finish time is shown in Section 3.3.

Lemma 4. *Let G be a graph in the equal layers model with level sizes $N_i = n/k$, $i = 1, \dots, k$, where $2 \le k \le k^* = \sqrt{\log n}/\log\log n$.*

(1) *Let $\gamma_k = \mathrm{e}^{-1}(1 + O(\log k/k))$, then*

$$T_f = n^{k/(k+1)}\gamma_k, \tag{22}$$

and, thus, $T_f \ll n/k = N_k$.

(2) *For $1 \le j \le k$, let $t_1(k-j)$ be as given in (5), then*

$$t_1(k-j) = ((j+1)!)^{1/(j+1)}\,(n/k)^{j/(j+1)} = \Theta(j)(n/k)^{j/(j+1)},$$

and

$$t_1(k-j) = t_1(k-(j-1)) \cdot \Theta(1)(n/k)^{1/(j(j+1))}.$$

If $k \le k^*$ then $t_1(k-j) \gg t_1(k-(j-1))$ in the equal layers model, as claimed in the note below Lemma 1.

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(3) *Let t be $t_1(k-j)$ rounded to an integer. For $j \geq 1$, $j/t_1(k-j) = o(1)$ and, thus, $\mu_{k-j}(t) \sim 1$, as asserted in the note under (6).*

Proof. Note that

$$\frac{(k+1)!}{k^k} = \mathrm{e}^{O(1/k)} \frac{1}{k^k} \sqrt{2\pi}\, \mathrm{e}^{-(k+1)} k^{k+3/2} (1+1/k)^{k+3/2} = \mathrm{e}^{-(k+1)} \Theta_k,$$

where $\Theta_k \sim (\mathrm{e}^{1+O(1/k)} \sqrt{2\pi} k^{3/2})$. Thus,

$$(\Theta_k)^{1/k+1} = \mathrm{e}^{O(\log k/k)} = 1 + O(\log k/k)$$

is bounded for $k \geq 2$ and tends to one as $k \to \infty$. From (21)

$$T_f = \left(\frac{(k+1)!}{k^k} n^k \right)^{1/(k+1)} = n^{\frac{k}{k+1}} \mathrm{e}^{-1} (1 + O(\log k/k)). \tag{23}$$

Thus, $kT_f/n = O(kn^{1/(k+1)}) = o(1)$ for $k \leq k^*$.

The second part follows by direct calculation using $((j+1)!)^{1/(j+1)} = j\Theta(1 + j^{3/(2j)}) = j\Theta(1)$. The third part follows from $k/t_1(k-1) = k/(n/k)^{1/2} = o(1)$, and $k = o(\sqrt{\log n})$. □

3.1. Evolution of the state vector $\widehat{L}$ in the equal layers model

We prove there is a large gap in the number of steps between the time when $\mathrm{E}\,\widehat{L}_{k-j} = 1$ with all lower values zero, and the time when $\mathrm{E}\,\widehat{L}_{k-(j+1)} = 1$. The gap allows $\widehat{L}_{k-j}$ to increase and become concentrated around μ_{k-j}, whilst all values with a lower index $i < k-j$ remain zero. This confirms the inductive assumption stated below (5) in Lemma 3.

Let $t_1(k-j)$ and $t_{k-j}(\omega)$ be as given by (5) and (6). We list various assumptions used in this and the next section:

$$1 \leq k \leq k^* = \frac{\sqrt{\log n}}{\log\log n}, \quad \omega = 6\log n, \quad \beta = \frac{N_k}{\omega T_f}. \tag{24}$$

Let $\widehat{L} = (\widehat{L}_0, \widehat{L}_1, \ldots, \widehat{L}_k)$ be the state vector of the upper-blocked process. The entries in $\widehat{L}$ are non-negative integers, and if $\widehat{L}_i = 0$ then $\widehat{L}_{i-1} = 0$.

The following argument for $t \leq \omega T_f$ proves there is a large enough gap $t'' - t$ between $\mu_{k-j}(t) = 1$ and $\mu_{k-(j+1)}(t'') = 1$ for $\widehat{L}_{k-j}(t'')$ to be concentrated, as assumed in Lemma 3. In particular w.h.p. at $t_{k-j}(\omega)$, where $t'' \gg t_{k-j}(\omega) > t'$ we will have (32), as is to be shown.

From (22),

$$\beta = \frac{N_k}{\omega T_f} = \frac{n}{\omega k} \frac{1}{\gamma_k n^{k/(k+1)}} = \frac{1}{\gamma_k} \frac{1}{\omega k} n^{1/(k+1)} \geq (6\log n)^{4+k/2}. \tag{25}$$

This is true as $\omega = 6\log n$ and we assumed that $k \leq \sqrt{\log n}/\log\log n$.

As $N_{k-j} = N_k = n/k$ in the equal layers model, for any $t \leq \omega T_f$,

$$\mu_{k-(j+1)}(t) = \frac{t}{(j+2)N_{k-j}} \cdot \mu_{k-j}(t) \leq \frac{1}{(j+1)\beta} \mu_{k-j}(t). \tag{26}$$

For $j + \ell \leq k$, we can iterate this to give

$$\mu_{k-(j+\ell)}(t) \leq \mu_{k-j}(t) \frac{1}{\beta^\ell} \frac{1}{(j+\ell+1)(j+\ell)\cdots(j+2)} \leq \mu_{k-j}(t) \frac{1}{((j+1)\beta)^\ell}. \tag{27}$$

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Consider $\mathrm{E}\,\widehat{L}_{k-j}(t)$. At $t=t_1(k-j)$ when $\mu_{k-j}(t)=1$, then by Lemmas 1 and 2, $\mathrm{E}\,\widehat{L}_{k-j}(t)\sim 1$. The Markov inequality implies that w.h.p. $\widehat{L}_{k-j}(t)\in I_\omega=[0,1,\dots,\omega]$. From (26) and (27),

$$\mu_{k-(j+\ell)}(t)\le \frac{1+o(1)}{((j+1)\beta)^\ell},\quad \text{for } 1\le \ell\le k-j, \tag{28}$$

and, thus, w.h.p. $\widehat{L}_0=0$, $\widehat{L}_1=0,\dots,\widehat{L}_{k-(j+1)}=0$. Using (26) and (27), we see that

$$\mu_{k-j+\ell}(t)\ge \mu_{k-j}(t)j(j-1)\cdots(j-\ell+1)\beta^\ell,\quad \text{for } \ell\le j. \tag{29}$$

Thus, if $\mu_{k-j}(t)=1$ then for $1\le\ell\le j$, $t\ge t_{k-j+\ell}(\omega)$, and

$$\mu_{k-j+\ell}(t)\ge \beta^\ell\ge (6\log n)^{4+k/2}. \tag{30}$$

It now follows that Lemma 1(2) holds, and w.h.p. $\widehat{L}_{k-j+\ell}$ is equal to $(1+o(1))\mathrm{E}\,\widehat{L}_{k-j+\ell}\sim \mu_{k-j+\ell}(t)$.

In summary, at time t such that $\mu_{k-j}(t)\sim 1$ (implying that $t\le T_f$, see Lemma 4(2)), w.h.p., the state vector $\widehat{L}$ is such that $\mu_{k-\ell}\to\infty$ for $\ell\le j-1$, and

$$\widehat{L}(t)=(0,\dots,0,\widehat{L}_{k-j}\in I_\omega,(1+o(1))\mu_{k-j+1},(1+o(1))\mu_{k-j+2},\dots,\mu_k). \tag{31}$$

Let $t=t_1(k-j)$, let $t'=t_{k-j}(\omega)=(4\omega^3)^{1/(j+1)}t_1(k-j)$, so $\mu_{k-j}(t')\sim 4\omega^3$. By Lemma 3 it holds w.h.p. that $\widehat{L}_{k-j}(t')\sim\mu_{k-j}(t')$.

Let $t''=t_1(k-(j+1))$. By Proposition 4(2), $t_1(k-(j+1))=\Theta(1)(n/k)^{1/((j+1)(j+2))}t_1(k-j)$. As $t'=(4\omega^3)^{1/((j+1)}t_1(k-j)$, it can be checked that $t''\gg t'$. In addition, by (27)

$$\sum_{\ell\ge 1}\mu_{k-(j+\ell)}(t')=O\left(\frac{\mu_{k-j}(t')}{\beta}\right)=O\left(\frac{\omega^3}{\beta}\right)=O\left(\frac{1}{\log^{1+k/2}n}\right).$$

Thus, applying the Markov inequality to the above, at $t'=t_{k-j}(\omega)$, w.h.p.

$$\widehat{L}(t_{k-j}(\omega))=(0,\dots,0,(1+o(1))\mu_{k-j},(1+o(1))\mu_{k-j+1},\dots,\mu_k), \tag{32}$$

so that $\widehat{L}_{k-j}(t')$ is concentrated around its mean, and all lower levels are unoccupied, and thus the claimed gap exists. This condition persists w.h.p. until around $t_1(k-(j+1))=t''\gg t'=t_{k-j}(\omega)$, when $\mu_{k-(j+1)}(t_1(k-(j+1)))=1$ at which point $\widehat{L}(t'')$ resembles (31) and the induction continues.

3.2. Lower bound on occupancy in the equal layers model

We prove that, for $t\ge t_i(\omega)$, we have $\mathrm{E}\,\widetilde{L}_i(t)\sim\mathrm{E}\,\widehat{L}_i(t)\sim\mu_i(t)$. As with the upper bounds, we have that $\mathrm{E}\,\widetilde{L}_k\gg\mathrm{E}\,\widetilde{L}_{k-j}$ for $j\ge 1$. The first step is to draw a line between these quantities.

Let $t^-=t_1(k-1)/\omega$, so that $\mu_{k-1}(t^-)=1/\omega^2$. Then (14) and the Markov inequality imply that w.h.p. $\widehat{L}_{k-1}(t^-)=0$ and as $\widehat{L}_j(t)$ is monotone non-decreasing in j, w.h.p. $\widehat{L}_j(t)=0$ for all $j\le k-1$ and $t\le t^-$. As $\widehat{L}_k(t)=L_k^*(t)=t$ deterministically, this simplifies (20) for $t\le t^-$.

As before let $\beta=N_k/\omega T_f$ where ω, k satisfy (24). Provided $k\ge 2$, if $t\ge t^-$ then $t/\beta\gg\omega^3$. Indeed using (25),

$$t^-=\frac{t_1(k-1)}{\omega}=\frac{1}{\omega}\sqrt{\frac{2n}{k}}\quad\Longrightarrow\quad\frac{t^-}{\beta}=\frac{\gamma_k\omega k}{n^{1/(k+1)}}\cdot\frac{1}{\omega}\sqrt{\frac{2n}{k}}\ge 2\gamma_k n^{1/6}\gg\omega^3.$$

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For $t \geq t^-$ and $1 \leq i \leq k-1$, we first consider the product term in (19). Recalling that $N_i = N_k = n/k$, we next prove that

$$\prod_{j=0}^{i} \left(1 - \frac{L_j^*(t)}{N_j}\right) \geq 1 - \sum_{j=0}^{i} \frac{L_j^*(t)}{N_j} = 1 - O\left(\frac{t}{\beta N_k}\right) = 1 - o(1). \tag{33}$$

By (27), $\mu_{k-j} \leq \mu_k/\beta^j$. Assume $t \geq t^-$ and that for some $\ell \leq k-1$, $t_{\ell+1}(\omega) \leq t < t_\ell(\omega)$. Lemma 3 gives concentration of $\widehat{L}_j(t)$ for $j \geq \ell+1$, which we use along with (31) and (32).

If $i < \ell$ then $L_j^*(t) = 0$, for $j \leq i$. Next, if $i = \ell$, $L_j^*(t) = 0$ for $j < i$ and $L_i^*(t) \leq 2\mu_i(t_i(\omega)) \leq 8\omega^3$, since $t < t_i(\omega)$, and again Lemma 3 gives this upper bound at $t_i(\omega)$. Finally assume $i \geq \ell+1$. Then (as $t \geq t^-$) using (27),

$$\sum_{j=0}^{i} \frac{L_j^*(t)}{N_j} \leq \frac{8\omega^3}{N_k} + 2\sum_{j=\ell+1}^{i} \frac{t}{\beta^{k-j}N_k} \leq \frac{O(1)}{N_k}\left(\omega^3 + \frac{t}{\beta}\right) = O\left(\frac{t}{\beta N_k}\right). \tag{34}$$

This confirms (33).

We turn now to the solution of (19) and (20). Consider first $\mathrm{E}\,\widetilde{L}_k(t)$. For all $t \geq 0$, $\widehat{L}_k(t) = t$, and $\sum_{j<k} \widehat{L}_j(t) = O(t/\beta)$ follows from (34). Thus, from (20)

$$\mathrm{E}\,\widetilde{L}_k(t) = t - \sum_{s=0}^{t} O\left(\frac{s}{\beta N_k}\right) = t - O\left(\frac{t^2}{\beta N_k}\right) = t\left(1 - O\left(\frac{t}{\beta N_k}\right)\right).$$

For $t \geq t_0$ where $t_0 \to \infty$ arbitrarily slowly we have $\widetilde{L}_k(t) \sim t$ w.h.p., initializing an induction for $\mathrm{E}\,\widetilde{L}_i(t)$ using arguments equivalent to those in Lemma 1 for $\mathrm{E}\,\widehat{L}_i(t)$.

At step $t+1$, (33) implies that, for the lower bounds on the process, with probability $(1-o(1))$ particle $t+1$ will arrive at level i. If $i < k$, it halts at this level with probability $\widetilde{L}_{i+1}(t)/N_{i+1}$. Thus,

$$\mathrm{E}\,\widetilde{L}_i(t+1) = \mathrm{E}\,\widetilde{L}_i(t) + \left(1 - O\left(\frac{t}{\beta N_k}\right)\right) \cdot \frac{\mathrm{E}\,\widetilde{L}_{i+1}(t)}{N_{i+1}}. \tag{35}$$

Arguing as in (8) on the inductive assumption that

$$\mathrm{E}\,\widetilde{L}_{i+1}(t) = \mu_{i+1}(t)\left(1 - O\left(\frac{t(k-(i+1)+1)}{\beta N_k}\right)\right),$$

which is true when $i+1 = k$, using (35) we find

$$\begin{aligned}
\mathrm{E}\,\widetilde{L}_i(t) &= \frac{1}{N_{i+1}} \sum_{s=0}^{t-1} \mathrm{E}\,\widetilde{L}_{i+1}(s)\left(1 - O\left(\frac{s}{\beta N_k}\right)\right) \\
&= \frac{1}{N_{i+1}} \sum_{s=0}^{t-1} \mu_{i+1}(s)\left(1 - O\left(\frac{s(k-(i+1)+1)}{\beta N_k}\right)\right)\left(1 - O\left(\frac{s}{\beta N_k}\right)\right) \\
&= \frac{1}{N_{i+1}} \sum_{s=0}^{t-1} \frac{s^{k-i}}{(k-i)!N_k \cdots N_{i+2}}\left(1 - O\left(\frac{s(k-i+1)}{\beta N_k}\right)\right) \\
&= \mu_i(t) - \frac{t\mu_i(t)}{N_k} O\left(\frac{k-i+1}{\beta}\right) = \mu_i(t)\left(1 - O\left(\frac{(k-i+1)t}{\beta N_k}\right)\right).
\end{aligned}$$

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Provided $t \le \omega T_f$, which is true w.h.p. by Lemma 5, then $kt/\beta N_k \le k(\omega T_f/N_k)^2 = o(1)$, see (24) and Lemma 4; in this case

$$\mathrm{E}\,\widetilde{L}_i(t) \sim \mathrm{E}\,L_i(t) \sim \mathrm{E}\,\widehat{L}_i(t) \sim \mu_i(t) = \frac{t^{i+1}}{(i+1)!N_k \cdots N_{i+1}} \tag{36}$$

as required.

Let $t_i(\omega)$ be given by (6). For those $i \le k$, such that $t \ge t_i(\omega)$, then $\mu_i(t) \to \infty$ suitably fast and the concentration results of Lemma 3 hold. The gaps inherited from the upper-bound argument of Section 3.1 are essentially unaltered.

This completes the proof of Proposition 1 for the equal layers model.

3.3. Finish time of DLA in the equal layers model

A lower bound on the finish time follows from the upper-blocked process, and an upper bound from the lower bound estimates for DLA. The range of k requires us to split the finish time analysis into two cases; k constant, and $k \to \infty$ with n.

Lemma 5. *For $2 \le k \le k^* = \sqrt{\log n}/\log\log n$, let G be an equal layers graph with level sizes $N_i = n/k$, $i = 1, \ldots, k$. Let T_f be given by (21). If k is constant then the finish time t_f of the DLA process in G satisfies $T_f/\omega' \le t_f \le \omega' T_f$, with probability $1 - O(1/\omega')$, where $\omega' \to \infty$ arbitrarily slowly. However, if $k \to \infty$ then $t_f = \Theta(T_f)$.*

Proof. By Lemma 1, $\mathrm{E}\,\widehat{L}_0(t) \le \mu_0(t)$. As usual let $t_1(0)$ denote the step t when $\mu_1(t) \sim 1$. From Lemma 4(2), with $j = k$ we have $t_1(0) = \Theta(k)n^{k/(k+1)}$, which follows because $k^{1/(k(k+1))} = \Theta(1)$. On the other hand, from Lemma 4(1), we have that $T_f = \gamma_k n^{k/(k+1)}$, so $t_1(0) = CT_f$ for some constant $C > 0$.

Case k constant. Let $t' = t_1(0)/\omega'$, where $\omega' \to \infty$ arbitrarily slowly. Then w.h.p., $\mathrm{E}\,\widehat{L}_0(t') \le \mu_0(t') = O(1/(\omega')^{k+1})$ and, thus, $\mathrm{P}(\widehat{L}_0(t') > 0) = O(1/\omega')$.

We next investigate the concentration of $L_1(T_f)$. By Lemma 4(1), as $T_f = t_1(0)$, $\mu_0(T_f) = 1$ and, thus,

$$\mu_1(T_f) = \mu_0(T_f)\frac{N_1(k+2)}{T_f} \ge \frac{n}{\gamma_k n^{k/(k+1)}} = \Theta(1)n^{1/(k+1)} \gg 4\omega^3, \tag{37}$$

where $\omega = 6\log n$ as usual. Thus, $T_f \gg t_1(\omega)$ and, hence, Lemma 3 holds for $\widetilde{L}_1(T_f)$. Suppose that $L_0(T_f) = 0$ at step T_f. By (37) the expected waiting time τ for a particle to hit $L_1(T_f)$ is

$$\tau \le (1 + o(1))\frac{N_1}{\mu_1(T_f)} = \Theta(1)\frac{n}{kn^{1/(k+1)}} = \Theta(1)\frac{n^{k/(k+1)}}{k} = \Theta(T_f/k) = O(T_f). \tag{38}$$

Thus, by step $\omega' T_f$, w.h.p. $L_0(\omega' T_f) = 1$.

Case $k \to \infty$ with n. Let $t' = (1-\varepsilon)t_1(0)$, where $0 < \varepsilon < 1$ constant, then $t' = C(1-\varepsilon)T_f$ and $\mu_0(t') \sim (1-\varepsilon)^{k+1}$. Thus, $\mathrm{P}(\widehat{L}_0(t') > 0) \le \mathrm{e}^{-\varepsilon k}$ and so w.h.p. $t_f > t'$, lower bounding the finish time. On the other hand, for the upper bound, the inequality (37) still holds. By (38), the expected waiting time for the source to be occupied satisfies $\tau = \Theta(T_f/k) = o(T_f)$. Taking $\varepsilon = o(1)$ we see that $t_f \sim CT_f$ for some constant C.

This completes the proof of Lemma 5 and, hence, Theorem 1(1). □

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3.4. Existence of a unique connecting path component

We now prove Theorem 2 for the equal layers model. We must show w.h.p. that at step t_f, the finish time, the path of occupied vertices connecting the source to level k (and, hence, to the sink), has no off-path neighbours in C_{t_f}.

At a given step t, the edge-induced component C_t is obtained from C_{t-1} by adding a newly occupied vertex which points to the neighbour in C_{t-1} which halted the particle: Thus, a particle halts at vertex u in level i if it chooses an edge uw to an occupied neighbour w in level $i+1$. We consider this edge uw as being directed from u to w in the component C_t rooted at the sink. An arborescence is a rooted tree with all edges directed towards the root vertex. Thus C_t is an arborescence with root z. On deletion of the z, $D_t = C_t \backslash \{z\}$ becomes a directed forest of arborescences each rooted at a vertex in level k.

Let B_i be the set of occupied vertices in level i, where $L_i = |B_i|$. Given that a particle at u chooses a vertex in B_{i+1} as a neighbour, then this neighbour is chosen uniformly at random (u.a.r.) from the set B_{i+1}.

We regard vertices occupied by halted particles as coloured either red or blue, with all occupied vertices in level k coloured blue. If u is the first in-neighbour of w then u is coloured blue. If however w already has an in-neighbour u' then u, u' and all other in-neighbours are (re)coloured red. At any step, the red vertices in a level are those with siblings and the blue vertices are the unique in-neighbour of some vertex in the next level. The choice of w by the particle at u is independent of the colour of w at this step.

The process halts when there is a directed path of occupied vertices $v = u_0 u_1 \cdots u_k z$ from the source to sink. The source vertex v is blue at t_f as it is the first in-neighbour of u_1.

Lemma 6. *With high probability, the path $v = u_0 u_1 \cdots u_k = w$ from the source to level k is blue, and thus the arborescence of halted particles rooted at $w = u_k$ is exactly this path.*

Proof. As before, let B_i be the set of occupied vertices in level i, and $L_i = |B_i|$. The subset $\mathrm{Out}(B_i)$ of B_{i+1} with at least one in-neighbour in B_i has size at most L_i. This is because $u \in B_i$ has a unique occupied out-neighbour in C_t.

Let $\mathbb{1}_{\{k-j,s\}}$ be the indicator that particle s halts in level $k-j$ and is coloured red due to a pre-existing sibling. In this case particle s has chosen an out-neighbour in the existing set $\mathrm{Out}(B_{k-j}) \subseteq B_{k-j+1}$ and, thus, as $|\mathrm{Out}(B_{k-j})| \le \widehat{L}_{k-j}(s)$,

$$\mathrm{E}\,\mathbb{1}_{\{k-j,s\}} \le \frac{\mathrm{E}\,\widehat{L}_{k-j}(s)}{(n/k)} \sim \frac{\mu_{k-j}(s)}{(n/k)}.$$

Let $Z_{k-j}(t)$ be the number of red vertices in level $k-j$. We associate the number of possibly red vertices at each step with a super-process $\widehat{Z}_{k-j}(t) = Z_{k-j}(t) + Q_{k-j}(t)$, where $Q_{k-j}(t)$ is Bernoulli with parameter $(\widehat{L}_{k-j} - |(\mathrm{Out})B_{k-j}|)$. Thus, $Z_{k-j}(t) \le \widehat{Z}_{k-j}(t)$ and

$$\mathrm{E}\left(\widehat{Z}_{k-j}(t+1) \mid \widehat{\mathcal{H}}(t)\right) = \widehat{Z}_{k-j}(t) + \frac{\widehat{L}_{k-j}(t)}{N_{k-j}}.$$

The recurrence for $\mathrm{E}\,\widehat{Z}$ mirrors the recurrence (4) for $\widehat{L}_i(t)$ with $k-j$ replacing $i+1$. The variable $\widehat{Z}_{k-j}(t)$ occurs in place of $\widehat{L}_i(t)$, and will grow as $\widehat{L}_{k-(j+1)}(t)$. Specifically,

$$\begin{aligned}\mathrm{E}\,\widehat{Z}_{k-j}(t) &\sim \frac{1}{(n/k)} \sum_{s=1}^{t} \mu_{k-j}(s) = \frac{1}{(n/k)} \sum_{s=1}^{t} \frac{s^{j+1}}{(j+1)!(n/k)^j} \\ &\sim \frac{t^{j+2}}{(j+2)!(n/k)^{j+1}} = \mu_{k-(j+1)}(t). \end{aligned} \tag{39}$$

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We note that $\widehat{Z}_{k-j}(t)$ is the sum of independent indicator variables, and will be concentrated for $t \geq t_{k-(j+1)}(\omega)$, where $\omega = 6 \log n$ as before. The total number of red vertices is at most $2Z_{k-j}(t) \leq 2\widehat{Z}_{k-j}(t)$, where the factor 2 covers the case where the pre-existing sibling was blue but is recoloured red.

Assume $T_f/\omega' \leq t_f \leq \omega' T_f$, where ω' from Lemma 5 tends to infinity arbitrarily slowly with n. Denote the path connecting $v = u_0$ to level k by $u_0 u_1 u_2 \cdots u_k$, where by definition the source $v = u_0$ is blue at t_f. For $i \geq 1$, let $R_i(s)$ be the red vertices in level i at step s. Let s_i denote the step at which u_i became occupied. Consider the colour of u_1 at t_f. As $T_f = t_1(0)$, and $t_f \geq T_f/\omega' \gg t_1(\omega)$ by the gap property, so we have that $L_1(t_f) \sim \mu_1(t_f)$. Let $t = \omega' T_f$, be a w.h.p. upper bound on t_f, then by (3) $\mu_0(t) = (\omega')^{k+1}$. On the other hand, $\widehat{Z}_1(t_f)$ may not be concentrated, but we can assume $\widehat{Z}_1(t_f) \leq \omega \mu_0(t)$ where $\omega = 6 \log n$ will be large enough to use in the Markov inequality for $k = o(\sqrt{\log n})$ levels.

Vertex u_1 was chosen u.a.r. from the occupied vertices in level one by the particle halting at u_0, thus,

$$\mathrm{P}(u_1 \in R_1(t)) \leq \frac{2\widehat{Z}_1(t)}{L_1(t)} \leq \frac{2\omega\mu_0(t)}{\mu_1(t)} \leq \frac{2\omega t}{(k+1)(n/k)} \leq \frac{2\omega t}{n}. \tag{40}$$

Next consider the colour of u_2. If u_2 is red at step t then either (i) it became red before step s_1 when it was chosen u.a.r. by u_1 or (ii) it became red at some later step.

In the first case,

$$\mathrm{P}(u_2 \in R_2(s_1)) \leq \frac{2\widehat{Z}_2(s_1)}{L_2(s_1)} \leq 2\omega \frac{\mu_1(s_1)}{\mu_2(s_1)} = \frac{2\omega s_1}{k(n/k)} = \frac{2\omega s_1}{n}.$$

By the gap property, $L_2(s_1)$ is already concentrated, for otherwise $L_1(s_1)$ would be empty, a contradiction. The ω covers possible lack of concentration of $\widehat{Z}_2(s_1)$ as explained previously.

In the second case, as u_3 is the chosen out-neighbour of u_2, then u_2 will turn red if some particle chooses u_3 after time s_1, and thus

$$\mathrm{P}(u_2 \in R_2(t) \backslash R_2(s_1)) \leq \frac{1}{(n/k)} \sum_{\tau=s_1+1}^{t} \mathrm{E}\ \mathbb{1}_{\{u_3 \text{ chosen at } \tau\}} = \frac{t - s_1}{(n/k)}.$$

Thus

$$\mathrm{P}(u_2 \in R_2(t)) \leq \frac{2\omega k t}{n},$$

and similarly for vertices $u_3, \ldots, u_{k-1}$. Thus

$$\mathrm{P}(\text{path } v u_1 \cdots u_k \text{ is blue}) \geq 1 - \sum_{i=1}^{k-1} \frac{2\omega k t}{n} = 1 - O\left(\frac{\omega k^2 t}{n}\right).$$

By Lemma 5, $t = \omega' T_f \leq \omega' n^{k/(k+1)}$, where $\omega' \to \infty$ arbitrarily slowly. As $k \leq \sqrt{\log n}/\log\log n$, then $\omega k^2 \omega' T_f/n \leq \omega^3/n^{1/(k+1)} = o(1)$, by upper bounding $\omega' \leq \log\log n$. Thus the path from the source to vertex $w = u_k$ level k is blue w.h.p., and thus the arborescence rooted at w is exactly this path. □

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4. Analysis of DLA in the growing layers model

In the growing layers model, each layer is larger than the previous one by a factor of d. Thus, $N_j = d^j$ for $j = 0, 1, \ldots, k$ and we take $N_k = d^k = n$. Many of the properties of this model such as a gap property and unique connecting path are similar to the equal layers model.

The main, and most striking difference, is that there is a well-defined level j^* at a distance $\sqrt{2k+2} - 1$ from the end (ignoring rounding) at which agglomerative growth stops and from which a single path grows back towards the source. Moreover at the end, *except for the last* $j^* - 1$ levels, the connecting path is the *only occupied vertex* in the layer. This is in contrast to the equal layers model where all levels have significant occupancy, and even that of the first level at the end is $\Theta(n^{1/(k+1)})$.

Proposition 2. *Let G be a growing layers graph with level sizes $N_i = d^i$, for $i = 0, \ldots, k$, where $d \to \infty$, $k \to \infty$, and $k \le (\log d/(\log\log d)^2$. Let*

$$T_f = \sqrt{k} d^{k+3/2-\sqrt{2k+2}}. \tag{41}$$

(1) *The finish time t_f of DLA on G satisfies $d^{-O(1/\sqrt{k})} T_f \le t_f \le d^{O(1/\sqrt{k})} T_f$.*

(2) *At t_f, levels $i = 1, \ldots, k - \lceil \sqrt{2k+2} - 1 \rceil$ contain a single occupied vertex, the vertex u_i of the connecting path.*

Proof. The size N_i of layer i is $N_i = d^i$. It follows that the product of the set sizes in the denominator of $\mu_{k-j}(t)$ in (3) is given by

$$M_{j-1} = N_k N_{k-1} \cdots N_{k-j+1} = d^k d^{k-1} \cdots d^{k-j+1} = d^{kj - j(j-1)/2}, \tag{42}$$

and, thus, (3) becomes

$$\mu_{k-j}(t) = \frac{t^{j+1}}{(j+1)! d^k d^{k-1} \cdots d^{k-j+1}} = \frac{t^{j+1}}{(j+1)! d^{kj-j(j-1)/2}}. \tag{43}$$

Assume d is sufficiently large. The upper bound $\mathrm{E}\,\widehat{L}_{k-j}$ is obtained in Section 2. However, a problem can arise in the growing layers model in the upper bound calculations. The value of μ_{k-j} can decrease with increasing j and then (anomalously) increase again. This is because the recurrence used to establish it assumes $\mu_{k-\ell} \to \infty$ for all $\ell < j$, which is not the case. We next locate where this happens; this is where the unique path back to the source begins. □

4.1. An important level

We next show the existence of a level $i \sim k + 1 - \sqrt{2k+2}$, such that the first occupancy of this level effectively determines the finish time of the process.

From (43),

$$\mu_{k-j}(t_1) = \frac{t_1^{j+1}}{(j+1)! d^{kj-j(j-1)/2}} = 1 \quad \Longrightarrow \quad t_1(k-j) = [(j+1)!]^{1/(j+1)} d^{\frac{2kj-j(j-1)}{2(j+1)}}. \tag{44}$$

Before proceeding we note that rounding $t_1(k-j)$ up or down to the nearest step has negligible effect on $\mu_{k-j}(t_1)$, as asserted in the note under (6). Indeed

$$\max_{j \ge 1} \frac{j}{t_1(k-j)} \le \frac{k}{t_1(k-1)} = \frac{k}{d^{k/2}} = o(1). \tag{45}$$

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We next determine the maximum value of $t_1(k-j)$. From (3), for any t,

$$\frac{\mu_{k-j+1}(t)}{\mu_{k-j}(t)} = \frac{(j+1)d^{k-j+1}}{t}, \tag{46}$$

so setting $\mu_{k-j}(t_1(k-j)) = 1$ gives

$$\begin{aligned}\mu_{k-j+1}(t_1) &= \frac{j+1}{[(j+1)!]^{1/j+1}} d^{k-j+1-\frac{2kj-j(j-1)}{2(j+1)}} \\ &= \frac{\Theta(1)}{(2\pi/e(j+1))^{1/2(j+1)}} d^{\frac{2k+2-j(j+1)}{2(j+1)}}.\end{aligned} \tag{47}$$

The leading term on the right-hand side of (47) is bounded, and the exponent of d on the right-hand side is positive provided $2k+2 > j(j+1)$, which ensures that $\mathrm{E}\,\widehat{L}_{k-j+1}(t)$ is sufficiently large close to $t_1(k-j)$.

What value of $k-j$ maximizes the value of $t_1 = t_1(k-j)$ at which $\mu_{k-j}(t_1) = 1$? Write the exponent of d on the right-hand side of (44) as $f(j)/2$ where

$$f(j) = \frac{2kj - j(j-1)}{(j+1)} = (2k+2) - j - \frac{2k+2}{j+1}. \tag{48}$$

The maximum of $f(j)$ occurs at j^* with solution

$$(j^*+1)^2 = (2k+2), \quad \text{giving } f(j^*) = (j^*)^2. \tag{49}$$

The (not necessarily integer) values of j^*, $k-j^*$ and $d^{f(j^*)/2}$ are

$$j^* = \sqrt{2k+2} - 1, \quad k - j^* = k + 1 - \sqrt{2k+2}, \quad d^{f(j^*)/2} = d^{k+3/2-\sqrt{2k+2}}. \tag{50}$$

In the case where j^* is not an integer, the effect of the rounding error is addressed in Section 4.4.

Ignoring rounding effects, we evaluate $t_1 = t_1(k-j)$ at $j = j^*$, in which case $f(j) = j^2$, to find

$$\begin{aligned}t_1 &= [(j+1)!]^{1/(j+1)} d^{f(j)/2} \\ &\sim \mathrm{e}^{-1}(\sqrt{2\pi})^{1/(j+1)}(j+1)^{1+1/2(j+1)} d^{j^2/2} \\ &= C_{k-j}\sqrt{2k+2}\, d^{k+3/2-\sqrt{2k+2}},\end{aligned} \tag{51}$$

on inserting the values from (50), and where $C_{k-j} = \mathrm{e}^{-1}(1 + O(1/j))$. Note that $t_1 = \Theta(T_f)$. From (46),

$$\frac{\mu_{k-(j+1)}(t)}{\mu_{k-j}(t)} = \frac{t}{(j+2)d^{k-j}}$$

so that at t_1, for some C, $C' = \Theta(1)$,

$$\mu_{k-j^*}(t_1) = 1, \quad \mu_{k-(j^*-1)}(t_1) = Cd^{1/2}, \quad \mu_{k-(j^*+1)}(t_1) = C'd^{1/2}. \tag{52}$$

At first this seems confusing, as one might expect to have $\mu_{k-(j^*+1)}(t_1) = o(1)$ by analogy with the equal layers model. Assuming $d^{1/2} \to \infty$, level $k - j^* + 1$ is the last level at which the condition $\mu_{k-j+1} \to \infty$ is valid in the recurrence from $k-j+1$ to $k-j$ given in (4). Indeed $j^* - 1 = \sqrt{2k+2} - 2$ is the last value of j for which we claim Proposition 1 holds.

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Finally, we need to check $t_1(k-j+1) \ll t_1(k-j)$ for all $j \le j^*-1$ as required in Lemma 1(2). Using (44)

$$\begin{aligned}\frac{t_1(k-j)}{t_1(k-j+1)} &= \frac{[(j+1)!]^{1/(j+1)}}{[j!]^{1/j}} d^{\left(\frac{2kj-j(j-1)}{2(j+1)} - \frac{2k(j-1)-(j-1)(j-2)}{2j}\right)} \\ &= \Theta(1) d^{\frac{2k-(j-1)(j+2)}{2j(j+1)}} \\ &\ge \Theta(1) d^{\frac{3\sqrt{2k+2}-2}{2(2k-3\sqrt{2k+2}+4)}} = \Theta(d^{3/\sqrt{8k}}). \end{aligned} \tag{53}$$

The last line is evaluated at $j = j^*-1 = \sqrt{2k+2}-2$, and this is the largest value of j for which Proposition 1 is claimed to hold. By Theorem 1(2), $k \le ((\log d)/(\log\log d))^2$ so the right-hand side of (53) is $\Theta((\log d)^{3/\sqrt{8}})$ which tends to infinity with d, thus validating the of concentration condition of Lemma 1(2) for $\widehat{L}_{(k-j^*-1)}(t)$.

4.2. Gap property of $\widehat{L}$

Let $\omega = 6\log n$ as before. From the definition of $t_1 = t_1(k-j^*)$, at t_1/ω, we have $\mu_{k-j^*}(t_1/\omega) = O(\omega^{-(j^*+1)})$, where $j^* \sim \sqrt{2k} \to \infty$. Thus, w.h.p. $\widehat{L}_{k-j^*}(t_1/\omega) = o(1)$ in the upper process. Consequently all levels $i = k-\ell$, $\ell \ge j^*$, have $\widehat{L}_i(t) = 0$, w.h.p., for $t \le t_1/\omega$.

In what follows we only consider gaps between indices $k-\ell$ where $0 \le \ell \le k-j^*+1$.

By (50) we have $k-j^*+1 = k+2-\sqrt{2k+2}$. We see from (52) that $\mu_{k-j^*+1}(t_1(k-j^*)) = Cd^{1/2}$. As $d \to \infty$, then $\widehat{L}_{k-j^*+1}$ will be concentrated around $\mu_{k-j^*+1}(t_1)$, although we cannot expect it to be as strong as in Lemma 3. Fortunately this will not matter as $k-j^*$ is the last level to which we apply a gap argument. For $\ell \le j^*-1$ at $t_1(k-\ell)$, the value of $\widehat{L}_{k-\ell+1}$ obeys Lemma 3. In particular, it can be checked that $t_1(k-j^*+1) = \Theta(1)\sqrt{k} d^{k+5/2-(3/2)\sqrt{2k+2}}$. Using (46), we obtain $\mu_{k-j^*+2}(t_1(k-j^*+1)) = \Theta(d^{(\sqrt{2k+2}+1)/2})$, establishing a gap property between $\widehat{L}_{k-j^*+1}$ and $\widehat{L}_{k-j^*+2}$; the situation being better for smaller ℓ.

4.3. Lower bound on L

Turning to the lower bound $\widetilde{L}$ as given in (19) and (20), we need to prove that (33) holds. This requires that $\sum_{j=0}^{i} L_j^*(t)/N_j = o(1)$ for $1 \le i \le k-1$ and $t \le t_f$. We only consider this sum up to the time (in order of magnitude) when level $k-j^*$ first becomes occupied, and thus levels $0 \le i \le k-j^*-1$ are empty.

The main task is to choose a value of β for the growing layers model analogous to (25) which we can use with the arguments given in Section 3.2 for the equal layers model. There is not so much room for manoeuvre here if we do not want to restrict the value of d. Let ϖ to be a function tending to infinity more slowly than $d^{1/4}$, and assume $t_f \le \varpi T_f$; a value which satisfies Proposition 2. We choose $\beta = \sqrt{d}/\varpi$. As we assume $d \to \infty$, then also $\beta \to \infty$.

For $1 \le \ell \le j^*-1$, we use (41), (43), and $N_{k-\ell} = d^{k-\ell}$. As $d^\ell/(\ell+1)$ is monotone increasing, the worst case is $\ell = j^*-1$, so, assuming $t_f \le \varpi T_f$

$$\begin{aligned}\frac{\mu_{k-\ell}}{N_{k-\ell}} &= \frac{t}{(\ell+1)d^{k-\ell}} \frac{\mu_{k-\ell+1}}{N_{k-\ell+1}} \qquad (54)\\ &\le \frac{\mu_{k-\ell+1}}{N_{k-\ell+1}} \frac{\varpi\sqrt{k}}{\sqrt{2k+2}-1} \frac{d^{k+3/2-\sqrt{2k+2}}}{d^{k+2-\sqrt{2k+2}}} \\ &\le \frac{\varpi}{\sqrt{d}} \frac{\mu_{k-\ell+1}}{N_{k-\ell+1}} = \frac{1}{\beta} \frac{\mu_{k-\ell+1}}{N_{k-\ell+1}} \le \left(\frac{1}{\beta}\right)^\ell \frac{t}{N_k} = o(1), \qquad (55)\end{aligned}$$

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as $\varpi T_f/d^k = o(1)$ and $\beta = \sqrt{d}/\varpi$. Equation (34) becomes

$$\sum_{\ell=j^*+1}^{i} \frac{L^*_{k-\ell}}{N_{k-\ell}} \le \frac{O(\omega^3)}{N_{k-j^*}} + 2\sum_{\ell=j^*+1}^{i} \left(\frac{1}{\beta}\right)^{\ell} \frac{t}{N_k} = \frac{O(\omega^3)}{N_{k-j^*}} + O\left(\frac{t}{\beta^i N_k}\right)$$
$$\le \left(\frac{\log^3 n}{n^{1-o(1)}}\right) + \left(\frac{\varpi T_f}{d^k}\right)$$
$$\le O\left(\frac{\varpi\sqrt{k}}{d^{\sqrt{2k+2}-3/2}}\right) = O\left(\frac{\sqrt{k}}{d^{\sqrt{2k+2}-2}}\right) = o(1).$$

The ω^3 term comes from using Lemma 3 to bound the size of $L^*_{k-\ell}(t)$ for $t \le t_{k-\ell}(\omega)$. The last line follows on assuming $k \le (\log d/\log\log d)^2$.

It now follows from the proof in Section 3.2 that if $k - j^* + 1 \le i \le k$ and $t \ge t_i(\omega)$ then $\mu_i(t) \to \infty$ suitably fast. We hence obtain that $\mathrm{E}\,\widetilde{L}_i(t) \sim \mu_i(t) \sim \mathrm{E}\,\widehat{L}_i(t)$, and so we have $L_{k-\ell}(t) \sim \widehat{L}_{k-\ell}(t) \sim \mu_{k-\ell}(t)$.

4.4. The finish time in the growing layers model.

We now turn to the proof of (41). We first give a bound ignoring rounding errors on the value of j^*, and subsequently give a correction. We show that, w.h.p., a path (of occupied vertices) grows back to the source from the *first vertex to be occupied* in level $k - j^*$, thus halting the process; and, moreover, this occurs *before a second vertex becomes occupied* in level $k - j^*$.

4.4.1. *Proof ignoring rounding.* Using (51) we know that $t_1(k - j^*) = CT_f$ for some constant C. Let $t' = (1-\varepsilon)t_1(k-j^*)$, then arguing as in Proposition 5, we have that at t', as by assumption $k \to \infty$,

$$\mathrm{E}\,\widehat{L}_{k-j^*}(t') \le \mu_{k-j^*}(t') \sim (1-\varepsilon)^{k+1} \le \mathrm{e}^{-\varepsilon k} = o(1).$$

Thus, at $t' = C'T_f$, w.h.p. level $k - j^*$ is unoccupied.

Let t_0 be the first step t for which $L_{k-j^*}(t) = 1$, where w.h.p. $t_0 \ge t'$. Then either $t_0 \le t_1(k-j^*)$, or if not we upper bound t_0 as follows. Consider the probability ϕ that a particle halts at level $k - j^*$. This is given by

$$\phi = \frac{L_{k-(j^*-1)}(t_1)}{N_{k-(j^*-1)}} \sim \frac{Cd^{1/2}}{d^{k-j^*+1}} = \frac{C}{d^{k+3/2-\sqrt{2k+2}}}.$$

Then the probability no particle halts there in a further $t_1(k-j^*) = O(T_f)$ steps is (see (51))

$$(1-\phi)^{t_1} \le \mathrm{e}^{-t_1\phi} = \mathrm{e}^{-C'\sqrt{2k+2}} = o(1),$$

where $C' \sim C/e$ and we assume $k \to \infty$.

Let u be the vertex in level $k - j^*$ containing the unique particle halted at t_0. Construct a path back from u to the source as follows. Wait until a particle halts at w_{k-j^*-1} in level $k - j^* - 1$ by choosing edge $w_{k-j^*-1}u$. The expected time for this is d^{k-j^*}. In a further expected time d^{k-j^*-1}, the path will extend backwards, as a particle will halt in level $k - j^* - 2$ by choosing an edge to w_{k-j^*-1}, etc. Thus, in a further

$$T = d^{k-j^*} + d^{k-j^*-1} + \cdots + d = d^{k-j^*}\left(\frac{1-1/d^{k-j^*}}{1-1/d}\right) = \Theta(d^{k-j^*}) = \Theta\big(d^{k+1-\sqrt{2k+2}}\big)$$

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expected steps there will be a path $vw_1 \cdots w_{k-j^*-1}u$ of halted particles extending from the source v to vertex u, thus stopping the DLA process (if it has not already halted).

The value of T given above is $O(T_f/\sqrt{kd})$, and we conclude from the Markov inequality that the source will be occupied in at most $t_0 + (kd)^{1/4}T = C''T_f$ steps, with probability $1 - O(1/(kd)^{1/4})$. This completes the proof that $t_f = \Theta(T_f)$ w.h.p.

We next give the proof of Proposition 2(2). The expected number of steps (after t_0) needed to create another halted particle in level $k - j^*$ is

$$\frac{1}{\phi} = \Theta\big(d^{k+3/2-\sqrt{2k+2}}\big) = \Theta\big(Td^{1/2}\big),$$

whereas, w.h.p., on the assumption that $d \to \infty$, it is in at most $(kd)^{1/4}T$ further steps after t_0 the process has halted as claimed. This occurs before a second vertex can become occupied in level $k - j^*$.

4.4.2. *The effect of rounding error on T_f in the growing layers model.* Recall that j^* is the solution to $(j^*+1)^2 = (2k+2)$ as given in (49). Integer solutions exist for some values of k. To see this, let a be a positive integer, and put $j^* + 1 = 2a$ so that $k = 2a^2 - 1$; this holds for example for $k = 1, 5, 17, 31$.

In the case that j^* is not an integer, we require the value of j given by $\lfloor j^* \rfloor$ or $\lceil j^* \rceil$ which maximizes $t_1(k-j)$. Expand $f(j)$ in (48) about j^*, giving

$$\begin{aligned} f(j^*+h) &= f(j^*) + hf'(j^*) + \frac{h^2}{2}f''(j^*+\theta h) \\ &= (j^*)^2 + 0 - h^2\frac{2k+2}{(j^*+1+\theta h)^3} \\ &= (j^*)^2 - \frac{h^2}{\sqrt{2k+2}}\left(1 + O\left(\frac{h}{(j^*)^3}\right)\right). \end{aligned}$$

Let $\varepsilon = \min(j^* - \lfloor j^* \rfloor, \lceil j^* \rceil - j^*)$, then by the above T_f is altered to $T_f(\varepsilon) = T_f d^{-\varepsilon^2/\sqrt{2k+2}(1+o(1))}$, other arguments about t_f being unchanged.

4.5. Theorem 2: existence of a unique connecting path at t_f.

Finally, we prove that the directed tree rooted at level k containing the connecting path from source to sink, consists uniquely of that path. The proof is similar to Lemma 6 for the equal layers model. At $t = t_f$, w.h.p. there is a unique path from level $k - j^*$ to level zero, so that level $k - j^* + 1$ plays the role of level one. By analogy with Section 3.4 and (40), etc., and with $\omega = 6\log n$,

$$\mathrm{P}(u_{k-j^*+1} \in R_{k-j^*+1}(t)) = \frac{Z_{k-j^*+1}(t)}{L_{k-j^*+1}(t)} \le \frac{\omega}{\mu_{k-j^*+1}(t)} \le O\left(\frac{\omega^2}{d^{1/2}}\right),$$

where we used an earlier result from (52) that $\mu_{k-j^*+1}(t_1) = Cd^{1/2}$.

Thus, as $t_f \le \varpi t_1(j^*)$, where $\varpi = o(d^{1/4})$, $j^* = \sqrt{2k+2} - 1$, and $t_1(k-j^*)$ is given by (51), then

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$$\mathrm{P}((v, u_k)\text{-path is blue}) \geq 1 - O\left(\frac{\omega\varpi}{d^{1/2}} + \sum_{j=1}^{j^*-2} \frac{\varpi t_f}{d^{k-j}}\right)$$
$$= 1 - O\left(\frac{\omega\varpi}{d^{1/2}}\right) - O\left(\frac{\varpi\sqrt{k}}{d^{3/2}}\right) = 1 - o(1),$$

as $k \leq (\log d/\log\log d)^2$.

5. Analysis of DLA on Cayley trees

Things are easier from now on for several reasons. The values of the parameters needed in the analysis of DLA on trees have been derived in the previous section. Furthermore, the DLA process corresponds in some sense to a percolation on the corresponding infinite tree which is halted at a fixed level. Finally, for brevity, we omit any structural results about the connecting path.

5.1. Trees with large branching factor: proof of Theorem 3(1)

Let $G = G(k, d)$ be a labelled tree with branching factor d and final level k, where $d^k = n$. The source of the particles is the unique vertex v at level zero. An artificial sink vertex z placed at level $k+1$ is attached to the vertices at the final level k.

We borrow several ideas from the growing layers model. Let $j^* = \sqrt{2k+2} - 1$ as in (50), and as in (41), let T_f be given by

$$T_f = \sqrt{k} d^{k+3/2-\sqrt{2k+2}}. \tag{56}$$

We first ignore rounding on j^* and prove w.h.p. that $t_f = \Theta(T_f)$. Correcting for rounding then gives Theorem 3(1).

By symmetry, the expected number of particle walks arriving at a given vertex in level $k - j^*$ by step T_f, is $T_f/d^{(k-j^*)} = \sqrt{kd}$. However, the expected number arriving at a vertex w at level $k - j^* + 1$ is

$$\frac{T_f}{d^{k-j^*+1}} = \sqrt{\frac{k}{d}} = o(1), \tag{57}$$

provided $k \ll d$, which we assume to be true. In order for w to be occupied, the subtree rooted at w must contain a path from w to level k on which all j^* vertices are occupied. By the Markov inequality, the event that $j^* = \Theta(\sqrt{k})$ particles arrive at w by step T_f, has probability $O(1/\sqrt{d})$. Thus, there should be few vertices in level $k - j^* + 1$ with such a path to level k containing j^* halted particles.

Consider t particles percolating downwards from the root of the *infinite d-ary tree* $G(d, \infty)$. Particle s starts at step s and each particle transitions one edge at each step, so that they never collide.

We approximate the distribution of the number of particles arriving at a vertex w of layer $k - j$ in t steps, where eventually we assume $j \leq j^* - 1$. Let $X_w = X_w(k - j, \ell, t)$ be the indicator for the event that exactly ℓ particles arrive at vertex w of layer $k - j$ at or before step t, then

$$\mathrm{E}\, X_w(k - j, \ell, t) = \binom{t}{\ell} \left(\frac{1}{d^{k-j}}\right)^{\ell} \left(1 - \frac{1}{d^{k-j}}\right)^{t-\ell}. \tag{58}$$

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Let $X = X(k-j, \ell, t)$ where $X = \sum_{w \in S_{k-j}} X_w$ is the number of vertices at level $k-j$ which have exactly ℓ particles reaching them. Thus $\mathrm{E}\, X = d^{k-j}\mathrm{E}\, X_w$, and

$$\begin{aligned}\mathrm{E}\, X(X-1) &= \left(d^{k-j}\right)\left(d^{k-j}-1\right)\binom{t}{2\ell}\binom{2\ell}{\ell}\left(\frac{1}{d^{k-j}}\right)^{\ell}\left(\frac{1}{d^{k-j}-1}\right)^{\ell}\left(1-\frac{2}{d^{k-j}}\right)^{t-2\ell} \\ &= \left(1+O\left(\frac{\ell}{d^{k-j}}\right)-O\left(\frac{t}{d^{2(k-j)}}\right)-O\left(\frac{\ell^2}{t}\right)\right)(\mathrm{E}\, X)^2,\end{aligned}$$

and so

$$\mathrm{Var}\, X = \mathrm{E}\, X + O\left(\frac{\ell}{d^{k-j}}\right)(\mathrm{E}\, X)^2.$$

Next let $\ell = j+1$ so that $X(k-j, \ell, t) = X(k-j, j+1, t)$. As $\mathrm{E}\, X = d^{k-j}\mathrm{E}\, X_w$, using (58),

$$\mathrm{E}\, X = \frac{t^{j+1}}{(j+1)!}\left(\frac{1}{d^{k-j}}\right)^{j}\left(1-O\left(\frac{j^2}{t}\right)\right)\left(1-O\left(\frac{t-j}{d^{k-j}}\right)\right).$$

In what follows we choose $j = j^* - 1$ and $k^2 \le t \le CT_f$ where $C = o(\sqrt{d/k})$ so that, by (57), $t/d^{k-j^*+1} \le C\sqrt{k/d} = o(1)$. In addition, $(j^*)^2/t = O(1/k) = o(1)$.

With these choices, *it will be the case that*

$$\mathrm{E}\, X \sim \frac{t^{j+1}}{(j+1)!}\left(\frac{1}{d^{k-j}}\right)^{j}. \tag{59}$$

Furthermore, again by (57)

$$\frac{j}{d^{k-j}}\mathrm{E}\, X \sim \frac{1}{j!}\left(\frac{t}{d^{k-j}}\right)^{j+1} = o(1),$$

so that

$$\mathrm{Var}\, X = \mathrm{E}\, X + O\left(\frac{j}{d^{k-j}}\right)(\mathrm{E}\, X)^2 = (1+o(1))\mathrm{E}\, X. \tag{60}$$

This being the case, by the Chebychev inequality we have that w.h.p. $X \sim \mathrm{E}\, X$.

Continuing our discussion in the infinite d-ary tree $G(d, \infty)$, let w be a vertex in level $k-j$ of $G(d, \infty)$. Let u be a vertex at level k contained in the subtree R_w rooted at w, and let $H(w, u)$ be the unique path $w = w_0 w_1 \cdots w_j = u$ from w in level $k-j$ to u in level k. Let $1 \le s_0 \le s_1 \le s_2 \le \cdots \le s_j \le t$ be the labels of particles which have arrived at w at these steps. Assume that on entering the subtree R_w, the particles behave as follows. Particles s_0 and s_1 transition all edges of H, particle s_2 transitions (at least) edges $w_0 w_1 \cdots w_{j-1}$; in general, s_i transitions (at least) $w_0 w_1 \cdots w_{j-i+1}$, and s_j transitions $w_0 w_1$. The probability of this is

$$P(s_0, s_1, \ldots, s_j) = \frac{1}{d^j}\frac{1}{dd^2 \cdots d^j}. \tag{61}$$

As there are d^j vertices at level k which belong to the subtree R_w rooted at w, then provided exactly $j+1$ particles have arrived at w, the probability $P_w(j+1)$ that such a transition ordering exists is

$$P_w(j+1) = \frac{1}{dd^2 \cdots d^j} = \frac{1}{d^{j(j+1)/2}}.$$

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For distinct vertices w, w' in level $k-j$ which contribute to X (i.e. exactly $j+1$ particles enter the given vertex in the given time period), the subsequent ordered transition events which occur in the subtrees R_w and $R_{w'}$ are independent. Let $Z(k-j, j+1, t)$ be the number of vertices in level $k-j$ of $G(d,\infty)$ for which exactly $j+1$ particles enter the vertex and which satisfy the ordering. Then $\mathrm{E}\, Z = P_w(j+1)\mathrm{E}\, X$ and

$$\mathrm{E}\, Z(k-j, j+1, t) \sim \frac{t^{j+1}}{(j+1)!}\left(\frac{1}{d^{k-j}}\right)^j \frac{1}{d^{j(j+1)/2}} = \frac{t^{j+1}}{(j+1)!}\frac{1}{d^{kj-j(j-1)/2}} = \mu_{k-j}(t), \qquad (62)$$

where $\mu_{k-j}(t)$ is given by (43). We choose $j = j^* - 1$ in $P_w(j+1)$, and use the values of $\mathrm{E}\, X$ and $\mathbf{Var}X$ derived via (59) and (60). As remarked previously, with these values, w.h.p. $X \sim \mathrm{E}\, X$. Thus, as long as $\mu_{k-j}(t)$ is sufficiently large, w.h.p. $Z \sim \mu_{k-j}(t)$.

Return now to the finite Cayley tree $G(k,d)$. For some vertex in level $k-j$ to be occupied at step t there must be some sequence $(s_0, s_1, \ldots, s_j)$, which satisfies the ordered transition construction given above for $G(d,\infty)$. Indeed, on contact with the sink z, particle s_0 will halt in level k, causing s_1 to halt in level $k-1$ and so on, until s_j halts in level $k-j$.

Say a path of occupied vertices from vertex w to level k is *an exact path*, if it is the unique occupied path in the subtree R_w rooted at w. Thus, the paths from level $k-j$ to level k counted by Z are *exact* in $G(d,k)$. Informally, as t tends to T_f, w.h.p. there will be many exact paths at level $k-j = k-j^*+1$ waiting to extend backwards towards the source.

5.1.1. *Lower bound on t_f.* The above calculations are used to establish an upper bound on the finish time. For a lower bound, a similar but cruder approach can be used. The total number of $j+1$ tuples which percolate a given vertex in level $k-j$ and form an ordered transition event in $G(d,\infty)$ is

$$d^{k-j}\binom{t}{j+1}\left(\frac{1}{d^{k-j}}\right)^{j+1}\frac{1}{d^{j(j+1)/2}} \sim \mu_{k-j}(t). \qquad (63)$$

Here $\mu_{k-j}(t)$ is as defined in (43); this latter calculation being made without any regard for exactness.

Let $j = j^*$ and $t' = (1-\varepsilon)t_1(k-j^*)$ where ε is a small positive constant. Using (63), and as by definition $\mu_{k-j^*}(t_1(k-j^*)) \sim 1$, at t' the expected number of paths between level $k-j^*$ and level k consisting of halted particles is upper bounded by

$$\mu_{k-j^*}(t') = (1-\varepsilon)^{j^*+1}\mu_{k-j^*}(t_1(k-j^*)) \le \mathrm{e}^{-\varepsilon j^*} = o(1). \qquad (64)$$

We conclude that at t' no such path exists w.h.p. and levels $0, 1, \ldots, k-j^*$ are empty. By (51), $t_1(k-j^*) = CT_f$, for some constant $C > 0$, so w.h.p. the finish time $t_f \ge t' \ge C'T_f$, for some constant C'.

Note. This calculation is valid for any $d \ge 2$, and will be reused in the next section for the case where d is finite.

5.1.2. *Upper bound on t_f.* Let $t_1 = t_1(k-j^*)$. Using (52), $\mu_{k-j^*+1}(t_1) = C_1\sqrt{d}$, for some positive constant C_1. Thus, w.h.p., $Z(k-j^*+1, j^*, t') = \Theta(\sqrt{d})$ which is large, provided $d \to \infty$.

A path of occupied vertices back from the sink to level i, is a *leading path at level i* if all vertices at level $i-1$ are unoccupied. If we increase t above t_1, then either some leading

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path is already at level $i \le k - j^*$, or level $k - j^*$ is still unoccupied. If the latter, how long does it take in $G(d, \infty)$ for a particle to hit a vertex with an exact path in level $k - j^* + 1$ as counted by $Z(k - j^* + 1, j^*, t_1)$? The hitting probability of such vertices at a given step is $p_h = Z(t_1)/d^{k-j^*+1}$, so

$$\mathrm{E}\, T(\text{wait}) = \frac{1}{p_h} \sim \frac{d^{k-j^*+1}}{\mathrm{E}\, Z(t_1)} = \Theta\left(\frac{T_f}{\sqrt{k}}\right) = o(T_f), \tag{65}$$

assuming $k \to \infty$. Thus, the probability we wait a further T_f steps is $o(1)$. Once a leading path is at $i \le k - j^*$ the same waiting time argument as used in Section 4.4 applies, the expected waiting times reducing at each step. In conclusion, w.h.p. the source vertex is now occupied in $\Theta(T_f)$ steps after t_1, completing the proof of Theorem 3(1).

Note. We found t_f on the assumption that j^* is an integer. In the case that it is not an integer, we use the rounding correction of (at most) $d^{O(1/\sqrt{k})}$ derived previously for the growing layers model. The value of $t_f \le CT_f$ used above with $C = o(\sqrt{d/k})$ is large enough to absorb any rounding errors.

5.2. Trees with constant or slowly growing branching factor: proof of Theorem 3(2)

In the previous section we assumed that $k \ll d$. In this section the assumption $k \ll d$ may no longer hold. For example, when $d = 2$, $k = \log n/\log 2$.

We always assume $k \to \infty$ with n, even if arbitrarily slowly, which places an upper limit on d. However, if $k \ge \log^2 d$, for example, then the rounding error is $d^{1/\sqrt{k}} = \Theta(1)$, so for d constant or growing suitably slowly (say $d \le k^2$) we can ignore rounding.

Proposition 3. *For $d \ge 2$, let $G = G(k, d)$ be the Cayley tree with branching factor d and height k, where $d^k = n$. Let T_f be given by*

$$T_f = \sqrt{k} d^{k+3/2-\sqrt{2k+2}}.$$

Let t_f be the finish time of DLA on G. Then w.h.p. $T_f/C \le t_f \le CT_f$ for some constant $C \ge 1$.

Proof. The lower bound on the finish time t_f follows directly from the lower bound proof in the previous section, Section 5.1. The upper bound in the previous section used a result from (52) that when $\mu_{k-j^*}(t_1) \sim 1$, then $\mu_{k-j^*+1}(t_1) = Cd^{1/2}$; this latter value being large when d is. However, then d is constant $\mu_{k-j^*}(t)$ and $\mu_{k-j^*+1}(t)$ are of the same order, so we need to compare two levels whose distance apart, h, is larger than one. Thus, for the upper bound, let $j = j^* - h$ where $h = \lceil 1/2 + \log_d k \rceil$. When $k \ll d$ as in Section 5.1, then $h = 1$, which led to the argument based on the occupancy of levels $k - j^*$ and $k - j^* + 1$. The largest value of k satisfying $h = 1$ is when $k = d^{1/2}$; in which case as $d^k = n$ we have $d \to \infty$ and, thus, $k \ll d$, the case of the previous section. We assume henceforth that $h \ge 2$.

The value of μ_{k-j*} given by (46), satisfies $\mu_{k-j+1}(t) = [(j+1)d^{k-j+1}/t] \cdot \mu_{k-j}(t)$. Iterate this, and evaluate at $t = t_1(k - j^*) = (1 + O(1/j))\mathrm{e}^{-1}\sqrt{2k+2}\, d^{k-j^*+1/2}$ from (51). As $j^* + 1 = \sqrt{2k+2}$, then on the assumption that $h^2 \ll j^*$,

$$\frac{\mu_{k-j^*+h}(t)}{\mu_{k-j^*}(t)} = \frac{(j^*+1)\cdots(j^*+h)}{t^h} d^{k-j^*+1} \cdots d^{k-j^*+h} = \Theta(1)\mathrm{e}^h d^{h^2/2}. \tag{66}$$

This ratio is $\omega(1)$, as either (i) $d \to \infty$ or (ii) d is constant and as $h = \lceil 1/2 + \log_d k \rceil$ and $d^k = n$ then $k = \log_d n$, this implies that $h = \omega(1)$.

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In either case, the calculations for X, Z leading to (62) in the previous section hold for the cases given here, with $j^* - h$ now replacing $j^* - 1$. At $t_1 = t_1(k - j^*)$, where $t_1 = \Theta(T_f)$ then $\mu_{k-j^*+h}(t_1) = \Theta(\mathrm{e}^h d^{h^2/2}) \to \infty$, by the comments below (66). Thus, w.h.p. $Z(k - j^* + h, j^*, t_1) = (1 - o(1))\mu_{k-j^*+h}(t_1)$, and there are $Z(t_1) = \Theta(\mathrm{e}^h d^{h^2/2})$ exact paths from level $k - j^* + h$ to level k.

Choose a step $t' \geq T_f > t_1$. Then either some leading path of halted particles already extends to a level i where $i < k - j^* + h$, or all vertices in these levels are still unoccupied. In the latter case, as in (65), in expectation, it takes at most

$$\mathrm{E}\, T(\text{wait}) = \tau = \frac{d^{k-j^*+h}}{Z(t_1)} = \Theta(1)\frac{d^{k-j^*+h}}{\mathrm{e}^h d^{h^2/2}}$$

further steps for one of these paths to extend back to the source.

The proof in the previous section depended on the ratio $T_f/d^{k-j^*+1} = o(1)$ (see (57)). Recall that $j^* = \sqrt{2k+2} - 1$, and the generic level $k - j$ in the proof between (58) and (62) is now $k - j^* + h$, so that

$$\rho_h = \frac{T_f}{d^{k-j^*+h}} = \frac{\sqrt{k}d^{k+3/2-\sqrt{2k+2}}}{d^{k-j^*+h}} = \frac{\sqrt{kd}}{d^h}. \tag{67}$$

The final step is now to prove that $\tau = o(T_f)$. From the expression for ρ_h in (67), we have that

$$d^{k-j^*+h} = \frac{d^h}{\sqrt{kd}}T_f,$$

and as we assumed $h \geq 2$,

$$\tau = \Theta(1)\frac{d^{k-j^*+h}}{\mathrm{e}^h d^{h^2/2}} = \Theta(1)\left(\frac{d}{ed^{h/2}}\right)^h \frac{1}{\sqrt{kd}}T_f \leq \Theta(1)\frac{1}{\sqrt{kd}}T_f = o(T_f),$$

as required. □

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