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RESEARCH ARTICLE

# A gluing operation for dimer quivers

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**Abstract**
In this article we introduce a gluing operation on dimer models. This allows us to construct dimer quivers on arbitrary surfaces. We study how the associated dimer and boundary algebras behave under the gluing and how to determine them from the gluing components. We also use this operation to construct homogeneous dimer quivers on annuli.

## 1. Introduction

Dimer models on surfaces are oriented graphs embedded in surfaces which arise from gluing oriented discs along common arrows. They have been studied in different contexts, including statistical mechanics [10], integrable systems [6] and mirror symmetry [2]. More recently, dimer models on surfaces with a boundary have been considered; cf. [1, 5] in the context of cluster algebras and cluster categories, as they arise as quivers of clusters with frozen variables or as quivers of cluster-tilting objects with projective-injective summands.

For dimer quivers in the disc, Postnikov diagrams provide a convenient way to describe the cluster structure associated with the Grassmannian $\mathrm{Gr}(k,n)$ of $k$-dimensional subspaces of $\mathbb{C}^n$. Postnikov diagrams are a collection of $n$ oriented curves between the $n$ vertices of the disc, inducing a permutation of the $n$ vertices. If this permutation sends vertex $i$ to vertex $i+k$ (reducing modulo $n$), we say it is a $(k,n)$-diagram; these diagrams are used in the description of the cluster algebra structure on the coordinate ring of the Grassmannian [9] or of the cluster-tilting objects in the Grassmannian cluster category [1]. A natural question is how to generalise these constructions to other surfaces while retaining control of their algebraic properties. The annulus is the first non-simply connected case, and in this setting, it is known that gluing along boundary intervals changes the topology of the ambient surface and turns boundary arrows into internal arrows along the resulting seams [1].

The first aim of this paper is to introduce a gluing operation for dimer quivers and to describe how the associated dimer and boundary algebras behave under gluing. Proposition 5.14 shows that the glued dimer algebra is determined by the components, and Theorem 5.16 shows that, under thinness hypotheses, the glued boundary algebra can also be recovered from the components. In particular, Corollary 5.17 gives a recursive way to determine the boundary algebra of a Postnikov diagram in the disc.

The second aim is to use this gluing construction to build homogeneous dimer quivers on annuli in the sense that the associated strand diagram induces a permutation of the form $i \mapsto i+k$ along each of the two components of the boundary; we say that such a diagram is of degree $k$. With this in mind, we introduce a family of bridge quivers $\Theta_k$ and glue them to $(k,n)$-diagrams in the disc. Theorem 7.2 shows that the resulting quivers on the annulus correspond to weak Postnikov diagrams of degree $k$, and

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Theorem 7.14 gives an explicit presentation of the resulting boundary algebra. Our gluing construction can be used to construct dimer quivers on general surfaces from those on "simpler" surfaces. It would be interesting to do this in a way such that the associated (weak) Postnikov diagrams are also homogenous in the above sense, that is, of degree $k$.

The paper is organised as follows. Sections 2 and 3 contain the necessary background on dimer quivers, Postnikov diagrams and their associated algebras. In Section 4, we recall weak Postnikov diagrams on arbitrary surfaces. Section 5 introduces the gluing operation and studies its effect on dimer and boundary algebras. Section 6 introduces the bridge quivers $\Theta_k$. In Section 7, we use these quivers to construct weak Postnikov diagrams of degree $k$ on annuli and to determine their boundary algebras.

In particular, we show that the dimer algebras (Definition 2.7) obtained through gluing are determined by the components used in the gluing (Proposition 5.14). Additionally, when certain consistency conditions are met, we also determine the glued boundary algebra from the components in Theorem 5.16. Notably, this allows the boundary algebra associated arbitrary Postnikov diagrams on the disc to be calculated recursively using Corollary 5.17. In Section 6, we introduce a family of consistent dimer quivers on the disc, called $k$-bridges. These will be instrumental in the construction of dimer quivers for the annulus in Section 7. In this final section, we glue bridge quivers with certain homogeneous dimer quivers on the disc, called $(k, n)$-diagrams. We prove that the resulting dimer quivers on the annulus correspond to weak Postnikov diagrams of degree $k$ in Theorem 7.14.

## 2. Dimer quivers

A quiver is an oriented graph $Q = (Q_0, Q_1)$, with a finite vertex set $Q_0$ and a finite set of arrows (oriented edges) $Q_1$. We denote by $Q_{cyc}$ the set of oriented cycles in $Q$.

**Definition 2.1.** *A quiver with faces is a quiver $Q = (Q_0, Q_1)$ together with a set of faces $Q_2$ and a map $\partial : Q_2 \to Q_{cyc}$ sending a face $F \in Q_2$ to its boundary $\partial F \in Q_{cyc}$.*

We consider a special class of quivers with faces obtained from gluing a collection of oriented cycles along arrows in a consistent way. They were first formalised in [1] as dimer models with a boundary. We will call them *dimer quivers*.

We first recall the notion of the incidence graph of a vertex of $Q$: For vertex $i \in Q_0$ of the quiver $Q$, the *incidence graph* of $Q$ at $i$ has as vertices the set of arrows incident with $i$ and contains an edge between vertices $v_\alpha$ and $v_\beta$ if the path

$$\bullet \xrightarrow{\alpha} i \xrightarrow{\beta} \bullet$$

is part of the boundary $\partial F$ of a face $F \in Q_2$.

If $\alpha \in Q_1$, the *face multiplicity of* $\alpha$ is the number of faces such that $\alpha$ belongs to their boundary.

**Definition 2.2.** *A dimer quiver is a quiver with faces $(Q_0, Q_1, Q_2)$ where $Q_2$ can be expressed as the disjoint union of two sets $Q_2^+$ and $Q_2^-$ and which satisfies the additional properties:*

*(1) $Q$ has no loops.*
*(2) Every arrow $\alpha$ in $Q_1$ appears with face multiplicity one or two. In the latter case, if $F_1$ and $F_2$ are the two faces $F_1, F_2$, such that $\alpha \in \partial F_i$ for $i = 1, 2$, then $F_1 \in Q_2^+$ and $F_2 \in Q_2^-$ (or vice versa).*
*(3) The incidence graph of every vertex of $Q_0$ is (non-empty and) connected.*

Notice that a dimer quiver partitions the set of faces $Q_2$ according to the orientation of their boundary cycles in a way that the boundaries of any two adjacent faces have opposite orientations. See Figure 1 for an illustration.

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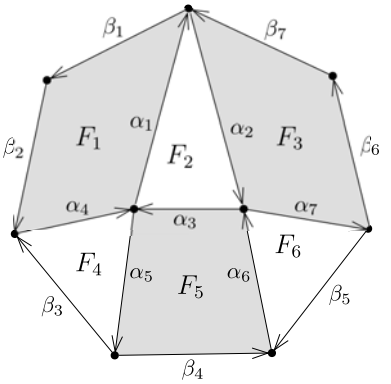



***Figure 1.*** *A dimer quiver with six faces.*

**Remark 2.3.** *Dimer quivers are the dual graphs to certain well-studied bipartite graphs on surfaces called dimer models (see [3, 7, 10], e.g.).*

If $\alpha$ has face multiplicity one in the dimer quiver $Q$, we call $\alpha$ a *boundary arrow;* otherwise, $\alpha$ is an *internal arrow*. A vertex $i \in Q_0$ is a *boundary vertex* if it is incident with a boundary arrow. Otherwise, $i$ is called *internal*.

**Example 2.4.** *Figure 1 shows an example of a dimer quiver with six faces. The partition of $Q_2$ is indicated by the shading:*

$$Q_2^+ = \{F_1, F_3, F_5\} \quad \text{and} \quad Q_2^- = \{F_2, F_4, F_6\}.$$

**Remark 2.5.** *Given a dimer quiver $Q$, we can form a topological space $\Sigma_Q$ by gluing the arrows bounding each face as directed by their orientation. The resulting space $\Sigma_Q$ is an oriented surface with boundary, where the faces in $Q_2^+, Q_2^-$ can be declared to have positive and negative orientations, respectively. The quiver $Q$ is said to be a dimer quiver in a disc, if $\Sigma_Q$ is homeomorphic to a disc.*

A *path* $p$ in a quiver $Q$ is a composition of arrows $p = \alpha_1\alpha_2 \cdots \alpha_s$ such that the endpoint of $\alpha_i$ is equal to the starting point of $\alpha_{i+1}$ for $i = 1, \ldots, s-1$ (writing the composition from left to right). We allow paths of length 0, that is, paths starting and ending at a vertex, without using any arrows. Such a path is called a *trivial path*. To any quiver $Q$, one can associate an algebra over $\mathbb{C}$, the so-called path algebra $\mathbb{C}Q$ of $Q$: it has as a basis the set of all paths of $Q$, including the trivial ones, with multiplication given by the concatenation of paths. Note that since the dimer quivers we consider contain-oriented cycles, the associated path algebras are infinitely dimensional.

For any oriented face $F \in Q_2$, the element $\partial F$ determines a cycle in $Q$, and so the set of all cycles defines an element

$$W_Q = \sum_{F \in Q_2^+} \partial F - \sum_{F \in Q_2^-} \partial F$$

in $\mathbb{C}Q_{cyc}$ called the *potential* of $Q$.

Internal arrows impose relations on the path algebra, given by the so-called cyclic derivatives of the potential:

**Definition 2.6.** *Let $\alpha$ be an internal arrow of $Q$, and let $\alpha\beta_1 \cdots \beta_s = \partial F_1 \in Q_{cyc}$ and $\alpha\gamma_1 \cdots \gamma_t = \partial F_2 \in Q_{cyc}$ be the two boundary cycles of the faces in $Q_2$ containing $\alpha$, with $F_1 \in Q_2^+$ and $F_2 \in Q_2^-$. Let $p_{F_1} := \beta_1 \cdots \beta_s$ and $p_{F_2} := \gamma_1 \cdots \gamma_t$ denote the complements of $\alpha$ in $\partial F_1$ and $\partial F_2$, respectively. Then the cyclic*

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*derivative of $W_Q$ with respect to $\alpha$ is defined by*

$$\partial_\alpha(W_Q) = p_{F_1} - p_{F_2}.$$

*The potential $W_Q$ determines an ideal of relations. We write*

$$I_W := \langle \partial_\alpha(W_Q) : \alpha \in Q_1, \quad \alpha \text{ internal} \rangle$$

*for the ideal of $\mathbb{C}Q$ generated by the cyclic derivatives with respect to internal arrows.*

**Definition 2.7.** *Let $Q$ be a dimer quiver; the dimer algebra $A_Q$ is defined to be the quotient of the path algebra by the relations arising from internal arrows $\alpha \in Q_1$:*

$$A_Q := \mathbb{C}Q/I.$$

Note that while many paths get identified in $A_Q$, the dimer algebra is still infinite dimensional in general. Indeed, by Lemma 2.8 below, the element $t = \sum_{i \in Q_0} u_i$ generates a polynomial subalgebra $\mathbb{C}[t] \subseteq Z(A_Q)$.

Any boundary cycle of a face is called a unit cycle. As a consequence of the definition, for any vertex $I \in Q_0$, the different unit cycles going through a vertex $i$ are all equivalent in $A_Q$. We write $u_i$ to denote the corresponding element in $A_Q$.

**Lemma 2.8.** *Let $t$ be the element of $A_Q$ given by $t := \sum_{i \in Q_0} u_i$, then $\mathbb{C}[t] \subseteq Z(A_Q)$, where $Z(A_Q)$ denotes the centre of $A_Q$.*

*Proof.* It suffices to show that $t$ commutes with any arrow $\alpha \in Q_1$. Suppose $\alpha$ is an arrow starting at vertex $i$ and terminating at vertex $j$. Then $t\alpha = u_i\alpha$ and $\alpha t = \alpha u_j$; therefore, we only have to show $u_i\alpha = \alpha u_j$. Let $F \in Q_2$ be a face whose boundary is given by $\partial F = \alpha p_F$. As $\alpha p_F$ is a unit cycle at $i$, we have $u_i\alpha = \alpha p_F \alpha$. Similarly, $p_F\alpha$ is a unit cycle at $j$, and so $\alpha p_F \alpha = \alpha u_j$. □

**Definition 2.9.** *Let $Q$ be a dimer quiver and $A_Q$ its dimer algebra. Let $e_1, \ldots, e_n$ be the idempotent elements of $A_Q$ corresponding to the trivial paths at the boundary vertices of $Q$ and set $e = \sum_{i=1}^n e_i$. Then the boundary algebra of $Q$ is defined to be the algebra given by:*

$$B_Q = eA_Qe.$$

This is often called the 'idempotent subalgebra of $A_Q$' as we pre- and postcompose all paths with the idempotent element $e$ of $A_Q$.

Note that the boundary algebra of a dimer quiver is also infinite dimensional in general.

## 3. Postnikov diagrams

In this section, we consider certain diagrams in the disc: the surface $\Sigma$ is a disc with $n$ marked points on the boundary. These diagrams correspond to dimer quivers in the disc that exhibit global properties which will later be used to determine the corresponding boundary algebra.

**Definition 3.1.** *A Postnikov diagram (in the disc) consists of a collection of oriented curves (strands) in the disc with marked points on its boundary, where each marked point is the source of exactly one strand and the target of exactly one strand. The curves are considered up to isotopy, fixing endpoints. The diagram must satisfy the following axioms.*

*(1) There are finitely many intersections, and each intersection is a transversal of order two.*
*(2) Following a fixed strand, the strands that cross it must alternate between crossing from the left and crossing from the right.*

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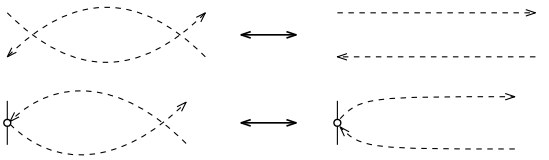


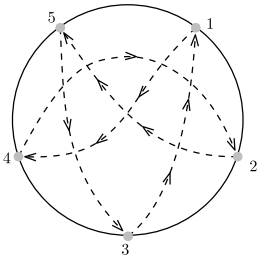
***Figure 2.*** *Local twisting and untwisting moves.*

*(3) Strands have no self-intersections.*
*(4) If two strands cross at distinct points, the corresponding bounded region forms an oriented disc.*

*The label of a strand is the number i of the marked point where it starts.*

It is clear from the definition that a Postnikov diagram determines a permutation of the boundary vertices by $i \mapsto j$ if the strand starting at vertex $i$ terminates at vertex $j$. We call this the strand permutation $\pi_D$. Postnikov diagrams whose strand permutation takes the form $i \mapsto i + k \ (mod\ n)$ are of particular interest as they describe a cluster in the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(k, n)$ [10, Theorem 3]. We refer to such a diagram as a $(k, n)$-diagram. Note that such diagrams always exist: in the paper mentioned above, Scott gave an explicit construction of a $(k, n)$-diagram for every $k < n$.

**Example 3.2.** *The following is an example of a* $(3, 5)$*-diagram. The starting and endpoints of the strands are indicated by grey circles.*




Two diagrams are said to be *equivalent* if one can be obtained from the other through a sequence of local twisting or untwisting moves, which are illustrated in the following diagram. The lower diagram represents the boundary case, the boundary indicated on the left.

We call a Postnikov diagram *reduced* if it may not be simplified via untwisting moves (moving left to right in Figure 2). The diagram in Example 3.2 is a reduced Postnikov diagram.

Each diagram divides the interior of the disc into regions, which are the connected regions in the complement of the union of all strands. In the definition of a Postnikov diagram, we required that strands must alternate between crossing from left to right and right to left when following a particular strand. As a consequence, the interior regions can be classified as either:

(1) alternating if the strands forming its boundary alternate in orientation, or
(2) oriented if the strands are all oriented in the same direction (all clockwise or all anticlockwise).

We associate a label with each alternating region obtained from the Postnikov diagram. These labels are subsets of $[n] = \{1, 2, \ldots, n\}$ given by the strands: each strand divides the disc into a left and right region with respect to the strand orientation. This allows us to attach labels to the alternating regions: an alternating region is labelled by $i \in [n]$ if the strand starting at $i$ keeps the given region on its left.

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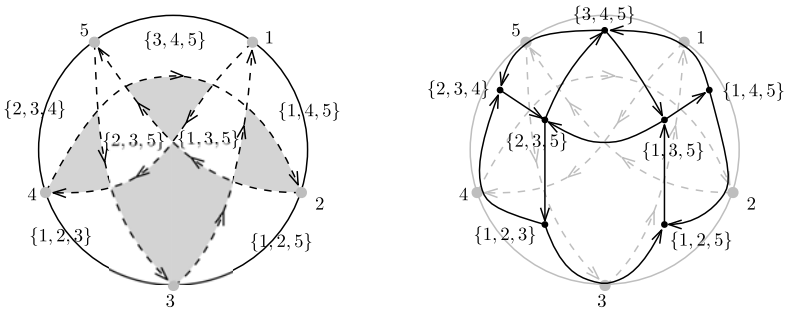



***Figure 3.*** *Postnikov diagram with its labels (left) and with the quiver (right).*



***Figure 4.*** *Associating arrows to intersections of strands.*

**Example 3.3.** *The Postnikov diagram from Example 3.2 has six oriented regions (shaded in grey on the left of Figure 3) and seven alternating regions, with labels as shown in both pictures. One can see that labels of the alternating regions on the boundary are of the form $i, i+1, i+2$ modulo $n$, for $n = 5$.*

We want to associate with every Postnikov diagram a dimer quiver. Its vertices are given by the alternating regions. The arrows arise from crossings: to each intersection point between strands of alternating regions, we associate an arrow between the corresponding vertices, with the direction described in the following diagram in Figure 4. The figure on the right represents the case where both regions are on the boundary.

**Definition 3.4.** *The quiver $Q_D$ associated with a Postnikov diagram $D$ has a vertex set $Q_0$ given by the alternating regions. The arrows are given by the intersection points of strands between alternating regions, with orientation as above.*

Notice that we allow arrows between boundary vertices.

**Example 3.5.** *The quiver $Q_D$ obtained from the Postnikov diagram of Example 3.2 recovers the dimer quiver from Figure 2.4. The quiver $Q_D$ is shown on the right of Figure 3. It is drawn on top of the Postnikov diagram, using solid lines, with the vertices corresponding to the alternating regions in $D$.*

We have seen how a Postnikov diagram gives rise to a quiver $Q_D$. In fact, $Q_D$ is a dimer quiver in the disc. See [1, Section 2] for more details.

The Postnikov diagram $D$ can be recovered from $Q_D$ by taking its strands to be the collection of *zig–zag paths* of the dimer model [6], as we recall here.

**Definition 3.6.** *Let $Q$ be a dimer quiver in the disc, with $n$ vertices on the boundary. We draw two short crossing-oriented segments on every arrow of $Q$, in the orientation shown on the left and in the middle of Figure 5. Inside each face, we connect each segment to the next one encountered when moving around the face according to its orientation. In this way we obtain $n$ oriented curves starting and ending at the boundary of the disc, as on the right of the figure. We write $D_Q$ for the resulting diagram. It is sometimes called the diagram of zig–zag paths in $Q$.*

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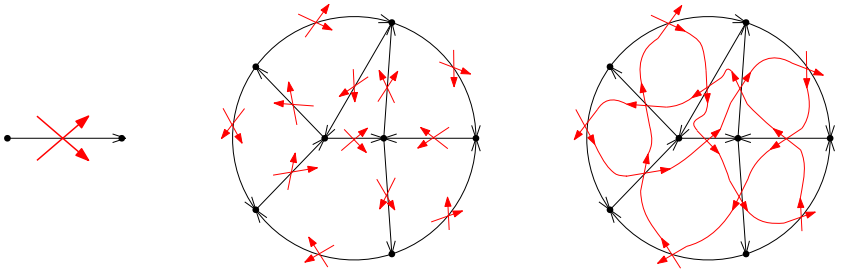


***Figure 5.*** *Associating a strand diagram to a dimer quiver.*

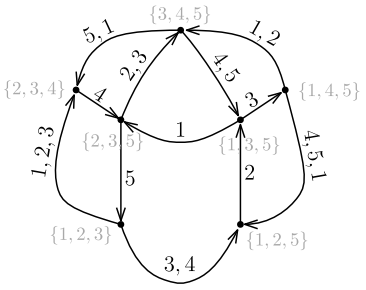



***Figure 6.*** *The weights on the arrows of a dimer quiver.*

In Section 4, we will lift this correspondence to the more general set-up of dimer quivers on arbitrary surfaces and of so-called 'weak strand diagrams'.

**Definition 3.7** (1, Section 4). *For any arrow $\alpha: I \to J$ in $Q_1(D)$, let $c$ be the label of the strand crossing $\alpha$ from right to left and let $d$ be the label of the strand crossing $\alpha$ from left to right, when looking in the direction of $\alpha$. In other words, $J = I \setminus \{c\} \cup \{d\}$. The weight of $\alpha$ is then defined to be the cyclic interval from $c$ to $d$ in $[n]$*

$$w_\alpha = [c, d).$$

*The weight of a path $p = \alpha_1 \cdots \alpha_r$ in $Q_D$ is the multiset union of the weights of the arrows in the path; equivalently, it is the sum of the indicator functions of the intervals $w_{\alpha_1}, \ldots, w_{\alpha_r}$. We write* $\mathrm{supp}\,(w_p) \subseteq [n]$ *for its support. A path is called sincere if* $\mathrm{supp}\,(w_p) = [n]$ *and insincere otherwise.*

**Example 3.8.** *To determine the weight of the arrow $\{1, 3, 5\} \to \{2, 3, 5\}$ of the quiver from Figure 6, observe that $c = 1$ and $d = 2$. So the weight is $[1, 2) = \{1\}$. The weights of all arrows of that dimer quiver are shown in Figure 6. Any arrow is an insincere path. However, as one can check, any unit cycle is a sincere path. The path from $\{3, 4, 5\}$ to $\{1, 2, 5\}$ via $\{1, 3, 5\}$ and $\{1, 4, 5\}$ has 4 and 5 with multiplicity two in its weight. It is insincere, since 2 does not lie in the support.*

**Definition 3.9.** *A dimer algebra $A_Q$ is called thin if for any pair of vertices $a, b \in Q$, there is a path $h \in e_a A_Q e_b$ such that $e_a A_Q e_b = \mathbb{C}[[t]]h$. In other words, there is a path $h : a \to b$ such that for any path $g: a \to b$ in $A_Q$, one has $g = ht^m$ for some $m \geq 0$. Such a path will be called minimal.*

**Remark 3.10.** *For a Postnikov diagram $D$ in the disc, insincere paths in $A_{Q_D}$ are minimal, cf. [BKM16, Corollary 9.4]. We will use this in the proof of Lemma 3.14(2).*

The following two results on the thinness of dimer algebras will be useful when working with the dimer algebras of Postnikov diagrams.

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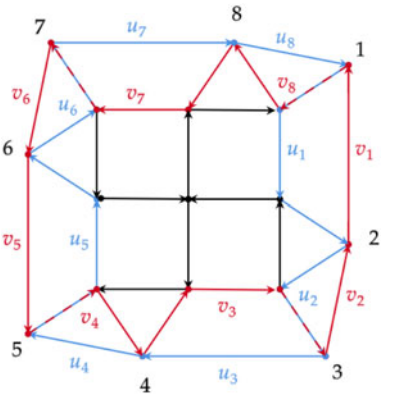



***Figure 7.*** *Paths $u_i$ (in blue) and $v_i$ (in red).*

**Proposition 3.11** (8, Proposition 2.11). *Let $D$ be a Postnikov diagram in the disc. If the associated quiver $Q_D$ is connected, then $A_{Q_D}$ is thin.*

**Lemma 3.12** (8, Lemma 2.13). *Let $Q$ be a connected dimer quiver such that $A_Q$ is thin. If $g{:}a \to b$ and $h{:}b \to c$ are paths in $Q$ such that the composition $gh$ is minimal, then both $g$ and $h$ are minimal.*

We now introduce some notation which will be useful when describing paths between boundary vertices.

**Notation 3.13.** *Let $D$ be a $(k,n)$-diagram in the disc, and let $\alpha$ be a boundary arrow in $Q_D$. As each boundary arrow $\alpha$ in a dimer quiver $Q$ belongs to precisely one face $F_\alpha \in Q_2$, we can uniquely refer to the complement $\widehat{\alpha}$ of $\alpha$ around $F_\alpha$ in the dimer algebra $A_Q$ (i.e., $\alpha\widehat{\alpha} = \partial F_\alpha$).*

*For each $i \in [n]$, we define paths $u_i : i \to i+1$ and $v_i : i+1 \to i$ as follows: let $\alpha$ be the boundary arrow between $i$ and $i+1$. If $\alpha$ is oriented clockwise (that is, going from $i$ to $i+1$, we set $u_i$ to be this boundary arrow. If $\alpha$ is oriented anticlockwise (i.e. $i+1 \to i$), we set $u_i$ to be $\hat{\alpha}$. Then $v_i : i+1 \to i$ is defined to be the complement of $u_i$ in the unit cycle at vertex $i$. Figure 7 gives an illustration of the paths $u_i$ and $v_i$ on a dimer quiver.*

*Take $m \in [n]$ and $h \geq 0$. Going forward, we will use the following shorthand for composing $h$ consecutive $u_i$'s ($v_i$'s) starting at the vertex $m$ on the boundary of $Q$:*

- $u_m^h := u_m u_{m+1} .. u_{m+h-1} : m \to m+h$
- $v_m^h := v_{m-1} v_{m-2} \ldots v_{m-h} : m \to m-h$

*where addition and subtraction in the indices are done modulo $[n]$. When the starting point is clear, we will drop the index $m$ from the notation. In particular, in Lemma 3.14(2), the path $u^h$ is $u_a^h$, starting at $a$ and ending at $b \equiv a+h$ and $v^{h'}$ stands for a path starting at $a$ and ending at $b \equiv a-h'$.*

**Lemma 3.14.** *Let $D$ be a $(k,n)$-diagram in the disc. The paths defined above satisfy:*

*(1)* $v_m^k = u_m^{n-k}$ *in $A_{Q_D}$ for any $m \in [n]$.*
*(2) For any pair of boundary vertices $a, b$ in $Q_D$, a minimal path from $a$ to $b$ in $A_{Q_D}$ is either $u_a^h$ or $v_a^{h'}$, where $0 \leq h, h' \leq n$, $b \equiv a+h \equiv a-h' \pmod n$, and $h+h' = n$.*

To obtain this result, one can explicitly determine the image of arrows under the inverse of the isomorphism given in [1, Theorem 11.2] or by analysis of perfect matching modules [Claim 3.4, [4]]. However, we provide a direct argument by associating weights to the arrows in $Q_D$.

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*Proof of Lemma* 3.14. (1) The vertices of $Q_D$ arise from $k$-subsets. For $a \in [n]$, we will write $I_a$ for the $k$-subset $\{a, a+1, \ldots, a+k-1\}$. Adjacent boundary vertices in $D$ are assigned the $k$-element subsets $I_a$ and $I_{a+1}$ of $[n]$ for some $a \in [n]$. If the boundary arrow between them is oriented clockwise, then $u_a$ is that arrow and has support $[a, a+k)$. If the boundary arrow is oriented anticlockwise, then it has support $[a+k, a)$ while the unit cycle through it is sincere; hence, its complement $u_a$ has support $[a, a+k)$. Similarly, every path $v : I_{a+1} \to I_a$ has support $[a+k, a)$. It is clear that the weight of each path $v : I_{a+1} \to I_a$ has the support $[a+k, a)$, and the weight of each path $u : I_a \to I_{a+1}$ has the support $[a, a+k)$. Therefore, the support of $v^k$ (ending at $I_a$) will be

$$\bigcup_{j=0}^{k-1} [a+k+j, a+j).$$

The cardinality of this set is $(n-k)+(k-1) = n-1$. Similarly, the weight of $u^{n-k}$ (starting at $I_a$) is

$$\bigcup_{j=0}^{n-k-1} [a+j, a+k+j)$$

with cardinality $k+(n-k-1) = n-1$. Thus, we have shown that $v^k$ and $u^{n-k}$ are insincere paths between fixed vertices in $A_Q$. Since there is only a unique insincere path between any two vertices by [1, Corollary 9.4], the claim follows.

(2) Set $d := |b-a|$. If $d \in \{k, n-k\}$, then the paths constructed in the proof of (1) are minimal between $I_b$ and $I_a$. If $d < k$, then $v^d$ is a subpath of $v^k$ and therefore insincere. By Remark 3.10, $v^d$ is minimal. If $d > k$, then we have $n-d < n-k$, and similarly, $u^{n-d}$ is a subpath of $u^{n-k}$ and therefore insincere and once again minimal. □

## 4. Diagrams on surfaces

The notion of a weak Postnikov diagram was introduced in [1], relaxing the conditions of Definition 3.1 and allowing arbitrary surfaces. Let $\Sigma$ be an oriented surface with marked points on its boundary. We assume that every connected component of the boundary contains at least one marked point. The surface $\Sigma$ may consist of several connected components.

**Definition 4.1.** *A weak Postnikov diagram on a surface $\Sigma$ consists of oriented curves in $\Sigma$, where each marked point is the source of a unique strand and the target of a unique strand. The diagram must also satisfy the following axioms:*

*(1) There are finitely many intersections, and each intersection is a transversal of order two.*
*(2) Following a fixed strand, the strands that cross it must alternate between crossing from the left and from the right.*

In particular, we allow strands to have self-intersections, and we do not ask for condition (4) of Definition 3.1 to hold – unoriented lenses may occur.

Weak Postnikov diagrams are considered up to isotopy that preserves crossings. Notice that a weak Postnikov diagram determines a permutation of the marked points of $\Sigma$.

Let $C_1, \ldots, C_t$ be the connected components of the boundary of $\Sigma$, for some $t > 0$. We label the marked points of $\Sigma$ clockwise around each connected component $C_j$ of the boundary of $\Sigma$ by $p_{j,1}, \ldots, p_{j,n_j}$.

**Example 4.2.** *Figure 8 shows two weak Postnikov diagrams on an annulus with three points on the outer boundary and two points on the inner boundary. In cycle notation, the permutations these diagrams induce are* $(132)$ *for the one on the left and* $(13\widetilde{12})$ *for the one on the right. Both permutations have fixed points.*

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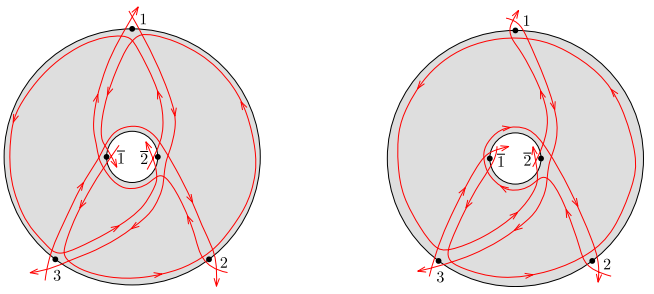



***Figure 8.*** *Two Postnikov diagrams on an annulus.*

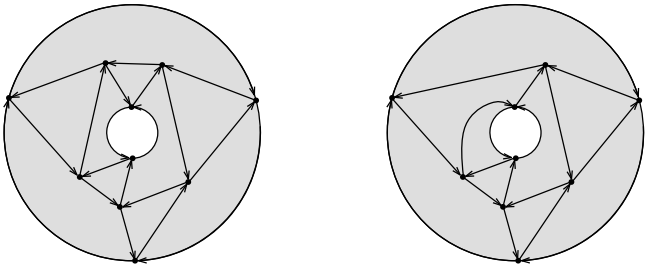


***Figure 9.*** *Dimer quivers of weak Postnikov diagrams.*

**Definition 4.3.** *A (weak) Postnikov diagram $D$ on a surface $\Sigma$ is said to be of degree $k$ if the permutation induced by $D$ is of the form $p_{j,i} \mapsto p_{j,i+k}$ (reducing modulo $n_j$) at every connected component $C_j$ of the boundary of $\Sigma$.*

The second picture in Figure 8 shows a weak Postnikov diagram of degree 2. To any weak Postnikov $D$ diagram on a surface, we can associate a quiver $Q_D$, as for the disc case (Definition 3.4). This is also a dimer quiver, following the reasoning of [1, Remark 3.4], so we have:

**Lemma 4.4.** *Let $D$ be a weak Postnikov diagram on a surface $\Sigma$. Then $Q_D$ is a dimer quiver on $\Sigma$.*

**Example 4.5.** *The dimer quivers of the weak Postnikov diagrams from Example 4.2 are in Figure 9.*

**Remark 4.6.** *Recall that for any dimer quiver on the disc, the collection of its zig–zag paths (Definition 3.6) is a Postnikov diagram. The zig–zag paths can be drawn for dimer quivers on arbitrary surfaces, as this only relies on the quiver and not on the surfaces. This results in a collection of oriented curves (strands) which will be denoted by $D_Q$ (as in the disc case, Definition 3.6). We now show that $D_Q$ is a weak Postnikov diagram.*

**Lemma 4.7.** *Let $Q$ be a dimer quiver on a surface $\Sigma$. Then the collection of alternating strands $D_Q$ is a weak Postnikov diagram on $\Sigma$.*

*Proof.* The collection of zig–zag paths of a dimer model gives a strand diagram satisfying the local conditions of a Postnikov diagram (i.e., a weak Postnikov diagram) but not necessarily the global conditions, arguing as in [1, Remark 3.4]. □

By Lemma 4.4, every weak Postnikov diagram $D$ defines a dimer algebra $A_Q = A_{Q_D} = \mathbb{C}Q_D/I$ where $I$ is the ideal given by the relations from the internal arrows, cf. Definition 2.7.

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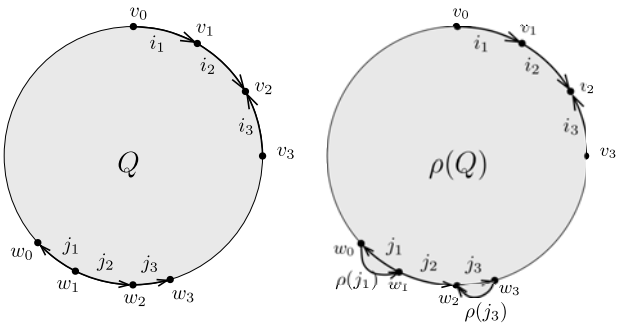



***Figure 10.*** *The quivers $Q$ and $\rho(Q)$: adding boundary arrows.*

Setting $e$ to be the sum of idempotents of $A_Q$ corresponding to boundary vertices in $Q_D$, we thus obtain the boundary algebra of $Q_D$

$$eA_Qe,$$

(Definition 2.9). The boundary algebra associated with a weak Postnikov diagram on a general surface is invariant under twisting/untwisting and geometric exchange, where twisting and untwisting refer to the local moves from Figure 2, while geometric exchange is the local square move from [1, Section 12]. Let $D$ and $D'$ be two weak Postnikov diagrams on $\Sigma$ that can be linked by a sequence of geometric exchanges and (un)twisting moves. Let $Q_D$ and $Q_{D'}$ be the associated dimer quivers. These have the same boundary vertices (i.e., the same $k$-subsets at the alternating regions at the boundary of $\Sigma$). So the associated algebras $A_D$ and $A_{D'}$ have the same boundary idempotents.

Arguing similarly as in [1, Section 12] (in particular, Lemma 12.1 and Corollary 12.4), now in the setup of weak Postnikov diagrams, one gets the following result:

**Lemma 4.8.** *Let $D, D'$ be weak Postnikov diagrams on $\Sigma$, let $e$ and $e'$ be the sum of the idempotents corresponding to boundary vertices of $Q_D$, and $e'$ be the sum of the idempotents corresponding to the boundary vertices of $Q_{D'}$. Assume that $D'$ can be obtained from $D$ by a sequence of geometric exchanges and twisting or untwisting moves. Then the corresponding boundary algebras $eA_De$ and $e'A_{D'}e'$ are isomorphic.*

## 5. Dimer gluing

In this section we allow dimer quivers to have several connected components. If $Q$ is a dimer quiver which is not connected, then the associated surface $\Sigma = \Sigma_Q$ (see Remark 2.5) has several connected components.

We will choose pairs of connected sets of arrows along the boundary of the dimer quiver and glue $Q$ to itself by identifying these arrows pairwise.

This gluing operation will be done in such a way that important properties of the dimer quivers are preserved.

**Definition 5.1.** *Let $Q$ be a dimer quiver and $\Sigma = \Sigma_Q$ the underlying surface. Let $I$ and $J$ be two disjoint connected sets of $s > 0$ boundary arrows of $Q$, and let $\{v_0, \dots, v_s\}$ and $\{w_0, \dots, w_s\}$ be the vertices of $I$ and of $J$, respectively, where $v_m \neq w_{m'}$ for any $m, m'$ in $\{1, \dots, s-1\}$. We write $I = \{i_1, \dots, i_s\}$ and $J = \{j_1, \dots, j_s\}$ for the corresponding boundary arrows, ordered compatibly with these vertex labellings. The sets $I$ and $J$ may belong to different connected components of the boundary of $Q$ or to two different connected components of $Q$. See Figures 10, 11 and 12 with Example 5.11(1). If the sets $I$ and $J$ are on the same connected component of the boundary of $Q$ or if they belong to two different connected*

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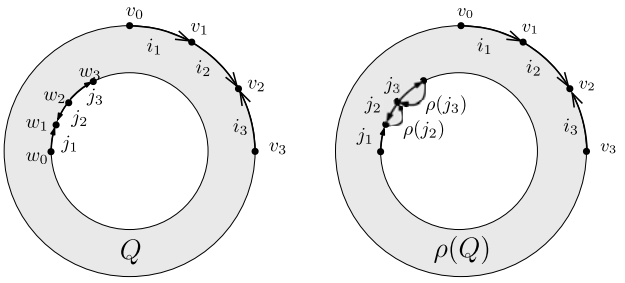



***Figure 11.*** *The quivers $Q$ and $\rho(Q)$.*

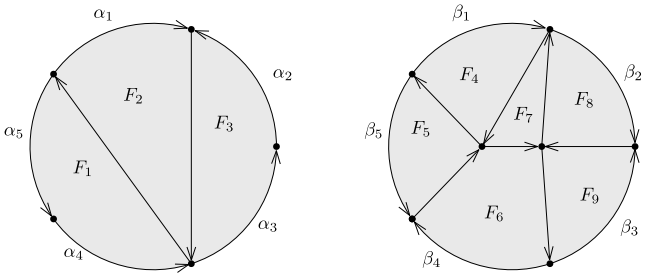



***Figure 12.*** *A dimer quivers with two connected components.*

*components of the dimer quiver, we label the vertices and arrows of $I$ clockwise and the vertices and arrows of $J$ anticlockwise along the boundary of $Q$. If they belong to different connected components of the boundary of the same connected component of $Q$, we label both clockwise along the boundary (see Figure 11). For every $m \in \{1, \dots, s\}$, $i_m$ is an arrow $v_{m-1} \to v_m$ or $v_m \to v_{m-1}$ and $j_m$ is an arrow $w_{m-1} \to w_m$ or $w_m \to w_{m-1}$. See Figures 10 and 11 for two examples, one where $I$ and $J$ are on the same boundary component and one where they belong to different components.*

*We say that the two arrows $i_m$ and $j_m$ are parallel if $i_m : v_m \to v_{m-1}$ and $j_m : w_m \to w_{m-1}$ or if $i_m : v_{m-1} \to v_m$ and $j_m : w_{m-1} \to w_m$.*

We now define a map $\rho = \rho_{I,J}$ on dimer quivers with chosen sets $I, J$ of arrows along the boundary. This map adds new boundary arrows between the vertices of $J$ whenever the arrows of $I$ and $J$ are not parallel, keeping the set of vertices as well as the number of boundary arrows fixed. For every added arrow, the resulting quiver $\rho(Q)$ has a two-cycle.

**Definition 5.2.** *Let $I$ and $J$ be as in Definition 5.1. Let $m$ be in $\{1, \dots, s\}$. If the arrows $i_m$ and $j_m$ are parallel, we set $\rho(j_m) := \rho_{I,J}(j_m) := j_m$. If $i_m$ and $j_m$ are not parallel, we add a new arrow $\rho(j_m) := \rho_{I,J}(j_m)$ in the opposite direction to $j_m$, thereby creating an oriented digon between the two endpoints $w_{m-1}$ and $w_m$ of $j_m$. Note that when $\beta \neq \rho(\beta)$, the new arrow $\rho(\beta)$ replaces $\beta$ as a boundary arrow, and $\beta$ becomes an internal arrow.*

*Let $\rho(J) = \rho_{I,J} = \{\rho(j_1), \dots, \rho(j_s)\}$. We define $\rho(Q)$ to be the quiver $Q$ obtained by adding the arrows $\rho(j_m)$ for every occurrence of non-parallel arrows.*

**Lemma 5.3.** *Let $Q$ be a dimer quiver. Then $\rho(Q)$ is a dimer quiver.*

*Proof.* The new quiver has no loops. For condition (2) of Definition 2.2, it is enough to consider arrows $j$ with $j \neq \rho(j)$. Each such arrow becomes an internal arrow of $\rho(Q)$, and by construction it is incident with two faces of opposite sign. The new arrows $\rho(j)$ are boundary arrows. Adding the arrows $\rho(j)$ increases the connectivity of the incidence graph, so condition (3) of Definition 2.2 is also satisfied. □

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Note that the surface $\Sigma_{\rho(Q)}$ is the same as $\Sigma_Q$, up to homotopy. In Section 6, we will use the quiver $\rho(Q)$ to create dimer models on annuli.

**Lemma 5.4.** *Let $Q$ and $\rho(Q)$ be as in Definition 5.2. Then $A_Q \cong A_{\rho(Q)}$.*

*Proof.* Notice that the canonical inclusion of $Q$ into $\rho(Q)$ induces a map $\mathbb{C}Q \hookrightarrow \mathbb{C}(\rho(Q))$ that descends to a homomorphism $\psi : A_Q \to A_{\rho(Q)}$. Let $J' \subseteq J$ be the set of arrows in $\rho(Q)$ for which $\rho(j_m) \neq j_m$ and let $K := \{\rho(j) : j \in J'\}$ be the corresponding subset of boundary arrows of $\rho(Q)$.

Each $\alpha \in J'$ is an internal arrow of $\rho(Q)$ and so belongs to the boundary of faces $F_1, F_2 \in \rho(Q)_2$, where $\partial F_1 = j\rho(j)$ and $\partial F_2 = jp_{F_2}$, where $p_{F_2}$ is the complement of $j$ in $\partial F_2$. The corresponding relation $\partial_\alpha(W)$ implies $\rho(j) = p_{F_2}$.

Define $\varphi : \mathbb{C}\rho(Q) \to \mathbb{C}Q$ to be the identity on vertices in $\rho(Q)_0$ (viewed as vertices of $Q$) and the identity on arrows in $\rho(Q)_1 \backslash K$ (viewed as arrows of $\rho(Q)$). For each $\alpha \in K$ define $\varphi(\alpha) := p_{F_2}$. Then $\varphi$ descends to a homomorphism $A_{\rho(Q)} \to A_Q$ that is the inverse of $\psi$. □

**Remark 5.5.** *As a consequence of Lemma 5.4, the dimer and boundary algebras of $Q$ are the same as those of $\rho(Q)$. This will allow us to replace $Q$ with $\rho(Q)$ in the gluing construction that follows so that we can assume the arrows in $I$ and $J$ are parallel. We will sometimes do this.*

**Definition 5.6** (The quiver $Q_{I\equiv J}$ obtained from gluing). *Let $Q$ be a dimer quiver. Let $I = \{i_1, \ldots, i_s\}$ and $J = \{j_1, \ldots, j_s\}$ be disjoint connected sets of boundary arrows of $Q$ and let $\rho(j_m)$, $1 \leq m \leq s$, be as in Definition 5.2.*

*The quiver $Q_{I\equiv J}$ obtained from $\rho(Q)$ by identifying $v_m$ with $w_m$ for $m = 0, \ldots, s+1$ and identifying $i_m$ with $\rho(j_m)$ for every $1 \leq m \leq s$ is called the quiver arising from $Q$ by gluing $I$ with $J$.*

**Lemma 5.7.** *Let $Q$ be a dimer quiver, and let $I$ and $J$ be as in Definition 5.6. Then the quiver $Q_{I\equiv J}$ is a dimer quiver.*

*Proof.* By Lemma 5.3, $\rho(Q)$ is a dimer quiver. Condition (3) of Definition 2.2 is satisfied as the effect of gluing is to increase the connectivity of the incidence graphs of the arrows in $I$ or $J$. (1) is satisfied, as gluing does not introduce any loops, as no arrows are contracted. For (2): any arrow of $I$ and of $\rho(J)$ is an internal arrow of $\rho(Q)$, between two faces of opposite orientations. □

**Remark 5.8.** *(1) The resulting quiver $Q_{I\equiv J}$ lives on the surface obtained by gluing the parts of the boundary of $\Sigma$ along the identified arrows.*

*(2) The quiver $Q_{I\equiv J}$ need not be free of internal two-cycles, even if $Q$ is; for every arrow $j \in J$ with $j \neq \rho(j)$, the construction introduces an internal two-cycle.*

We can iterate the construction from Definition 5.6 and glue along several intervals of arrows:

**Definition 5.9.** *Let $Q$ be a dimer quiver. Let $I_1, \ldots, I_m$ and $J_1, \ldots, J_m$ be disjoint connected sets of boundary arrows of $Q$ (as in Definition 5.1). For $t \in \{2, \ldots, m\}$ set*

$$Q_{I_1\equiv J_1,\ldots,I_t\equiv J_t} := (Q_{I_1\equiv J_1,\ldots,I_{t-1}\equiv J_{t-1}})_{I_t\equiv J_t}$$

**Remark 5.10.** *One can check that iterated gluing is independent of the order:*

$$Q_{I_{\sigma(1)}\equiv J_{\sigma(1)},\ldots,I_{\sigma(t)}\equiv J_{\sigma(t)}} = Q_{I_1\equiv J_1,\ldots,I_m\equiv J_m}$$

*for any permutation $\sigma$ of $\{1, \ldots, m\}$.*

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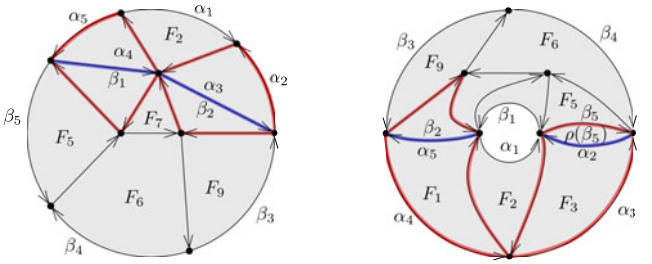



***Figure 13.*** *Two dimer quivers were obtained from the quiver of Figure 12 by gluing. The red and blue indicate the seams – the red paths correspond to complements of arrows of the sets of glued arrows.*

In Section 7, we will consider dimer quivers consisting of two discs and glue them along two sets of arrows to construct dimer quivers on annuli.

**Example 5.11.** *Let $Q$ be the dimer quiver with two connected components as in Figure 12.*

*(1) Let $I = \{\alpha_4, \alpha_3\}$ and $J = \{\beta_1, \beta_2\}$. Then $\rho(J) = J$ and $Q_{I\equiv J}$ are dimer quivers on a disc, as on the left of Figure 13.*

*(2) Let $I_1 = \{\alpha_2\}$, $I_2 = \{\alpha_5\}$, $J_1 = \{\beta_5\}$ and $J_2 = \{\beta_2\}$. Here, $\beta_5 \neq \rho(\beta_5)$, so the new boundary arrow is now $\rho(\beta_5)$; we then have $\rho(J_1) = \{\rho(\beta_5)\}$, and $\rho(J_2) = J_2$ and $Q_{I_1\equiv J_1, I_2\equiv J_2}$ is a dimer quiver on an annulus, as on the right of Figure 13.*

Recall that if $\beta$ is a boundary arrow of a dimer quiver, we write $\widehat{\beta}$ for the complement of $\beta$ around the unique face $\beta$ belongs to (Notation 3.13).

**Definition 5.12.** *Let $r \in \{1, \dots, m\}$ and write*

$$I_r = \{i_{r,1}, \dots, i_{r,s_r}\}, \qquad J_r = \{j_{r,1}, \dots, j_{r,s_r}\}, \qquad s_r := |I_r| = |J_r|.$$

*We refer to the subquiver $I_r = J_r$ of $Q_{I_1\equiv J_1, \dots, I_m\equiv J_m}$ on the paths*

$$\{i_{r,m}, \rho(j_{r,m}), \widehat{i_{r,m}}, \widehat{\rho(j_{r,m})} : 1 \leq m \leq s_r\}$$

*as the seam at $I_r = J_r$.*

To illustrate the notion of a seam: the disc in Figure 13 has a single seam given by the paths $\{\alpha_4, \alpha_3, \beta_1, \beta_2, \widehat{\alpha_4}, \widehat{\alpha_3}, \widehat{\beta_1}, \widehat{\beta_2}\}$. The annulus in 13 has a seam at $\alpha_5 \equiv \beta_2$ given by the paths $\{\alpha_5, \beta_2, \widehat{\alpha_5}, \widehat{\beta_2}\}$ and a seam at $\alpha_2 \equiv \beta_5$ given by $\{\alpha_2, \beta_5, \widehat{\alpha_2}, \widehat{\beta_5}\}$.

**Definition 5.13.** *Let $g$ be a path in $Q_{I\equiv J}$. We say that $g$ crosses the seam $I \equiv J$ if $g$ contains a subpath $\sigma = g_0 g_1 g_2$ where $g_1 : v_0 \to v_2$ is a path of arrows completely contained in $I \equiv J$ and $v_0, v_2$ are the only vertices of $g_0$ and $g_2$ respectively belonging to $I \equiv J$, where both $g_0$ and $g_2$ are not trivial paths.*

We fix the following notation for the remainder of the section. Let $Q$ be a dimer quiver, and $I = \{i_1, \dots, i_s\}$ and $J = \{j_1, \dots, j_s\}$ be disjoint connected sets of boundary arrows of $Q$ as in Definition 5.1. Let $h, t : Q_1 \to Q_0$ denote the source and target maps of the quiver $Q$. Let $\mathcal{R} = \mathcal{R}(I, J)$ denote the following set of paths in $\mathbb{C}(\rho(Q))$:

$$\mathcal{R} := \{i_m - \rho(j_m), \widehat{i_m} - \widehat{\rho(j_m)}, e_{h(i_m)} - e_{h(\rho(j_m))} : 1 \leq m \leq s\} \cup \{e_{t(i_s)} - e_{t(\rho(j_s))}\}. \tag{5.1}$$

In this setting, the glued dimer algebra can be determined from the initial dimer algebra in the following sense.

**Proposition 5.14.** *The dimer algebra $A_{Q_{I\equiv J}}$ is isomorphic to $A_{\rho(Q)}/\langle \mathcal{R} \rangle$.*

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*Proof.* Let $\mathcal{T}$ denote the set of paths

$$\mathcal{T} := \{i_m - \rho(j_m), e_{h(i_m)} - e_{h(\rho(j_m))} : 1 \leq m \leq s\} \cup \{e_{t(i_s)} - e_{t(\rho(j_s))}\}$$

in $\mathbb{C}(\rho(Q))$. Then $\langle \mathcal{R} \rangle = \langle \mathcal{T} \rangle + \langle \widehat{i_m} - \widehat{\rho(j_m)} : 1 \leq m \leq s \rangle$. Observe that we have an isomorphism $\psi : \mathbb{C}Q_{I \equiv J} \cong \mathbb{C}(\rho(Q))/\langle \mathcal{T} \rangle$ that is the identity on arrows and on the trivial paths at vertices.

Let $V$ be the ideal of relations of $\mathbb{C}Q_{I \equiv J}$ defined by the internal arrows of $Q_{I \equiv J}$ (Definition 2.6), and let $U$ be the ideal of relations of $\mathbb{C}(\rho(Q))$ defined by the internal arrows of $\rho(Q)$, so

$$A_{Q_{I \equiv J}} = \mathbb{C}Q_{I \equiv J}/V \quad \text{and} \quad A_{\rho(Q)} = \mathbb{C}(\rho(Q))/U.$$

Since every internal arrow of $\rho(Q)$ remains internal in $Q_{I \equiv J}$, every relation generating $U$ also lies in $\psi(V)$. The only additional internal arrows of $Q_{I \equiv J}$ are the images of the arrows in $I$ (equivalently, of the arrows in $\rho(J)$). For each such arrow, the corresponding cyclic derivative is $\widehat{i_m} - \widehat{\rho(j_m)}$. Hence,

$$\psi(V) = U + \langle \widehat{i_m} - \widehat{\rho(j_m)} : 1 \leq m \leq s \rangle.$$

Combining this with the quotient by $\langle \mathcal{T} \rangle$, we obtain

$$(\mathbb{C}(\rho(Q))/\langle \mathcal{T} \rangle)/\psi(V) \cong (\mathbb{C}(\rho(Q))/U)/\langle \mathcal{R} \rangle.$$

Putting everything together yields

$$\begin{aligned} A_{Q_{I \equiv J}} &= \mathbb{C}Q_{I \equiv J}/V \\ &\cong (\mathbb{C}\rho(Q)/\langle \mathcal{T} \rangle)/V \\ &\cong (\mathbb{C}\rho(Q)/U)/\langle \mathcal{R} \rangle \\ &= A_{\rho(Q)}/\langle \mathcal{R} \rangle \end{aligned}$$

as claimed. □

**Remark 5.15.** *Note that the boundary algebra of the dimer quiver $Q_{I \equiv J}$ is by definition equal to $e'Q_{I \equiv J}e'$ where $e'$ is the sum of the idempotent elements of $A_{Q_{I \equiv J}}$ corresponding to the boundary vertices of the quiver $Q_{I \equiv J}$. In terms of the boundary of $Q$, the element $e'$ is the sum $\sum_{i \in X} e_i$ where $X$ is the union of the boundary vertices of $Q$ excluding those which are both a source and target of an arrow of $I$ or of $J$, as these vertices are internal vertices of the surface obtained from gluing.*

The following result explains how the boundary algebra of a glued dimer quiver can be obtained from the two boundary algebras of the individual quivers when the individual dimer algebras as well as the glued dimer algebra are thin. In the statement of the following two results, we will write $e_\alpha$ (respectively $e_\beta$) for the sum of all idempotents corresponding to the boundary vertices of the quiver $Q_\alpha$ (respectively of $Q_\beta$). As before, fix $I$ and $J$ to be two disjoint sets of boundary arrows with $|I| = |J|$ and with relations $\mathcal{R}$ as in (5.1). Let $e'$ be the sum of idempotent elements corresponding to the boundary vertices of $Q_{I \equiv J}$.

**Theorem 5.16.** *Let $Q, I, J, \mathcal{R}$ and $e'$ be as above. Assume that $Q$ can be expressed as a disjoint union $Q = Q_\alpha \amalg Q_\beta$ of connected dimer quivers $Q_\alpha, Q_\beta$ whose dimer algebras are thin. Let $e_\alpha A_{Q_\alpha} e_\alpha = \langle \alpha_1, .., \alpha_{s_1} \rangle / \langle \mathcal{R}_1 \rangle$ and $e_\beta A_{Q_\beta} e_\beta = \langle \beta_1, .., \beta_{s_2} \rangle / \langle \mathcal{R}_2 \rangle$. If $A_{Q_{I \equiv J}}$ is thin then we have an isomorphism:*

$$e' A_{Q_{I \equiv J}} e' \cong e' \left( \langle \{\alpha_i\}_i, \{\beta_j\}_j \rangle / \langle \mathcal{R}_1, \mathcal{R}_2, \mathcal{R} \rangle \right) e'.$$

In other words, the boundary algebra of $Q_{I \equiv J}$ is determined by those of $A_{Q_\alpha}$ and $A_{Q_\beta}$.

*Proof.* Viewing $\langle \{\alpha_i\}_i, \{\beta_j\}_j \rangle$ as a subalgebra of $\mathbb{C}Q$ we inherit the homomorphism

$$\varphi : e' \left( \langle \{\alpha_i\}_i, \{\beta_j\}_j \rangle \right) e' \to e' A_{Q_{I \equiv J}} e'.$$

given by the composition of homomorphisms

$$\varphi : e' \langle \{\alpha_i\}_i, \{\beta\}_j \rangle e' \hookrightarrow e' \mathbb{C}Q e' \twoheadrightarrow e' \mathbb{C}Q_{I \equiv J} e' \twoheadrightarrow e' A_{Q_{I \equiv J}} e'.$$

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We first show that $\varphi$ is surjective: as $A_{Q_{I\equiv J}}$ is thin, for any pair of boundary vertices $a, b$ in $A_{Q_{I\equiv J}}$, we have a (unique) minimal path $g : a \to b$. It suffices to show that each such minimal path $g$ is in the image of $\varphi$. If $g$ can be viewed as a path contained entirely in $Q_\alpha$ or $Q_\beta$ then $g \in \langle \alpha_1, .., \alpha_s \rangle$ or $g \in \langle \beta_1, .., \beta_t \rangle$ respectively. In either case, we find $g' \in e' \langle \{\alpha_i\}_i, \{\beta\}_j \rangle e'$ with $\varphi(g') = g$.

Next consider the case where $g : a \to b$ crosses the seam $I \equiv J$; that is, $g = g_0 g_1$, where $g_0$ is a path completely contained in $Q_\alpha$ or $Q_\beta$. Without loss of generality, assume $g_0$ is a path in $Q_\alpha$. By Lemma 3.12, $g_0$ is a minimal between boundary vertices of $Q_\alpha$ and can be expressed in terms of $g \in \langle \alpha_1, .., \alpha_s \rangle$. By iterating this procedure for each crossing of the seam, we obtain an element $g' \in e' \langle \{\alpha_i\}_i, \{\beta\}_j \rangle e'$ with $\varphi(g') = g$.

Moreover, the unit cycle at any boundary vertex of $Q_{I\equiv J}$ is contained entirely in either $Q_\alpha$ or $Q_\beta$, so its class also lies in the image of $\varphi$. Hence, the boundary version of the central element $t$ is in the image. Since every path between boundary vertices is a minimal path multiplied by a power of this central element, surjectivity follows.

Now we determine the kernel of $\varphi$. By Proposition 5.14, the dimer algebra of $Q_{I\equiv J}$ can be written as $\mathbb{C}(\rho(Q))/\langle \{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}\} \rangle$, yielding the claim. □

A useful application of Theorem 5.16 is that the boundary algebra of any Postnikov diagram in the disc can be determined by recursively splitting it into smaller Postnikov diagrams.

**Corollary 5.17.** *Let $D$ be any Postnikov diagram on the disc with dimer quiver $Q_D$. If $Q_D = Q_{I\equiv J}$, where $Q = Q_\alpha \sqcup Q_\beta$ is a decomposition of $Q$ into disjoint, connected dimer quivers, then*

*(1) The strand diagrams $D_\alpha$ and $D_\beta$ corresponding to $Q_\alpha$ and $Q_\beta$ are Postnikov diagrams in the disc, and*

*(2) the boundary algebra of $Q_D$ may be explicitly determined from the boundary algebras of $Q_\alpha$ and $Q_\beta$.*

*Proof.* (1) As $Q_\alpha$ and $Q_\beta$ are dimer quivers, the associated strand diagrams $D_\alpha$ and $D_\beta$ will satisfy the local properties of a Postnikov diagram. To see they also satisfy the global properties, any self-intersection or oriented lens in $D_\alpha$ or $D_\beta$ would necessarily appear in $D$ as $Q_\alpha$ and $Q_\beta$ are connected subquivers of $Q$.

(2) By part (1) and Proposition 3.11, the dimer algebras $A_{Q_\alpha}$ and $A_{Q_\beta}$ are thin, so Theorem 5.16 can be applied. □

**Remark 5.18.** *In Section 7, we illustrate a more general procedure for determining the boundary algebra of*

$$A_{(Q_\alpha \sqcup Q_\beta)_{I_1 \equiv J_1, J_2 \equiv J_2, \ldots, J_m \equiv J_m}}$$

*from presentations of $e_\alpha A_{Q_\alpha} e_\alpha$ and $e_\beta A_{Q_\beta} e_\beta$ under the additional assumption that $A_{(Q_\alpha \sqcup Q_\beta)_{I_q \equiv J_q}}$ is thin for any $1 \le q \le m$.*

## 6. Bridge quivers

In [1, Section 13], weak Postnikov diagrams arising from surface triangulations were studied. These were shown to be examples of degree 2 diagrams. In particular, [1, Proposition 13.5] gives an explicit presentation of the boundary algebra of a weak Postnikov diagram arising from a triangulation of the annulus.

Our goal is to produce degree $k$ diagrams on surfaces. In [9, Section 4], Scott gives a Postnikov diagram of degree $k$ for any $k, n$, the so-called rectangular arrangements. At present, a general construction of (weak) Postnikov diagrams of degree $k$ is not known, apart from $k = 2$ as mentioned above.

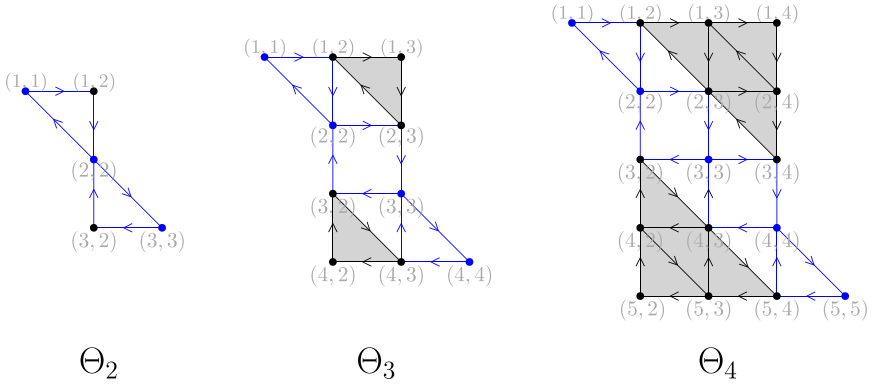
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In this section, we give a construction of dimer quivers on annuli whose associated (weak) Postnikov diagrams are of degree $k$. We will do so by gluing certain dimer quivers along a collection of arrows, using dimer quivers of degree $k$ together with the so-called *bridge* quivers we introduce below.

**Definition 6.1.** *Take $k \geq 2$. For $k = 2$, let $\Theta_2$ be the quiver shown in Example 6.2. For $k \geq 3$, we define a quiver $\Theta_k$ called the $k$-bridge as follows. In order to describe the arrows of $\Theta_k$, we label the vertices of the quiver by lattice points. Now, the quiver $\Theta_k$ is formed by two triangular regions with a right angle at positions $(1, k)$ and $(k+1, 2)$ and $k-2$ arrows horizontally, vertically and diagonally (for the hypotenuse) forming their sides (for $k = 2$, we do not have these triangles but two single vertices). One of these right-angled triangles has the vertices $(k+1, k)$, $(k+1, 2)$, $(3, 2)$. The other triangle has vertices $(1, k)$, $(k-1, k)$ and $(1, 2)$. This is completed by a triangle $(1, 1) \to (1, 2) \to (2, 2) \to (1, 1)$, a sequence of $k-2$ squares going diagonally between the vertices $(2, 2)$ and $(k, k)$ and a triangle on the vertices $(k+1, k+1) \to (k+1, k) \to (k, k) \to (k+1, k+1)$. Each of the $k-2$ intermediate squares is oriented clockwise, and its diagonal arrow is directed from the north-west vertex to the south-east vertex.*

See Example 6.2 for $k \leq 4$, and Figures 15 and 19 for an illustration. In the figure of Example 6.2, the vertices and arrows of two triangles with right angles at $(1, k)$ and at $(k+1, 2)$ are drawn in black, with grey shading. The arrows of the triangles and squares completing the bridge quiver are drawn in blue.

**Example 6.2.** *The $k$-bridges for $2 \leq k \leq 4$ are given below.*




**Remark 6.3.** *For $k > 1$, the quiver $\Theta_k$ is a dimer quiver. For $k > 2$, the associated surface $\Sigma_{\Theta_k}$ is a disc; for $k = 2$, it is two discs meeting at a single vertex. For any $k$, $\Theta_k$ has a rotational symmetry about the centre of the quiver.*

Recall from Definition 3.6 that $D_{\Theta_k}$ is the collection of strands arising from the zig–zag paths in $\Theta_k$.

**Remark 6.4.** *One can use the explicit embedding of $\Theta_k$ in the plane to check that the diagram $D_{\Theta_k}$ is homotopic to a collection of strands where each strand is either a straight line between endpoints or is of an L-shape, containing a single turn to the left in the direction of the strand. See Figure 14. In particular, any two strands starting from the western boundary of $D_{\Theta_k}$, drawn in blue in Figure 14 (i.e., starting at arrows $s_i$ and $s_j$ in Figure 15), will never cross. Symmetrically, any two strands starting from the eastern boundary of $D_{\Theta_k}$, drawn in red (i.e., starting at arrows $r_i$ and $r_j$ in Figure 15), will also never cross.*

**Lemma 6.5.** *For $k > 2$, let $D_{\Theta_k}$ be the strand diagram associated to $\Theta_k$ as in Definition 3.6. Then $D_{\Theta_k}$ is a Postnikov diagram on the disc.*

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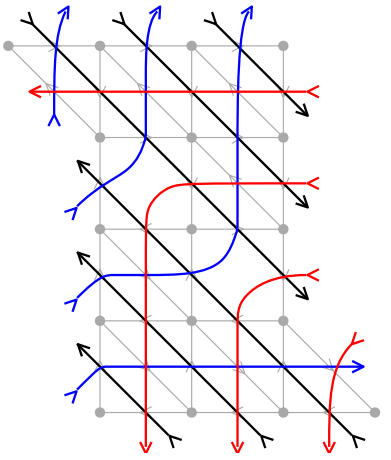


***Figure 14.*** *Strands for* $\Theta_4$.

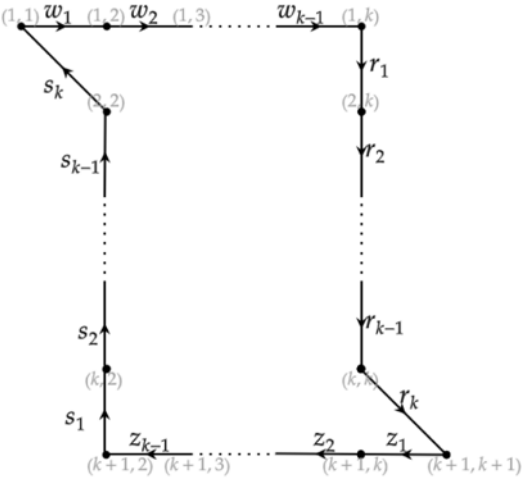



***Figure 15.*** *Labelled boundary arrows of* $\Theta_k$.

*Proof.* By Lemma 4.7, $D_{\Theta_k}$ is a weak Postnikov diagram on the disc, so it remains to show the global axioms ((3) and (4) of Definition 3.1) hold. From Remark 6.4, we can see that (i) there are no self-intersections of strands, and (ii) the lenses formed by double-intersections of the $L$-shaped strands are all oriented, and thus (3) and (4) are satisfied. □

**Lemma 6.6.** *For* $k \geq 2$*, the dimer algebra* $A_{\Theta_k}$ *associated to* $\Theta_k$ *is thin.*

*Proof.* When $k = 2$, $A_{\Theta_k} \cong \mathbb{C}\Theta_2$ as $\Theta_k$ has no internal arrows. Thus, there is a single path between any pair of vertices in $\theta_k$ up to a power of cycles.

Applying Lemma 6.5 and Proposition 3.11 to $\Theta_k$ for $k > 2$ gives the result. □

The following technical result gives a condition for paths between two vertices in the bridge quiver to be minimal.

**Lemma 6.7.** *Let* $g = \alpha_1 \cdots \alpha_n$ *be a path in* $\Theta_k$ *where* $\alpha_i{:}(a_i, b_i) \to (a_{i+1}, b_{i+1})$ *is an arrow between vertices* $(a_i, b_i), (a_{i+1}, b_{i+1}) \in [k+1] \times [k+1]$ *for each* $1 \leq i \leq n$*. Assume that* $(a_1, b_1) \neq (a_{n+1}, b_{n+1})$*. If one of the following conditions holds, then* $g$ *is a minimal path from* $(a_1, b_1)$ *to* $(a_{n+1}, b_{n+1})$ *in* $A_{\Theta_k}$.

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*(1)* *If* $a_1 < a_{n+1}$ *then* $a_i \leq a_{i+1}$ *for all* $1 \leq i \leq n$.
*(2)* *If* $a_1 > a_{n+1}$ *then* $a_i \geq a_{i+1}$ *for all* $1 \leq i \leq n$.
*(3)* *If* $b_1 < b_{n+1}$ *then* $b_i \leq b_{i+1}$ *for all* $1 \leq i \leq n$.
*(4)* *If* $b_1 > b_{n+1}$ *then* $b_i \geq b_{i+1}$ *for all* $1 \leq i \leq n$.

*Proof.* Let $g$ be such a path. Then $g$ doesn't contain any arrows in a given direction determined by the hypothesis. By construction, any path equivalent to $g$ in the dimer algebra $A_{\Theta_k}$ will also be missing an arrow in this direction and so can't contain a cycle. The result then follows from Lemma 6.6. □

**Remark 6.8.** *In practice, we will apply Lemma 6.7 to paths that are weakly monotone in at least one coordinate. By Lemma 6.7, any such path is minimal in the associated dimer algebra.*

**Notation 6.9.** *The following notations for paths in the bridge quiver will be useful in Section 7, when we need to refer to arrows and paths along the boundary of the bridge quiver* $\Theta_k$ *(see Figure 15).*

*The boundary of* $\Theta_k$ *is a clockwise-oriented cycle. If* $\gamma$ *is a boundary arrow of* $\Theta_k$ *from a to b, we write* $\widehat{\gamma}:b \to a$ *for the complementary path around the unique face incident with* $\gamma$ *(cf. Notation 3.13). More generally, if* $g = \gamma_1 \cdots \gamma_t : a \to b$ *is a path of boundary arrows, we write*

$$\widehat{g} := \widehat{\gamma}_t \cdots \widehat{\gamma}_1 : b \to a.$$

- $r_i : (i,k) \to (i+1,k)$ *for* $1 \leq i \leq k-1$
- $r_k : (k,k) \to (k+1,k+1)$
- $z_{i+2} : (k+1, k-i) \to (k+1, k-(i+1))$ *for* $-1 \leq i \leq k-3$
- $s_i : (k-i, 2) \to (k-(i+1)), 2)$ *for* $-1 \leq i \leq k-3$
- $s_k : (2,2) \to (1,1)$
- $w_i : (1,i) \to (1, i+1)$ *for* $1 \leq i \leq k-1$.

*We will also frequently use*

- $r := r_1 r_2 \cdots r_k$
- $s := s_1 s_2 \cdots s_k$
- $\widehat{r} := \widehat{r_k}\widehat{r_{k-1}} \cdots \widehat{r_1}$
- $\widehat{s} := \widehat{s_k}\widehat{s_{k-1}} \cdots \widehat{s_1}$

*for the composition of all k arrows* $r_i$, $s_i$, $\widehat{r_i}$ *or* $\widehat{s_i}$ *respectively.*

With this, the following notation makes sense:

If $g = \gamma_1, .., \gamma_t : a \to b$ is a path composed of boundary arrows $\gamma_i$ in $\Theta_k$, we write $\widehat{g} := \widehat{\gamma_t}\widehat{\gamma_{t-1}} \ldots \widehat{\gamma_1}:b \to a$ for the path in the reverse direction formed by the complements.

For $1 \leq a < k$ and $1 \leq l \leq k-a-1$, we use the following abbreviations for paths in the $z_i$s and $w_i$s, along the northern/southern boundary of the bridge quiver (as indicated by the coordinates defining $\Theta_k$):

- $w^l(a) := w_a w_{a+1} \cdots w_{a+l-1}:(1,a) \to (1, a+l)$
- $z^l(a) := z_a z_{a+1} \cdots z_{a+l-1}:(k+1, (k+2)-a) \to (k+1, k+2-(a+l))$

## 7. Gluing dimer quivers for annuli

In this section, we will glue the dimer quiver of a $(k,n)$-diagram and the bridge quiver $\Theta_k$ along two sets of arrows to obtain a dimer quiver on the annulus which in turn corresponds to a weak Postnikov diagram on the new surface. We show that this diagram is in fact of degree $k$, permuting the vertices along the boundary in a homogeneous way.

Recall that if $Q$ is a dimer quiver on an arbitrary surface, we can associate a collection of strands to it which we denote by $D_Q$ (Remark 4.6).

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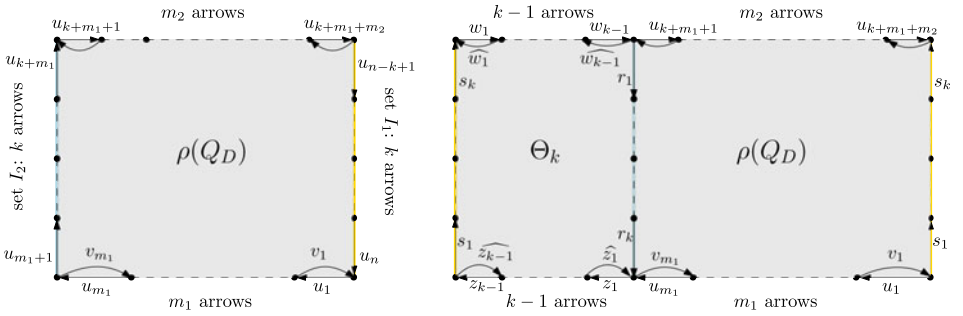



***Figure 16.*** *The quivers $Q_D$ and $Q_{I_1\equiv J_1, I_2\equiv J_2}$.*

**Definition 7.1.** *Let $D$ be a weak Postnikov diagram on a surface $\Sigma$. Label all endpoints of the strands of $D$ clockwise around each boundary component: if $B$ is a connected component of the boundary of $\Sigma$, label the endpoints by $1_B, \dots, (n_B)_B$, where $n_B$ is the number of endpoints of strands on $B$. Let $N := \sum_B n_B$. We say that $D$ has degree $k$ if all strands of $D$ start and end at the same boundary and if the permutation of $N$ is of the form $i_B \mapsto (i+k)_B$ (reducing modulo $n_B$) for every component.*

**Theorem 7.2.** *Let $D$ be a $(k,n)$-diagram in the disc with $n \geq 2k$ and $Q_D$ its dimer quiver. Let $Q = Q_D \cup \Theta_k$. Let $I_1$ and $I_2$ be two disjoint sets of $k$ consecutive boundary arrows along $Q_D$, and let $J_1$ and $J_2$ be the sets $J_1 = \{s_1, \dots, s_k\}$ and $J_2 = \{r_1, \dots, r_k\}$ of consecutive boundary arrows along the boundary of $\Theta_k$. Then*

*(1) $Q_{I_1\equiv J_1, I_2\equiv J_2}$ is a dimer quiver on an annulus and*
*(2) The collection $D_{Q_{I_1\equiv J_1, I_2\equiv J_2}}$ is a weak Postnikov diagram of degree $k$.*

*Proof.* (1) We know by Lemma 5.7 that $Q_{I_1\equiv J_1, I_2\equiv J_2}$ is a dimer quiver. The choice of $I_1$ and $I_2$ ensures that the underlying surface of $Q_{I_1\equiv J_1, I_2\equiv J_2}$ is an annulus.

(2) By Lemma 4.7, the collection $D_{Q_{I_1\equiv J_1, I_2\equiv J_2}}$ of strands is a weak Postnikov diagram. It remains to compute the degree of this diagram. We will show that the strand diagram results in a degree $k$ permutation of boundary vertices on a fixed boundary component, with the case for the other boundary component being symmetric.

Strands in the glued dimer quiver $Q_{I_1\equiv J_1, I_2\equiv J_2}$ arise from composing each strand whose target is in an arrow in $I_t$ with the strand whose source is the glued arrow in $J_t$ (and similarly for strands whose source is in $I_t$ and target is in $J_t$) for $t \in \{1,2\}$. Consequently, the degree of the strand permutation on $D_{Q_{I_1\equiv J_1, I_2\equiv J_2}}$ can be calculated by composing the permutation on $D$ with the strand permutation on $\Theta_k$ along the corresponding seams $I_1 \equiv J_1$ or $I_2 \equiv J_2$.

For $1 \leq m \leq k-1$, the strand of $D_{\Theta_k}$ starting at the arrow $s_{k-m+1}$ is one of the $L$-shaped strands from Remark 6.4. After gluing, it composes with a strand of $D$ terminating at the $m$-th arrow of $I_1$ and with a strand of $D$ starting at the corresponding arrow of $I_2$. Along the relevant boundary component of the annulus, these two pieces contribute shifts by $m$ and by $k-m$, respectively, so the total effect is a shift by $k$. The remaining strand, starting at $s_1$, is straight across the bridge and contributes $1 + (k-2) + 1 = k$. The argument for the other seam and for the other boundary component is symmetric. Hence, the induced permutation on each boundary component is the shift by $k$. □

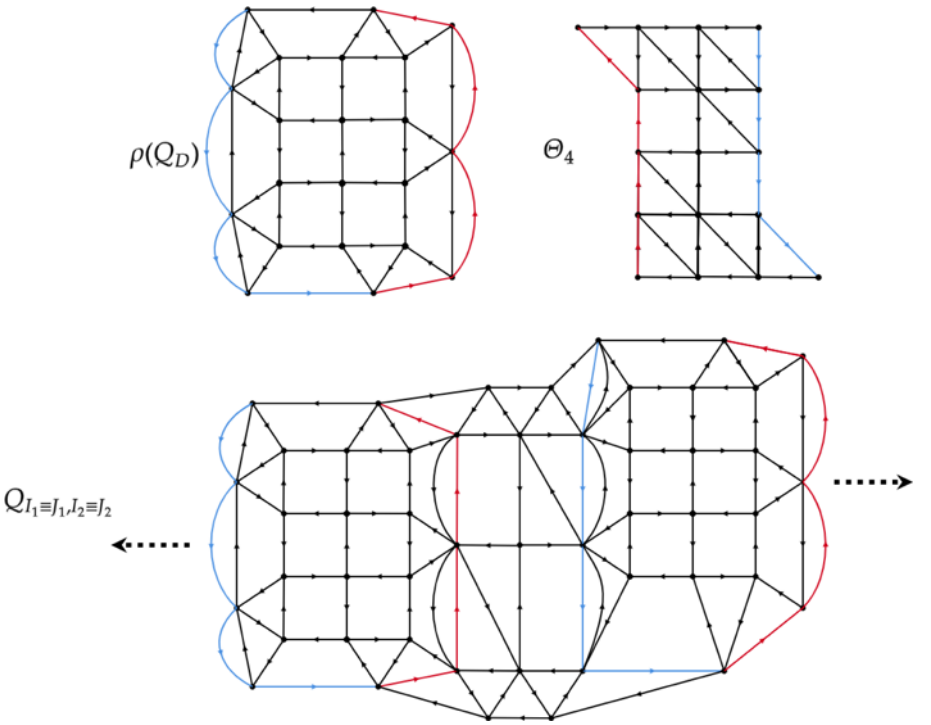
**Remark 7.3.** *The quiver $Q_{I_1\equiv J_1, I_2\equiv J_2}$ has $n-2$ vertices (or arrows) on its boundary. The partition of these boundary vertices depends on the choices of the sets $I_1$ and $I_2$, more precisely, on the proximity of the two sets. Let $m_1 \geq 0$ (and $m_1 < n$) denote the number of consecutive arrows on the boundary of $Q_D$ between the last arrow of $I_1$ and the first arrow of $I_2$, and let $m_2 \geq 0$ denote the number of consecutive arrows between the last arrow of $I_2$ and the first arrow of $I_1$, so that $n = m_1 + m_2 + 2k$. These sets of $m_1$ (respectively $m_2$) arrows lie on the two different components of the boundary of the resulting quiver. We refer to them as the outer and inner boundary components, respectively. See Figure 16 for an illustration.*

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We illustrate the theorem with two examples, the first one in the case $k = 2$, where our construction recovers the decomposition of a triangulation of the annulus into a triangulation of the disc used in the proof of [1, Proposition 13.5] as a special case.

**Example 7.4.** *We let $Q_D$ be the dimer model arising from a triangulation of a disc (and therefore corresponding to a $(2, n)$-diagram) and take the bridge quiver $\Theta_2$. Figure 17 shows an example for $n = 6$. The disc (hexagon) is shown with purple shading and the diagonals of the triangulation with the purple dashed lines. Two sets, $I_1$ and $I_2$, of two arrows along the boundary of $\rho(Q_D)$ are chosen, indicated in red (for $I_1$) and blue (for $I_2$) in the figure. The sets $J_1$ and $J_2$ of arrows of the bridge quiver are each the two arrows on the left/on the right, respectively. Here, $m_1 = 0$ and $m_2 = 2$. The bridge quiver is indicated with grey shading. The dotted arrows indicate the identification of the two ends of the strip obtained by cutting the annulus open.*

**Example 7.5.** *In this example, $Q$ corresponds to a $(4, 9)$-diagram $D$ in the disc. Two disjoint subsets of 4 consecutive boundary arrows in $Q$ are chosen (illustrated in red and blue, respectively) and are identified with the corresponding coloured arrows in $\Theta_k$. In the picture, $n = 9$, $m_1 = 0$ and $m_2 = 1$.*




**Lemma 7.6.** *Let $D$ be any Postnikov diagram in the disc with $n \geq 2k$ marked points. Set $Q = Q_D \cup \Theta_k$ and let $I_1, I_2$ be disjoint sets of $k$-consecutive boundary arrows of $Q_D$, with $J_1 = \{s_1, \ldots, s_k\}$, $J_2 = \{r_1, \ldots, r_k\}$. Then both $D_{Q_{I_1\equiv J_1}}$ and $D_{Q_{I_2\equiv J_2}}$ are Postnikov diagrams on the disc.*

*Proof.* By Lemma 5.7, $D_{Q_{I_1\equiv J_1}}$ is a weak Postnikov diagram on the disc. Recall from Remark 6.4 that the set of strands in $D_{\Theta_k}$ with source in $J_1$ are mutually non-intersecting, and the set of strands in $D_{\Theta_k}$ with target in $J_1$ are also mutually non-intersecting. Consequently, the strands in $D_{Q_{I_1\equiv J_1}}$ that cross the seam $I_1 = J_1$ extend the strands in $D$ without introducing crossings, and so the global axioms of Definition 3.1 are satisfied. Similarly for $D_{Q_{I_2\equiv J_2}}$. □

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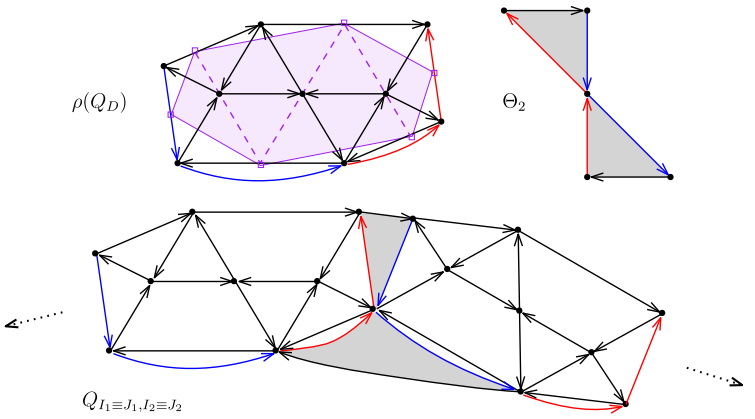



***Figure 17.*** *Gluing the dimer quiver of a triangulation with $\Theta_2$.*

Our goal is now to characterise the boundary algebra of the degree $k$ dimer quiver on the annulus from Theorem 7.2.

Let $D$ be a $(k,n)$-diagram in the disc where $2k \leq n$ and let $I_1$ and $I_2$ be disjoint sets of $k$ consecutive arrows along the boundary of the dimer quiver $Q_D$ as in Theorem 7.2. We fix a labelling of the boundary vertices of $\rho(Q_D)$ by $\{1, .., n\}$ so that the vertices incident with $I_1$ are labelled $n-k+1, n-k+2, \ldots, 1$. Under this labelling and with the notation of Remark 7.3, the vertices incident with $I_2$ are $m_1+1$, $m_1+2, \ldots, m_1+k+1$.

Now let $u_i : i \to i+1$ and $v_i : i+1 \to i$ denote the minimal paths between consecutive boundary vertices in $\rho(Q_D)$ as in Lemma 3.14.

**Remark 7.7.** *Recall that $n = m_1 + m_2 + 2k$. Here $\widehat{s_i}$ and $\widehat{r_i}$ are taken in the bridge quiver $\Theta_k$ before gluing, and we then view their images in the glued quiver $Q_{I_1\equiv J_1, I_2\equiv J_2}$. With this convention we have:*

- $s_1 = v_n$, $s_2 = v_{n-1}, \ldots, s_k = v_{n-k+1}$,
- $\widehat{s_1} = u_n$, $\widehat{s_2} = u_{n-1}, \widehat{s_k} = u_{n-k+1}$,
- $r_1 = v_{m_1+k}, r_2 = v_{m_1+k-1}, \ldots, r_k = v_{m_1+1}$,
- $\widehat{r_1} = u_{m_1+k}, \widehat{r_2} = u_{m_1+k-1}, \ldots, \widehat{r_k} = u_{m_1+1}$.

**Remark 7.8.** *Note that any (connected) path in $\mathbb{C}(\rho(Q) \cup \Theta_k)$ may be regarded as a path in $\mathbb{C}Q_{I_1\equiv J_1}$ and as a path in $\mathbb{C}Q_{I_2\equiv J_2}$. Also, any path in $\mathbb{C}Q_{I_1\equiv J_1}$ or $\mathbb{C}Q_{I_2\equiv J_2}$ that is solely contained in either $\mathbb{C}(\rho(Q))$ or $\mathbb{C}\Theta_k$ may be regarded as a path in $\mathbb{C}(\rho(Q) \cup \Theta_k)$. By Proposition 5.14, $A_{Q_{I_1\equiv J_1}}$ and $A_{Q_{I_2\equiv J_2}}$ are quotients of $A_{\rho(Q)\cup\Theta_k}$, so if $g = h$ in $A_{\rho(Q)\cup\Theta_k}$ then equality holds for the corresponding paths in $A_{Q_{I_1\equiv J_1}}$ and $A_{Q_{I_2\equiv J_2}}$. Similarly, as $A_{Q_{I_1\equiv J_1, I_2\equiv J_2}}$ is a quotient of $A_{Q_{I_1\equiv J_1}}$ and a quotient of $A_{Q_{I_2\equiv J_2}}$, if $g = h$ in $A_{Q_{I_1\equiv J_1}}$ or $A_{Q_{I_2\equiv J_2}}$, then equality holds for the corresponding paths in $A_{Q_{I_1\equiv J_1, I_2\equiv J_2}}$.*

A convenient generating set for the boundary algebra consists of the boundary arrows on the two boundary components of $Q_{I_1\equiv J_1, I_2\equiv J_2}$, their complementary boundary paths in the bridge piece, the seam-crossing paths $r$ and $s$, and the boundary idempotents. This is the set recorded in Notation 7.9.

**Notation 7.9.** *Let $Q_{I_1\equiv J_1, I_2\equiv J_2}$ as above. We denote by $\mathcal{A}$ the following collection of elements of $\mathbb{C}Q_{I_1\equiv J_1, I_2\equiv J_2}$:*

$$\begin{aligned}\mathcal{A} := &\{r, s\} \cup \{w_i, \widehat{w_i}, z_i, \widehat{z_i} : 1 \leq i \leq k-1\} \\ &\cup \{v_i, u_i : 1 \leq i \leq m_1\} \\ &\cup \{v_i, u_i : m_1 + k + 1 \leq i \leq m_1 + k + m_2\} \\ &\cup \{e_i : i \text{ is a boundary vertex of } Q_{I_1\equiv J_1, I_2\equiv J_2}\}.\end{aligned}$$

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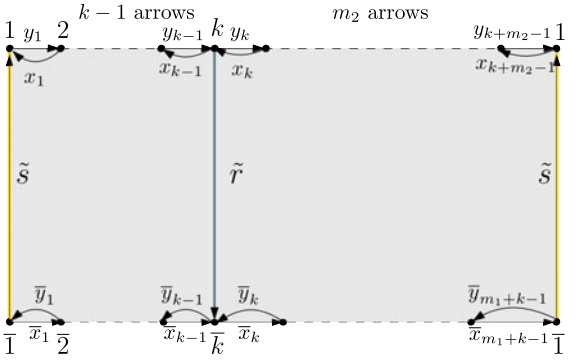



***Figure 18.*** *The quiver* $\Gamma_{m_1,m_2,k}$.

**Lemma 7.10.** *Let* $Q_{I_1\equiv J_1,I_2\equiv J_2}$ *and* $\mathcal{A}$ *be as above. Then we have:*
*The boundary algebra* $eA_{Q_{I_1\equiv J_1,I_2\equiv J_2}}e$ *is generated by* $\mathcal{A}$.

*Proof.* Let $g : a \to b$ be a path between boundary vertices of $Q_{I_1\equiv J_1,I_2\equiv J_2}$. We first treat paths with no seam crossings.

Suppose $g$ is contained in $\rho(Q_D)$. By Lemma 3.14, a minimal path from $a$ to $b$ is of the form $u_a^\ell$ or $v_a^\ell$. If $a$ and $b$ lie on the same boundary component of $Q_{I_1\equiv J_1,I_2\equiv J_2}$, then this path is a product of arrows $u_i$ and $v_i$ belonging to $\mathcal{A}$. If $a$ and $b$ lie on different boundary components, then the path crosses exactly one glued interval, and by Remark 7.7 it can be written as one of

$$u^i s u^j, \qquad u^i r u^j, \qquad v^i s v^j, \qquad v^i r\, v^j$$

for suitable $i, j \geq 0$. Hence every path in $\rho(Q_D)$ between boundary vertices is generated by $\mathcal{A}$.

Now suppose that $g$ is contained in $\Theta_k$. If both endpoints lie on the outer boundary, Lemma 6.7 gives a minimal path of the form $w^\ell(a)$ or $\widehat{w^\ell(a)}$. If both endpoints lie on the inner boundary, we similarly obtain a path of the form $z^\ell(a)$ or $\widehat{z^\ell(a)}$. If the endpoints lie on different boundary components, then a minimal path is of the form $w^{k-i} r z^{j-1}$ or $\widehat{z^{j-1}}\, s\, \widehat{w^{k-i}}$. In every case the path is a product of elements of $\mathcal{A}$.

Moreover, the boundary unit cycles are generated by $\mathcal{A}$, so the boundary version of the central element $t$ is also generated by $\mathcal{A}$. Since every path is a minimal path multiplied by a power of this central element, the claim holds for all paths with no seam crossings.

We now argue by induction on the number $q$ of seam crossings of $g$, where seam crossing is understood in the sense of Definition 5.13. The case $q = 0$ was just proved.

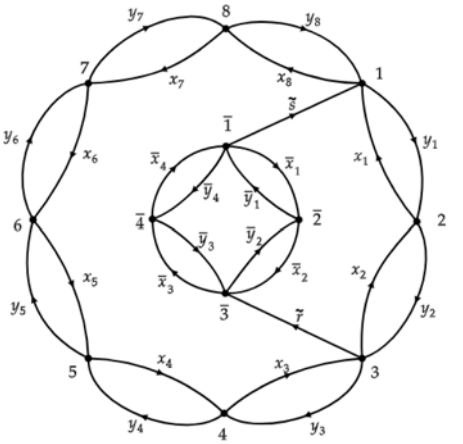
Assume $q > 0$ and write $g = \sigma_1 \cdots \sigma_q$, where each $\sigma_\ell$ is a maximal subpath crossing a seam. If two consecutive crossings use the same seam, then the intermediate subpath has endpoints on that seam, and, by the no-crossing case together with Remarks 7.8 and 7.7, it is equivalent to a path with no seam crossings. Repeating this reduction, we may assume that successive seam crossings alternate between $I_1 \equiv J_1$ and $I_2 \equiv J_2$.

Consider the initial segment of $g$ up to and including the first seam crossing. Its endpoints are boundary vertices, and it crosses at most one seam, so by the base case it is generated by $\mathcal{A}$. The remaining tail of $g$ has $q - 1$ seam crossings, and therefore it is generated by $\mathcal{A}$ by the induction hypothesis. Hence $g$ itself lies in the subalgebra generated by $\mathcal{A}$. This proves that every path between boundary vertices is generated by $\mathcal{A}$, so $eA_{Q_{I_1\equiv J_1,I_2\equiv J_2}}e$ is generated by $\mathcal{A}$. □

**Notation 7.11.** *We define a quiver on an annulus: For* $k \geq 2$ *and* $m_1, m_2 \geq 0$, *let* $\Gamma = \Gamma_{m_1,m_2,k}$ *be the quiver with* $2k - 2 + m_1 + m_2$ *vertices arranged as follows.* $\Gamma$ *has* $m_2 + k - 1$ *vertices on the outer boundary, arranged clockwise. It has* $m_1 + k - 1$ *vertices on the inner boundary, also arranged clockwise. The arrows of* $\Gamma_{m_1,m_2,k}$ *are given by* $y_i : i \to i + 1, x_i : i + 1 \to i$ *on the outer boundary, for* $1 \leq i \leq k + m_2 - 1$, $\overline{x_i} : \overline{i} \to \overline{i+1}, \overline{y_i} : \overline{i+1} \to \overline{i}$ *on the inner boundary,* $1 \leq i \leq k + m_1 - 1$ *and arrows* $\tilde{s} : \overline{1} \to 1$ *and* $\tilde{r} : k \to \overline{k}$ *between the boundary components. A depiction of the quiver* $\Gamma_{m_1,m_2,k}$ *is given in Figure 18.*

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**Example 7.12.** *The following is an illustration of the quiver* $\Gamma_{2,6,3}$.




We will drop the indices and refer to the arrows as $x, y, \overline{x}, \overline{y}$ whenever the indices are clear from composition.

**Definition 7.13.** *Let* $\Lambda_{m_1,m_2,k}$ *be the quotient of the path algebra* $\mathbb{C}\Gamma_{m_1,m_2,k}$ *by the ideal generated by the following relations.*

(1) $y_i x_i - x_{i-1} y_{i-1}$ $\quad i = 1, \ldots, k + m_2 - 1$
(2) $\overline{x}_i \overline{y}_i - \overline{y}_{i-1} \overline{x}_{i-1}$ $\quad i = 1, \ldots, k + m_1 - 1$

(3) $x^k - y^{j_1} \tilde{r} \overline{y}^{2k+m_1-2} \tilde{s} y^{k+m_2-j_1-2}$ $\quad (k + m_2 - 1$ *relations*)
(4) $\overline{x}^k - \overline{y}^{j_2} \tilde{s} y^{2k+m_2-2} \tilde{r} \overline{y}^{k+m_1-j_2-2}$ $\quad (k + m_1 - 1$ *relations*)

(5) $\tilde{s} - \overline{y}^{m_1+k-1} \tilde{s} y^{m_2+k-1}$
(6) $\tilde{r} - y^{m_2+k-1} \tilde{r} \overline{y}^{m_1+k-1}$

(7) $x_{k-1} y_{k-1} \tilde{r} - \tilde{r} \overline{y}_{k-1} \overline{x}_{k-1}$
(8) $\overline{x}_{m_1+k-1} \overline{y}_{m_1+k-1} \tilde{s} - \tilde{s} y_{k+m_2-1} x_{k+m_2-1}$

*where there are relations of type (1) and (3) for each of the* $k + m_2 - 1$ *vertices on the outer boundary and where there are relations of type (2) and (4) for each of the* $k + m_1 - 1$ *vertices on the inner boundary. The exponent* $j_1 \geq 0$ *in (3) is minimal such that the path* $y^{j_1}$ *ends at the starting point of* $\tilde{r}$ *and the exponent* $j_2 \geq 0$ *in (4) is the minimum such that the path* $\overline{y}^{j_2}$ *ends at the starting point of* $\tilde{s}$*. Indices are reduced modulo* $k + m_2 - 1$ *on the outer boundary and modulo* $k + m_1 - 1$ *on the inner boundary.*

Note that replacing $\tilde{r}$ and $\tilde{s}$ in the presentation of $\Lambda_{m_1,m_2,k}$ by any pair of arrows linking the two double cycles of $m_2 + k - 1$ and $m_1 + k - 1$ arrows in a way to keep the distances between these two arrows fixed (the distance between the starting point of $\tilde{s}$ and the endpoint of $\tilde{r}$ as well as the distance between the starting point of $\tilde{r}$ and the endpoint of $\tilde{s}$) and modifying relations (3)–(8) accordingly gives an isomorphic algebra. Further, similar presentations may be derived from an arbitrary choice of $\tilde{r}, \tilde{s}$ between the two double cycles.

**Theorem 7.14.** *Let* $Q_{l_1 \equiv J_1, J_2 \equiv J_2}$ *be a dimer quiver on the annulus as in Theorem 7.2, and let* $m_1, m_2$ *be as in Remark 7.3. Let* $e$ *be the idempotent corresponding to the sum of the boundary vertices. Then we have an isomorphism*

$$eA_{Q_{l_1 \equiv J_1, J_2 \equiv J_2}} e \cong \Lambda_{m_2,m_1,k}.$$

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*Proof.* Set

$$Q := \rho(Q_D) \amalg \Theta_k,$$

where $I_1$ and $I_2$ are the two chosen sets of $k$ consecutive boundary arrows of $Q_D$.

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the relations derived from the potentials on $\rho(Q_D)$ and $\Theta_k$ respectively and let $R_i = \langle \mathcal{R}_i \rangle$ be the corresponding ideals, that is $A_{\rho(Q_D)} = \mathbb{C}(\rho(Q_D))/R_1$ and $A_{\Theta_k} = \mathbb{C}\Theta_k/R_2$. Throughout the proof we write $\Gamma$ for $\Gamma_{m_1,m_2,k}$ and $\Lambda$ for $\Lambda_{m_1,m_2,k}$ to abbreviate and we often use $Q_{I\equiv J}$ as a short notation for $Q_{I_1\equiv J_1, I_2\equiv J_2}$. Let $\mathcal{A}$ be the collection of paths of $Q_{I\equiv J}$ from Notation 7.9.

Our goal is to define a map $\mathbb{C}\Gamma \to e(A_{Q_{I\equiv J}})e$ and to show that its kernel is the ideal given by the relations (1)-(8) of Definition 7.13.

We first define $\psi : \mathbb{C}\Gamma \to \mathbb{C}Q_{I\equiv J}$ on arrows of $\Gamma$ as follows (recalling that we write $s$ for the composition $s_1 \cdots s_k$ and $r$ for $r_1 \cdots r_k$, Notation 6.9).

$$\begin{aligned} \psi(\tilde{r}) &= r \\ \psi(\tilde{s}) &= s \end{aligned}$$

$$\psi(x_i) = \begin{cases} \widehat{w}_i & \text{if } 1 \le i \le k-1 \\ v_{m_1+i+1} & \text{if } k \le i \le k+m_2-1 \end{cases}$$

$$\psi(y_i) = \begin{cases} w_i & \text{if } 1 \le i \le k-1 \\ u_{m_1+i+1} & \text{if } k \le i \le k+m_2-1 \end{cases}.$$

$$\psi(\overline{x_i}) = \begin{cases} \widehat{z}_{k-i} & \text{if } 1 \le i \le k-1 \\ v_{m_1+k-i} & \text{if } k \le i \le k+m_1-1 \end{cases}$$

$$\psi(\overline{y_i}) = \begin{cases} z_{k-i} & \text{if } 1 \le i \le k-1 \\ u_{m_1+k-i} & \text{if } k \le i \le k+m_1-1 \end{cases}$$

On vertices $\psi$ is determined from the mappings on arrows. We extend $\psi$ to all paths in $\Gamma$ and $\mathbb{C}$-linearly to the algebra $\mathbb{C}\Gamma$.

We then compose $\psi$ with the quotient map $\mathbb{C}Q_{I\equiv J} \to A_{Q_{I\equiv J}}$ to the dimer algebra and call the resulting map $\varphi$.

By construction, the image of $\varphi$ lies in $eA_{Q_{I\equiv J}}e$: the map $\psi$ sends arrows of $\Gamma$ to paths between boundary vertices of $Q_{I\equiv J}$ in such a way that the latter agrees with the vertices of $\Gamma$. The map $\varphi$ is a homomorphism of algebras. Since $\mathcal{A}$ is a generating set for $e(A_{Q_{I\equiv J}})e$ by Lemma 7.10, $\varphi$ is surjective. By the first isomorphism theorem, we thus have $e(A_{Q_{I\equiv J}})e \cong \mathbb{C}\Gamma/\ker\varphi$. To see that the latter is equal to $\Lambda$, it remains to show that the kernel of $\varphi$ is equal to the ideal of $\mathbb{C}\Gamma$ given by the relations (1)–(8) of Definition 7.13.

We first express $\ker\varphi$ in terms of the ideals $R_1$, $R_2$ and of the relations arising from the seam. Let $I_1 = \{i_{1,1}, \ldots, i_{1,k}\}$ and $I_2 = \{i_{2,1}, \ldots, i_{2,k}\}$ be the arrows of $Q_D$ giving the seam together with the sets $J_1 = \{s_1, \ldots, s_k\}$ and $J_2 = \{r_1, \ldots, r_k\}$ of arrows of $\Theta_k$. We define

$$\mathcal{W}_1 := \left\{ \begin{array}{ll} s_m - \rho(i_{1,m}) & : 1 \le m \le k, \\ \widehat{s_m} - \widehat{\rho(i_{1,m})} & : 1 \le m \le k, \\ e_{h(s_m)} - e_{h(i_{1,m})} & : 1 \le m \le k, \\ e_{t(s_k)} - e_{t(i_{1,k})} & \end{array} \right\}$$

and

$$\mathcal{W}_2 := \left\{ \begin{array}{ll} r_m - \rho(i_{2,m}) & : 1 \le m \le k, \\ \widehat{r_m} - \widehat{\rho(i_{2,m})} & : 1 \le m \le k, \\ e_{h(r_m)} - e_{h(i_{2,m})} & : 1 \le m \le k, \\ e_{t(r_k)} - e_{t(i_{2,k})} & \end{array} \right\}.$$

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The first and third/fourth types of generators in $\mathcal{W}_1$ and $\mathcal{W}_2$ record the identifications made in the gluing construction. The second type are the additional dimer relations created when the seam arrows become internal after gluing. Let $W_1 := \langle \mathcal{W}_1 \rangle$, $W_2 := \langle \mathcal{W}_2 \rangle$, and set

$$R_3 := W_1 + W_2.$$

By abuse of notation, we view $W_1$, $W_2$, $R_1$, $R_2$ and $R_3$ as ideals of $\mathbb{C}Q$. Since $Q_{I\equiv J}$ can be obtained from $Q_{I_1\equiv J_1}$ by gluing $I_2 \equiv J_2$, and also from $Q_{I_2\equiv J_2}$ by gluing $I_1 \equiv J_1$, Proposition 5.14 gives

$$A_{Q_{I\equiv J}} \cong A_Q/R_3.$$

Furthermore, as $Q$ is the disjoint union of $\rho(Q_D)$ and $\Theta_k$, we have

$$A_Q \cong \mathbb{C}Q/\langle R_1, R_2\rangle.$$

Hence

$$A_{Q_{I\equiv J}} \cong \mathbb{C}Q/\langle R_1, R_2, R_3\rangle.$$

It remains to show that $\ker(\varphi)$ is given by the relations (1)–(8). We check relations (1), (3), (5) and (7) hold as the rest is symmetric.

Relations (1):

$$\varphi(y_i x_i - x_{i-1}y_{i-1}) = \begin{cases} w_1\widehat{w}_1 - v_{k+m_1+m_2}u_{k+m_1+m_2} & \text{if } i = 1, \\ w_i\widehat{w}_i - \widehat{w}_{i-1}w_{i-1} & \text{if } 1 < i \le k-1, \\ u_{k+m_1+1}v_{k+m_1+1} - \widehat{w}_{k-1}w_{k-1} & \text{if } i = k, \\ u_{m_1+i+1}v_{m_1+i+1} - v_{m_1+i-2}u_{m_1+i} & \text{if } k < i \le k+m_2-1. \end{cases}$$

The second and fourth of these cases are relations in $\mathcal{R}_2$ and $\mathcal{R}_1$, respectively. To see that the relation holds for $i = 1$:

$$w_1\widehat{w}_1 \overset{\mathcal{R}_2}{=} \widehat{s}_k s_k \overset{\mathcal{R}_3}{=} \widehat{\rho(i_{1,1})}i_{1,1} \overset{\mathcal{R}_1}{=} v_{k+m_1+m_2}u_{k+m_1+m_2}.$$

For the relation in case $i = k$:

$$u_{k+m_1+1}v_{k+m_1+1} \overset{\mathcal{R}_1}{=} \widehat{\rho(i_{2,k})}i_{2,k} \overset{\mathcal{R}_3}{=} r_1\widehat{r}_1 \overset{\mathcal{R}_2}{=} \widehat{w}_{k-1}w_{k-1}.$$

So in all cases, the term $x_iy_i - y_{i-1}x_{i-1}$ is in the kernel of $\varphi$.

Relations (3): The proof of these relations is in Lemma 7.15 below.

Relation (5): We have to check $\varphi(\tilde{s} - \overline{y}^{m_1+k-1}\tilde{s}y^{m_2+k-1}) = 0$.

$$\begin{aligned} \varphi(\tilde{s}) = s &\overset{\mathcal{R}_3}{=} v_n v_{n-1}\dots v_{n-k+1} \overset{\mathcal{R}_1}{=} u_1u_2\dots u_{k+m_1+m_2} \\ &\overset{\mathcal{R}_3}{=} u_1u_2\dots u_{m_1}\hat{r}u_{k+m_1+1}u_{k+m_1+2}\dots u_{k+m_1+m_2} \\ &\overset{\mathcal{R}_2}{=} u_1u_2\dots u_{m_1}z_1z_2\dots z_{k-1}sw_1w_2\dots w_{k-1}u_{k+m_1+1}u_{k+m_1+2}\dots u_{k+m_1+m_2} \\ &= \varphi(\overline{y}^{m_1+k-1}\tilde{s}y^{m_2+k-1}). \end{aligned}$$

Relation (7): We have to check $\varphi(x_{k-1}y_{k-1}\tilde{r} - \tilde{r}\,\overline{y}_{k-1}\,\overline{x}_{k-1}) = 0$.

$$\varphi(x_{k-1}y_{k-1}\tilde{r}) = r\widehat{w_{k-1}}w_{k-1}.$$

As unit cycles are central in the dimer algebra we have

$$\begin{aligned} \widehat{w_{k-1}}w_{k-1}r &= r_1\widehat{r}_1r_1r_2\cdots r_k \\ &= r_1r_2\widehat{r}_2r_2\dots r_k \\ &= \cdots \\ &= r_1r_2\dots r_k\widehat{r}_kr_k \\ &= r\widehat{r}_kr_k. \end{aligned}$$

Under the gluing identifications this becomes

$$r\widehat{r}_kr_k \overset{\mathcal{R}_3}{=} r\rho(i_{2,k})\widehat{\rho(i_{2,k})},$$

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where $\rho(i_{2,k})\widehat{\rho(i_{2,k})}$ is a unit cycle at the target of $r$. Therefore we obtain

$$r\rho(i_{2,1})\widehat{\rho(i_{2,1})} \overset{\mathcal{R}_1}{=} rv_{m_1}u_{m_1} \overset{\mathcal{R}_3}{=} rz_1\widehat{z_1} = \varphi(\tilde{r}\,\bar{y}_{k-1}\,\bar{x}_{k-1}).$$

Conversely, every generator of $\ker\varphi = \psi^{-1}(R_1 + R_2 + R_3)$ comes from one of the local dimer relations in $R_1$, $R_2$ or $R_3$. Relations supported entirely on the outer boundary give the relations of type (1), those supported on the inner boundary give the relations of type (2), the length-$k$ boundary paths meeting $\tilde{r}$ or $\tilde{s}$ give the relations of type (3) and (4), and the remaining seam relations at the two distinguished vertices are exactly the relations of type (5)–(8). Hence $\ker\varphi$ is precisely the ideal generated by the relations in Definition 7.13. Therefore we obtain the desired isomorphism. □

**Lemma 7.15.** *In the situation of Theorem 7.14, the* $m_2 + k - 1$ *relations*

$$\varphi(x^k) = \varphi(y^{j_1}\tilde{r}\bar{y}^{2k+m_1-2}\tilde{s}y^{k+m_2-j_1-2})$$

*hold for any of the* $m_2 + k - 1$ *paths of type* $x^k$ *starting on the outer boundary.*

*Proof.* We write the term $x^k$ depending on its starting position between the starting point of $\tilde{r}$ and the ending point of $\tilde{s}$, as $x^{a_1}x^{a_2}x^{a_3}x^{a_4}$ with $a_i \geq 0$: the first $a_1$ arrows are from the set $\{x_1, \ldots, x_{k-1}\}$, the second $a_2$ arrows from the set $\{x_k, \ldots, x_{k+m_2-1}\}$, the third $a_3$ arrows from the set $\{x_1, \ldots, x_{k-1}\}$ and the last $a_4$ arrows are again from the set $\{x_k, \ldots, x_{k+m_2-1}\}$ (see Figure 18 for the labelling of the arrows in $\Gamma$). The exponents $a_i$ are all non-negative and add up to $k$. Furthermore, $a_1$ and $a_3$ are strictly smaller than $k$. We may assume that $a_4 = 0$ (if in the expression for $x^k$, we had $a_4 > 0$, we would have $a_2 = a_1 = 0$. Such a case can be covered by $a_2 > 0$, with $a_4 = 0$).

The triples $(a_1, a_2, a_3)$ are therefore one of the following:

$$\begin{array}{ll} (i)\ (a_1, m_2, k - a_1 - m_2) & \text{where } 0 < a_1 < k,\ a_1 + m_2 \leq k \\ (ii)\ (0, a_2, k - a_2) & \text{where } 0 < a_2 < k \\ (iii)\ (0, k, 0) & \end{array}$$

Note that $m_2$ or $k - a_1 - m_2$ may be 0. We obtain

$$x^k = \begin{cases} \overbrace{x_{a_1}x_{a_1-1}\cdots x_1}^{a_1 \text{ terms}}\overbrace{x_{k+m_2-1}x_{k+m_2-2}\cdots x_k}^{m_2 \text{ terms}}\overbrace{x_{k-1}x_{k-2}\cdots x_{a_1+m_2}}^{k-a_1-m_2 \text{ terms}} & \text{in case (i),} \\ \overbrace{x_{k+a_2-1}x_{k+a_2-2}\cdots x_k}^{a_2 \text{ terms}}\overbrace{x_{k-1}x_{k-2}\cdots x_{a_2}}^{k-a_2 \text{ terms}} & \text{in case (ii),} \\ \overbrace{x_{2k+b-1}x_{2k+b-2}\cdots x_{k+b}}^{k \text{ terms}} \text{ with } 0 \leq b \leq m_2 - k & \text{in case (iii).} \end{cases}$$

Define

$$\eta := y^{j_1}\tilde{r}\bar{y}^{2k+m_1-2}\tilde{s}y^{k+m_2-j_1-2}.$$

Then the exponent $j_1$ in $\eta$ is equal to $k - 1 - a_1$ in the first case, to $m_2 - a_2 + k - 1$ in the second case and to $m_2 - b - 1$ in the third case, giving do we still need the $*$ here?

$$\eta = \begin{cases} \overbrace{y_{a_1+1}y_{a_1+2}\cdots y_{k-1}}^{k-1-a_1 \text{ terms}}\,\tilde{r}\,\bar{y}^{2k+m_1-2}\,\tilde{s}\,\overbrace{y_1y_2\cdots y_{a_1+m_2-1}}^{a_1+m_2-1 \text{ terms}} & \text{case (i),} \\ \overbrace{y_{k+a_2}y_{k+a_2+1}\cdots y_{k+m_2-1}}^{m_2-a_2 \text{ terms}}\overbrace{y_1y_2\cdots y_{k-1}}^{k-1 \text{ terms}}\,\tilde{r}\,\bar{y}^{2k+m_1-2}\,\tilde{s}\,\overbrace{y_1y_2\cdots y_{a_2-1}}^{a_2-1 \text{ terms}} & \text{case (ii),} \\ \overbrace{y_{2k+b}y_{2k+b+1}\cdots y_{k+m_2-1}}^{m_2-k-b \text{ terms}}\overbrace{y_1y_2\cdots y_{k-1}}^{k-1 \text{ terms}}\,\tilde{r}\,\bar{y}^{2k+m_1-2}\,\tilde{s}\,\overbrace{y_1y_2\cdots y_{k+b-1}}^{k+b-1 \text{ terms}} & \text{case (iii).} \end{cases}$$

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Accordingly, the terms $\varphi(x^k)$ and $\varphi(\eta)$ are as follows (see Figure 16 for the labelling), abbreviating $r = r_1 \cdots r_k$ and $s = s_1 \cdots s_k$ as before:

$$\varphi(x^k) = \begin{cases} \widehat{w_{a_1}} \cdots \widehat{w_1}\, v_{k+m_1+m_2} \cdots v_{k+m_1+1}\, \widehat{w_{k-1}} \cdots \widehat{w_{a_1+m_2}} & \text{in case (i)}, \\ v_{k+m_1+a_2} v_{k+m_1+a_2-1} \cdots v_{k+m_1+1}\, \widehat{w_{k-1}} \cdots \widehat{w_{a_2}} & \text{in case (ii)}, \\ v_{2k+m_1+b} v_{2k+m_1+b-1} \cdots v_{k+m_1+b+1} & \text{in case (iii)}. \end{cases}$$

Recall that $\widehat{r}$ and $\widehat{s}$ are the compositions of all $k$ arrows $\widehat{r_i}$ and all $k$ arrows $\widehat{s_i}$, respectively, see Notation 6.9.

To write the image of the term $(\eta)$ under $\varphi$, we use $U$ to abbreviate the composition $u_1 \cdots u_{m_1}$, $Z$ for $z_1 \cdots z_{k-1}$ and $W$ for $w_1 w_2 \cdots w_{k-1}$ (we *only* use these abbreviations in the situation when there are all $k-1$ arrows $z_1, \ldots, z_{k-1}$, all arrows $u_1, \ldots, u_{m_1}$ and all arrows $w_1, \ldots, w_{k-1}$, respectively). Then we have:

$$\varphi(\eta) = \begin{cases} w_{a_1+1} w_{a_1+2} \cdots w_{k-1}\, r\, ZUZ\, s w_1 \cdots w_{m_2+a_1-1} & \text{(i)}, \\ u_{k+m_1+a_2+1} \cdots u_{k+m_1+m_2} WrZUZ\, s w_1 \cdots w_{a_2-1} & \text{(ii)}, \\ u_{2k+b+1+m_1} \cdots u_{k+m_1+m_2} WrZUZsW\, u_{k+m_1+1} u_{k+m_1+2} \cdots u_{k+m_1+b} & \text{(iii)}. \end{cases}$$

For all three cases, the paths $\varphi(x^k)$ and $\varphi(\eta)$ are illustrated in Figure 20.

We first consider case (iii),

Case (iii)

$$\begin{aligned} \varphi(x^k) &= v_{2k+b-1} v_{2k+b-2} \cdots v_{k+b} \\ &= u_{2k+b} u_{2k+b+1} \cdots u_{k+b-1} && \text{Lemma 3.14} \\ &= u_{2k+b-1} \cdots u_{k+m_1+m_2} \widehat{s}\, U\, \widehat{r} u_{k+m_1+1} \cdots u_{k+b-1} && \text{Remark 7.7} \end{aligned}$$

By Lemma 6.7, $\widehat{s}$ and $WrZ$ are minimal paths between the same boundary vertices in the bridge (cf. Remark 6.8). From Lemma 6.6 they are necessarily equal under $\mathcal{R}_3$. Similarly, $\widehat{r}$ is equal to $ZsW$, so we obtain:

$$\begin{aligned} &= u_{2k+m_1+b+1} \cdots u_{k+m_1+m_2} WrZU\, \widehat{r} u_{k+m_1+1} \cdots u_{k+b+1} && \text{Lemma 6.7} \\ &= u_{2k+m_1+b+1} \cdots u_{k+m_1+m_2} WrZUZsW u_{k+m_1+1} \cdots u_{k+b+1} && \text{Lemma 6.7} \end{aligned}$$

as claimed.

Case (ii)

We first consider $a_2 = 1$. Define $h$ to be the (horizontal) path of length $k-1$ in $\Theta_k$, starting at $(k,k)$, going to $(k,k-1)$, etc., ending at $(k,2)$ (as shown in Figure 19). Recall that $R_2$ is the ideal of relations arising from the potential on $\Theta_k$ and $R_1$ the ideal of relations arising from the potential on $Q_D$.

$$\begin{aligned} \varphi(x^k) &= v_{k+m_1+1}\, \widehat{w_{k-1}} \cdots \widehat{w_1} \\ &= v_{k+m_1+1} r_k \cdots r_2 h s_2 \cdots s_k && \text{using } R_2 \\ &= v_{k+m_1+1} v_{m_1+k} \cdots v_{m_1+2} h s_2 \cdots s_k && \text{Remark 7.7} \\ &= u_{k+m_1+2} u_{k+m_1+3} \cdots u_{m_1+1} h s_2 \cdots s_k && \text{using } R_1 \text{ (Lemma 3.14)} \\ &= u_{k+m_1+2} \cdots u_{k+m_1+m_2}\, \widehat{s}\, u_1 \cdots u_{m_1+1} h s_2 \cdots s_k && \text{Remark 7.7} \\ &= u_{k+m_1+2} \cdots u_{k+m_1+m_2} WrZU u_{m_1+1} h s_2 \cdots s_k && \text{Lemma 6.7} \\ &= u_{k+m_1+2} \cdots u_{k+m_1+m_2} WrZU r_k h s_2 \cdots s_k && \text{Remark 7.7} \\ &= u_{k+m_1+2} \cdots u_{k+m_1+m_2} WrZUZs && \text{seam relations and } R_2. \end{aligned}$$

This proves (ii) for $a_2 = 1$. If $1 < a < k$, we write $d_{[a]}$ for the following (diagonal) path of length $k-a$ in $\Theta_k$: $d_{[a]}$ starts at $(a,k)$ and ends at $(1, k-a+1)$, using the arrows $(a+1-j, k+1-j) \to (a-j, k-j)$ for $1 \le j \le a-1$. See Figure 19.

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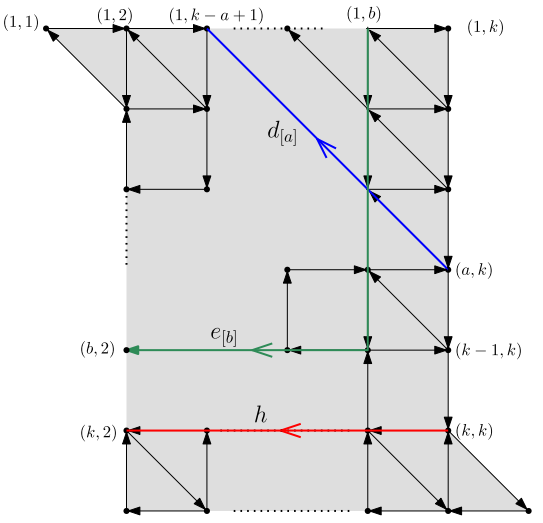



***Figure 19.*** *Shape of $\Theta_k$, paths $h, d_{[a]}, e_{[b]}$.*

The general situation is then as follows (with $1 < a_2 < k$):

$$\begin{aligned}
\varphi(x^k) &= v_{k+m_1+a_2} v_{k+m_1+a_2-1} \cdots v_{k+m_1+1} \widehat{w_{k-1}} \cdots \widehat{w_{a_2}} \\
&= v_{k+m_1+a_2} \cdots v_{k+m_1+1}\, r_1 \cdots r_{k-a_2} d_{[k-a_2]} && \text{using } R_2 \\
&= \underbrace{v_{k+m_1+a_2} \cdots v_{k+m_1+1}\, v_{k+m_1} \cdots v_{m_1+a_2+1}}_{k \text{ terms}} d_{[k-a_2]} && \text{Remark 7.7} \\
&= u_{k+m_1+a_2+1} u_{k+m_1+a_2+2} \cdots u_{m_1+a_2} d_{[k-a_2]} && \text{using } R_1 \\
&= u_{k+m_1+a_2+1} \cdots u_{k+m_1+m_2}\, \widehat{s}\, U u_{m_1+1} \cdots u_{m_1+a_2} d_{[k-a_2]} \\
&= u_{k+m_1+a_2+1} \cdots u_{k+m_1+m_2}\, WrZU u_{m_1+1} \cdots u_{m_1+a_2} d_{[k-a_2]} \\
&= u_{k+m_1+a_2+1} \cdots u_{k+m_1+m_2}\, WrZUZs w_1 \cdots w_{a_2-1} && \text{seam and } R_2,
\end{aligned}$$

as claimed.

Case (i) For $1 < b < k$, define $e_{[b]}$ to be the path in $\Theta_k$ starting from $(1, b)$, going down via $(2, b)$, $(3, b)$, etc. to $(b, b)$, then left to $(b-1, b)$, etc. to $(2, b)$.

Then $\varphi(x^k)$ is equal to

$$\begin{aligned}
&\widehat{w_{a_1}} \cdots \widehat{w_1}\, v_{k+m_1+m_2} \cdots v_{k+m_1+1} \widehat{w_{k-1}} \cdots \widehat{w_{a_1+m_2}} \\
&= \widehat{w_{a_1}} \cdots \widehat{w_1}\, v_{k+m_1+m_2} \cdots v_{k+m_1+1} r_1 \cdots r_{k-a_1-m_2} d_{[a_1]} && \text{using } R_2 \\
&= e_{[a_1]} \underbrace{s_{k-a_1+1} \cdots s_k}_{a_1 \text{ terms}} \underbrace{v_{k+m_1+m_2} \cdots v_{k+m_1+1}}_{m_2 \text{ terms}} \underbrace{r_1 \cdots r_{k-a_1-m_2}}_{k-a_1-m_2 \text{ terms}} d_{[k-a_1-m_2+1]} && \text{using } R_2 \\
&= e_{[a_1]} v_{k+m_1+m_2+a_1} \cdots v_{m_1+m_2+a_1+1} d_{[k-a_1-m_2+1]} && \text{Remark 7.7} \\
&= e_{[a_1]} u_{k+m_1+m_2+a_1+1} \cdots u_{m_1+m_2+a_1} d_{[k-a_1-m_2+1]} && \text{using } R_1 \\
&= e_{[a_1]} \widehat{s_{k-a_1}} \cdots \widehat{s_1}\, U\, \widehat{r_k} \cdots \widehat{r_{k-a_1-m_2-1}} d_{[k-a_1-m_2+1]} && \text{Remark 7.7} \\
&= w_{a_1+1} w_{a_1+2} \cdots w_{k-1}\, r\, ZU\, \widehat{r_k} \cdots \widehat{r_{k-a_1-m_2-1}} d_{[k-a_1-m_2+1]} && \text{using } R_2 \\
&= w_{a_1+1} w_{a_1+2} \cdots w_{k-1}\, r\, ZUZ\, s w_1 \cdots w_{m_2+a_1-1} && \text{using } R_2,
\end{aligned}$$

as claimed. □

**Remark 7.16.** *By Lemma 4.8 if $D'$ is a weak Postnikov diagram on the annulus that can be obtained from $D$ by a series of geometric exchange moves, then the boundary algebra of $Q_{D'}$ is isomorphic to $\Lambda_{m_1,m_2,k}$.*

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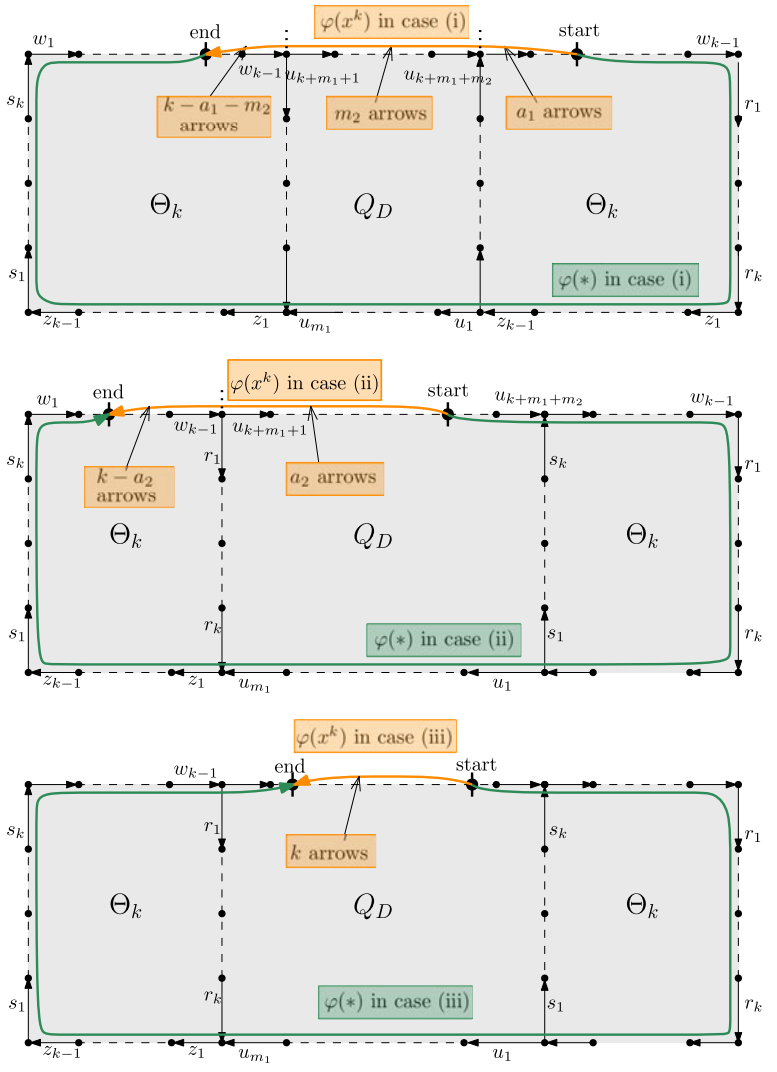



***Figure 20.*** *The images of the paths from relation (3) under $\varphi$ on a 'cover' of the annulus, cases (i), (ii) and (iii).*

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**Competing interests.** The authors declare none.

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