Acknowledgement

I thank the referee for helpful comments that improved the article, any remaining faults being all my own work.

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100.21 Generalising the arbelos

The shape shown in Figure 1 is known as the *arbelos*, the name given by Archimedes. The arbelos is bounded by three semicircles; the diameter of the larger semicircle is the sum of the diameters of the two smaller ones. As a result, the semicircles are mutually tangent.

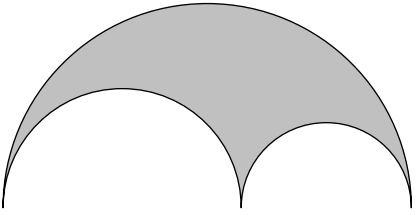


FIGURE 1

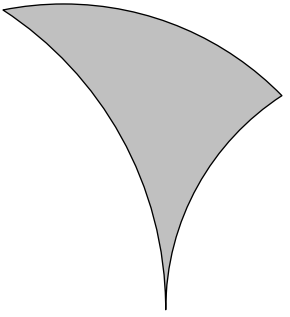


FIGURE 2

The arbelos has been extensively studied, originally by Archimedes (in his *Book of Lemmas*, see [1]), later by Pappus, and in a flurry of recent activity (see, for example, [2, 3]).

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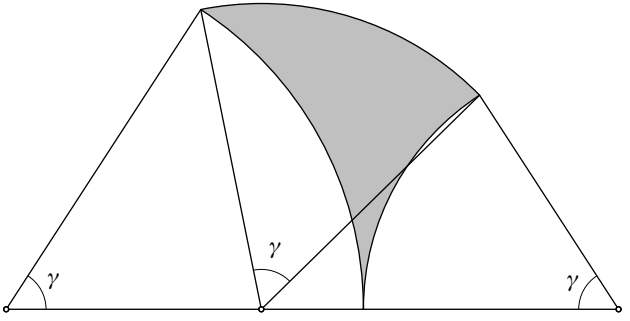


FIGURE 3

In this note we consider a generalisation of the arbelos, shown in Figure 2. The shape is bounded by three circular arcs subtending the same angle at each of their centres. Two of the arcs are tangential, and the three centres are collinear, as shown in Figure 3, where the common angle has value γ.

We shall refer to this shape simply as the *general arbelos*. Setting $\gamma = \pi$, we recover Archimedes' shape, which we shall refer to as an *Archimedean arbelos*.

Radii

Let the radii of the three arcs be a, b and c. Since the centres of the circular arcs are collinear, the two triangles in Figure 3 have the same angles (see Figure 4). It follows that the triangles are congruent.

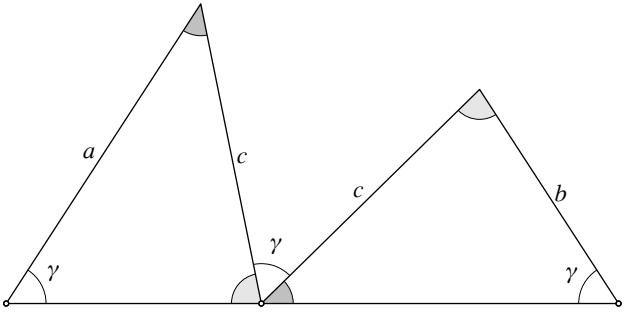


FIGURE 4

Each triangle therefore has sides a, b and c, so that $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

Note that the general arbelos is actually part of an Archimedean arbelos; the three arcs of the general arbelos have the same centres as the three semicircles of the Archimedean arbelos (see Figure 5).

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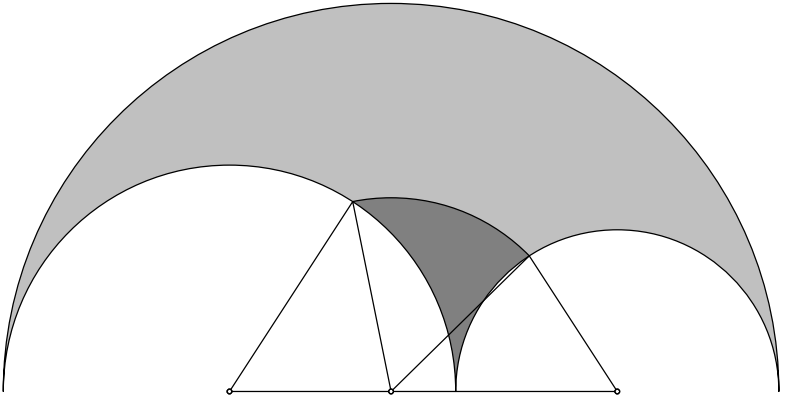


FIGURE 5

The incircle

Archimedes found the radius of the inscribed circle, that is, the circle which touches each of the three semicircular arcs that bound the Archimedean arbelos, shown in Figure 6.

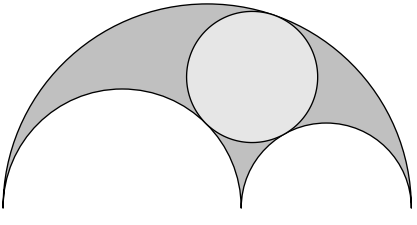


FIGURE 6

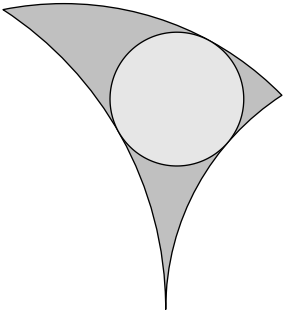


FIGURE 7

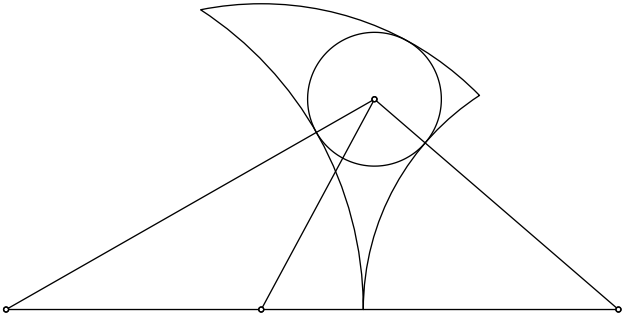


FIGURE 8

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We may find the radius r of the incircle of the general arbelos (see Figure 7) by joining the centres of all the circles, as shown in Figure 8. The key information about lengths is shown in Figure 9.

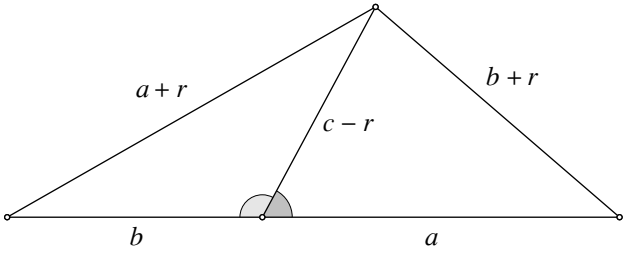


FIGURE 9

One way to proceed now is to use the cosine rule in the two smaller triangles. Noting that the two marked angles are supplementary, so that their cosines add up to zero, we get

$$\frac{(a+r)^2-(c-r)^2-b^2}{2b(c-r)}+\frac{(b+r)^2-(c-r)^2-a^2}{2a(c-r)}=0,$$

from which we obtain

$$r=\frac{a\left(b^2+c^2-a^2\right)+b\left(a^2+c^2-b^2\right)}{2\left(a^2+b^2+c(a+b)\right)}.$$

In the case of an Archimedean arbelos we have $c = a + b$, so that

$$r=\frac{ab(a+b)}{a^2+ab+b^2}.$$

Archimedean twins

Figure 10 shows two circles, each of which touches two of the bounding semicircular arcs of an Archimedean arbelos and also touches the common tangent of the two smaller semicircular arcs. Another result of Archimedes (Proposition 5 of the *Book of Lemmas*) is that these two circles have the same radius, hence they are known as *Archimedean twins*.

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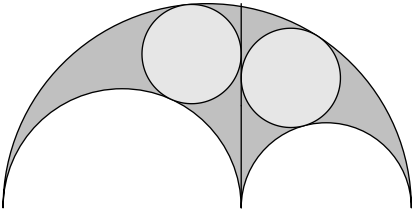


FIGURE 10

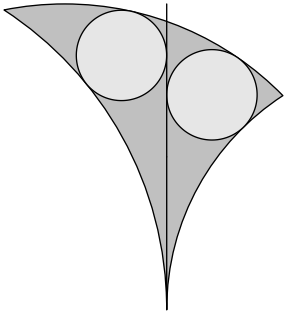


FIGURE 11

As we shall see, the corresponding result for the general arbelos also holds, so the two circles shown in Figure 11 are equal. Consider the left-hand circle. Joining the centres of three circles and drawing the perpendiculars from these centres to the tangent line, we obtain Figure 12.

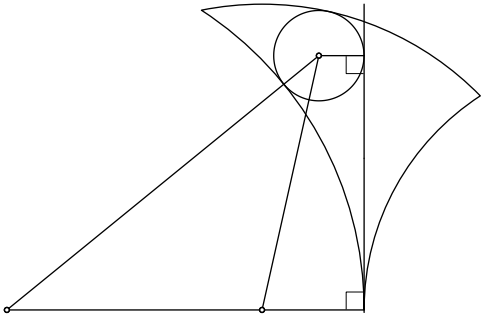


FIGURE 12

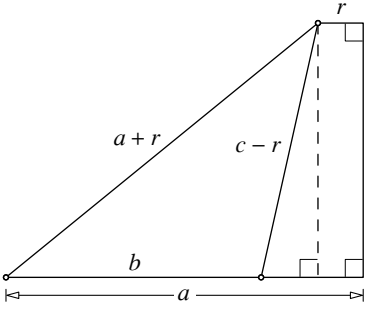


FIGURE 13

Letting the radius of the left-hand circle be r, we have the lengths shown in Figure 13. Drawing the dashed line in Figure 13 and applying Pythagoras' theorem twice, we obtain

$$(a + r)^2 - (a - r)^2 = (c - r)^2 - (a - b - r)^2,$$

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so that

$$r = \frac{c^2 - (a - b)^2}{2(a + b + c)}. \tag{1}$$

To find the radius of the other circle, we would repeat this process on the other side, which is equivalent to interchanging a and b. However, expression (1) remains the same if a and b are interchanged, and it follows that the radii of the two circles shown in Figure 11 are equal. In other words, the general arbelos also has Archimedean twins.

Area

In Proposition 4 of the *Book of Lemmas*, Archimedes proved that the area of an (Archimedean) arbelos is equal to the area of the dashed circle in Figure 14, whose diameter is that part of the common tangent of the smaller arcs that lies inside the arbelos. A glance at Figure 15 should be enough to convince the reader that the corresponding result is *not* true for the general arbelos.

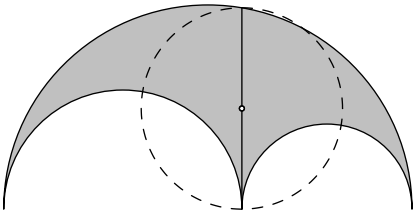


FIGURE 14

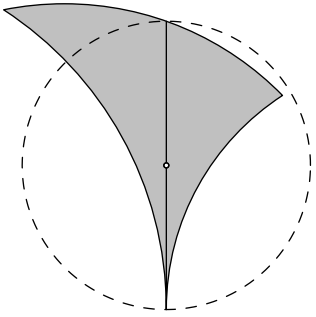


FIGURE 15

However, it is possible to find an expression for the area of the general arbelos. From Figure 3 we see that the area of the general arbelos together with two sectors is the same as the area of the other sector together with two copies of the triangle. Letting A be the area of the general arbelos, we therefore obtain

$$A + \tfrac{1}{2}\left(a^2 + b^2\right)\gamma = \tfrac{1}{2}c^2\gamma + ab\sin\gamma,$$

so that

$$A = \tfrac{1}{2}\left(c^2 - a^2 - b^2\right)\gamma + ab\sin\gamma.$$

We leave the reader to investigate which, if any, of the many other known results about the Archimedean arbelos carry across to the general arbelos.

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100.22 The maximal area property of cyclic quadrilaterals

It is well known that among all quadrilaterals with given side lengths, the cyclic quadrilaterals have maximal area. Proofs of this can be found, for example, in [1, p. 5] and [2, p. 49], and these proofs use Brahmagupta's formula

$$16K^2 = 4(ad + bc)^2 - \left(a^2 + d^2 - b^2 - c^2\right)^2 - 16abcd\cos^2\tfrac{1}{2}(x + y)$$

that gives the area K of a convex quadrilateral $ABCD$ whose side lengths and angles are given by

$$AB = a,\ BC = b,\ CD = c,\ DA = d,\ \angle DAB = x,\ \angle BCD = y;$$

see Figure 1. A proof that uses a different approach is given in [3]. In this note, we give a short proof using Lagrange's multipliers. Neither the existing proofs nor this one are Euclidean, in the sense that Euclid would not understand, let alone enjoy, reading them.

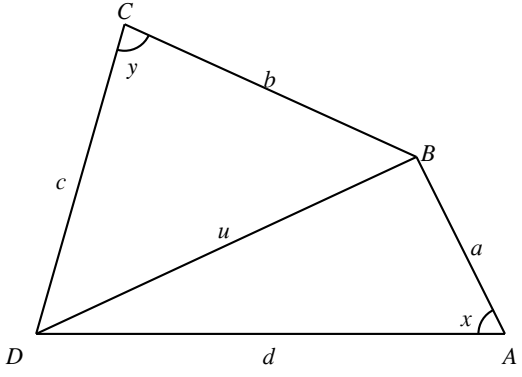


FIGURE 1

Referring to Figure 1, we use the law of cosines to obtain $u^2 = a^2 + d^2 - 2ad\cos x$ and $u^2 = b^2 + c^2 - 2bc\cos y$. Subtracting, we obtain

$$f(x, y) = \left(a^2 + d^2 - 2ad\cos x\right) - \left(b^2 + c^2 - 2bc\cos y\right) = 0.$$

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