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# Matched asymptotic solutions for turbulent plumes

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Recent analytical investigations have shown that the vertical evolution of turbulent plumes variables can be derived straightforwardly from the knowledge of a single function $\Gamma(z)$ (called the plume function) which is the solution of a nonlinear differential equation. This article presents matched asymptotic solutions of this equation in the cases corresponding to highly lazy or highly forced plumes. First, it is shown that, far from the source, the asymptotic expression of the plume function can be derived by means of a perturbation method based on a Padé-like approximation. The resulting outer solution is invariant under translation (with respect to the vertical coordinate) so that we are led to the classical problem concerning the location of the plume (asymptotic) virtual origin. In order to determine this virtual origin location as a function of the conditions at the source, the far-field asymptotic solution is matched to an inner expansion of the solution which is valid near the source. Comparisons between these asymptotic solutions and numerical results are finally made in order to test their validity.

**Key words:** plumes/thermals

## 1. Introduction

Turbulent plumes are widely encountered in many industrial and environmental problems (pollutant release and dispersion, propagation and control of fire-induced smoke, heating and cooling of buildings) and, as a result, the literature in this area is extensive. We shall not attempt a detailed review here but the interested reader can refer to the survey articles by Turner (1969), List (1982) and recently Woods (2010). Let us simply mention that, in general, following the pioneer article by Morton, Taylor & Turner (1956), the great majority of theoretical approaches concerning vertical plumes uses a formalism based on time-averaged mono-dimensional conservation equations in which the plume is described by its fluxes or alternatively by its radius $b$, its mean density $\rho$ and its mean vertical velocity $w$. Assuming top-hat velocity and density profiles in a cross-section of the plume, conservation equations for mass, momentum and buoyancy can be expressed, in the general non-Boussinesq case, as follows:

$$\frac{\mathrm{d}}{\mathrm{d}z}(\rho w b^2) = 2\rho_0 b u_e, \quad \frac{\mathrm{d}}{\mathrm{d}z}(\rho w^2 b^2) = (\rho_0 - \rho) g b^2, \quad \frac{\mathrm{d}}{\mathrm{d}z}((\rho_0 - \rho) w b^2) = 0, \quad (1.1)$$

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where $z$ is the vertical coordinate, $\rho_0$ is the (constant) density of ambient fluid, $g$ is the gravity constant and $u_e$ is the radial velocity at which the ambient fluid is entrained at the edge of the plume. In such an approach, the problem is not closed and a model that links the entrainment velocity to the other variables has to be defined. Under the Boussinesq approximation, it is generally admitted that this velocity is proportional to the local vertical velocity: $u_e = \alpha w$, where $\alpha$ is the entrainment coefficient (Morton *et al.* 1956). In the non-Boussinesq case, the entrainment coefficient is usually corrected by the square root of the local density: $u_e = \sqrt{\rho/\rho_0}w$ (Morton 1965).

Although simple, the set of equations (1.1) do not admit direct analytical solutions and in many engineering problems where plumes are involved, it is preferred to use similarity solutions instead of solving them numerically. Similarity solutions (introduced first by Morton *et al.* (1956), under the Boussinesq approximation) are obtained by considering a point source of buoyancy, which allows the plume variables to be sought in the form of $z$-power laws. As it was shown later by Delichatsios (1980) and Rooney & Linden (1996) that such similarity solutions can also be obtained in the non-Boussinesq case, providing that the 'density square root' corrected entrainment model is used in the conservation equations.

In order to account for the fact that real sources have finite area with initial radius, velocity, and density, the similarity solutions are usually modified by introducing a virtual origin correction (see for instance Morton 1959 or Morton & Middleton 1973). The determination of the virtual origin location has been the subject of many studies over the last few decades (for a review, see the introduction of the article by Hunt & Kaye 2001).

According to their simplicity, similarity solutions are often used in spite of the fact that they correctly predict the plume variables only far from the source (far field). Indeed, in the near field (i.e. at a few diameters distance over the source), they can lead to non-physical estimations of the variables (in particular for density or temperature).

In a similar spirit to Carlotti & Hunt (2005), the main objective of this paper is then to derive convenient and accurate matched asymptotic solutions for both near and far fields of a turbulent plume. To do so, we use the results of Michaux & Vauquelin (2008) who have shown that the variables of the plume could be derived from the knowledge of a single function, generally called the plume function $\Gamma(z)$, which is a solution of a nonlinear differential equation. This function was originally defined by Hunt & Kaye (2005) from a local flux combination, and later by Michaux & Vauquelin (2008) as the normalized value of the local Froude number. It is defined as follows:

$$\Gamma(z) = \frac{5g\eta\beta}{8\alpha w^2}, \tag{1.2}$$

where the new variables $\eta = (\rho_0 - \rho)/\rho$ and $\beta = b\,(\rho/\rho_0)^{1/2}$ were introduced for mathematical convenience in order to allow the conservation equations to be written in a similar form in both Boussinesq and non-Boussinesq cases.

By introducing $\Gamma(z)$ in the conservation equations, it can be shown (after some algebra) that $w(z)$, $\eta(z)$ and $\beta(z)$ can be explicitly expressed as

$$\left.\begin{aligned} \frac{\beta}{\beta_i} &= \left(\frac{\Gamma}{\Gamma_i}\right)^{1/2}\left(\frac{1-\Gamma_i}{1-\Gamma}\right)^{3/10}; \quad \frac{w}{w_i} = \left(\frac{\Gamma_i}{\Gamma}\right)^{1/2}\left(\frac{1-\Gamma}{1-\Gamma_i}\right)^{1/10}; \\ \frac{\eta}{\eta_i} &= \left(\frac{\Gamma_i}{\Gamma}\right)^{1/2}\left(\frac{1-\Gamma}{1-\Gamma_i}\right)^{1/2}, \end{aligned}\right\} \tag{1.3}$$

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where $w_i$, $\eta_i$, $\beta_i$ and $\Gamma_i$ are initial values (assumed to be known) of the release at the source. In particular, $\Gamma_i$ is the source flux balance parameter originally defined by Morton (1959) to discriminate buoyancy dominated ($\Gamma_i > 1$) and momentum dominated ($\Gamma_i < 1$) plumes, which are also called, respectively, lazy and forced plumes. The vertical evolution of $\beta$, $w$ and $\eta$ is then obtained from that of $\Gamma(z)$ which is the solution of the following differential equation (respectively for lazy and forced plumes):

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}z} = -\frac{1}{\Lambda_i}\Gamma^{1/2}(\Gamma-1)^{13/10} \quad \text{or} \quad \frac{\mathrm{d}\Gamma}{\mathrm{d}z} = \frac{1}{\Lambda_i}\Gamma^{1/2}(1-\Gamma)^{13/10} \quad \text{with } \Gamma(0)=\Gamma_i, \tag{1.4}$$

and where

$$\Lambda_i = \frac{\beta_i|\Gamma_i - 1|^{3/10}}{4\alpha\Gamma_i^{1/2}}. \tag{1.5}$$

The paper is organized as follows: in § 2, asymptotic solutions of (1.4) are sought. In a first step, an outer solution is drawn by means of a perturbation method which relies on the fact that in the far field, the value of the plume function naturally tends to 1, whatever the conditions at the source. In a second step, inner expansions (i.e. in the near field) of the solution are derived respectively for highly lazy (i.e. $\Gamma_i \gg 1$) and highly forced plumes (i.e. $\Gamma_i \ll 1$) and, in a third step, these outer and inner solutions are matched together. Comparisons between these matched analytical solutions and numerical results are presented in § 3.

## 2. Analytical investigations

In order to present an approach that is valid for both lazy and forced plumes, let us first note that making the following change of variable $\Gamma(z) = 1/\Gamma'(z)$ in the second equation of (1.4) (forced plume) leads to a differential equation which is similar to that written for a lazy plume, except that the exponent $1/2$ must be replaced by $1/5$. For the sake of generality, let us then consider a differential equation of the form

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta} = -\Gamma^q(\Gamma-1)^r \quad \text{where } \zeta = \frac{z}{\Lambda_i} \tag{2.1}$$

and where $q$ and $r$ are strictly positive rational numbers.

### 2.1. *Asymptotic solutions when* $\Gamma \to 1$

According to the negative sign of the derivative of $\Gamma$ in (2.1), this function only decreases with respect to $\zeta$, and whatever the values of $q$ and $r$ (if they are positive), we have at the limit $\Gamma \to 1$ as $\zeta \to \infty$. Thus, the most immediate perturbation scheme that may be used to determine an asymptotic solution of (2.1) is

$$\Gamma \simeq 1 + \varepsilon(\zeta) \quad \text{where it is assumed that } \varepsilon(\zeta) \ll 1. \tag{2.2}$$

By injecting this perturbation scheme into (2.1), and according to a series expansion (in terms of $\varepsilon$), we are led to

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\zeta} = -\varepsilon^r + O(\varepsilon^{r+1}) \quad \text{with } \varepsilon \to 0 \text{ when } \zeta \to \infty. \tag{2.3}$$

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Since (2.3) is a constant-coefficient differential equation, any change of variable of the form $\zeta' = \zeta - \zeta_v$ leaves it unchanged. As a result, the solution of (2.3) reads as

$$\varepsilon = [(r-1)(\zeta - \zeta_v)]^{1/(1-r)} + O(\varepsilon^r), \tag{2.4}$$

where $\zeta_v$ can be interpreted as the virtual origin location (here divided by $\Lambda_i$) of an equivalent point source plume. It may be worth noting that if this result is injected in (2.2) then we recover the asymptotic expression of the plume function, obtained with a different approach, by Van den Bremer & Hunt (2010).

It is actually possible, in this problem, to do a little better without a great deal of work by using a Padé-like approximation scheme of the form

$$\Gamma = \frac{1 + a\varepsilon(\zeta)}{1 - b\varepsilon(\zeta)}. \tag{2.5}$$

By injecting this expression into (2.1) we are led to

$$(a+b)^{1-r} \frac{\mathrm{d}\varepsilon}{\mathrm{d}\zeta} = -\frac{(1+a\varepsilon)^q}{(1-b\varepsilon)^{q+r-2}} \varepsilon^r, \tag{2.6}$$

and according to the following series expansion

$$\frac{(1+a\varepsilon)^q}{(1-b\varepsilon)^{q+r-2}} = 1 + (qa + (r+q-2)b)\varepsilon + O(\varepsilon^2), \tag{2.7}$$

we observe that if the constants $a$ and $b$ are set in order to satisfy the following relations

$$a + b \neq 0 \quad \text{and} \quad qa + (r+q-2)b = 0 \tag{2.8}$$

then (2.6) can be written as

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\zeta} = -(a+b)^{r-1} \varepsilon^r + O(\varepsilon^{r+2}). \tag{2.9}$$

In comparison with (2.3), the order of accuracy has been increased by one and the general solution of this equation reads as

$$(a+b)\varepsilon = [(r-1)(\zeta - \zeta_v)]^{(1/1-r)} + O(\varepsilon^{r+1}). \tag{2.10}$$

By using (2.8) in order to determine separately the two parts $a\varepsilon$ and $b\varepsilon$ appearing in (2.5), we are led to

$$\Gamma \simeq \frac{(2-r) + (2-r-q)\,[(r-1)(\zeta - \zeta_v)]^{(1/1-r)}}{(2-r) - q\,[(r-1)(\zeta - \zeta_v)]^{(1/1-r)}}, \tag{2.11}$$

where the virtual origin coordinate $\zeta_v$ has not been determined yet. Note that in Hunt & Kaye (2001) (see also Van den Bremer & Hunt 2010), the authors present a method which allows $\zeta_v$ to be determined in the form of an infinite sum whose terms are functions of the initial condition of the plume. As we shall see shortly, when we are dealing with highly lazy or highly forced plumes, it is actually possible to approximate $\zeta_v$ in a compact form, by means of matched asymptotic expansions.

### 2.2. *Asymptotic expansion when* $\Gamma \gg 1$

When $\zeta$ is close to 0, the order of magnitude of $\Gamma$ (respectively $\Gamma'$) naturally scales as $\Gamma_i$ (respectively $\Gamma_i'$) which is assumed to be, from now, much greater than unity.

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Let us then introduce the small parameter $\epsilon = 1/\Gamma_i \ll 1$, and let us also introduce the normalized plume function $\gamma$ defined as $\gamma = \Gamma/\Gamma_i = \epsilon\Gamma$. This normalized plume function is a solution of

$$\epsilon^{q+r-1}\frac{\mathrm{d}\gamma}{\mathrm{d}\zeta} = -\gamma^q(\gamma-\epsilon)^r \quad \text{with } \gamma(0)=1, \tag{2.12}$$

and for later convenience, we write

$$s = q + r \quad \text{and} \quad s - 1 = \frac{n}{m} \quad \text{where } (n,m) \in \mathbb{N}^2. \tag{2.13}$$

In regular differential problems involving a small parameter $\epsilon$, a classical way to determine an asymptotic expansion of a differential equation solution consists in seeking it in the form (see for instance Hinch 1991)

$$\gamma = \sum_{k=0} \gamma_k \epsilon^k. \tag{2.14}$$

In our problem, however, (2.12) involves a non-integer exponent of the small-parameter $\epsilon$ on its left-hand side, and this is a source of difficulty. We need first to make a change of variable which consists in seeking the series (2.14) in terms of functions

$$\gamma_k(\chi) \quad \text{where } \chi = \left(\frac{m}{m + n\epsilon^{-n/m}\zeta}\right)^{1/n}, \tag{2.15}$$

instead of functions $\gamma_k(\zeta)$. Indeed, in terms of this new (stretched) variable, equation (2.12) becomes

$$\frac{\chi^{n+1}}{m}\frac{\mathrm{d}\gamma}{\mathrm{d}\chi} = \gamma^q(\gamma-\epsilon)^r \quad \text{with } \gamma(1)=1. \tag{2.16}$$

Expanding now the right-hand side of this equation yields

$$\gamma^q(\gamma-\epsilon)^r = 1 + \left(\frac{s\gamma_1 - r}{\gamma_0}\right)\epsilon + \left(\frac{s\gamma_2}{\gamma_0} + \frac{1}{2}q(q-1)\left(\frac{\gamma_1}{\gamma_0}\right)^2 + \frac{1}{2}r(r-1)\left(\frac{\gamma_1-1}{\gamma_0}\right)^2 + qr\frac{\gamma_1(\gamma_1-1)}{\gamma_0^2}\right)\epsilon^2 + O(\epsilon^3), \tag{2.17}$$

and comparing coefficients of $\epsilon^k$ provides us with a set of differential equations (associated with the initial conditions $\gamma_0(1)=1$ and $\gamma_k(1)=0$ for $k \geqslant 1$) which can be solved one after the other. For the four first functions, which are enough for our purpose, we obtain

$$\left.\begin{aligned} &\gamma_0 = \chi^m; \quad \gamma_1 = \frac{r}{s}(1-\chi^{n+m}); \\ &\gamma_2 = -\frac{r(1+r)}{2(1+s)}\chi^{n+m} + \frac{r^2}{2s}\chi^{m+2n} + \frac{qr}{2s(1+s)}\frac{1}{\chi^m} \end{aligned}\right\}; \tag{2.18}$$

$$\gamma_3 = \frac{r}{2s^2}\left(\frac{r^2(2s-1)}{3} - \frac{2q(q-r)}{3(2+s)} - \frac{r(s^2(1+r)+q)}{(1+s)}\right)\chi^{n+m} + \frac{r^2(1+r)}{2(1+s)}\chi^{n+2m} - \frac{r^3(2s-1)}{6s^2}\chi^{n+3m} + \frac{r^2q}{2s^2(s+1)}\chi^{n-m} + \frac{rq(q-r)}{3s^2(s+2)}\frac{1}{\chi^{2m}}, \tag{2.19}$$

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and by returning to the plume function, we are led to

$$\Gamma = \sum_{k=0}^{k=3} \gamma_k(\chi)\epsilon^{k-1} + O(\epsilon^3). \tag{2.20}$$

An interesting feature of the functions $\gamma_2$ and $\gamma_3$ is that they contain diverging terms as $\zeta$ tends to infinity (since in this case $\chi$ tends to zero). This is not so unusual in this kind of problem since such a series is assumed to be valid only when $\gamma \gg \epsilon$, that is for finite values of $\zeta$. In this particular problem, however, it turns out that these diverging terms play a key role in the matching procedure that will be used to determine the value of $\zeta_v$ involved in (2.11).

### 2.3. *Matching procedure*

So far, two asymptotic solutions have been obtained, the first which is valid (rigourously) when $\Gamma \gg 1$, that we shall now denote as $\Gamma_{inner}$, and the other one which is valid when $\varepsilon^{r+1} \ll 1$, that we shall now denote as $\Gamma_{outer}$. A simple method to determine $\zeta_v$ is to assume that the inner and the outer solutions are both valid in an overlap region and, then, to match them at some particular $\zeta$ which belongs to this zone. This is indeed what we propose to do, but in order to avoid any arbitrariness in this choice, we suggest to chose the highest value of $\zeta$ for which the inner expansion remains valid, since for large $\zeta$ the outer expansion is more accurate.

According to the existence of diverging terms in the inner expansion (2.20), it can be shown that this maximal value is given by $\chi^m/\epsilon \to 1 \Leftrightarrow \chi \to \epsilon^{1/m}$. The corresponding value for $\zeta$ is $(m/n)(1-\epsilon^{n/m})$ and the matching procedure then reads as

$$\lim_{\chi \to \epsilon^{1/m}} \Gamma_{inner} = \lim_{\zeta \to (m/n)(1-\epsilon^{n/m})} \Gamma_{outer}. \tag{2.21}$$

Considering the inner expansion, we obtain

$$\lim_{\chi \to \epsilon^{1/m}} \left( \frac{\gamma_0}{\epsilon} + \gamma_1 + \epsilon\gamma_2 + \epsilon^2\gamma_3 + \cdots \right) = 1 + \frac{r}{s} + \frac{qr}{2s(1+s)} + \frac{rq(q-r)}{3s^2(s+2)} + O(\epsilon^s) + \cdots \tag{2.22}$$

For the sake of simplicity, this limit is denoted by $l$, and if it is rewritten in the form

$$l = 1 + \frac{r}{s} + \frac{r}{s}\left(\frac{q}{2(1+s)}\right) + \frac{r}{s}\left(\frac{q}{2(1+s)}\right)\left(\frac{2(1+s)}{3(2+s)}\frac{q-r}{s}\right) + O(\text{neglected terms}) \tag{2.23}$$

it can be observed that the successive terms are decreasing, since $r$ and $q$ are both positive and $s = q + r$. As we shall see shortly, in this problem, considering only the four first terms of the inner expansion is enough to determine $l$ with an acceptable accuracy. The matching procedure (2.21) therefore yields an algebraic equation whose solution is

$$\zeta_v = \frac{m}{n}\left(1-\epsilon^{n/m}\right) - \frac{1}{r-1}\left(\frac{(2-s)+ql}{(2-r)(l-1)}\right)^{r-1}. \tag{2.24}$$

## 3. Comparisons with numerical results

In the case of a lazy plume rising from a circular source, we have $r = 13/10$ and $q = 1/2$ (so that $n = 4$, $m = 5$) and the small parameter $\epsilon$ is given by $1/\Gamma_i$.

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Accordingly, the inner solution reads

$$\Gamma_{inner} = \Gamma_i\gamma_0(\chi) + \gamma_1(\chi) + \frac{1}{\Gamma_i}\gamma_2(\chi) + \left(\frac{1}{\Gamma_i}\right)^2\gamma_3(\chi) + O\left(\left(\frac{1}{\Gamma_i}\right)^3\right) \tag{3.1}$$

where $\chi = (1 + (4/5)\Gamma_i^{4/5} z/\Lambda_i)^{-1/4}$ and where the functions $\gamma_k(\chi)$ are given by (2.18) and (2.19). Concerning the outer solution, the matching procedure provides us with the value $l \simeq 1.7726$. According to the definitions of $\Lambda_i$ and $\zeta$ given by (1.5) and (2.1), respectively, we are led to

$$\left.\begin{aligned} &\frac{z_v}{\beta_i} \simeq \frac{1}{4\alpha}\frac{(\Gamma_i - 1)^{3/10}}{\Gamma_i^{1/2}}\left(\frac{5}{4}\left(1 - \frac{1}{\Gamma_i^{4/5}}\right) - 4.1091\right) \\ &\text{and} \quad \Gamma_{outer} \simeq \frac{7 + 2\left(\dfrac{10\Lambda_i}{3(z - z_v)}\right)^{10/3}}{7 - 5\left(\dfrac{10\Lambda_i}{3(z - z_v)}\right)^{10/3}} \end{aligned}\right\}. \tag{3.2}$$

In the case of a forced plume, we still have $r = 13/10$, but this time $q = 1/5$ (so that $n = 1$, $m = 2$) and the small parameter $\epsilon$ is straightforwardly $\Gamma_i$. The inner solution is then given by

$$\Gamma_{inner} = \frac{1}{\dfrac{1}{\Gamma_i}\gamma_0(\chi) + \gamma_1(\chi) + \Gamma_i\gamma_2(\chi) + \Gamma_i^2\gamma_3(\chi)} + O(\Gamma_i^3) \tag{3.3}$$

where $\chi = (1 + (1/2)\Gamma_i^{-1/2} z/\Lambda_i)^{-1}$ and the far-field solution, in which $l \simeq 1.8892$, reads as

$$\Gamma_{outer} \simeq \frac{7 - 2\left(\dfrac{10\Lambda_i}{3(z - z_v)}\right)^{10/3}}{7 + 5\left(\dfrac{10\Lambda_i}{3(z - z_v)}\right)^{10/3}} \quad \text{where} \quad \frac{z_v}{\beta_i} \simeq \frac{1}{4\alpha}\frac{(1 - \Gamma_i)^{3/10}}{\Gamma_i^{1/2}}(2(1 - \Gamma_i^{1/2}) - 3.6954). \tag{3.4}$$

The validity of the outer solutions (3.2) and (3.4) depends crucially on the accuracy in which the virtual origin coordinate is determined. Let us then compare the values of $z_v$ provided by (3.2) and (3.4) to those obtained by solving numerically (1.4) (Runge–Kutta 4). Figures 1 and 2 show the results concerning lazy and forced plumes, respectively. In figure 1 (lazy plumes) it can be observed that the theoretical virtual origin location is correctly determined as soon as the values of $\Gamma_i$ become greater than approximately 5, for which it can be shown that the relative error falls under 1 % and stabilizes around 0.4 % for greater values of $\Gamma_i$. This irreducible error is actually due to the accuracy in which $l$ has been determined here. Similarly, in figure 2 (forced plume), we can observe that the theoretical values start to differ from the numerical results when the values of $\Gamma_i$ become greater than approximately 0.2 (below this value, the relative error remains of the order of 1 %).

Figure 3 shows the evolution (obtained numerically) of the plume function for which $\Gamma_i$ has been set equal to 10 (lazy plume), together with the inner solution (3.1) and the outer solution (3.2). By construction, these two asymptotic solutions match at the value $l = 1.7726$. It can be observed that below $z/\beta_i \approx 3$ (respectively, above $z/\beta_i \approx 1$) the

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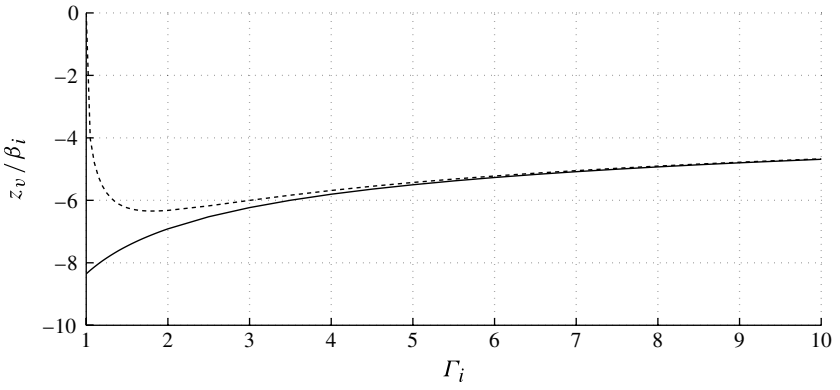



FIGURE 1. Virtual origin location (divided by $\beta_i$) versus $\Gamma_i$ for lazy plumes. Continuous line: virtual origin location obtained by a numerical resolution of the differential equation (1.4). Dashed line: virtual origin location given by (3.2).

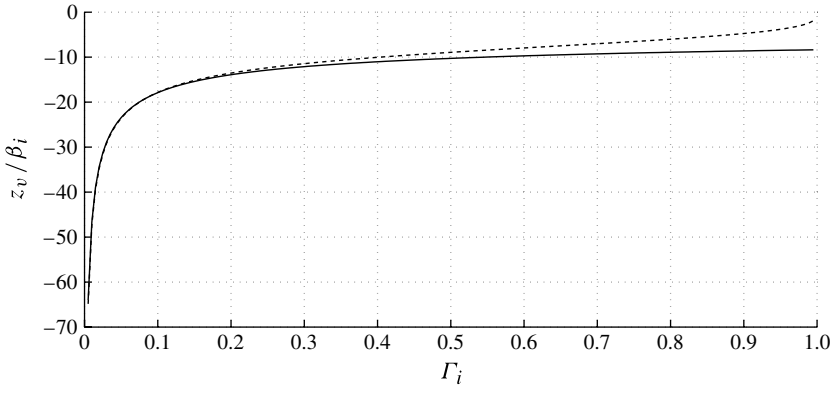



FIGURE 2. Virtual origin location (divided by $\beta_i$) versus $\Gamma_i$ for forced plumes. Continuous line: virtual origin location obtained by a numerical resolution of the differential equation (1.4). Dashed line: virtual origin location obtained by using (3.4).

inner solution (respectively, the outer solution) and the numerical solution are almost undistinguishable. In a such a lazy case then, the overlap region between the inner and the outer solution is quite large and the plume function is well approximated whatever the values of the vertical coordinates.

Such a comparison is also performed in a case where $\Gamma_i$ has been set equal to 0.1 (forced plume) and results are presented in figure 4. The overlap region of the two asymptotic solutions is a little more narrow than in the previous case, but the plume function is still entirely well approximated.

## 4. Discussion and conclusion

This paper provides asymptotic solutions for the non-Boussinesq plume conservation equations. These solutions, which have been obtained by means of matched asymptotic expansions, allow the plume variables to be determined analytically in a simple manner. Rigorously speaking, according to the required assumptions, these solutions should be valid only for highly lazy or highly forced plumes. Nevertheless,

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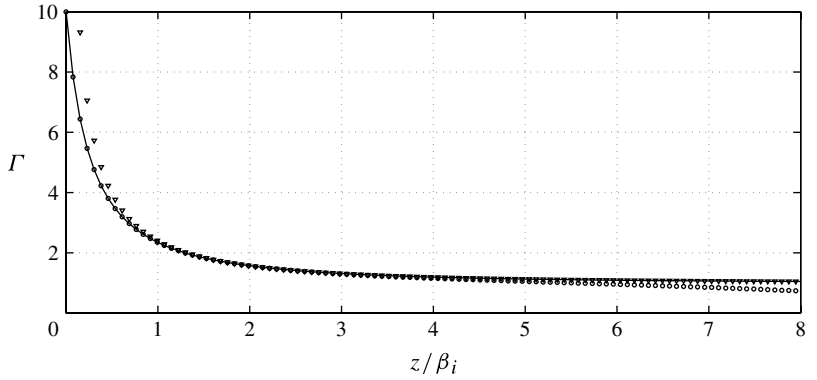



FIGURE 3. Vertical evolution of $\Gamma$ for a lazy plume ($\Gamma_i = 10$). Continuous line: numerical resolution of the differential equation (1.4). Symbols ∘, inner solution (3.1). Symbols ▽, outer solution given by (3.2).

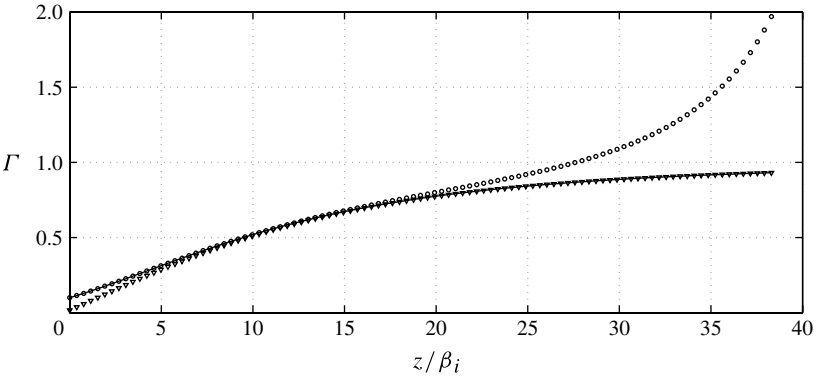



FIGURE 4. Vertical evolution of $\Gamma$ for a forced plume ($\Gamma_i = 0.1$). Continuous line: numerical resolution of the differential equation (1.4). Symbols ∘, inner solution (3.3). Symbols ▽, outer solution given by (3.4).

comparisons with numerical results have shown that they can be confidently used provided that the initial value of the plume function lies within the intervals $\Gamma_i \lesssim 0.2$ (forced plumes) or $\Gamma_i \gtrsim 5$ (lazy plumes). When such a condition is satisfied, the evolution of the plume variables can be obtained by using the inner solution while $z < \Lambda_i(m/n)(1 - \epsilon^{n/m})$ and by using the outer solution over this value.

As mentioned in the introduction, in most engineering applications where plumes are involved, similarity solutions (including a virtual origin correction) are used, for the sake of simplicity, as for instance in the relations for fire plumes provided by Heskestad (1984) or in the filling box model due to Baines & Turner (1969), to name but a few. It is in such situations that the advantages of the present results become apparent since the asymptotic solutions presented here are valid everywhere, whereas similarity solutions are valid only far from the source.

Furthermore, it is even possible to use only the outer solutions (3.2) or (3.4) to predict the $z$-evolution of the plume variables with an accuracy much greater than by using the usual similarity solutions. As an example, let us consider the vertical release of helium ($\rho_i = 0.17$ kg m$^{-3}$) in the ambient air ($\rho_0 = 1.2$ kg m$^{-3}$) from a circular

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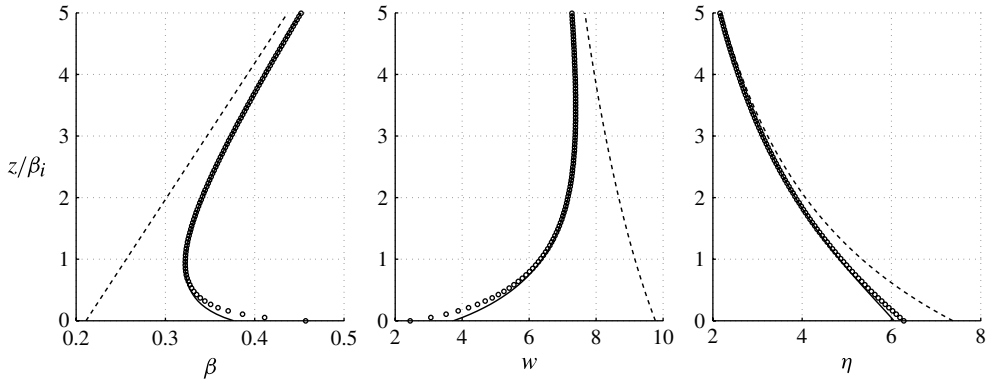



FIGURE 5. Vertical evolutions of the plume variables in a lazy case ($\Gamma_i = 10$, with $\beta_i = 0.3764$ m, $w_i = 3.7393$ m s$^{-1}$ and $\eta_i = 6.0588$). Continuous line, numerical solutions. Symbols ∘, relations (1.3) obtained by using the outer solution (3.2). Dashed line, similarity solutions.

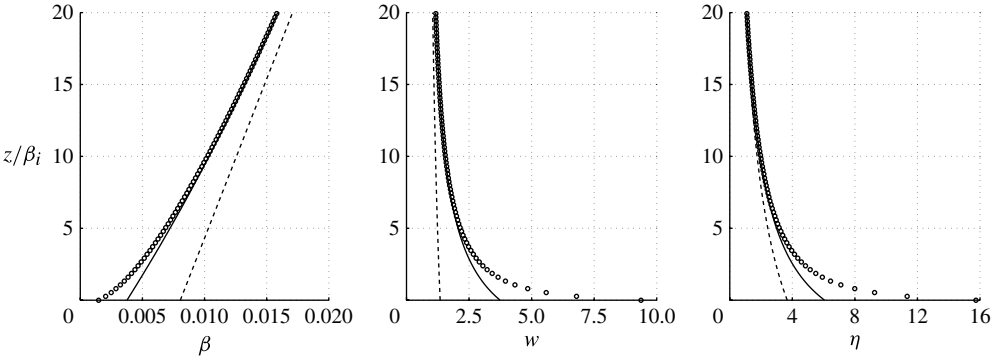



FIGURE 6. Vertical evolutions of the plume variables in a forced case ($\Gamma_i = 0.1$ with $\beta_i = 0.0038$ m, $w_i = 3.7393$ m s$^{-1}$ and $\eta_i = 6.0588$). Continuous line: numerical solutions. Symbols ∘, relations (1.3) obtained by using the outer solution (3.4). Dashed line, similarity solutions.

source whose radius is $b_i = 1$ m, with an initial velocity set in order to satisfy the condition $\Gamma_i = 10$ (i.e. $w_i = 3.74$ m s$^{-1}$). Results concerning the vertical evolution of $\beta$, $w$ and $\eta$ are presented in figure 5. It is observed that values obtained by using (3.2) are accurately determined at a distance from the source given by $z/\beta_i \approx 1$ which corresponds here to $z \approx 0.38 b_i$. At such a distance from the source, it can be seen that the results provided by the similarity solutions are far from being accurate. This figure also shows that, by using the outer solution (3.2), we are able to correctly describe the region where the plume diameter becomes minimal (neck) and where the velocity becomes maximal.

In order to deal now with a forced plume, we consider the same release of helium in the ambient air, but this time, from a source whose radius is set to 0.01 m, so that $\Gamma_i = 0.1$. Results are presented in figure 6 and it can be observed that the variables obtained with the outer solution are correctly predicted at a distance from the source given by $z/\beta_i \approx 10$ or equivalently $z \approx 3.76 b_i$. This distance is greater than

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that observed in the previous case concerning a lazy plume, but the results obtained with this outer solution still remain much better than those provided with the similarity solutions.

To conclude the present investigation, let us mention that the only condition required to obtain the outer solution (2.11) is that $\Gamma(z)$ tends to 1. In the particular case where $\Gamma_i$ is already close to 1, it turns out that this asymptotic expression constitutes actually a correct approximation of the exact solution whatever the $z$ coordinate, and hence also at the vicinity of the source. As a result, the virtual origin, which cannot be obtained by the matching procedure described in § 2, can be nevertheless simply determined by applying the real boundary condition

$$\Gamma_{outer}(\zeta = 0) = \Gamma_i \quad \text{which leads us to} \quad \frac{z_v}{\beta_i} = -\frac{5}{6\alpha}\frac{1}{\Gamma_i^{1/2}}\left(\frac{2+5\Gamma_i}{7}\right)^{3/10}, \tag{4.1}$$

for both lazy and forced plumes. Surprisingly (because it is not mathematically justified), the outer solutions involving this virtual origin allow the evolution of the plume variables to be predicted accurately not only for values of $\Gamma_i$ close to 1, but also for values of the order of a few units. This latter result therefore complements the previous ones.

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