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Nonlinear low-frequency waves in dusty self-gravitating plasmas

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Abstract. Nonlinear electrostatic waves in self-gravitating dusty plasmas are considered in two limiting cases, according to whether the charged-particle dynamics is governed mostly by electrostatic forces or mostly by gravitation. This shows a significant difference between these two plasma media with respect to the envelope dynamics in the nonlinear regime. In the former case, when $\omega_{p\alpha}^2 > \omega_{J\alpha}^2$, the amplitude perturbations are longitudinally unstable only in the short-wave range, and the nonlinear effects can result in the formation of longitudinal dust-acoustic solitary waves. But even weak self-gravitational effects can lead to the existence of a long-wavelength range, where self-gravitation prevents the formation of dust-acoustic solitons, and only transverse solitary structures are possible. In the other limiting case ($\omega_{p\alpha}^2 < \omega_{J\alpha}^2$), there is always a transverse modulational instability, which can lead to transverse solitary waves. In both cases, there is a threshold for solitary-wave formation.

Dedication

It is a pleasure to dedicate this paper to Dr J. P. Dougherty on the occasion of his 65th birthday.

1. Introduction

The investigation of plasmas containing heavy dust particles is very important for the proper understanding of many space and astrophysical phenomena, as well as for many technological and laboratory plasma processes. Charged dust grains can significantly modify the properties of collective modes and their evolution. Moreover, the presence of such particles results in the appearance of new modes, such as dust-acoustic waves (Rao et al. 1990), which involve oscillations at such low frequencies that the electrons and ions remain in equilibrium and the dynamics is mainly due to the dust particles.

A large number of authors have discussed various aspects of linear and nonlinear plasma waves in dusty plasmas, and we refer to a recent review (Verheest 1996) for more details. One of the most interesting subjects is the formation of nonlinear low-frequency solitary waves (i.e. ion-acoustic and dust-acoustic solitons). At the

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same time, in real dusty plasmas, self-gravitation might become important, and may even be comparable in magnitude to electromagnetic forces, owing to the very broad spectrum of dust masses. Although this problem has been studied in great detail at the linear level, little has been done for nonlinear waves in a dusty plasma when self-gravitating effects are included (Verheest and Shukla 1997). Since particles in dusty plasmas can interact through both electric and gravitational forces, it is physically relevant to investigate the effect of self-gravitation on nonlinear longitudinal waves. Indeed, keeping in mind the modification of the wave spectrum in self-gravitating dusty plasmas already within the framework of linear theory (Bliokh and Yaroshenko 1985; Avinash and Shukla 1994; Pandey et al. 1994; Bliokh et al. 1995; Meuris et al. 1997; Verheest et al. 1997, 1999), one would expect a host of interesting results to follow from the inclusion of self-gravitation in the nonlinear analysis.

The plan of this paper is as follows. In Sec. 2, we recall the general formalism and dispersion law for longitudinal waves in a two-component model of dusty self-gravitating plasmas, and sketch the broad outline of the 'envelope' approach used in the nonlinear analysis. This method is subsequently applied to a mixture of charged heavy dust grains and inertialess light particles, in order to treat the modification of dust-acoustic solitons due to self-gravitation (Sec. 3). Nonlinear waves in a mixture of two very heavy charged dust species are considered in Sec. 4. Our discussion reveals the outstanding distinction between soliton-like solutions for the two above-mentioned models of a dusty self-gravitating plasma. Finally, our conclusions are summarized in Sec. 5.

2. Basic equations

We consider an infinite, homogeneous and unmagnetized self-gravitating plasma. In the low-frequency regime, the particle motion is adiabatic, and hence we can use the usual model to describe the particle pressure by

$$\boldsymbol{\nabla} p = T_\alpha \boldsymbol{\nabla} n_\alpha, \tag{1}$$

where T_α and n_α are the kinetic temperature and number density respectively for the particles of species α.

The governing equations for such a system can be written as

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}(v_\alpha n_\alpha) = 0, \tag{2a}$$

$$\frac{\partial v_\alpha}{\partial t} + v_\alpha \frac{\partial v_\alpha}{\partial x} + \frac{q_\alpha}{m_\alpha}\frac{\partial \psi_E}{\partial x} + \frac{\partial \psi_G}{\partial x} + \frac{v_{T\alpha}^2}{n_\alpha}\frac{\partial n_\alpha}{\partial x} = 0, \tag{2b}$$

$$\frac{\partial^2 \psi_E}{\partial x^2} + \frac{1}{\varepsilon_0}\sum_\alpha q_\alpha n_\alpha = 0, \tag{2c}$$

$$\frac{\partial^2 \psi_G}{\partial x^2} - 4\pi G \sum_\alpha m_\alpha n_\alpha = 0. \tag{2d}$$

The notation is standard: q_α and m_α denote the particle charges and masses, v_α and $v_{T\alpha} = (T_\alpha/m_\alpha)^{1/2}$ the fluid and thermal velocities, G the gravitational constant, and ψ_E the electric and ψ_G the gravitational potentials.

In general, the number of species to be considered in (2) can be very large, and

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then a straightforward analysis is hardly possible. Nevertheless, a distribution over different dust species is needed, because we know that dust occurs in a broad spectrum of masses and sizes. So, in order to have a tractable model for some of the important dusty plasmas, we represent the charged dust by a limited number of discrete species. By way of example, consider a dusty self-gravitating plasma consisting of two species of particles with different masses and other characteristics.

For a small-amplitude solution, the nonlinear effects can first be neglected, and then (2) admits approximate wave solutions of the form

$$\psi_E = \varphi \exp[i(kx - \omega t)], \tag{3}$$

with a constant amplitude φ. The frequency ω and wavenumber k are related by the linear dispersion law

$$1 - \frac{\omega_{p1}^2 - \omega_{J1}^2}{\omega^2 - k^2 v_{T1}^2} - \frac{\omega_{p2}^2 - \omega_{J2}^2}{\omega^2 - k^2 v_{T2}^2} - \frac{(\omega_{p1}\omega_{J2} - \omega_{p2}\omega_{J1})^2}{(\omega^2 - k^2 v_{T1}^2)(\omega^2 - k^2 v_{T2}^2)} = 0, \tag{4}$$

where $\omega_{p\alpha} = (N_\alpha q_\alpha^2/\varepsilon_0 m_\alpha)^{1/2}$ and $\omega_{J\alpha} = (4\pi G N_\alpha m_\alpha)^{1/2}$ are the plasma and Jeans frequency respectively, and N_α corresponds to the equilibrium number density for the particles of species α. This and others dispersion laws will be represented in the form $D(k, \omega) = 0$.

To analyse nonlinear waves in dusty self-gravitating plasmas, we shall use the standard approach, based on ideas from nonlinear optics and known as the method of 'envelopes' (Kadomtsev and Karpman 1971; Kadomtsev 1976). According to this approach, the approximate solution of the nonlinear equations (2) will be sought in the same form as (3), although the complex wave amplitude can no longer be constant but is slowly modulated in space and time, i.e. $\varphi = \varphi(x, t)$. This assumes that it varies only slightly with distances of the order of the fundamental wavelength. It is then expected that the amplitudes of higher harmonics, arising owing to nonlinear effects, will be small. In a general way, dispersion prevents the growth of higher harmonics – at least when wave amplitudes are not too large (Kadomtsev 1976). The nonlinearity, however, is expected to affect the dispersion of the carrier wave. The frequency and phase velocity of the wave thus become dependent on the wave amplitude, which leads us to replace the linear dispersion law by a nonlinear one:

$$D(k, \omega, |\varphi|^2) = 0. \tag{5}$$

With the help of this dispersion law, the nonlinear set (2) is reduced to a so-called nonlinear parabolic equation (Kadomtsev and Karpman 1971), which governs the behaviour of the $\varphi(x, t)$:

$$i\left(\frac{\partial \varphi}{\partial t} + v_g \frac{\partial \varphi}{\partial x}\right) + \frac{v_g'}{2}\frac{\partial^2 \varphi}{\partial x^2} + \frac{v_g}{2k_0}\nabla_\perp^2 \varphi - \beta |\varphi|^2 \varphi = 0. \tag{6}$$

Here $v_g = (\partial\omega/\partial k)_{\varphi=0}$ is the group velocity, and then $v_g' = (\partial^2\omega/\partial k^2)_{\varphi=0}$, the second derivative of the frequency with respect to k, refers to the group velocity dispersion. Also, $\nabla_\perp^2$ is the Laplacian with respect to directions orthogonal to the propagation of the wave, and describes the wave diffraction. The coefficient β is associated with the expansion in powers of the wave intensity in the nonlinear dispersion law (5) and is given by $\beta = (\partial\omega/\partial|\varphi|^2)_{\varphi=0}$. Hence, the nonlinear analysis requires that first the nonlinear dispersion law (5) be obtained.

Before applying this approach to a specific case of wave processes in self-

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gravitational plasmas, it should be noted that we shall be dealing here only with stable solutions of the corresponding linear dispersion law (4). In what follows, the appropriate conditions will be formulated for two different two-component plasma models. The first corresponds to the usual model of a dusty plasma, with inertialess light particles (electrons and/or ions) in the presence of one representative heavy dust-grain species. As an alternative to this model, we shall treat a two-component dusty plasma with two sufficiently heavy species, taking into account their different thermal velocities.

3. Dust-acoustic solitons in a plasma containing self-gravitating particles

Consider a dusty plasma consisting of two species of particles with radically different masses. We think of the often used, simplest model where all plasma electrons have been absorbed by dust grains, and the dusty plasma just consists of ions and grains of a single species. Subscript 1 will relate to the lighter and subscript 2 to the heavier kind, i.e. $m_1 \ll m_2$. For the lighter particles (e.g. ions), we shall assume that $\omega_{p1} \gg \omega_{J1}$, and moreover let $\omega_{J1} \to 0$. Besides this, we treat these as being Boltzmann-distributed:

$$n_1 = N_1 \exp\left(-\frac{q_1\psi_E}{T_1}\right). \tag{7}$$

It is also reasonable to neglect the thermal effects of the heavier particles, i.e. to assume $v_{T1} \gg v_{T2} \to 0$.

As for the number density n_2 of the dust grains, its expression in terms of the potentials ψ_E and ψ_G is to be determined from the nonlinear set (2). This set of partial differential equations admits solutions depending on the combination $\zeta = kx - \omega t$, representing waves propagating without variations in waveform. By introducing the new independent variable ζ instead of x and t, we transform (2) into a set of ordinary differential equations.

In order to analyse the influence of weak nonlinear effects, for small but finite-amplitude disturbances, we expand all variables v_2, n_2, ψ_E and ψ_G in powers of a small parameter $\mu \ll 1$, measuring the magnitude of the wave amplitude, of the form

$$a = A + \mu a^{(1)}(\zeta) + \mu^2 a^{(2)}(\zeta) + \mu^3 a^{(3)}(\zeta) + \ldots \quad . \tag{8}$$

All the equilibrium quantities A are zero, except the density N_2. Besides this, we also expand (7) in $q_1\psi_E/T_1$, and get

$$n_1 = N_1\left(1 - \frac{q_1\psi_E}{T_1} + \frac{q_1^2\psi_E^2}{2T_1^2} - \ldots\right). \tag{9}$$

Substituting (8) into (2) and (9), we equate the coefficients of the same powers of μ (in other words, of the same harmonics) to zero.

At order μ, we recover the solution of the linear problem

$$\psi_E^{(1)} = \varphi\cos\zeta, \tag{10a}$$

$$\psi_G^{(1)} = -\frac{4\pi G N_2 q_2}{\omega^2 + \omega_{J2}^2}\varphi\cos\zeta, \tag{10b}$$

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$$n_2^{(1)} = \frac{k^2 N_2 q_2}{m_2(\omega^2 + \omega_{J2}^2)} \varphi \cos\zeta, \tag{10c}$$

$$v_2^{(1)} = \frac{k\omega q_2}{m_2(\omega^2 + \omega_{J2}^2)} \varphi \cos\zeta. \tag{10d}$$

The values for ω and k occurring in (10) are related through the linear dispersion law (4), which, in the case considered here, can be rewritten as

$$1 + \frac{1}{k^2\lambda_{D1}^2} - \frac{\omega_{p2}^2}{\omega^2 + \omega_{J2}^2} = 0, \tag{11}$$

where the Debye length for the light particles is defined through $\lambda_{D1} = v_{T1}/\omega_{p1}$.

With $k^2\lambda_{D1}^2 \ll 1$, one obtains the analogue of the dust-acoustic mode in self-gravitating plasmas (Avinash and Shukla 1994; Pandey et al. 1994):

$$\omega^2 = k^2 v_{da}^2 (1 - k^2\lambda_{D1}^2) - \omega_{J2}^2, \tag{12}$$

where $v_{da} = \lambda_{D1}\omega_{p2}$ denotes the dust-acoustic velocity. As can be seen from (12), a self-gravitating plasma can support stable modifications of the dust-acoustic waves when $k^2\lambda_{D1}^2\omega_{p2}^2 \gg \omega_{J2}^2$. Therefore, in contrast to the usual dust-acoustic condition $k^2\lambda_{D1}^2 \ll 1$, we arrive at the double inequality

$$\frac{\omega_{J2}^2}{\omega_{p2}^2} \ll k^2\lambda_{D1}^2 \ll 1, \tag{13}$$

and the value of k is also bounded from below. In so doing, we obtain the approximate solution

$$\omega = k v_{da} \left(1 - \frac{k^2\lambda_{D1}^2}{2} - \frac{\omega_{J2}^2}{2k^2 v_{da}^2}\right). \tag{14}$$

One further distinguishing feature of the dust-acoustic mode in a self-gravitating plasma is its dispersion, which is controlled by the last two terms in the parentheses. It then follows that the 'short-wave' disturbances in the acceptable range (13) obey

$$\frac{\omega_{J2}}{\omega_{p2}} < k^2\lambda_{D1}^2 \ll 1, \tag{15}$$

and these propagate in almost the same manner as the conventional dust-acoustic mode, with very small corrections due to self-gravitation. Using (14), the missing coefficients v_g and v_g' in (6) can in this case be written as

$$v_g = v_{da}\left(1 - \frac{3}{2}k^2\lambda_{D1}^2 + \frac{\omega_{J2}^2}{2k^2 v_{da}^2}\right), \tag{16a}$$

$$v_g' = -v_{da}\left(3k\lambda_{D1}^2 + \frac{\omega_{J2}^2}{k^3 v_{da}^2}\right). \tag{16b}$$

However, the situation is quite different when we are dealing with 'long-wave' disturbances, namely when

$$\frac{\omega_{J2}^2}{\omega_{p2}^2} \ll k^2\lambda_{D1}^2 < \frac{\omega_{J2}}{\omega_{p2}}, \tag{17}$$

where the dispersion properties of (14) are completely specified by the self-gravitation. For these disturbances of a sufficiently large scale, the magnitudes

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of v_g and v_g' become

$$v_g = v_{da}\left(1 + \frac{\omega_{J2}^2}{2k^2 v_{da}^2}\right), \tag{18a}$$

$$v_g' = -\frac{\omega_{J2}^2}{k^3 v_{da}}. \tag{18b}$$

This case is of special interest, and will be considered further in more detail – if for no other reason than that all dusty plasmas have a range of parameters (17) where the gravitational term prevails.

At order μ^2, we have a set of inhomogeneous equations for the functions $a^{(2)}$ due to terms such as $v_2^{(1)} dv_2^{(1)}/d\zeta$ and $d(v_2^{(1)} n_2^{(1)})$. The latter are responsible for the generation of the second harmonic $a^{(2)}(\zeta) \propto \cos 2\zeta$, but we shall not give their bulky expressions explicitly.

At the next order, one gets nonlinear inhomogeneous terms such as $v_2^{(1)} dv_2^{(2)}/d\zeta$, $v_2^{(2)} dv_2^{(1)}/d\zeta$ and $d(v_2^{(2)} n_2^{(1)})/d\zeta$, $d(v_2^{(1)} n_2^{(2)})/d\zeta$ proportional to combinations such as $C_i \cos 3\zeta + D_i \cos\zeta$. The terms in $\cos 3\zeta$ are associated with the third harmonics in $a^{(3)}(\zeta)$.

As is to be expected, regular perturbation calculations indeed lead to resonant terms in the hierarchy of equations arising at higher orders, which results in secular terms in the perturbative expansion of the solution. The classic way of dealing with such a situation is by expanding the frequency ω itself in terms of the same parameter μ (Whitham 1974):

$$\omega(k) = \omega^{(0)}(k) + \mu\omega^{(1)}(k) + \mu^2\omega^{(2)}(k) + \ldots, \tag{19}$$

where $\omega^{(0)}(k)$ is determined from the linear dispersion law. Because difficulties arise only in third order, we can, without loss of generality, put the frequency correction $\omega^{(1)}(k)$ equal to zero, $\omega^{(1)}(k) = 0$.

Substituting (19) into the third-order equations, one then obtains the solvability condition that eliminates the resonant terms proportional to $\cos\zeta$. Consequently, we determine $\omega^{(2)} = \beta\,|\varphi|^2$, prescribing thereby the nonlinear term in the dispersion relation (5).

Strictly speaking, to analyse the influence of self-gravitation, we have to consider two different envelope equations (6) for the nonlinear dust-acoustic modes. In the 'short-wave' case (15), the coefficients of the envelope equation (6) can be estimated by letting $\omega_{J2}^2 \to 0$ in (16). Straightforward calculations then give

$$\beta_D = \frac{q_2^2}{12 m_2^2 k \lambda_{D1}^5 \omega_{p2}^3}. \tag{20}$$

On the other hand, for the 'long-wave' case (17), we get

$$\beta_J = -\frac{k^3 q_2^2}{3 m_2^2 \lambda_{D1} \omega_{p2} \omega_{J2}^2}. \tag{21}$$

Constant-amplitude solutions of the envelope equations correspond to plane waves, the frequency of which is slightly shifted by the nonlinearity. In the 'short-wave' case, since β is positive and v_g' negative, such constant-amplitude solutions are always modulationally unstable with respect to longitudinal perturbations (Kadomtsev and Karpman 1971). The modulational instability of dust-acoustic waves was considered by Salahuddin (1993), but without taking any self-gravitational effects

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into account. Disturbances with wavenumbers κ obeying $\kappa^2 < -4\beta\,|\varphi_0|^2\,/v_g'$, where φ_0 is the initial wave amplitude, are linearly unstable, while larger wavenumbers are stabilized by dispersion. The stabilizing role of dispersion results in the possibility of having a localized traveling wave soliton.

Indeed, if we put $\nabla_\perp^2 = 0$ in (6), we get the nonlinear Schrödinger (NLS) equation with an attracting nonlinearity (Sulem and Sulem 1999). This admits an exact solution of the form $\varphi(x,t) = \exp(-i\beta\varphi_0^2 t)u(x - v_g t)$. Substituting such a solution into the NLS equation, we obtain the equation prescribing the shape of the wave profile:

$$\frac{\partial^2 u}{\partial x^2} + \nu u^3 - \gamma u = 0, \tag{22}$$

where $\nu = -2\beta/v_g'$ and $\gamma = -2\beta\varphi_0^2/v_g'$.

When $\gamma > 0$, (22) is the typical equation for cnoidal waves with solutions

$$u = \left(\frac{2\gamma}{\nu}\right)^{1/2} \frac{1}{\cosh[\gamma^{1/2}(x - v_g t)]}. \tag{23}$$

We thus obtain localized solutions in the form of solitary waves in dusty self-gravitating plasmas:

$$\varphi = \left(\frac{2\gamma}{\nu}\right)^{1/2} \frac{1}{\cosh[\gamma^{1/2}(x - v_g t)]} \cos[kx - t(\omega + \beta\varphi_0^2)], \tag{24}$$

where the initial amplitude φ_0 is related to the amplitude, width, and frequency of the nonlinear wave. The coefficients ν and γ are defined by the relations (16) and (20) for the usual dust-acoustic wave.

The NLS equation in one dimension is integrable by the inverse scattering transform (IST) (Zakharov and Shabat 1972; Novikov et al. 1984). In this method, the solution of the nonlinear equation is associated with linear eigenvalue equations

$$\frac{\partial \psi_1}{\partial x} = i\lambda\psi_1 + i\varphi\left(-\frac{\partial\omega/\partial\,|\varphi|^2}{\partial^2\omega/\partial k^2}\right)^{1/2}\psi_2, \tag{25a}$$

$$\frac{\partial \psi_2}{\partial x} = -i\lambda\psi_2 + i\varphi\left(-\frac{\partial\omega/\partial\,|\varphi|^2}{\partial^2\omega/\partial k^2}\right)^{1/2}\psi_1. \tag{25b}$$

These equations describe a scattering process for the potential $\varphi(x,t)$, which in turn is the solution of the initial NLS equation.

The details of this method are clearly beyond the scope of this paper. Nevertheless, integrability determined from the IST leads in particular to the conclusion that soliton-like solutions, localized in space, preserving their form and propagating with constant velocity, correspond to those initial potentials $\varphi(x,0)$ of the eigenvalue problem (25) that have a single eigenvalue λ_1. The number of eigenvalues λ_n of the set (25), associated with an initial perturbation $\varphi(x,0)$, corresponds to the number of solitons when $t \to \infty$. More specifically, an initial pulse of constant amplitude and width τ (i.e. the pulse length is $b = \tau v_g$) will be completely dissipated due to dispersion if its amplitude is below a critical magnitude, namely

$$(\varphi_{\rm cr} b)^2 = \frac{\pi^2}{4}\left|\frac{\partial^2\omega/\partial k^2}{\partial\omega/\partial\,|\varphi|^2}\right|. \tag{26}$$

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This is due to the fact that the first eigenvalue λ_1 in the set (25) appears for a rectangular potential only when $\varphi_0 \geqslant \varphi_{\rm cr}$.

At this stage, it is convenient to estimate the energy required to transform the initial disturbance into a soliton. The energy flux density and the amplitude of electrostatic waves are connected by the well-known relation (Stix 1992)

$$S = \tfrac{1}{2}\varepsilon_0 k^2 v_g \langle \psi_E^2 \rangle \left| \frac{\partial}{\partial \omega} \left[\varepsilon(\omega)\omega \right] \right|_{\varepsilon=0}. \tag{27}$$

Here $\langle\ \rangle$ corresponds to values averaged over one oscillation period, and $\varepsilon(\omega)$ is identified with the dielectric constant given by the left-hand side of (11). Using (11) and (18) or (16), one then obtains approximately the same value of S for the two dust-acoustic waves (12):

$$S \simeq \frac{\varepsilon_0 \omega_{p2}}{2\lambda_{D1}} \varphi^2. \tag{28}$$

Combining (28) with (26) and taking (16) into account, we obtain the threshold values of the power flux density for the conventional dust-acoustic wave in a self-gravitating plasma, namely

$$S_{{\rm cr}D} \simeq \frac{9\pi^2 \varepsilon_0 \omega_{p2}^5 m_2^2 k^2 \lambda_{D1}^7}{2q_2^2 b^2}. \tag{29}$$

As the rectangular potential pulse propagates in the self-gravitating plasma, it can only be transformed into a soliton when its power flux density exceeds the value (29), i.e. $S > S_{{\rm cr}D}$.

Let us now take up the nonlinear dust-acoustic waves in the long-wave limit (17), where the dispersion is specified by self-gravitational effects. In this case, the value v_g' remains negative as in (16), but, in contrast to (20), β_J is negative also, and constant-amplitude solutions of the envelope equations will be unstable to transverse perturbations (Kadomtsev 1976). This instability, known as self-focusing, allows the appearance of a solitary wave localized in the transverse direction with respect to the direction of wave propagation. Thus, even weak self-gravitational effects can prevent the formation of the usual longitudinal dust-acoustic solitary waves in the long-wave limit, and lead instead to the existence only of transverse solitary structures.

Indeed, let us consider again the envelope equation (6), assuming that the wave has a constant amplitude in the transverse direction, i.e. $\partial\varphi/\partial x = 0$. Then (6) can be written in the form of an NLS equation

$$i\frac{\partial \varphi}{\partial t} + \frac{v_g}{2k}\frac{\partial^2 \varphi}{\partial y^2} - \beta_J |\varphi|^2 \varphi = 0, \tag{30}$$

where y is the transverse coordinate. Equation (30) represents wave diffraction. It admits solutions of the form $\varphi = \varphi(y)\exp(i\Omega t)$, where $\varphi(y)$ is a real function of y that tends to zero when $|y| \to \infty$. Substituting this in (30), one gets

$$\varphi = \psi \frac{\exp(-\frac{1}{2} i\psi^2 \beta_J t)}{\cosh[\psi y(-k\beta_J/v_g)^{1/2}]}. \tag{31}$$

The soliton amplitude ψ is related to the frequency through

$$\Omega = -\tfrac{1}{2}\psi^2 \beta_J. \tag{32}$$

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The width of the self-focusing region also depends upon the wave amplitude:

$$\Delta \propto \frac{1}{\psi}\left|-\frac{k\beta_J}{v_g}\right|^{-1/2}. \tag{33}$$

As a result of diffraction, an initially uniform disturbance of width $b_\perp$ will be transformed into a stable soliton-type wave only if its amplitude satisfies the inequality

$$\varphi \geqslant \psi_{\mathrm{cr}J} = \frac{\pi}{2b_\perp}\left|\frac{v_g}{2k\beta_J}\right|^{1/2}. \tag{34}$$

This condition is found in much the same way as the condition (26) for the longitudinal soliton. Hence, we have a critical amplitude for the dust-acoustic mode in the 'long-wave' range (17):

$$\psi_{crJ} \simeq \frac{\pi\sqrt{3}\lambda_{D1}m_2\omega_{p2}\omega_{J2}}{2\sqrt{2}b_\perp k^2 q_2}. \tag{35}$$

The next step then consists in estimating the required energy for the formation of the transverse soliton. Combining (35) with (28), we obtain the threshold values of the power flux density for the transverse solitary wave in a self-gravitating plasma:

$$S_{\mathrm{cr}J} \simeq \frac{3\pi^2\varepsilon_0\omega_{p2}^3\omega_{J2}^2 m_2^2\lambda_{D1}}{16q_2^2 b_\perp^2 k^4}. \tag{36}$$

A comparison of (29) and (36) for both cases, assuming $b_\perp \simeq b$, reveals the effect due to self-gravitation, and gives

$$\frac{S_{\mathrm{cr}D}}{S_{\mathrm{cr}J}} \simeq \frac{24\omega_{p2}^2 k^6\lambda_{D1}^6}{\omega_{J2}^2}. \tag{37}$$

Keeping the 'long-wave' range (17) in mind, (37) becomes $S_{\mathrm{cr}D}/S_{\mathrm{cr}J} \ll 1$, and we see that the threshold value (36) for the transverse soliton in a self-gravitating plasma is much greater than for the longitudinal dust-acoustic solitary wave (29) without considering ω_{J2}. Thus, self-gravitation would have to be fairly large in order to give rise to the formation of dust-acoustic solitons in the 'long-wave' part of spectrum.

4. Nonlinear waves in a self-gravitating medium with charged particles

So far, it has been assumed that $\omega_{p\alpha}^2 > \omega_{J\alpha}^2$, i.e. the particle dynamics is predominantly controlled by electrostatic forces. For the sake of generality, we now consider an alternative limiting case of a self-gravitating plasma in which $\omega_{p\alpha}^2 < \omega_{J\alpha}^2$. We shall concentrate on a two-component dusty plasma with two heavy species, assume $m_1 \ll m_2$, and take their different thermal velocities $v_{T\alpha} \neq 0$ into account. This amounts to having a system that is almost completely controlled by self-gravitation, with small corrections due to the plasma effects through the charges.

In what follows, we shall use the same standard method of envelope modulation as in the previous section. The variables of the nonlinear set (2) will again be expanded in powers of a small parameter $\mu \ll 1$ of the type (8). Substituting the perturbation expansions into (2) now gives to lowest non-zero order that

$$\psi_E^{(1)} = \varphi\cos\zeta, \tag{38a}$$

$$\psi_G^{(1)} = \frac{P(4\pi\varepsilon_0 G)^{1/2}}{K}\varphi\cos\zeta, \tag{38b}$$

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$$n_\alpha^{(1)} = \frac{k^2 N_\alpha}{\omega^2 - k^2 v_{T\alpha}^2}\left[\frac{q_\alpha}{m_\alpha} + \frac{P(4\pi\varepsilon_0 G)^{1/2}}{K}\right]\varphi\cos\zeta, \tag{38c}$$

$$v_\alpha^{(1)} = \frac{k\omega}{\omega^2 - k^2 v_{T\alpha}^2}\left[\frac{q_\alpha}{m_\alpha} + \frac{P(4\pi\varepsilon_0 G)^{1/2}}{K}\right]\varphi\cos\zeta, \tag{38d}$$

where

$$P = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2 - k^2 v_{T\alpha}^2}, \tag{39a}$$

$$K = \sum_\alpha \frac{\omega_{p\alpha}\omega_{J\alpha}}{\omega^2 - k^2 v_{T\alpha}^2}. \tag{39b}$$

The solution (38) represents the linear approximation, and the wave variables obey the corresponding dispersion law (4). Since, as already mentioned, the roots of the dispersion law have to be real, we assume that $(\omega_{J\alpha}^2 - \omega_{p\alpha}^2)/k^2 v_{T\alpha}^2 \ll 1$ ($\alpha = 1, 2$). Equation (4) is then characterized by two real, approximate solutions

$$\omega_\alpha = k v_{T\alpha}\left(1 - \frac{1}{2k^2\lambda_{J\alpha}^2} + \frac{1}{2k^2\lambda_{D\alpha}^2}\right), \tag{40}$$

where, besides the Debye length $\lambda_{D\alpha} = v_{T\alpha}/\omega_{p\alpha}$, we have also introduced the Jeans length $\lambda_{J\alpha} = v_{T\alpha}/\omega_{J\alpha}$ for both species.

Going further through the algebra, to orders μ^2 and μ^3, and using the frequency expansion (19), one can obtain the solvability condition. Hence, we need to consider both solutions (40), and it is convenient to represent the nonlinear term in (19) in the form $\omega_\alpha^{(2)} = \beta_\alpha |\varphi|^2$, where β_α determines the appropriate nonlinear frequency shift and is given by

$$\beta_\alpha = -\frac{4\pi\varepsilon_0 G k^7 v_{T\alpha}^3}{3\omega_{p\alpha}^2\omega_{J\alpha}^4}. \tag{41}$$

Proceeding then as in the previous section, one gets the envelope equations (6) for the two types of waves (40), where the coefficients v_g and v_g' are given by

$$v_{g\alpha} = v_{T\alpha}\left(1 + \frac{1}{2k^2\lambda_{J\alpha}^2} - \frac{1}{2k^2\lambda_{D\alpha}^2}\right), \tag{42a}$$

$$v_{g\alpha}' = -\frac{v_{T\alpha}}{k^3\lambda_{J\alpha}^2} + \frac{v_{T\alpha}}{k^3\lambda_{D\alpha}^2}. \tag{42b}$$

The values $v_{g\alpha}'$, as well as (16) and (18), are both negative, as are now both values of β_α. Therefore, this result corresponds to the transverse modulational instability, and the NLS equation (6) has two soliton solutions, which can be represented in the same form as (31), by replacing $\beta_J \to \beta_\alpha$ and $v_g \to v_{g\alpha}$.

A critical amplitude for each mode (40) is determined exactly in the same way as given in the previous section, which leads to

$$\psi_{cr\alpha} \simeq \frac{\omega_{J\alpha}^2}{4b_\perp k^4 \lambda_{D\alpha}}\left(\frac{3\pi}{2G\varepsilon_0}\right)^{1/2}. \tag{43}$$

Finally, we have to estimate the energy required for the formation of transverse solitons in this self-gravitating medium. As before, we shall compute the wave energy through (27), using the left-hand side of (4) as the dielectric constant of the self-gravitating plasma and taking the condition $\omega_{J\alpha}^2 > \omega_{p\alpha}^2$ into account.

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This yields

$$S_\alpha \simeq \frac{\varepsilon_0 k^4 v_{T\alpha}^3}{\omega_{J\alpha}^2 - \omega_{p\alpha}^2}\varphi^2. \tag{44}$$

The threshold values for both transverse waves then become

$$S_{\mathrm{cr}\alpha} \simeq \frac{3\pi\omega_{J\alpha}^4 v_{T\alpha}^3}{32 b_\perp^2 G k^4 \lambda_{D\alpha}^2(\omega_{J\alpha}^2 - \omega_{p\alpha}^2)}. \tag{45}$$

One can now clearly see the role of the plasma effects in such a medium: the greater the plasma frequency $\omega_{p\alpha}$ is, the greater the threshold energy (45) becomes for the formation of solitary waves. Let us compare the values (45) for both possible waves (40), which gives

$$\frac{S_{\mathrm{cr}1}}{S_{\mathrm{cr}2}} \simeq \frac{(\omega_{J2}^2 - \omega_{p2}^2)\omega_{J1}^2 v_{T1}}{(\omega_{J1}^2 - \omega_{p1}^2)\omega_{J2}^2 v_{T2}}. \tag{46}$$

This relation is obtained with the help of the charge-neutrality condition $N_1 q_1 + N_2 q_2 = 0$.

In the case where $\omega_{J\alpha}^2 \gg \omega_{p\alpha}^2$ and the plasma terms can be neglected, it is reasonable to assume that $S_{\mathrm{cr}1}/S_{\mathrm{cr}2} > 1$ for $m_1 \ll m_2$. But with increasing plasma frequency of the lighter species ($\omega_{p1} \to \omega_{J1}$) the ratio (46) even becomes much larger than the previous one, i.e. $S_{\mathrm{cr}1}/S_{\mathrm{cr}2} \gg 1$. In any case, this means that the threshold power for the 'higher-frequency' mode ($\alpha = 1$) far exceeds the corresponding value for the 'lower-frequency' wave ($\alpha = 2$).

Thus, our approach demonstrates a significant distinction between two plasma media with respect to the envelope dynamics, but, in both cases, the nonlinear effects could give rise to the formation of soliton-like solutions. These are longitudinal solitons in the short-wave limit in a self-gravitating plasma, quite similar to the usual dust-acoustic solitons, when $\omega_{p\alpha}^2 > \omega_{J\alpha}^2$ and the dispersion of the medium is determined mainly by the plasma processes. However, only transverse solitons are possible in the other case, when self-gravitational effects become important, i.e. in the long-wavelength range case given by (17), when $\omega_{p\alpha}^2 > \omega_{J\alpha}^2$ or else when $\omega_{p\alpha}^2 < \omega_{J\alpha}^2$. In all these cases, there is a threshold before solitons can be formed.

5. Conclusions

In Secs 3 and 4, we have considered soliton-like solutions of the one-dimensional nonlinear Schrödinger equation for the envelope of electrostatic waves in self-gravitating dusty plasmas in two limiting cases, namely when the charged-particle dynamics is governed mostly by electrostatic forces ($\omega_{p\alpha}^2 > \omega_{J\alpha}^2$) or by gravitation ($\omega_{p\alpha}^2 < \omega_{J\alpha}^2$). This approach indicates a significant distinction between these two plasma media with respect to the envelope dynamics. In the former case, the amplitude perturbations are longitudinally unstable in the short-wave range, and the nonlinear effects can result in the formation of longitudinal dust-acoustic solitary waves. However, self-gravitational effects can crucially change the plasma stability to transverse modulation in the long-wave end of the spectrum, and can lead to the formation of transverse solitary waves. The point to make here is that even weak self-gravitational effects result in the presence of such a long-wavelength range, and then self-gravitation can prevent the formation of the usual dust-acoustic solitary waves and lead to the existence of transverse solitary waves. However, the latter solitons need a sufficiently high threshold power for their appearance.

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In the converse case, there is a transverse modulational instability in a self-gravitating plasma ($\omega_{p\alpha}^2 < \omega_{J\alpha}^2$), and only transverse solitons are possible here. There is a threshold before these transverse solitary waves can be generated, and this threshold can be quite appreciable if the plasma and Jeans frequencies are of the same order.

The appearance of such transverse solitary waves is similar to the phenomenon of nonlinear self-focusing in optical systems, when the transverse instability spatially modifies the refractive index, so that it provides a waveguide in the transverse direction. In this context, a self-gravitating plasma with $\omega_{p\alpha}^2 > \omega_{J\alpha}^2$ in the long-wavelength range and a plasma with $\omega_{p\alpha}^2 < \omega_{J\alpha}^2$ may be considered as focusing media, while a dusty plasma with $\omega_{p\alpha}^2 > \omega_{J\alpha}^2$ in the short-wavelength range is not.

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