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Electromagnetic diffraction by two coaxial discs

By D. P. THOMAS
Department of Applied Mathematics, University of Liverpool

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1. *Introduction.* The problems associated with low-frequency diffraction by a circular disc have attracted the attention of many research workers. A review article by Bouwkamp (1) gives an excellent account of the research that was done prior to 1953 and a report by Eggimann (3) provides a summary of the papers that were published between 1953 and 1960. References to work published since 1960 may be found in papers of Noble (10) and Williams (16), (17).

Little consideration has been given to low-frequency diffraction by two or more circular discs (or the equivalent problems for two or more holes in an infinite plane). The only papers to be devoted to this topic seem to be those of Collins (2) and Nomura and Osanai (11), (12). Collins discussed diffraction of axisymmetric scalar waves by two perfectly hard coaxial circular discs. A contour integral method was used to show that this diffraction problem depends on the solutions of Fredholm integral equations of the second kind, and an approximation to the scattering cross-section was obtained for an incident low-frequency plane wave and large separation between the discs. The corresponding problem for two perfectly soft coaxial circular discs was considered in less detail—only the Fredholm integral equations that govern the problem were obtained. Nomura and Osanai have solved the problem of diffraction of plane sound waves by two equal circular holes in an infinitely large rigid plane plate (11). They used a method of expansion in hypergeometrical polynomials to reduce the problem to an infinite set of linear equations. Formulae for the acoustic impedance and the transmission coefficient were derived and approximate values for these quantities were calculated in the case of normal incidence. These authors have since generalized their method of solution to cover the case of many equal circular holes arbitrarily distributed in an infinitely large rigid plane plate (12).

The present paper considers the solution of a problem of low-frequency electromagnetic diffraction by two perfectly conducting coaxial circular discs. The incident field is assumed to be an electromagnetic plane wave travelling along the axis of symmetry of the obstacle. A technique suggested by Lebedev and Skal'skaya (8) enables this electromagnetic diffraction problem to be formulated as two independent scalar problems. A simple extension of Williams's work on diffraction of arbitrary non-axisymmetric scalar waves by a circular disc (16) reduces each of these two independent scalar problems to a pair of simultaneous Fredholm integral equations of the second kind. The calculation of the scattering coefficient is considered in some detail

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and three methods are used to derive an expression for the scattering coefficient in terms of the solutions of the Fredholm integral equations.

In the special case of two equal discs each pair of the simultaneous integral equations is transformed into two ordinary Fredholm integral equations of the second kind. The latter equations are solved by an iterative process when the distance between the discs is large and the incident field is a low-frequency plane wave. These iterative solutions are used to determine an approximation to the scattering coefficient. It is shown that in the limit as the distance between the discs tends to infinity the calculated value of the scattering coefficient for two coaxial discs equals the known approximate value for a single disc.

2. *Formulation of the problem.* Circular cylindrical polar coordinates, (r, ϕ, z), are chosen such that the z-axis coincides with the common axis of the discs. The two discs, Σ_1 and Σ_2, are then defined by

$$z = 0, \quad 0 \leqslant r \leqslant a, \quad \text{all } \phi$$

and

$$z = L, \quad 0 \leqslant r \leqslant b, \quad \text{all } \phi,$$

respectively. It is also convenient to define a system of spherical polar coordinates, (R, θ, ϕ), so that

$$r = R\sin\theta \quad \text{and} \quad z = R\cos\theta.$$

The electric field vector, $\mathbf{E}$, and the magnetic field vector, $\mathbf{H}$, satisfy Maxwell's equations,

$$\operatorname{curl}\mathbf{E} + ik\mathbf{H} = 0, \quad \operatorname{div}\mathbf{E} = 0,$$
$$\operatorname{curl}\mathbf{H} - ik\mathbf{E} = 0, \quad \operatorname{div}\mathbf{H} = 0.$$

Gaussian units are used, a time dependence of the form $\exp(i\omega t)$ is assumed throughout and $k = \omega/c$ where c is the speed of the electromagnetic wave propagation. It is useful to define three dimensionless quantities, α, β and Λ, by the relationships

$$\alpha = ka, \quad \beta = kb, \quad \text{and} \quad a\Lambda = L.$$

The total field vector can be expressed as the sum of the incident field vector and the diffracted field vector, i.e.

$$\mathbf{E} = \mathbf{E}^{(1)} + \mathbf{E}^{(2)} \quad \text{and} \quad \mathbf{H} = \mathbf{H}^{(1)} + \mathbf{H}^{(2)},$$

where the suffices 1 and 2 refer to the incident and diffracted waves, respectively. The incident field is assumed to be a plane wave travelling in the positive z direction. Hence

$$\left.\begin{aligned} \mathbf{E}^{(1)} &= (E_r^{(1)}, E_\phi^{(1)}, E_z^{(1)}) = E_0(\cos\phi, -\sin\phi, 0)\exp(-ikz) \\ \text{and} \quad \mathbf{H}^{(1)} &= (H_r^{(1)}, H_\phi^{(1)}, H_z^{(1)}) = E_0(\sin\phi, \cos\phi, 0)\exp(-ikz). \end{aligned}\right\} \tag{2·1}$$

It is known ((13), page 28) that Maxwell's equations are satisfied by

$$\mathbf{E} = -ik\operatorname{curl}\boldsymbol{\Pi} \quad \text{and} \quad \mathbf{H} = k^2\boldsymbol{\Pi} + \operatorname{grad}\operatorname{div}\boldsymbol{\Pi},$$

where $\boldsymbol{\Pi}$, the magnetic Hertz vector, is a solution of

$$\nabla^2\boldsymbol{\Pi} + k^2\boldsymbol{\Pi} = 0.$$

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It is assumed that the magnetic Hertz vector that corresponds to the diffracted field has the form

$$\mathbf{\Pi}^{(2)} = (\pi_r^{(2)}, \pi_\phi^{(2)}, \pi_z^{(2)}) = \{\Phi(r,z)\sin\phi, \Phi(r,z)\cos\phi, \Psi(r,z)\sin\phi\}.$$

In order that the resulting field components satisfy Maxwell's equations the functions $\Phi(r,z)$ and $\Psi(r,z)\sin\phi$ must be solutions of the scalar equation

$$(\nabla^2 + k^2)\, u(r,\phi,z) = 0. \tag{2·2}$$

Then the electric and magnetic field vectors of the diffracted field written in terms of the new functions $\Phi(r,z)$ and $\Psi(r,z)$ are

$$\left.\begin{aligned} E_r^{(2)} &= -ik\left(\frac{1}{r}\Psi - \frac{\partial\Phi}{\partial z}\right)\cos\phi, & H_r^{(2)} &= \left(\frac{\partial^2\Psi}{\partial r\,\partial z} + \frac{\partial^2\Phi}{\partial r^2} + k^2\Phi\right)\sin\phi, \\ E_\phi^{(2)} &= -ik\left(\frac{\partial\Phi}{\partial z} - \frac{\partial\Psi}{\partial r}\right)\sin\phi, & H_\phi^{(2)} &= \left(\frac{1}{r}\frac{\partial\Psi}{\partial z} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + k^2\Phi\right)\cos\phi, \\ E_z^{(2)} &= -ik\frac{\partial\Phi}{\partial r}\cos\phi, & H_z^{(2)} &= \left(\frac{\partial^2\Phi}{\partial r\,\partial z} + \frac{\partial^2\Psi}{\partial z^2} + k^2\Psi\right)\sin\phi. \end{aligned}\right\} \tag{2·3}$$

On the discs the tangential component of the total electric field must vanish. Therefore, from equations (2·1) and (2·3),

$$\left.\begin{aligned} E_0\exp(-ikz) - ik\left(\frac{1}{r}\Psi - \frac{\partial\Phi}{\partial z}\right) &= 0 \\ E_0\exp(-ikz) + ik\left(\frac{\partial\Phi}{\partial z} - \frac{\partial\Psi}{\partial r}\right) &= 0 \end{aligned}\right\} \quad \text{on } \Sigma_1 \text{ and } \Sigma_2.$$

These equations simplify to

$$\left.\begin{aligned} \frac{\partial\Phi}{\partial z} &= C_1 + \frac{iE_0}{k} \\ \Psi &= C_1 r \end{aligned}\right\} \text{ on } \Sigma_1 \quad \text{and} \quad \left\{\begin{aligned} \frac{\partial\Phi}{\partial z} &= C_2 + \frac{iE_0}{k}\exp(-ikL) \\ \Psi &= C_2 r \end{aligned}\right\} \text{ on } \Sigma_2, \tag{2·4}$$

where C_1 and C_2 are two constants, which will be determined by the edge conditions of Σ_1 and Σ_2. The diffracted electromagnetic field must be continuous everywhere in free space and hence Φ, Ψ and their derivatives must be continuous off the discs.

The original vector problem has thus been reduced to two independent scalar problems, namely, the determination of two scalar functions that satisfy equations (2·2) and (2·4), the continuity conditions off the discs and the radiation conditions at infinity. In the next section each of these two scalar problems is reduced to the solution of a pair of simultaneous Fredholm integral equations of the second kind.

3. *Reduction of the boundary-value problems to integral equations.* The boundary-value problem for Ψ is the first to be considered. It follows from Green's theorem that

$$\Psi(r,z) = \int_0^a g_1(t)\, tK_1(t,r,z)\, dt + \int_0^b h_1(t)\, tK_1(t,r,z-L)\, dt, \tag{3·1}$$

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where
$$g_1(r) = \frac{1}{2}\left\{\frac{\partial \Psi}{\partial z}(r,z)\Big|_{z=0-} - \frac{\partial \Psi}{\partial z}(r,z)\Big|_{z=0+}\right\}, \tag{3·2}$$

$$h_1(r) = \frac{1}{2}\left\{\frac{\partial \Psi}{\partial z}(r,z)\Big|_{z=L-} - \frac{\partial \Psi}{\partial z}(r,z)\Big|_{z=L+}\right\}, \tag{3·3}$$

$$K_n(t,r,z) = \int_0^{2\pi} \cos n\chi \frac{\exp(-ikD)}{2\pi D}\, d\chi$$

$$= \int_0^{\infty} \frac{p}{\gamma} J_n(tp) J_n(rp) \exp(-|z|\gamma)\, dp, \tag{3·4}$$

$$D = (r^2 + t^2 - 2rt\cos\chi + z^2)^{\frac{1}{2}}$$

and
$$\gamma = \begin{cases} (p^2 - k^2)^{\frac{1}{2}} & (p \geqslant k), \\ i(k^2 - p^2)^{\frac{1}{2}} & (p \leqslant k). \end{cases}$$

It is evident from equations (3·1) and (3·4) and the boundary conditions that the functions $g_1(r)$ and $h_1(r)$ must satisfy the equations

$$\int_0^a g_1(t)\, tK_1(t,r,0)\, dt = C_1 r - \int_0^b h_1(t)\, tK_1(t,r,L)\, dt \quad (0 < r < a) \tag{3·5}$$

and

$$\int_0^b h_1(t)\, tK_1(t,r,0)\, dt = C_2 r - \int_0^a g_1(t)\, tK_1(t,r,L)\, dt \quad (0 < r < b). \tag{3·6}$$

It follows from (16) that equation (3·5) can be rewritten in the form

$$\frac{2}{\pi r}\int_0^r t^2 \frac{\cos k(r^2 - t^2)^{\frac{1}{2}}}{(r^2 - t^2)^{\frac{1}{2}}} \int_t^a g_1(w) \frac{\cos k(w^2 - t^2)^{\frac{1}{2}}}{(w^2 - t^2)^{\frac{1}{2}}}\, dw\, dt$$

$$= C_1 r - \int_0^a g_1(t)\, tG_1(t,r)\, dt - \int_0^b h_1(t)\, tK_1(t,r,L)\, dt \quad (0 < t < a),$$

where

$$G_1(t,r) = k^2 \int_0^{\infty} (\lambda^2 + k^2)^{-1} J_1\{r(k^2 + \lambda^2)^{\frac{1}{2}}\} J_1\{t(k^2 + \lambda^2)^{\frac{1}{2}}\}\, d\lambda$$

$$- i\int_0^k J_1\{r(k^2 - \lambda^2)^{\frac{1}{2}}\} J_1\{t(k^2 - \lambda^2)^{\frac{1}{2}}\}\, d\lambda.$$

The above equation is a special case of equation (A 2) with $n = 1$. Hence from the analysis of the Appendix equation (3·5) reduces to

$$W^{(1)}(r) = \frac{2C_1}{k}\sinh \alpha r - \int_0^1 W^{(1)}(t)\{L_1^{(1)}(t,r,\alpha) + L_2^{(1)}(t,r,\alpha)\}\, dt$$

$$- \int_0^1 X^{(1)}(t)\, L_3^{(1)}(t,r:a,b)\, dt \quad (0 < r < 1), \tag{3·7}$$

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where
$$W^{(1)}(r) = ar\int_r^1 g_1(at)\frac{\cos\alpha(t^2-r^2)^{\frac{1}{2}}}{(t^2-r^2)^{\frac{1}{2}}}\,dt,$$

$$g_1(at) = -\frac{2}{a\pi}\frac{d}{dt}\int_t^1 W^{(1)}(u)\frac{\cosh\alpha(u^2-t^2)^{\frac{1}{2}}}{(u^2-t^2)^{\frac{1}{2}}}\,du,$$

$$L_1^{(1)}(t,r,\alpha) = \alpha^2(rt)^{\frac{1}{2}}\int_0^\infty \lambda^{-1}J_{\frac{1}{2}}(r\lambda)\,J_{\frac{1}{2}}(t\lambda)\,d\lambda,$$

$$L_2^{(1)}(t,r,\alpha) = -i(rt)^{\frac{1}{2}}\int_0^\alpha(\alpha^2-\lambda^2)\,\lambda^{-1}I_{\frac{1}{2}}(r\lambda)\,I_{\frac{1}{2}}(t\lambda)\,d\lambda$$

and
$$L_3^{(1)}(t,r:a,b) = b(abrt)^{\frac{1}{2}}\int_0^\infty p^3\gamma^{-2}J_{\frac{1}{2}}(ar\gamma)\,J_{\frac{1}{2}}(bt\gamma)\exp(-L\gamma)\,dp.$$

A similar treatment reduces equation (3·6) to

$$X^{(1)}(r) = \frac{2C_2}{k}\sinh\beta r - \int_0^1 X^{(1)}(t)\{L_1^{(1)}(t,r,\beta)+L_2^{(1)}(t,r,\beta)\}\,dt - \int_0^1 W^{(1)}(t)\,L_3^{(1)}(t,r:b,a)\,dt \quad (0<r<1), \tag{3·8}$$

where
$$X^{(1)}(r) = br\int_r^1 h_1(bt)\frac{\cos\beta(t^2-r^2)^{\frac{1}{2}}}{(t^2-r^2)^{\frac{1}{2}}}\,dt$$

and
$$h_1(bt) = -\frac{2}{b\pi}\frac{d}{dt}\int_t^1 X^{(1)}(u)\frac{\cosh\beta(u^2-t^2)^{\frac{1}{2}}}{(u^2-t^2)^{\frac{1}{2}}}\,du.$$

The function Φ will now be considered. Green's theorem gives

$$\Phi(r,z) = -\frac{\partial}{\partial z}\int_0^a b_0(t)\,tK_0(t,r,z)\,dt - \frac{\partial}{\partial z}\int_0^b d_0(t)\,tK_0(t,r,z-L)\,dt, \tag{3·9}$$

where
$$b_0(r) = \tfrac{1}{2}\{\Phi(r,0_+)-\Phi(r,0_-)\}, \tag{3·10}$$

and
$$d_0(r) = \tfrac{1}{2}\{\Phi(r,L_+)-\Phi(r,L_-)\}. \tag{3·11}$$

The approach suggested by Bouwkamp (1) reduces the boundary-value problem for Φ to the solution of the differential-integral equations

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+k^2\right]\left\{\int_0^a b_0(t)\,tK_0(t,r,0)\,dt+\int_0^b d_0(t)\,tK_0(t,r,L)\,dt\right\} = C_1+\frac{iE_0}{k} \quad (0<r<a) \tag{3·12}$$

and
$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+k^2\right]\left\{\int_0^b d_0(t)\,tK_0(t,r,0)\,dt+\int_0^a b_0(t)\,tK_0(t,r,L)\,dt\right\} = C_2+\frac{iE_0}{k}\exp(-ikL) \quad (0<r<b). \tag{3·13}$$

The recurrence relation for Bessel functions gives

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}+k^2\right)K_n(t,r,z) = -\frac{1}{(rt)^{n+1}}\frac{\partial^2}{\partial r\,\partial t}H_n(t,r,z),$$

where
$$H_n(t,r,z) = (rt)^{n+1}\int_0^\infty\frac{\gamma}{p}J_{n+1}(tp)\,J_{n+1}(rp)\exp(-|z|\,\gamma)\,dp.$$

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It follows from this identity that an integration by parts enables equations (3·12) and (3·13) to be rewritten as

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\int_0^a u_0(t)\,H_0(t,r,0)\,dt+\int_0^b v_0(t)\,H_0(t,r,L)\,dt\right]=C_1+\frac{iE_0}{k}\quad(0<r<a)\qquad(3\cdot14)$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\int_0^b v_0(t)\,H_0(t,r,0)\,dt+\int_0^a u_0(t)\,H_0(t,r,L)\,dt\right]=C_2+\frac{iE_0}{k}\exp(-ikL)\quad(0<r<b),\qquad(3\cdot15)$$

respectively, where

$$u_0(t)=\frac{d}{dt}b_0(t)\quad\text{and}\quad v_0(t)=\frac{d}{dt}d_0(t).$$

It is assumed that $b_0(a)=d_0(b)=0$. Equations (3·14) and (3·15) are similar to a differential-integral equation that Williams(16) discussed in connexion with scalar diffraction by a rigid disc. The procedure presented in (16) and the analysis contained in the Appendix enable equations (3·14) and (3·15) to be converted into

$$S^{(0)}(r)=\frac{1}{k}\left(C_1+\frac{iE_0}{k}\right)\sinh\alpha r+\int_0^1 S^{(0)}(t)\,M_2^{(0)}(t,r,\alpha)\,dt-\int_0^1 T^{(0)}(t)\,M_3^{(0)}(t,r:a,b)\,dt\quad(0<r<1)\qquad(3\cdot16)$$

and

$$T^{(0)}(r)=\frac{1}{k}\left\{C_2+\frac{iE_0}{k}\exp(-ikL)\right\}\sinh\beta r+\int_0^1 T^{(0)}(t)\,M_2^{(0)}(t,r,\beta)\,dt-\int_0^1 S^{(0)}(t)\,M_3^{(0)}(t,r:b,a)\,dt\quad(0<r<1),\qquad(3\cdot17)$$

respectively, where

$$S^{(0)}(r)=ar\int_r^1 u_0(at)\frac{\cos\alpha(t^2-r^2)^{\frac{1}{2}}}{(t^2-r^2)^{\frac{1}{2}}}\,dt,$$

$$T^{(0)}(r)=br\int_r^1 v_0(bt)\frac{\cos\beta(t^2-r^2)^{\frac{1}{2}}}{(t^2-r^2)^{\frac{1}{2}}}\,dt,$$

$$u_0(at)=-\frac{2}{a\pi}\frac{d}{dt}\int_t^1 S^{(0)}(r)\frac{\cosh\alpha(r^2-t^2)^{\frac{1}{2}}}{(r^2-t^2)^{\frac{1}{2}}}\,dr,$$

$$v_0(bt)=-\frac{2}{b\pi}\frac{d}{dt}\int_t^1 T^{(0)}(r)\frac{\cosh\beta(r^2-t^2)^{\frac{1}{2}}}{(r^2-t^2)^{\frac{1}{2}}}\,dr,$$

$$b_0(at)=-\frac{2}{\pi}\int_t^1 S^{(0)}(r)\frac{\cosh\alpha(r^2-t^2)^{\frac{1}{2}}}{(r^2-t^2)^{\frac{1}{2}}}\,dr,$$

$$d_0(bt)=-\frac{2}{\pi}\int_t^1 T^{(0)}(r)\frac{\cosh\beta(r^2-t^2)^{\frac{1}{2}}}{(r^2-t^2)^{\frac{1}{2}}}\,dr,$$

$$M_2^{(0)}(t,r,\alpha)=-i(rt)^{\frac{1}{2}}\int_0^\alpha pI_{\frac{1}{2}}(tp)\,I_{\frac{1}{2}}(rp)\,dp$$

and

$$M_3^{(0)}(t,r:a,b)=b(abrt)^{\frac{1}{2}}\int_0^\infty pJ_{\frac{1}{2}}(ar\gamma)\,J_{\frac{1}{2}}(bt\gamma)\exp(-L\gamma)\,dp.$$

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The present boundary-value problem for Φ is similar to a boundary-value problem that Collins (2) has considered in connexion with scalar diffraction of axisymmetric waves by two hard coaxial circular discs.

4. *Determination of the constants, C_1 and C_2.* In order to evaluate the constants, C_1 and C_2, it is necessary to examine the boundary conditions that the electromagnetic field must satisfy at the edges of the discs. The diffracted field is necessarily infinite along the edge of each disc. However, the diffracted energy flow across the surface of a torus that just encloses the rim of a disc must be zero. This situation exists when the singularities of the electric and magnetic fields of the diffracted wave are at most $O(\rho^{-\frac{1}{2}})$ as ρ tends to zero, where ρ is the distance from the field point to the edge. It has been shown, (4), (5), that this statement is equivalent to the following edge conditions:

(i) the component of the current density that is normal to the tangent to the edge of the disc is $O(\rho^{\frac{1}{2}})$ as ρ tends to zero,

(ii) the component of the current density that is tangential to the edge is $O(\rho^{-\frac{1}{2}})$ as ρ tends to zero. The application of these edge conditions ensures a unique solution to the electromagnetic diffraction problem.

The current density, $\mathbf{j}$, is

$$\mathbf{j} = (j_r, j_\phi, j_z) = \frac{c}{4\pi}\left[\mathbf{i}\times\mathbf{H}^{(2)}\Big|_{z=+} - \mathbf{i}\times\mathbf{H}^{(2)}\Big|_{z=-}\right],$$

where $\mathbf{i}$ is a unit vector along the z-axis. It is obvious that the z-component of the current density is zero; hence the first edge condition is satisfied if $j_r = O(\rho^{\frac{1}{2}})$ as $\rho \to 0$. Now

$$j_r = \frac{c}{4\pi}\left(H_\phi^{(2)}\Big|_{z=-} - H_\phi^{(2)}\Big|_{z=+}\right).$$

From equations (2·3), (3·2), (3·3), (3·10) and (3·11) it follows that

$$j_r = \begin{cases} -\dfrac{c}{2\pi}\left(k^2 b_0(r) + \dfrac{1}{r}\dfrac{d}{dr}b_0(r) - \dfrac{1}{r}g_1(r)\right)\cos\phi & \text{on} \quad \Sigma_1, \\ -\dfrac{c}{2\pi}\left(k^2 d_0(r) + \dfrac{1}{r}\dfrac{d}{dr}d_0(r) - \dfrac{1}{r}h_1(r)\right)\cos\phi & \text{on} \quad \Sigma_2. \end{cases}$$

It can be shown that

$$b_0(r) = -(2/\pi a)\,S^{(0)}(1)\,(a^2-r^2)^{\frac{1}{2}} + O\{(a^2-r^2)^{\frac{3}{2}}\} \quad \text{as} \quad r \to a$$

and

$$g_1(r) = (2r/\pi a)\,W^{(1)}(1)\,(a^2-r^2)^{-\frac{1}{2}} + O\{(a^2-r^2)^{\frac{1}{2}}\} \quad \text{as} \quad r \to a.$$

Similar expressions exist for $d_0(r)$ and $h_1(r)$. Therefore the first edge condition requires that

$$W^{(1)}(1) = S^{(0)}(1) \quad \text{and} \quad X^{(1)}(1) = T^{(0)}(1). \tag{4·1}$$

The above conditions ensure that the second edge condition is satisfied also, and consequently equation (4·1) enables C_1 and C_2 to be found.

5. *Scattering coefficient.* The scattering coefficient, ν, is defined as the ratio of the average value (over a period) of the electromagnetic power diffracted by the discs to the average power incident on the discs, i.e.

$$\nu = P_2/P_1,$$

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where $$P_1 = \frac{c}{8\pi}\Re\int_{\Sigma_1+\Sigma_2}(\mathbf{E}^{(1)}\times\mathbf{H}^{(1)*}).\mathbf{i}\,dS$$

and $$P_2 = \frac{c}{8\pi}\Re\int_{\Sigma_3}(\mathbf{E}^{(2)}\times\mathbf{H}^{(2)*}).\mathbf{dS};$$

the asterisk denotes a complex conjugate and Σ_3' is any union of closed surfaces that encloses the two discs.

When the incident field is defined by equation (2·1) it is easily calculated that

$$P_1 = cE_0^2(a^2+b^2)/8.$$

If Σ_3 is taken to be the sphere at infinity,

$$P_2 = \frac{c}{8\pi}\Re\int_0^{2\pi}\int_0^{\pi}(E_\theta^{(2)}H_\phi^{(2)*} - E_\phi^{(2)}H_\theta^{(2)*})\,R^2\sin\theta\,d\theta\,d\phi. \tag{5·1}$$

In the limit as R tends to infinity

$$H_\theta^{(2)} = -E_\phi^{(2)} = k^2\pi_\theta^{(2)} + O(R^{-2})$$

and $$H_\phi^{(2)} = E_\theta^{(2)} = k^2\pi_\phi^{(2)} + O(R^{-2}),$$

where $$\pi_\theta^{(2)} = \Phi(R\sin\theta, R\cos\theta)\cos\theta\sin\phi - \Psi(R\sin\theta, R\cos\theta)\sin\theta\sin\phi$$

and $$\pi_\phi^{(2)} = \Phi(R\sin\theta, R\cos\theta)\cos\phi.$$

From equations (3·1) and (3·9) it can be shown that

$$\Phi(R\sin\theta, R\cos\theta) = ik\cos\theta\left\{\int_0^a b_0(t)\,tJ_0(kt\sin\theta)\,dt + \exp(ikL\cos\theta)\int_0^b d_0(t)\,tJ_0(kt\sin\theta)\,dt\right\}\frac{\exp(-ikR)}{R} + O(R^{-2}) \quad \text{as} \quad R\to\infty$$

and

$$\Psi(R\sin\theta, R\cos\theta) = i\left\{\int_0^a g_1(t)\,tJ_1(kt\sin\theta)\,dt + \exp(ikL\cos\theta)\int_0^b h_1(t)\,tJ_1(kt\sin\theta)\,dt\right\}\frac{\exp(-ikR)}{R} + O(R^{-2}) \quad \text{as} \quad R\to\infty.$$

Sonine's second finite integral and some manipulation involving the interchange in the order of integration reduce these far field expressions for Φ and Ψ to more convenient forms: the final result is

$$\Phi(R\sin\theta, R\cos\theta) = -\frac{2i}{\pi}F(a,b,\cos\theta)\frac{\exp(-ikR)}{R} + O(R^{-2}) \quad \text{as} \quad R\to\infty$$

and

$$\Psi(R\sin\theta, R\cos\theta) = \frac{2i}{\pi}\tan\theta\,G(a,b,\cos\theta)\frac{\exp(-ikR)}{R} + O(R^{-2}) \quad \text{as} \quad R\to\infty,$$

where $$F(a,b,\mu) = a\int_0^1 S^{(0)}(t)\sinh\alpha\mu t\,dt + b\exp(ikL\mu)\int_0^1 T^{(0)}(t)\sinh\beta\mu t\,dt$$

and $$G(a,b,\mu) = a\int_0^1 W^{(1)}(t)\sinh\alpha\mu t\,dt + b\exp(ikL\mu)\int_0^1 X^{(1)}(t)\sinh\beta\mu t\,dt.$$

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When the integration with respect to ϕ in equation (5·1) has been performed it is found that

$$P_2 = \frac{ck^4}{2\pi^2}\int_{-1}^{1}\left\{|F(a,b,\mu)|^2+\left|\mu F(a,b,\mu)+\left(\frac{1-\mu^2}{\mu}\right)G(a,b,\mu)\right|^2\right\}d\mu.$$

Instead of taking Σ_3 to be the sphere at infinity it is possible to let Σ_3 be the union of two closed surfaces each of which encloses one of the two discs. A similar method of determining P_2 has been employed by Lebedev and Skal'skaya (8) in the case of electromagnetic diffraction by a single perfectly conducting circular disc. A different expression for P_2 is obtained when this method is used; namely,

$$P_2 = \frac{cE_0 k}{\pi}\Re\{F(a,b,1)\}.$$

An alternative way of calculating P_2 would be to use a theorem derived by Jones (6). When this approach is adopted the second formula for P_2 is rederived.

Hence the three methods of obtaining the scattering coefficient have produced two different formulae, namely,

$$\nu = \frac{4k^4}{\pi^2 E_0^2(a^2+b^2)}\int_{-1}^{1}\left\{|F(a,b,\mu)|^2+\left|\mu F(a,b,\mu)+\left(\frac{1-\mu^2}{\mu}\right)G(a,b,\mu)\right|^2\right\}d\mu$$

and

$$\nu = \frac{8k}{\pi E_0(a^2+b^2)}\Re\{F(a,b,1)\}.$$

One formula has little merit over the other. The amount of energy that has to be expended to calculate an expression for the scattering coefficient using one formula is roughly the same as that using the other formula. However, the first formula has a marginal advantage over the second because less accurate solutions of the Fredholm integral equations are required when the first formula is employed to determine ν to any given accuracy.

6. *Two equal discs.* In the special case of two equal discs the simultaneous Fredholm integral equations (3·7) and (3·8) can be replaced by two ordinary Fredholm integral equations of the second kind; namely,

$$X^{\pm}(r) = \sinh\alpha r - \int_0^1 X^{\pm}(t)\{L_1^{(1)}(t,r,\alpha)+L_2^{(1)}(t,r,\alpha)\pm L_3^{(1)}(t,r:a,a)\}dt \quad (0<r<1), \quad (6\cdot1)$$

where

$$(2/k)(C_1\pm C_2)X^{\pm}(r) = W^{(1)}(r)\pm X^{(1)}(r)\Big|_{b=a}.$$

Similarly, equations (3·16) and (3·17) can be transformed to

$$T^{\pm}(r) = \sinh\alpha r + \int_0^1 T^{\pm}(t)\{M_2^{(0)}(t,r,\alpha)\mp M_3^{(0)}(t,r:a,a)\}dt \quad (0<r<1), \quad (6\cdot2)$$

where

$$\frac{1}{k}\left[\left\{C_1+\frac{iE_0}{k}\right\}\pm\left\{C_2+\frac{iE_0}{k}\exp(-ikL)\right\}\right]T^{\pm}(r) = S^{(0)}(r)\pm T^{(0)}(r)\Big|_{b=a}.$$

The kernels of the above two pairs of integral equations are small when the incident wave has a low frequency and the distance between the two discs is sufficiently large, that is, when α is small and $\alpha\Lambda$ is of the order of unity. (It can, in fact, be shown that

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each kernel is expressible as a power series in α and vanishes when α is zero. $L_1^{(1)}$ can be evaluated immediately; it is a particular case of the Weber–Shafheitlin integral. The other kernels are a little more complicated. However, the use of the series expansion for a product of Bessel functions and some elementary manipulation enable them to be written in the required form.) Hence, iterative solutions can be obtained for equations (6·1) and (6·2). These solutions are

(i) $$X^{\pm}(r) = \left\{\alpha r - \alpha^3\left(\frac{r}{2} - \frac{r^3}{3}\right) + \alpha^5\left(\frac{r}{6} - \frac{r^3}{12} + \frac{r^5}{40}\right)\right\} + \frac{i}{\pi}\left\{\frac{4}{9}\alpha^4 r - \alpha^6\left(\frac{26r}{75} - \frac{4r^3}{45}\right)\right\}$$
$$\mp \frac{i}{\pi}\exp(-i\alpha\Lambda)\left[\frac{\alpha^4}{3} r p_0 + \alpha^6\left\{r\left(\frac{p_1}{5} - \frac{4p_0}{15}\right) + r^3\left(\frac{p_0}{18} + \frac{p_1}{3}\right)\right\}\right] + O(\alpha^7),$$

where $$p_0 = \frac{4}{(\alpha\Lambda)^2} - \frac{4i}{(\alpha\Lambda)^3}$$

and $$p_1 = \frac{2}{3(\alpha\Lambda)^2} - \frac{10i}{3(\alpha\Lambda)^3} - \frac{8}{(\alpha\Lambda)^4} + \frac{8i}{(\alpha\Lambda)^5},$$

(ii) $$T^{\pm}(r) = \left(\alpha r + \frac{\alpha^3}{6} r^3 + \frac{\alpha^5}{120} r^5\right) - \frac{i}{\pi}\left\{\frac{2\alpha^4}{9} r + \alpha^6\left(\frac{8r}{225} + \frac{r^3}{45}\right)\right\}$$
$$\mp \frac{i}{\pi}\exp(-i\alpha\Lambda)\left[\frac{\alpha^4}{3} r q_0 + \alpha^6\left\{r\left(\frac{q_0}{30} + \frac{q_1}{5}\right) + r^3\frac{q_1}{3}\right\}\right] + O(\alpha^7),$$

where $$q_0 = \frac{2i}{\alpha\Lambda} + \frac{4}{(\alpha\Lambda)^2} - \frac{4i}{(\alpha\Lambda)^3}$$

and $$q_1 = \frac{i}{3\alpha\Lambda} + \frac{4}{3(\alpha\Lambda)^2} - \frac{4i}{(\alpha\Lambda)^3} - \frac{8}{(\alpha\Lambda)^4} + \frac{8i}{(\alpha\Lambda)^5}.$$

It is assumed that α and Λ^{-1} are of the same order of magnitude.

Collins(2) has shown that the solutions of equation (6·2) govern the problem of scalar diffraction of a normally incident plane wave by two hard equal coaxial circular discs.

When the edge conditions are applied to the solutions of the ordinary Fredholm integral equations (6·1) and (6·2) it is possible to determine the values that the constants, C_1 and C_2, must take: it is found that

$$C_1 \pm C_2 = \frac{iE_0 T^{\pm}(1)\{1 \pm \exp(-i\alpha\Lambda)\}}{k\{2X^{\pm}(1) - T^{\pm}(1)\}}.$$

Then the expressions for the scattering coefficient simplify to

$$\nu = \frac{2k^4}{\pi^2 E_0^2 a^2}\int_{-1}^{1}\left\{|F(a,a,\mu)|^2 + \left|\mu F(a,a,\mu) + \left(\frac{1-\mu^2}{\mu}\right) G(a,a,\mu)\right|^2\right\} d\mu$$

and $$\nu = \frac{4k}{\pi E_0 a^2}\,\Re\{F(a,a,1)\},$$

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where $F(a, a, \mu)$ and $G(a, a, \mu)$ are given by

$$F(a,a,\mu) = \frac{iE_0}{k^2}\{1+\exp(-i\alpha\Lambda)\}\{1+\exp(i\alpha\Lambda\mu)\}\left\{\frac{X^+(1)}{2X^+(1)-T^+(1)}\right\}\int_0^1 T^+(t)\sinh\alpha t\mu\,dt$$
$$+\frac{iE_0}{k^2}\{1-\exp(-i\alpha\Lambda)\}\{1-\exp(i\alpha\Lambda\mu)\}\left\{\frac{X^-(1)}{2X^-(1)-T^-(1)}\right\}\int_0^1 T^-(t)\sinh\alpha t\mu\,dt$$

and

$$G(a,a,\mu) = \frac{iE_0}{k^2}\{1+\exp(-i\alpha\Lambda)\}\{1+\exp(i\alpha\Lambda\mu)\}\left\{\frac{T^+(1)}{2X^+(1)-T^+(1)}\right\}\int_0^1 X^+(t)\sinh\alpha t\mu\,dt$$
$$+\frac{iE_0}{k^2}\{1-\exp(-i\alpha\Lambda)\}\{1-\exp(i\alpha\Lambda\mu)\}\left\{\frac{T^-(1)}{2X^-(1)-T^-(1)}\right\}\int_0^1 X^-(t)\sinh\alpha t\mu\,dt.$$

The first few terms in the expansion of the scattering coefficient in powers of α can be calculated from these results. Both formulae for ν give

$$\nu = \frac{128\alpha^4}{27\pi^2}\left(1+\frac{22\alpha^2}{25}\right)+\frac{64\alpha^4}{9\pi^2}\left[\frac{1}{(\alpha\Lambda)^2}\cos\alpha\Lambda+\left\{\frac{1}{\alpha\Lambda}-\frac{1}{(\alpha\Lambda)^3}\right\}\sin\alpha\Lambda\right]\cos\alpha\Lambda$$
$$+\frac{256\alpha^6}{15\pi^2}\left[\left\{\frac{2}{3(\alpha\Lambda)^2}-\frac{1}{(\alpha\Lambda)^4}\right\}\cos\alpha\Lambda+\left\{\frac{4}{9\alpha\Lambda}-\frac{1}{(\alpha\Lambda)^3}+\frac{1}{(\alpha\Lambda)^5}\right\}\sin\alpha\Lambda\right]\cos\alpha\Lambda+O(\alpha^8).$$

In the limit as Λ tends to infinity the above approximation to ν equals the first two terms in the known expansion of the scattering coefficient for a single disc.

7. *Observations.* It is possible to show that the electromagnetic diffraction problem that is discussed in the present paper can depend on different Fredholm integral equations from those that are presented. Instead of $K_n(t, r, 0)$ being split up as shown in this paper, $K_n(t, r, 0)$ can be written as the sum of a static kernel and a perturbation kernel. This is essentially the approach of Lebedev and Skal'skaya (8). However, when this alternative method is used the kernels of the Fredholm integral equations are more complicated than those that arise in the above analysis. The relationship between the two sets of integral equations is roughly that which exists between the integral equations derived by Jones (7) and by Magnus (9) for the problem of scalar diffraction of an axisymmetric wave by a hard circular disc.

It must be pointed out that we have made some assumptions about the form of the Hertz vector corresponding to the diffracted field and about the conditions on Φ and Ψ given in equation (2·4). However, it is important to stress that the form of the Hertz vector and the conditions on Φ and Ψ that we have selected are possible ones, but that they are not the most general. Other authors, (8) and (17), have found it necessary to make similar assumptions in their solutions of related boundary-value problems. Unfortunately, it is extremely difficult to present the deductive processes that lead one to the particular forms selected. Nevertheless, in view of the uniqueness of the solution of the electromagnetic diffraction problem (5), these assumptions are justifiable provided that we can show that they lead to a solution satisfying all the required conditions.

The above assumptions have enabled the electromagnetic diffraction problem to be reduced to the solution of two independent scalar problems. Each of these scalar

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problems involves two arbitrary constants, C_1 and C_2, and for given values of C_1 and C_2 each scalar problem has a unique solution (1). In section 3 the two scalar problems have been considered in turn and each has been formulated as two Fredholm integral equations of the first kind, which have then been reduced to two Fredholm integral equations of the second kind. It can be shown, by arguments similar to those of Collins (reference 5 in (2)), that for given values of C_1 and C_2 the latter integral equations possess unique solutions. Finally, it has been deduced in section 4 that C_1 and C_2 can be determined uniquely so that the edge conditions that are necessary for the uniqueness of the electromagnetic problem (5) are satisfied. Now, all the required conditions have been satisfied and, consequently, the solution that we have obtained has been shown to be the unique solution of the electromagnetic diffraction problem.

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APPENDIX

A general procedure of reducing a certain class of boundary-value problems to Fredholm integral equations of the second kind has been derived by Williams (15). An extension of this method is now presented.

Many boundary-value problems can be reduced to the solution of an integral equation of the form

$$e_1(r)\int_0^r K(t,r)\,e_2(t)\int_t^a g(w)\,e_3(w)\,K(t,w)\,dw\,dt = f(r) - \int_0^a G(t,r)\,g(t)\,dt - \sum_{s=0}^{\infty}\int_0^{b_s} H_s(t,r)\,j_s(t)\,dt \quad (0<r<a). \quad (A\,1)$$

It is assumed that $g(t)$ and $j_s(t)$ are the only unknown functions and that K is such that

$$\int_0^r K(t,r)\,g(t)\,dt = f(r) \quad (0<r<a); \quad \int_0^r K(t,r)\,g(t)\,dt = f(r) \quad (0<r<b_s);$$

$$\int_r^a K(r,t)\,g(t)\,dt = f(r) \quad (0<r<a); \quad \int_r^{b_s} K(r,t)\,g(t)\,dt = f(r) \quad (0<r<b_s);$$

possess explicit unique solutions for g in terms of f for all arbitrary differentiable functions f.

If functions $L(u,v)$, $n_s(t)$ and $m_s(v)\,M_s(u,v)$ are now defined by the relationships

$$G(t,r) = e_1(r)\,e_3(t)\int_0^r K(u,r)\int_0^t L(u,v)\,K(v,t)\,dv\,du$$

and

$$H_s(t,r) = e_1(r)\,n_s(t)\int_0^r K(u,r)\int_0^t m_s(v)\,M_s(u,v)\,K(v,t)\,dv\,du,$$

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then it follows, by some manipulation similar to that of (15), that equation (A 1) reduces to

$$e_2(u)\,S(u) = f_0(u) - \int_0^a S(v)\,L(u,v)\,dv - \sum_{s=0}^{\infty} \int_0^{b_s} T_s(v)\,M_s(u,v)\,dv \quad (0 < u < a),$$

where

$$S(v) = \int_v^a e_3(t)\,g(t)\,K(v,t)\,dt$$

$$T_s(v) = m_s(v) \int_v^{b_s} n_s(t)\,j_s(t)\,K(v,t)\,dt$$

and

$$f(r) = e_1(r) \int_0^r f_0(u)\,K(u,r)\,du.$$

$m_s(v)$ is an arbitrary function which is chosen so that $T_s(v)$ takes an appropriate form.

As an example of this procedure the following integral equation will be considered

$$\frac{2}{\pi r^n} \int_0^r t^{2n} \frac{\cos k(r^2-t^2)^{\frac{1}{2}}}{(r^2-t^2)^{\frac{1}{2}}} \int_t^a w^{1-n} g^{(n)}(w) \frac{\cos k(w^2-t^2)^{\frac{1}{2}}}{(w^2-t^2)^{\frac{1}{2}}}\,dw\,dt$$

$$= f(r) - \int_0^a t g^{(n)}(t)\,G_n(t,r)\,dt - \int_0^b t h^{(n)}(t)\,H_n(t,r)\,dt \quad (0 < r < a), \qquad \text{(A 2)}$$

where

$$G_n(t,r) = \int_0^{\infty} \left\{1 - \frac{\lambda^{2n}}{(\lambda^2+k^2)^n}\right\} J_n\{r(k^2+\lambda^2)^{\frac{1}{2}}\}\,J_n\{t(k^2+\lambda^2)^{\frac{1}{2}}\}\,d\lambda$$

$$- i \int_0^k J_n\{r(k^2-\lambda^2)^{\frac{1}{2}}\}\,J_n\{t(k^2-\lambda^2)^{\frac{1}{2}}\}\,d\lambda$$

and

$$H_n(t,r) = \int_0^{\infty} Q_n(p)\,J_n(tp)\,J_n(rp)\,dp.$$

Q_n and f are known functions and $g^{(n)}$ and $h^{(n)}$ are unknown functions. An equation of this form arises in the problem of scalar diffraction by two soft coaxial circular discs. The above technique reduces equation (A 2) to

$$\frac{2}{\pi} S_n(u) = f_0(u) - \int_0^a S_n(v)\,L(u,v)\,dv - \int_0^b T_n(v)\,M(u,v)\,dv \quad (0 < u < a),$$

where

$$S_n(u) = u^n \int_u^a t^{1-n} g^{(n)}(t) \frac{\cos k(t^2-u^2)^{\frac{1}{2}}}{(t^2-u^2)^{\frac{1}{2}}}\,dt,$$

$$T_n(v) = v^n \int_v^b t n_n(t)\,h^{(n)}(t) \frac{\cos k(t^2-v^2)^{\frac{1}{2}}}{(t^2-v^2)^{\frac{1}{2}}}\,dt$$

and

$$f_0(u) = \frac{2}{\pi u^n} \frac{d}{du} \int_0^u t^{n+1} f(t) \frac{\cosh k(u^2-t^2)^{\frac{1}{2}}}{(u^2-t^2)^{\frac{1}{2}}}\,dt.$$

$m_n(v)$ has been set equal to unity. The function $n_n(t)$ and the kernels $L(u,v)$ and $M(u,v)$ are defined by

$$G_n(t,r) = \frac{1}{(rt)^n} \int_0^r u^n \frac{\cos k(r^2-u^2)^{\frac{1}{2}}}{(r^2-u^2)^{\frac{1}{2}}} \int_0^t L(u,v)\,v^n \frac{\cos k(t^2-v^2)^{\frac{1}{2}}}{(t^2-v^2)^{\frac{1}{2}}}\,dv\,du$$

and

$$H_n(t,r) = \frac{n_n(t)}{r^n} \int_0^r u^n \frac{\cos k(r^2-u^2)^{\frac{1}{2}}}{(r^2-u^2)^{\frac{1}{2}}} \int_0^t M(u,v)\,v^n \frac{\cos k(t^2-v^2)^{\frac{1}{2}}}{(t^2-v^2)^{\frac{1}{2}}}\,dv\,du.$$

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The following three particular cases of Sonine's second finite integral ((14), page 376) are used to factorize $G_n(t,r)$ and $H_n(t,r)$ in the required way

$$J_n\{t(k^2+\lambda^2)^{\frac{1}{2}}\} = \left\{\frac{2\lambda}{\pi}(k^2+\lambda^2)^n\right\}^{\frac{1}{2}} \frac{1}{(t\lambda)^n}\int_0^t v^{n+\frac{1}{2}}J_{n-\frac{1}{2}}(\lambda v)\frac{\cos k(t^2-v^2)^{\frac{1}{2}}}{(t^2-v^2)^{\frac{1}{2}}}\,dv,$$

$$J_n\{t(k^2-\lambda^2)^{\frac{1}{2}}\} = \left\{\frac{2\lambda}{\pi}(k^2-\lambda^2)^n\right\}^{\frac{1}{2}} \frac{1}{(t\lambda)^n}\int_0^t v^{n+\frac{1}{2}}I_{n-\frac{1}{2}}(\lambda v)\frac{\cos k(t^2-v^2)^{\frac{1}{2}}}{(t^2-v^2)^{\frac{1}{2}}}\,dv$$

and

$$J_n(tp) = \left(\frac{2\gamma}{\pi}\right)^{\frac{1}{2}}\left(\frac{p}{r\gamma}\right)^n\int_0^t v^{n+\frac{1}{2}}J_{n-\frac{1}{2}}(\gamma v)\frac{\cos k(t^2-v^2)^{\frac{1}{2}}}{(t^2-v^2)^{\frac{1}{2}}}\,dv.$$

Hence

$$L(u,v) = \frac{2}{\pi}(uv)^{\frac{1}{2}}\int_0^\infty\{(\lambda^2+k^2)^n-\lambda^{2n}\}\frac{J_{n-\frac{1}{2}}(\lambda v)\,J_{n-\frac{1}{2}}(\lambda u)}{\lambda^{2n-1}}\,d\lambda$$

$$-i\frac{2}{\pi}(uv)^{\frac{1}{2}}\int_0^k(k^2-\lambda^2)^n\frac{I_{n-\frac{1}{2}}(\lambda v)\,I_{n-\frac{1}{2}}(\lambda u)}{\lambda^{2n-1}}\,d\lambda,$$

$$M(u,v) = \frac{2}{\pi}(uv)^{\frac{1}{2}}\int_0^\infty\frac{p^{2n}}{\gamma^{2n-1}}Q_n(p)\,J_{n-\frac{1}{2}}(\gamma v)\,J_{n-\frac{1}{2}}(\gamma u)\,dp$$

and $n_n(t) = t^{-n}$.

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