Ergod. Th. & Dynam. Sys. (2020), **40**, 751–762 doi:10.1017/etds.2018.56
© *Cambridge University Press, 2018*

Entropy ratio for infinite sequences with positive entropy

CHRISTIAN MAUDUIT† and CARLOS GUSTAVO MOREIRA‡

† *Université d'Aix-Marseille et Institut Universitaire de France, Institut de Mathématiques de Marseille, UMR 7373 CNRS, 163, avenue de Luminy, 13288 Marseille Cedex 9, France*

‡ *Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110, 22460-320 Rio de Janeiro, RJ, Brazil*

(*Received* 5 *September* 2017 *and accepted in revised form* 20 *June* 2018)

Abstract. The complexity function of an infinite word w on a finite alphabet A is the sequence counting, for each non-negative n, the number of words of length n on the alphabet A that are factors of the infinite word w. For any given function f with exponential growth, we introduced in [Complexity and fractal dimensions for infinite sequences with positive entropy. *Commun. Contemp. Math.* to appear] the notion of *word entropy* $E_W(f)$ associated to f and we described the combinatorial structure of sets of infinite words with a complexity function bounded by f. The goal of this work is to give estimates on the word entropy $E_W(f)$ in terms of the limiting lower exponential growth rate of f.

1. *Notation*

We denote by $\mathbb{N}$ the set of non-negative integers, by q a fixed integer greater than or equal to 2, by A the finite alphabet $A = \{0, 1, \ldots, q-1\}$, by $A^* = \bigcup_{n\geq 0} A^n$ the set of finite words on the alphabet A and by $A^{\mathbb{N}}$ the set of infinite words (or infinite sequences of letters) on the alphabet A. More generally, if $\Sigma \subset A^*$, we denote by $\Sigma^{\mathbb{N}}$ the set of infinite words obtained by concatenating elements of Σ. We denote by T the one-sided shift map defined on $A^* \cup A^{\mathbb{N}}$ and which deletes the first letter of a word. If $n \in \mathbb{N}$ and $v \in A^n$, we denote by $|v| = n$ the length of the word v, and if S is a finite set, we denote by $|S|$ the number of elements of S. If $w \in A^{\mathbb{N}}$ we denote by $L(w)$ the set of finite factors of w,

$$L(w) = \{v \in A^*, \exists\, (v', v'') \in A^* \times A^{\mathbb{N}},\ w = v'vv''\}$$

and, for any non-negative integer n, we write $L_n(w) = L(w) \cap A^n$. For any $n \in \mathbb{N}$, we say that $v \in L_n(w)$ is a special factor (of length n) of w if v is the prefix of more than one

https://doi.org/10.1017/etds.2018.56 Published online by Cambridge University Press

word in $L_{n+1}(w)$ (see [**CN10**] for a detailed study of these notions). If x is a real number, we write $\lfloor x \rfloor = \max\{n \in \mathbb{Z},\, n \leq x\}$, $\lceil x \rceil = \min\{n \in \mathbb{Z},\, x \leq n\}$ and $\{x\} = x - \lfloor x \rfloor$.

Let us recall the following classical lemma concerning sub-additive sequences due to Fekete [**Fek23**].

LEMMA 1.1. *If $(a_n)_{n\geq 1}$ is a sequence of real numbers such that $a_{n+n'} \leq a_n + a_{n'}$ for any positive integers n and n', then the sequence $(a_n/n)_{n\geq 1}$ converges to* $\inf_{n\geq 1} (a_n/n)$.

Definition 1.2. The *complexity function* of $w \in A^{\mathbb{N}}$ is defined for any non-negative integer n by $p_w(n) = |L_n(w)|$.

For any $w \in A^{\mathbb{N}}$ and for any $(n, n') \in \mathbb{N}^2$, we have $L_{n+n'}(w) \subset L_n(w)L_{n'}(w)$, so that

$$p_w(n + n') \leq p_w(n)p_w(n'),$$

and it follows then from Lemma 1.1 applied to $a_n = \log p_w(n)$ that, for any $w \in A^{\mathbb{N}}$, the sequence $((1/n) \log p_w(n))_{n\geq 1}$ converges to $\inf_{n\geq 1} (1/n) \log p_w(n)$.

We denote by

$$E(w) = \lim_{n\to\infty} \frac{1}{n} \log p_w(n) = h_{\text{top}}(X(w), T)$$

the topological entropy of the symbolic dynamical system $(X(w), T)$, where $X(w) = \overline{orb_T(w)}$ is the closure of the orbit of w under the action of T in $A^{\mathbb{N}}$ (see, for example, [**Fer99**] or [**PF02**] for a detailed study of the notions of complexity function and topological entropy).

If f is a function from $\mathbb{N}$ to $\mathbb{R}^+$, we consider the set

$$W(f) = \{w \in A^{\mathbb{N}},\, p_w(n) \leq f(n),\, \forall n \in \mathbb{N}\}$$

and we write

$$\mathcal{L}_n(f) = \bigcup_{w\in W(f)} L_n(w).$$

For any $(n, n') \in \mathbb{N}^2$, we have $\mathcal{L}_{n+n'}(f) \subset \mathcal{L}_n(f)\mathcal{L}_{n'}(f)$, so that we can deduce from Lemma 1.1 applied to $a_n = \log |\mathcal{L}_n(f)|$ that the sequence $((1/n) \log |\mathcal{L}_n(f)|)_{n\geq 1}$ converges to $\inf_{n\geq 1} (1/n) \log |\mathcal{L}_n(f)|$, which is the topological entropy of the subshift $(W(f), T)$:

$$h_{\text{top}}(W(f), T) = \lim_{n\to+\infty} \frac{1}{n} \log |\mathcal{L}_n(f)| = \inf_{n\geq 1} \frac{1}{n} \log |\mathcal{L}_n(f)|.$$

We denote by $E_0(f)$ the limiting lower exponential growth rate of f:

$$E_0(f) = \lim_{n\to\infty} \inf \frac{1}{n} \log f(n).$$

2. *Presentation of the results*

Our work concerns the study of infinite sequences w the complexity function of which is bounded by a given function f from $\mathbb{N}$ to $\mathbb{R}^+$. We studied in [**MM10**, **MM12**] the case $E_0(f) = 0$ and we considered in [**MM17**] the case of positive entropy, for which few results were known since the work of Grillenberger [**Gri73**]. We defined in [**MM17**] the notion of w-entropy (or word-entropy) of f as follows.

https://doi.org/10.1017/etds.2018.56 Published online by Cambridge University Press

Definition 2.1. If f is a function from $\mathbb{N}$ to $\mathbb{R}^+$, the *w-entropy* (or *word entropy*) of f is the quantity

$$E_W(f) = \sup_{w \in W(f)} E(w).$$

We gave in [**MM17**] a combinatorial proof of the fact that $E_W(f)$ is equal to the topological entropy of the subshift $(W(f), T)$ (notice that this can be also obtained as a consequence of the variational principle) and that it allows us to compute exactly the fractal dimensions of the set of real numbers from the interval [0, 1] the $q-$adic expansion of which has a complexity function bounded by f.

This paper is devoted to the study of the properties of the w-entropy E_W and its relations with the limiting lower exponential growth rate E_0. Infinite words whose complexity function has an exponential growth but low initial values play a special role in this study and we define the following important class of infinite words on the alphabet {0, 1} for which we provide in §3 a useful renormalization theorem (Theorem 3.2).

Definition 2.2. We say that $w \in \{0, 1\}^{\mathbb{N}}$ is a *pre-sturmian* infinite word of order $k \in \mathbb{N}$ if w is not ultimately periodic and if, for any non-negative integer $n \leq k$, we have $p_w(n) = n + 1$.

Remark 2.3. It follows from a classical result due to Coven and Hedlund [**CH73**] that non-ultimately periodic infinite words w with lowest possible complexity function p_w are those for which $p_w(n) = n + 1$ for any non-negative integer n. Such infinite words, called sturmian words, have been extensively studied since their introduction by Hedlund and Morse in [**HM38**, **HM40**] (see [**Lot02**, Ch. 2] and [**PF02**, Ch. 6] for surveys on sturmian words). It follows from Definition 3 that a sturmian word is a pre-sturmian word of any order.

Definition 2.4. If f is a function from $\mathbb{N}$ to $\mathbb{R}^+$, we call the *entropy ratio* of f the quantity

$$\rho(f) = \frac{E_W(f)}{E_0(f)}.$$

It follows from the definitions of E_0 and E_W that we always have $\rho(f) \leq 1$ and it is easy to give examples of the function f for which $\rho(f)$ can be made arbitrarily small (see the beginning of [**MM17**, §7]). On the other hand, if f is indeed a complexity function (i.e. $f = p_w$ for some $w \in A^{\mathbb{N}}$), then we clearly have $\rho(f) = 1$. But it seems difficult to find a set of simple conditions on f which hold for complexity functions and imply $\rho(f) = 1$ (see [**MM17**, Problem 2.5]).

We will suppose in this paper that functions f satisfy the following quite natural conditions $(\mathcal{C}^*)$ which hold for all unbounded complexity functions.

Definition 2.5. We say that a function f from $\mathbb{N}$ to $\mathbb{R}^+$ satisfies conditions $(\mathcal{C}^*)$ if:

(i) for any $n \in \mathbb{N}$, we have $f(n + 1) > f(n) \geq n + 1$;

(ii) for any $(n, n') \in \mathbb{N}^2$, we have $f(n + n') \leq f(n) f(n')$.

Remark 2.6. If there exists $n \in \mathbb{N}$ such that $f(n) \leq n$, then any $w \in A^{\mathbb{N}}$ such that $p_w \leq f$ is ultimately periodic, so that $W(f)$ is finite.

https://doi.org/10.1017/etds.2018.56 Published online by Cambridge University Press

Remark 2.7. Given any function f from $\mathbb{N}$ to $\mathbb{R}^+$ such that $f(n) \geq n+1$ for any $n \in \mathbb{N}$, it is possible to construct recursively a non-increasing, integer-valued function $\tilde{f}$ satisfying condition ($\mathcal{C}^*$(ii)) and a real-valued function $\tilde{\tilde{f}}$ satisfying conditions ($\mathcal{C}^*$) such that $W(f) = W(\tilde{f}) = W(\tilde{\tilde{f}})$ and such that $E_0(f) > 0$ implies $E_0(\tilde{f}) = E_0(\tilde{\tilde{f}}) > 0$ (see [**MM17**, Remark 7.3]).

Remark 2.8. If a function f from $\mathbb{N}$ to $\mathbb{R}^+$ satisfies conditions ($\mathcal{C}^*$) then, for any $n \in \mathbb{N}$, we have $f(n) \geq \max\{n+1, \exp(E_0(f)n)\}$ (see [**MM17**, Lemma 7.4]).

In [**MM17**] we showed that, even when the function f satisfies conditions ($\mathcal{C}^*$), it might happen that $\rho(f) < 1$. The main goal of this paper is to answer [**MM17**, Problem 6.5] by showing the following theorem.

THEOREM 2.9. *We have*

$$\inf\{\rho(f),\ f \text{ satisfies } (\mathcal{C}^*)\} = \tfrac{1}{2}.$$

The proof of Theorem 2.9 will follow from §4 where we prove that if f satisfies the conditions ($\mathcal{C}^*$), we always have $\rho(f) > \frac{1}{2}$ (Theorem 4.2), and from §5 where we prove that $\frac{1}{2}$ is the best possible constant (Theorem 5.1). The proof of Theorem 5.1 is based on the renormalization theorem concerning pre-sturmian infinite words given in §3 (Theorem 3.2).

Remark 2.10. It follows from Remark 2.8 that, in order to prove Theorem 4.2, it is enough to consider, for any $h > 0$, the functions f_h defined, for any $n \in \mathbb{N}$, by

$$f_h(n) = \max\{n+1, \exp(hn)\}. \tag{1}$$

In order to prove Theorem 5.1, we will show that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $h \in]0, \delta[$, we have $\rho(f_h) < \frac{1}{2} + \varepsilon$.

3. *Renormalization of pre-sturmian infinite words of order k*

LEMMA 3.1. *If a and b are words on the alphabet $\{0, 1\}$ with distinct first letters and $|a| \geq |b|$, s is a positive integer and w is a non-ultimately periodic word that belongs to $\{a, ba^s\}^{\mathbb{N}}$ such that $p_w(k) = k+1$ and $(s+1)|a| + |b| \leq k$ for some $k \in \mathbb{N}$ then, if we consider w as an infinite word on the two-letter alphabet $\{A, B\}$ with $A = a$ and $B = ba^s$, the words AA and BB cannot both be factors of w (except in the first positions). More precisely, if $w = \gamma_1\gamma_2 \ldots \gamma_i \ldots$ with $\gamma_i \in \{A, B\}$ for any $i \geq 1$, then there are no indexes i, j with $1 < i < j$, $\gamma_i = \gamma_{i+1} = B$ and $\gamma_j = \gamma_{j+1} = A$.*

Proof. If we put $r = (s+1)|a| + |b| \leq k$, we have $p_w(r) = r+1$, so that w has only one special factor of length $r-1$. Let us show by contradiction that the words aba^s and $ba^sa = ba^{s+1}$ (of size r) cannot be both special factors of w. Suppose that both words aba^s and ba^{s+1} are special factors of w. Then the words $T(a)ba^s$ and $T(b)a^{s+1}$ are also special factors of w and since their length is equal to $r-1$ they are equal, which implies $T(ab) = T(a)b = T(b)a = T(ba)$. Since ab and ba contain the same number of 0s and their first letters are different, $T(ab)$ and $T(ba)$ contain a different number of 0s, which is a contradiction.

https://doi.org/10.1017/etds.2018.56 Published online by Cambridge University Press

Let us remark that a factor ba^s that is not at the beginning of w is necessarily preceded by a factor a (since ba^s ends with a). Consider now the two possible cases.

(1) If aba^s is not a special factor, then
- if aba^s is never followed by ba^s, the factor ba^sba^s never appears (except in the first positions), which implies that BB is not a factor of $w \in \{A, B\}^{\mathbb{N}}$;
- if aba^s is never followed by a, it is necessarily followed by ba^s, so that w ends with $ba^sba^sba^s \ldots$, that is, w is ultimately periodic, so that this case does not occur.

(2) If aba^s is a special factor, then $ba^sa = ba^{s+1}$ is not a special factor, and thus
- if $ba^sa = ba^{s+1}$ (i.e. BA) is never followed by a (i.e. by A), that is, BAA never occurs in the renormalization of w, then the factor AA may only occur before the first B, which implies that AA is not a factor of $w \in \{A, B\}^{\mathbb{N}}$ (except in the first positions);
- if $ba^sa = ba^{s+1}$ is never followed by ba^s, it is necessarily followed by a. Let u be the word formed by the $|ba^s| = s|a| + |b|$ last letters of a^{s+1} (recall that $|b| \leq |a|$). Let us suppose that $u \neq ba^s$. Since ba^s is a special factor, u is not a special factor. Since ba^{s+1} (which ends with u) is always followed by a, the factor ua is allowed, hence u is always followed by a. Thus a^{s+1} (which ends with u) is always followed by a, that is, w is ultimately periodic, so that this case does not occur. □

THEOREM 3.2. *Any pre-sturmian infinite word $w \in \{0, 1\}^{\mathbb{N}}$ of order k is renormalizable in the following way: there are two words a and b belonging to $\{0, 1\}^*$ with distinct first letters and a positive integer s such that*

$$|a| \geq |b|, \quad (s+1)|a| + |b| > k,$$

and some iterate of w under the one-sided shift T belongs to $\{a, ba^s\}^{\mathbb{N}}$.

Proof. We first notice that, since $p_w(2) = 3$, the words 00 and 11 are not both factors of w (otherwise w would be ultimately periodic). This implies that w is renormalizable with words 0 and 10 (if w does not contain the factor 11) or with words 1 and 01 (if w does not contain the factor 00).

Assume now that w is renormalizable with words a and ba^s with distinct first letters and $(s+1)|a| + |b|$ maximal. If we suppose that $(s+1)|a| + |b| \leq k$, it follows from Lemma 3.1 that, as before, some iterate of w is renormalizable with words a and $ba^sa = ba^{s+1}$ or with words ba^s and $aba^s = a(ba^s)^1$, which would contradict in both cases the maximality of $(s+1)|a| + |b|$. □

4. *The entropy ratio is always bigger than* $\frac{1}{2}$

We begin by proving the following elementary lemma.

LEMMA 4.1. *If k is a fixed positive integer and $\beta > 1$ a solution of $\beta^{k+1} = \beta^k + 1$, then:*

(i) *for any $r \in \mathbb{N}$, we have $\beta^{k+r} > r + 1$;*

(ii) *for any $r \in \mathbb{N}$, we have $k + 1 + r(r+3)/2 < \beta^{2(k+r)}$.*

https://doi.org/10.1017/etds.2018.56 Published online by Cambridge University Press

Proof. When $r = 0$, we have $\beta^k > 1$ and, if we suppose that $\beta^{k+r} > r + 1$, it follows that $\beta^{k+r+1} = \beta^{k+r} + \beta^r > (r + 1) + 1$, which proves (i) by induction on r.

Then we can easily prove (ii) by induction on r. When $r = 0$, the assertion follows from (i) applied to $r = k$ and, if we suppose that $k + 1 + r(r + 3)/2 < \beta^{2(k+r)}$, it follows that $k + 1 + (r + 1)(r + 4)/2 = k + 1 + r(r + 3)/2 + r + 2 < \beta^{2(k+r)} + r + 2$. By (i) we have $\beta^{2(k+r)} + r + 2 < \beta^{2(k+r)} + \beta^{k+r+1}$, so that it remains to show that $\beta^{2(k+r)} + \beta^{k+r+1} < \beta^{2(k+r+1)}$, that is, that $\beta^{k+r} + \beta < \beta^{k+r+2} = \beta^{k+r+1} + \beta^{r+1} = \beta^{k+r} + \beta^r + \beta^{r+1}$. For $r = 0$ this follows from the inequality $\beta < 1 + \beta$ and for $r \geq 1$ from the fact that $1 < \beta^{r-1} + \beta^r$. □

THEOREM 4.2. *If f is a function from $\mathbb{N}$ to $\mathbb{R}^+$ such that $f(n) \geq \max\{n + 1, \exp(E_0(f)n)\}$ for any $n \in \mathbb{N}$, then $E_W(f) > \frac{1}{2}E_0(f)$.*

Remark 4.3. In particular, it follows from Remark 2.8 that Theorem 4.2 applies when f satisfies conditions $(\mathcal{C}^*)$. Moreover, it follows from Remark 2.10 that we can restrict ourselves in the proof of Theorem 4.2 to the functions f_h defined by (1).

Proof. We split the proof in two parts depending on the value of $h = E_0(f_h)$.

If $h \geq \log 2$ and $m = \lfloor e^h \rfloor \geq 2$, it follows from our hypothesis that $f(n) \geq m^n$ for any $n \in \mathbb{N}$. If w is a normal sequence on the alphabet $\{1, 2, \ldots, m\}$, we have $p_w(n) = m^n$ for any $n \in \mathbb{N}$, so that $w \in W(f)$ and thus

$$E_W(f) \geq E(w) = \lim_{n\to+\infty} \frac{1}{n} \log p_w(n) = \log m > \frac{1}{2} \log(m + 1) > \frac{h}{2}.$$

If $h < \log 2$, let us consider, for any positive integer k, the set $\Sigma_k = \{0, 10^k\}^{\mathbb{N}}$ (i.e. the set of infinite words over the alphabet $\{0, 1\}$ such that two occurrences of the letter 1 are always separated by at least k occurrences of the letter 0).

For any $n \in \mathbb{N}$, let $L_k(n)$ be the set of the words of length n over the alphabet $\{0, 1\}$ such that two occurrences of the letter 1 are always separated by at least k occurrences of the letter 0 and let $q_k(n) = |L_k(n)|$. For almost any $w \in \Sigma_k$ we have $p_w(n) = q_k(n)$ for any $n \in \mathbb{N}$ and it is not difficult to construct an infinite word $w = w^{(k)} \in \Sigma_k$ which satisfies these equalities: if we enumerate, for each $n \geq 1$, $L_k(n) = \{\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_{q_k(n)}^{(n)}\}$, and take

$$\begin{aligned} w^{(k)} = {} & \alpha_1^{(1)}0^k\alpha_2^{(1)}0^k \ldots \alpha_{q_k(1)}^{(1)}0^k\alpha_1^{(2)}0^k\alpha_2^{(2)}0^k \ldots 0^k\alpha_{q_k(2)}^{(2)}0^k \\ & \ldots \alpha_1^{(n)}0^k\alpha_2^{(n)}0^k \ldots 0^k\alpha_{q_k(n)}^{(n)}0^k \ldots, \end{aligned}$$

then we have $p_w^{(k)}(n) = q_k(n)$ for any $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, we have

$$L_k(n + k + 1) = L_k(n + k)0 \cup L_k(n)0^k1,$$

which implies that, for any fixed positive integer k, the sequence $(q_k(n))_{n\in\mathbb{N}}$ satisfies the following recurrence, valid for any $n \in \mathbb{N}$:

$$q_k(n + k + 1) = q_k(n + k) + q_k(n). \tag{2}$$

https://doi.org/10.1017/etds.2018.56 Published online by Cambridge University Press

Moreover, the sequence $(q_k(n))_{n\in\mathbb{N}}$ satisfies

$$q_k(n) = n+1 \quad \text{for } 0 \le n \le k+1, \tag{3}$$

because for $0 \le n \le k+1$ a word in $L_k(n)$ has at most one letter equal to 1. This implies by induction on r that, for any integer $r \in [0, k+1]$, we have

$$q_k(k+r+1) = k+2+r(r+3)/2.$$

Indeed, when $r=0$ it follows from (3) that $q_k(k+1) = k+2$ and if we suppose that $q_k(k+r+1) = k+2+r(r+3)/2$ for some integer $r \in [0, k]$, then it follows from (2) and (3) that $q_k(k+r+2) = k+2+r(r+3)/2+r+2 = k+2+(r+1)(r+4)/2$.

Let us denote by β_k the largest real root of the polynomial $x^{k+1} - x^k - 1$,

$$c_k = \min_{0\le n\le k} \frac{n+1}{{\beta_k}^n} \quad \text{and} \quad d_k = \max_{0\le n\le k} \frac{n+1}{{\beta_k}^n}. \tag{4}$$

We have

$$c_k.\beta_k^n \le q_k(n) = p_{w^{(k)}}(n) \le d_k.\beta_k^n \tag{5}$$

for any integer $n \in [0, k]$ by (4) and it follows by induction on n that (5) remains true for any $n \in \mathbb{N}$. Indeed it follows from (2) that if we have $c_k.\beta_k^n \le q_k(n) \le d_k.\beta_k^n$ and $c_k.\beta_k^{n+k} \le q_k(n+k) \le d_k.\beta_k^{n+k}$ for some $n \in \mathbb{N}$ (which is true for $n=0$ by (4)), then we have

$$\begin{aligned} c_k.\beta_k^{n+k+1} &= c_k.\beta_k^{n+k} + c_k.\beta_k^n \le q_k(n+k+1) \\ &= q_k(n+k) + q_k(n) \le d_k.\beta_k^{n+k} + d_k.\beta_k^n = d_k.\beta_k^{n+k+1}. \end{aligned}$$

Let us denote

$$\gamma_k = \max_{k+1\le n\le 2(k+1)} \frac{1}{n}\log q_k(n) = \max_{0\le r\le k+1} \frac{\log(k+2+r(r+3)/2)}{k+r+1}$$

and prove that $\gamma_k = \max_{n\ge k+1} (1/n)\log q_k(n)$. Indeed, if we write any integer $n \ge k+1$ as $n = b(k+1)+k+r+1$, with $b \in \mathbb{N}$ and $0 \le r \le k+1$, we get

$$\begin{aligned} q_k(n) &= q_k(b(k+1)+k+r+1) \le (q_k(k+1))^b q_k(k+r+1) \\ &\le \exp(b(k+1)\gamma_k)\exp((k+r+1)\gamma_k) = \exp(\gamma_k n). \end{aligned}$$

It follows that for any integer $k \ge 1$, we have $\log\beta_k \le \gamma_k$ and it follows from Lemma 4.1(ii) applied to $\beta = \beta_k$ that, for any integer $k \ge 2$, we have

$$\gamma_{k-1} = \max_{0\le r\le k} \frac{\log(k+1+r(r+3)/2)}{k+r} < 2\log\beta_k. \tag{6}$$

For any integer $n \ge k+1$, we have $L_k(n) \subset L_{k-1}(n)$, so that $q_k(n) < q_{k-1}(n)$ and

$$\gamma_k = \max_{n\ge k+1} \frac{1}{n}\log q_k(n) < \max_{n\ge k} \frac{1}{n}\log q_{k-1}(n) = \gamma_{k-1}.$$

Since

$$\gamma_k = \max_{0\le r\le k+1} \frac{\log(k+2+r(r+3)/2)}{k+r+1} < \frac{\log(k+2+(k+1)(k+4)/2)}{k+1},$$

https://doi.org/10.1017/etds.2018.56 Published online by Cambridge University Press

we have $\lim_{k\to+\infty}\gamma_k = 0$. Now, as $h < \log 2$ and $\gamma_0 = \log 2$, let $k_0 \geq 1$ be such that

$$\gamma_{k_0} \leq h < \gamma_{k_0-1}. \tag{7}$$

For any $n \in \mathbb{N}$, we have by (7) that

$$f_h(n) = \max\{n+1, \exp(hn)\} \geq \max\{n+1, \exp(\gamma_{k_0} n)\} \geq q_{k_0}(n),$$

so that $w^{(k_0)} \in W(f_h)$, which implies by (6) and (7) that

$$E_W(f_h) \geq \lim_{n\to+\infty} \frac{1}{n}\log q_{k_0}(n) = \log \beta_{k_0} > \frac{1}{2}\gamma_{k_0-1} > \frac{h}{2} = \frac{1}{2}E_0(f_h). \qquad \square$$

5. *The constant $\frac{1}{2}$ is optimal*

The constant $\frac{1}{2}$ in Theorem 4.2 cannot be improved, as the following theorem shows.

THEOREM 5.1. *For any $c > \frac{1}{2}$, there is a function f from $\mathbb{N}$ to $\mathbb{R}^+$ satisfying conditions $(\mathcal{C}^*)$ such that $\rho(f) < c$.*

Remark 5.2. By the proof of Theorem 4.2 it follows that, if c is very close to $\frac{1}{2}$ in Theorem 5.1, then $E_0(f)$ is necessarily very close to 0.

Proof. Let k be a large positive integer and consider the function f_h defined by (1) with

$$h = \frac{1}{k}\log k.$$

We have $E_0(f_h) = (1/k)\log k$ and our goal is to give, for any $w \in W(f_h)$, an upper bound for $E(w) = \lim_{n\to\infty}(1/n)\log p_w(n)$. As ultimately periodic infinite words have zero entropy, we can restrict ourselves to a non-ultimately periodic infinite word $w \in W(f_h)$. Such an infinite word is pre-sturmian of order k (we have $p_w(k) \leq f_h(k) = k+1$), so that if follows from Theorem 3.2 that $w \in \{0,1\}^{\mathbb{N}}$ is renormalizable with words $A = a$ and $B = ba^s$ with different first letters, $|a| \geq |b|$ and $(s+1)|a| + |b| > k$.

Any factor of length n of $w \in \{0,1\}^{\mathbb{N}}$ can be covered by a factor of length at most $\lceil n/|a|\rceil + 1$ in the representation of w as an infinite word on the new alphabet $\{A, B\}$. As there are at most $2^{|a|+|b|-1}$ possible choices for the words A and B, it follows that $p_w(n) \leq 2^{|a|+|b|+\lceil n/|a|\rceil}$ and

$$E(w) = \lim_{n\to+\infty}\frac{1}{n}\log p_w(n) \leq \frac{1}{|a|}\log 2.$$

It follows in particular that, if $E(w) \geq \frac{1}{2}\log\theta = (1/2k)\log k$, then $|a| < 2k\log 2/\log k$ and, since $(s+2)|a| \geq (s+1)|a| + |b| > k$, s should be large for k large.

The infinite word w can be written as

$$w = a^{s_0}ba^{s_1}\ldots ba^{s_j}\ldots, \tag{8}$$

with $s_0 \geq 0$ and $s_j \geq s$ for any integer $j \geq 1$. We will concentrate on the gaps of size less than $2k$ between two consecutive occurrences of the first letter of the word b in the

https://doi.org/10.1017/etds.2018.56 Published online by Cambridge University Press

representation (8). Let ε be a positive constant (which will be specified more precisely later) such that

$$\frac{2\log\log k}{\log k} \le \varepsilon \le \frac{1}{4} \tag{9}$$

and denote by

$$\{r_1, r_2, \dots, r_p\} = \{s_j|a| + |b|,\ j \in \mathbb{N}\} \cap \{(1-\varepsilon)k, \dots, 2k-1\}$$

the set (which could be empty) of the sizes of blocks that are not 'too large' (i.e. between $(1-o(1))k$ and $2k$). For each integer $r \in [0, \lceil 1/\varepsilon \rceil]$, let $X(r)$ be the set (which could be empty) of the indices j with

$$(1+(r-1)\varepsilon)k \le r_j < (1+r\varepsilon)k. \tag{10}$$

If we consider $\eta = \lfloor \varepsilon k/|a| \rfloor \le \lfloor \varepsilon(s+2) \rfloor$, it follows from (9) that $\eta \le \lfloor (s+2)/4 \rfloor \le s/2$ so that, for any $j \in X(r)$, the word $a^{2\eta}ba^{(r_j-|b|)/|a|}ba^{2\eta}$ is a factor of w. Let us write

$$a^{2\eta}ba^{(r_j-|b|)/|a|}ba^{2\eta} = e_1^{(j)}e_2^{(j)}\dots e_{\ell_j}^{(j)} \in \{0,1\}^{\ell_j}$$

(where $\ell_j = 4\eta|a| + |b| + r_j$).

If $t_r = \lfloor (1+(r+2)\varepsilon)k \rfloor$, we claim that the $\lceil \varepsilon k \rceil |X(r)| \le |a|(\eta+1)|X(r)|$ factors

$$e_i^{(j)}e_{i+1}^{(j)}\dots e_{t_r+i-1}^{(j)}, \quad 1 \le i \le \lceil \varepsilon k \rceil,\ j \in X(r),$$

of size t_r of the infinite word w are all distinct. Indeed,

- if we suppose that $j_1 < j_2$ and $e_i^{(j_1)}e_{i+1}^{(j_1)}\dots e_{t_r+i-1}^{(j_1)} = e_i^{(j_2)}e_{i+1}^{(j_2)}\dots e_{t_r+i-1}^{(j_2)}$, we have $e_{(2\eta-1)|a|+r_{j_1}+1}^{(j_1)}$ equal to the first letter of b, while $e_{(2\eta-1)|a|+r_{j_1}+1}^{(j_2)}$ is equal to the first letter of a, which is a contradiction;
- if we suppose that $i_1 < i_2$ and $e_{i_1}^{(j_1)}e_{i_1+1}^{(j_1)}\dots e_{t_r+i_1-1}^{(j_1)} = e_{i_2}^{(j_2)}e_{i_2+1}^{(j_2)}\dots e_{t_r+i_2-1}^{(j_2)}$, we have $e_{n-i_2+i_1}^{(j_1)} = e_n^{(j_2)}$ for $i_2 \le n \le t_r + i_2 - 1$ and, since $e_m^{(j_l)} = e_{m+|a|}^{(j_l)}$ for $l = 1, 2$ while $m + |a| < 2|a|\eta$, we have $e_{2|a|\eta+1}^{(j_2)} = e_{2|a|\eta+1-i_2+i_1}^{(j_1)} = e_{2|a|\eta+1-i_2+i_1-|a|}^{(j_1)} = e_{2|a|\eta+1-|a|}^{(j_2)}$, which leads to a contradiction, since $e_{2|a|\eta+1}^{(j_2)}$ is the first letter of b, and $e_{2|a|\eta+1-|a|}^{(j_2)}$ is the first letter of a, and they are distinct.

It follows that

$$\varepsilon k|X(r)| \le \lceil \varepsilon k \rceil |X(r)| \le p_w(t_r) \le \theta^{t_r} \le k^{1+(r+2)\varepsilon},$$

so that

$$|X(r)| \le \frac{1}{\varepsilon}k^{(r+2)\varepsilon}. \tag{11}$$

For a and b fixed, let us now consider, for any $n \in \mathbb{N}$, the number y_n of factors of length n of w of the form $v_1v_2\dots v_m$ with, for any $j \in \{1, \dots, m\}$, $v_j \in \{a, b\}$ and such that $v_m = b$. For k large enough, it follows from (9) that $\varepsilon \ge 2\log\log k/\log k \ge 2\log 2/\log k$, so that

$$(1-\varepsilon)k \le \left(1 - \frac{2\log 2}{\log k}\right)k \le k - |a| < s|a| + |b|$$

and

$$\inf_{j\ge 1} |ba^{s_j}| \ge (1-\varepsilon)k.$$

https://doi.org/10.1017/etds.2018.56 Published online by Cambridge University Press

By considering the largest suffix of $v_1 v_2 \dots v_m$ of the form $a^\tau b$, it follows that, for any integer $n \geq 2k$, we have

$$y_n \leq \sum_{j=1}^{p} y_{n-r_j} + \sum_{j=2k}^{n} y_{n-j}. \tag{12}$$

Let φ be the function defined on $[0, 1[$ by $\varphi(x) = \sum_{j=1}^{p} x^{r_j} + x^{2k}/(1-x)$. The function φ is increasing on $[0, 1[$ and satisfies $\varphi(0) = 0$ and $\lim_{x\to 1} \varphi(x) = +\infty$, so that there is exactly one real number $\hat{\lambda} > 1$ such that $\varphi(\hat{\lambda}^{-1}) = 1$. It follows that

$$\limsup_{n\to+\infty} \frac{1}{n} \log y_n \leq \log \hat{\lambda}, \tag{13}$$

where $\hat{\lambda} > 1$ satisfies the equation

$$1 = \sum_{j=1}^{p} \hat{\lambda}^{-r_j} + \sum_{j=2k}^{\infty} \hat{\lambda}^{-j} = \sum_{j=1}^{p} \hat{\lambda}^{-r_j} + \frac{\hat{\lambda}^{-2k}}{1-\hat{\lambda}^{-1}}. \tag{14}$$

Let us write

$$C = \max_{0\leq n\leq 2k} \frac{y_n}{\hat{\lambda}^n}. \tag{15}$$

We have

$$y_n \leq C\hat{\lambda}^n \tag{16}$$

for any integer $n \in [0, 2k]$ by (15), and it follows by induction on n that (16) remains true for any $n \in \mathbb{N}$. Indeed, if we assume that, for some integer $n > 2k$, we have $y_m \leq C\hat{\lambda}^m$ for any integer $m \in [0, n[$ (which is true for $n = 2k+1$ by (15)), then we have by (12) and (14) that

$$\begin{aligned} y_n &\leq \sum_{j=1}^{p} y_{n-r_j} + \sum_{j=2k}^{n} y_{n-j} \\ &\leq C\left(\sum_{j=1}^{p} \hat{\lambda}^{n-r_j} + \sum_{j=2k}^{n} \hat{\lambda}^{n-j}\right) < C\hat{\lambda}^n\left(\sum_{j=1}^{p} \hat{\lambda}^{-r_j} + \sum_{j=2k}^{\infty} \hat{\lambda}^{-j}\right) = C\hat{\lambda}^n. \end{aligned}$$

It follows from from (10), (11) and (14) that

$$\begin{aligned} 1 &\leq \sum_{r=1}^{\lceil 1/\varepsilon\rceil} |X(r)|\hat{\lambda}^{-(1+(r-1)\varepsilon)k} + \frac{\hat{\lambda}^{-2k}}{1-\hat{\lambda}^{-1}} \\ &\leq \sum_{r=1}^{\lceil 1/\varepsilon\rceil} \frac{1}{\varepsilon} k^{(r+2)\varepsilon} \hat{\lambda}^{-(1+(r-1)\varepsilon)k} + \frac{\hat{\lambda}^{-2k}}{1-\hat{\lambda}^{-1}}. \end{aligned}$$

It follows from (9) that

$$\begin{aligned} \frac{1}{\varepsilon} &\leq \frac{\log k}{2\log\log k} = \exp(\log\log k - \log\log\log k - \log 2) \\ &\leq \exp(\log\log k) = k^{\log\log k/\log k} \leq k^{\varepsilon/2}. \end{aligned}$$

Let us show by contradiction that, writing

$$\hat{\lambda} = \exp\left(\frac{\sigma}{k}\log k\right) \tag{17}$$

https://doi.org/10.1017/etds.2018.56 Published online by Cambridge University Press

with $\sigma > 0$, we have

$$\sigma < 1/2 + 2\varepsilon \tag{18}$$

for k large. Indeed, if we suppose that

$$\sigma \geq 1/2 + 2\varepsilon, \tag{19}$$

then

$$\begin{aligned} 1 &\leq \sum_{r=1}^{\lceil 1/\varepsilon \rceil} \frac{1}{\varepsilon} k^{(r+2)\varepsilon} k^{-(1+(r-1)\varepsilon)\sigma} + \frac{k^{-2\sigma}}{1 - \exp(-(\sigma/k)\log k)} \\ &= O\left(\frac{1}{\varepsilon} k^{-\sigma+\varepsilon(2+\sigma)} \sum_{r=1}^{\lceil 1/\varepsilon \rceil} k^{\varepsilon r(1-\sigma)} + \frac{k^{1-2\sigma}}{\log k} \right). \end{aligned} \tag{20}$$

If we suppose now that $\sigma \geq 9/10$, then we have by (9) that

$$\sum_{r=1}^{\lceil 1/\varepsilon \rceil} k^{\varepsilon r(1-\sigma)} \leq \lceil 1/\varepsilon \rceil k^{(\varepsilon/10)\lceil 1/\varepsilon \rceil} = O\left(\frac{1}{\varepsilon} k^{(1+\varepsilon)/10} \right) = O\left(\frac{1}{\varepsilon} k^{1/8} \right),$$

and by (20) that

$$\begin{aligned} 1 &= O\left(\frac{1}{\varepsilon^2} k^{1/8-\sigma+\varepsilon(2+\sigma)} + \frac{k^{1-2\sigma}}{\log k} \right) \\ &= O\left(\frac{1}{\varepsilon^2} k^{1/8-\sigma+(2+\sigma)/4} + \frac{k^{1-2\sigma}}{\log k} \right) = O((\log k)^2 k^{-1/20}) \end{aligned}$$

(since, for k large, we have $1/\log k < \varepsilon \leq 1/4$), which is a contradiction. On the other hand, if we suppose that $\sigma \leq 9/10$, then we have

$$\sum_{r=1}^{\lceil 1/\varepsilon \rceil} k^{\varepsilon r(1-\sigma)} = O(k^{\varepsilon \lceil 1/\varepsilon \rceil (1-\sigma)}) = O(k^{(1+\varepsilon)(1-\sigma)}),$$

and so we get by (19) and (20) that

$$\begin{aligned} 1 &= O\left(\frac{1}{\varepsilon} k^{-\sigma+\varepsilon(2+\sigma)} k^{(1+\varepsilon)(1-\sigma)} + \frac{k^{1-2\sigma}}{\log(k)} \right) \\ &= O\left(\frac{1}{\varepsilon} k^{1-2\sigma+3\varepsilon} \right) = O(k^{1-2\sigma+(7/2)\varepsilon}) = O(k^{-\varepsilon/2}) \end{aligned}$$

(since, for k large, we have $1/\varepsilon \leq k^{\varepsilon/2}$). It follows by (9) that we have

$$1 = O(k^{-\log\log k/\log k}) = O\left(\frac{1}{\log k} \right),$$

which is again a contradiction for k large.

If a factor of length n of w contains at least one occurrence of the word b, it is of the form $\beta_1 v_1 v_2 \dots v_m \beta_2$ where β_1 is a strict suffix of a or b, $v_j \in \{a, b\}$ for any $j \in \{1, \dots, m\}$, $v_m = b$ and β_2 a prefix of a^i for some $i \geq 0$. As there are $n - |b| + 1$ possible choices for the placement of the first letter of the word v_m, there are at most

$$2(n - |b| + 1) \sum_{n' \leq n} y_{n'}$$

https://doi.org/10.1017/etds.2018.56 Published online by Cambridge University Press

such factors of length n. On the other hand, there are at most $2|a|^2$ factors of length n of w without any occurrence of the word b (there are the words of the the kind $\beta_1 a^m \beta_2$, where β_1 is a strict suffix of a or b and β_2 a strict prefix of a or b). Finally, as there are at most $2^{|a|+|b|-1}$ possible words a and b, we have

$$p_w(n) \le 2^{|a|+|b|}\left((n-|b|+1)\sum_{n'\le n} y_{n'} + |a|^2\right),$$

and it follows by (13), (17) and (18) that

$$E(w) = \lim_{n\to+\infty} \frac{1}{n}\log p_n(w) = \limsup_{n\to+\infty} \frac{1}{n}\log y_n \le \log \hat{\lambda} = \frac{\sigma}{k}\log k < \frac{(1+4\varepsilon)}{2k}\log k.$$

If we now choose $\varepsilon = \frac{1}{2}(c-\frac{1}{2})$, where $c > \frac{1}{2}$ is given in the statement, we have

$$E_W(f_h) = \sup_{w\in W(f_h)} E(w) < \frac{c}{k}\log k,$$

which concludes the proof. □

Acknowledgements. We thank the referee for his comments and suggestions. This work was supported by CNPq, FAPERJ and the Agence Nationale de la Recherche project ANR-14-CE34-0009 MuDeRa.

References

[**CH73**] E. M. Coven and G. A. Hedlund. Sequences with minimal block growth. *Math. Systems Theory* **7** (1973), 138–153.

[**CN10**] J. Cassaigne and F. Nicolas. Factor complexity. *Combinatorics, Automata and Number Theory (Encyclopedia of Mathematics and its Applications, 135)*. Eds. V. Berth and M. Rigo. Cambridge University Press, Cambridge, 2010, pp. 163–247.

[**Fek23**] M. Fekete. Uber der Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. *Math. Z.* **17** (1923), 228–249.

[**Fer99**] S. Ferenczi. Complexity of sequences and dynamical systems. *Discrete Math.* **206**(1–3) (1999), 145–154.

[**Gri73**] C. Grillenberger. Construction of strictly ergodic systems I. Given entropy. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **25** (1973), 323–334.

[**HM38**] G. A. Hedlund and M. Morse. Symbolics dynamics. *Amer. J. Math.* **60** (1938), 815–866.

[**HM40**] G. A. Hedlund and M. Morse. Symbolics dynamics II. Sturmian trajectories. *Amer. J. Math.* **62** (1940), 1–42.

[**Lot02**] M. Lothaire. *Algebraic Combinatorics on Words (Encyclopedia of Mathematics and Its Applications, 90)*. Cambridge University Press, Cambridge, 2002.

[**MM10**] C. Mauduit and C. G. Moreira. Complexity of infinite sequences with zero entropy. *Acta Arith.* **142** (2010), 331–346.

[**MM12**] C. Mauduit and C. G. Moreira. Generalized Hausdorff dimensions of sets of real numbers with zero entropy expansion. *Ergod. Th. & Dynam. Sys.* **32** (2012), 1073–1089.

[**MM17**] C. Mauduit and C. G. Moreira. Complexity and fractal dimensions for infinite sequences with positive entropy. *Commun. Contemp. Math.* to appear.

[**PF02**] N. Pytheas Fogg. *Substitutions in Dynamics, Arithmetics and Combinatorics (Lecture Notes in Mathematics, 1794)*. Eds. V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel. Springer, Berlin, 2002.

https://doi.org/10.1017/etds.2018.56 Published online by Cambridge University Press