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ORIGINAL ARTICLE

Analysis of a network's asymptotic behavior via its structure involving its strongly connected components

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Abstract

In this paper, it is addressed how network structure can be related to asymptotic network behavior. If such a relation is studied, that usually concerns only strongly connected networks and only linear functions describing the dynamics. In this paper, both conditions are generalized. A couple of general theorems is presented that relates asymptotic behavior of a network to the network's structure characteristics. The network structure characteristics, on the one hand, concern the network's strongly connected components and their mutual connections; this generalizes the condition of being strongly connected to a very general condition. On the other hand, the network structure characteristics considered generalize from linear functions to functions that are normalized, monotonic, and scalar-free, so that many nonlinear functions are also covered. Thus, the contributed theorems generalize the existing theorems on the relation between network structure and asymptotic network behavior addressing only specific cases such as acyclic networks, fully, and strongly connected networks, and theorems addressing only linear functions. This paper was invited as an extended (by more than 45%) version of a Complex Networks'18 conference paper. In the discussion section, the differences are explained in more detail.

Keywords: asymptotic network behavior; strongly connected components; relating network structure and network behavior

1. Introduction

In many cases, the relation between network structure and emerging asymptotic diffusion behavior of a network is only studied by performing simulation experiments. In this paper, it is shown how within a certain context it is also possible to analyze mathematically how certain asymptotic behaviors relate to certain properties of the network structure. In Treur (2018a), this question was only addressed for the case of an acyclic network and the case of a strongly connected network. The current paper uses the strongly connected components of the network to develop a mathematical analysis for the general case. Tools were adopted from the area of Graph Theory, such as the manner to identify the connectivity structure within a graph by decomposition of the graph according to its (maximal) strongly connected components and the resulting (acyclic) condensation graph (Harary et al., 1965, Ch. 3) and, in addition, the notion of stratification of an acyclic directed graph (e.g., Chen, 2009).

Besides the connectivity structure, the theorems presented here also take into account the combination functions by which the impacts from multiple incoming connections are aggregated. It applies not to just one type (e.g., linear functions), but to a whole class of functions: those combination functions that are characterized as being monotonic, scalar-free, and normalized. This

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class includes not only the often used linear functions, but also nonlinear functions such as *n*th order Euclidean combination functions and normalized scaled geometric mean functions.

The theorems explain which are the relevant characteristics that make these combination functions contribute to certain asymptotic behavior. It will be shown how using the above-mentioned tools from Graph Theory, together with the characteristics of combination functions mentioned, enables to address the general case and obtain theorems about it. These theorems apply to arbitrary networks, but among the foci of application in particular are the types of example network models of which several are described in Treur (2016a):

1. Mental Networks describing the dynamics of mental processes such as the (usually cyclic) interaction of the mental states involved and behavior resulting from this.
2. Social Networks describing social contagion processes for example, for opinions, beliefs, and emotions,
3. Integrative Networks that integrate 1. and 2.

Note that especially Mental Networks are not often strongly connected, although some parts may be. Typically, they use sensory input that in general may not be affected by the behavior, and because of that such input is not on any cycle of the network. Therefore, they cannot be treated like strongly connected networks, but the theory developed here based on a decomposition by strongly connected components does apply (for applying the analysis to an example of such a Mental Network, see Section 7.2). Social Networks may often be strongly connected, but also in that case external nodes may be involved that affect them, which makes the whole not strongly connected. Therefore, for applicability on such types of networks, the generalization from strongly connected networks to general types of networks is important.

The foci of applicability on the three types of networks 1. to 3. also make that only addressing linear functions would be too limited. Especially for Mental Networks often nonlinear functions are used. Therefore, the challenge is also to stretch the type of analysis to at least certain types of nonlinear functions.

To apply the theorems introduced in this paper to any given network, first the decomposition of the network into its strongly connected components is determined. Multiple efficient algorithms are available to determine these strongly connected components (see, e.g., Bloem et al., 2006; Fleischer et al., 2000; Gentilini et al., 2003; Li et al., 2014; Tarjan, 1972; Wijs et al., 2016). The connections between these components are identified, as represented in an acyclic condensation graph, and a stratification of this graph is introduced. Based on this acyclic and stratified structure added to the original network, the theorems will show whether and which states within the network will end up in a common equilibrium value and more in general determine bounds for the equilibrium values of the states.

The research presented here has been initiated from the angle of mathematical analysis and verification of network models in comparison to simulations for these models. For more background on this angle, see, for example, Treur (2016b) or (Treur, 2016a, Ch 12). Just as verification in Software Engineering is very useful for the quality of developed software (e.g., Drechsler, 2004; Fisher, 2007), so too verification in network modeling is a useful means to get implementations of network models in accordance with the specifications of the models and eliminate implementation errors. If a simulation of an implemented network model contradicts one or more of the results presented in the current paper for the specification of the network model, then this pinpoints that something is wrong: a discrepancy between specification and implementation of the network model that needs to be addressed. Afterwards, it turned out that the contributions presented here also have some relations to research conducted from a different angle, namely on control of networks (e.g., Liu et al., 2011, 2012; Moschoyiannis et al., 2016; Haghighi & Namazi, 2015; Karlsen & Moschoyiannis, 2018). These relations will be discussed in the Discussion section.

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Table 1. Conceptual and numerical representations of a temporal–causal network.

Concepts	Notation	Explanation
States and connections	$X, Y, X \rightarrow Y$	Describes the nodes and links of a network structure (e.g., in graphical or matrix format)
Connection weight	$\boldsymbol{\omega}_{X,Y}$	*Connection weight* $\boldsymbol{\omega}_{X,Y} \in [-1, 1]$ represents the strength of the impact of state X on state Y through connection $X \rightarrow Y$
Aggregating multiple impacts	$\mathbf{c}_Y(..)$	For each state Y, a *combination function* $\mathbf{c}_Y(..)$ is chosen to combine the causal impacts of other states on state Y
Timing of the causal effect	$\boldsymbol{\eta}_Y$	For each state Y, a *speed factor* $\boldsymbol{\eta}_Y \geq 0$ is used to represent how fast a state is changing upon causal impact
Concepts	**Numerical representation**	**Explanation**
State values over time t	$Y(t)$	At each time point t, each state Y has a real number value in [0, 1]
Single causal impact	$\mathbf{impact}_{X,Y}(t) = \boldsymbol{\omega}_{X,Y} X(t)$	At t, state X with connection to state Y has an impact on Y, using weight $\boldsymbol{\omega}_{X,Y}$
Aggregating multiple impacts	$\mathbf{aggimpact}_Y(t) = \mathbf{c}_Y\left(\mathbf{impact}_{X_1,Y}(t), \ldots, \mathbf{impact}_{X_k,Y}(t)\right) = \mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y} X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y} X_k(t)\right)$	The aggregated impact of $k \geq 1$ states $X_1, \ldots, X_k$ on Y at t, is determined using combination function $\mathbf{c}_Y(..)$
Timing of the causal effect	$Y(t + \Delta t) = Y(t) + \boldsymbol{\eta}_Y \left[\mathbf{aggimpact}_Y(t) - Y(t)\right] \Delta t$ $Y(t + \Delta t) = Y(t) + \boldsymbol{\eta}_Y \left[\mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y} X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y} X_k(t)\right) - Y(t)\right] \Delta t$	The impact on Y is exerted over time gradually, using speed factor $\boldsymbol{\eta}_Y$

In Section 2, the basic definition of network used is discussed. Section 3 discusses asymptotic behavior, illustrated for an example network. Section 4 presents the definitions of the Graph Theory tools on connectivity; in Section 5, the identified characteristics of combination functions are defined. In Section 6, the main theorems are formulated and it is shown how they were proven. In Section 7, more in-depth analysis is added, and in particular applicability is illustrated for a type of network which is not a Social Network: a Mental Network describing sharing behavior based on emotional charge. Section 8 presents the final discussion.

2. Temporal–causal networks

This section describes the definition of the concept of network used: temporal–causal network model. This is a notion of network that covers all types of discrete or smooth continuous dynamical systems, as has been shown in Treur (2017), building further, among others, on Ashby (1960) and Port & van Gelder (1995).

A temporal–causal network model is based on three notions defining the network structure: connection weight, combination function, and speed factor (see Table 1, upper part). Here, the word temporal in temporal–causal refers to the causality. A library with a number (currently 33) of standard combination functions is available as options to choose from, but own-defined functions can also be used.

In the lower part of Table 1, it is shown how a conceptual representation of network structure defines a numerical representation of network dynamics (see also Treur, 2016a, Ch. 2 or Treur, 2019). Here, $X_1, \ldots, X_k$ with $k \geq 1$ are the states from which state Y gets incoming connections. This defines the detailed dynamic semantics of a temporal–causal network.

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Table 2. Connection weights for the example simulation.

Connection weights	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1		0.8		0.5						
X_2			1							
X_3				0.2				0.8		
X_4		0.6								
X_5							0.6		0.8	
X_6					0.7					
X_7						0.8				0.8
X_8									0.8	
X_9										0.7
X_{10}								0.6		
Speed factors										
	0	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
Initial values										
	0.9	0	0	0	0.9	0	0	0	0	0

The difference equations in the last row in Table 2 can be used for simulation and mathematical analysis. They can also be written in differential equation format:

$$\frac{\mathbf{d}Y(t)}{\mathbf{d}t} = \boldsymbol{\eta}_Y\left[\mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y}X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}X_k(t)\right) - Y(t)\right]$$

Note that combination functions are functions on the 0–1 interval within the real numbers: $[0, 1]^k \rightarrow [0, 1]$. Moreover, note that the condition $k \geq 1$ in Table 1 makes that by definition the above general format applies only to states Y with at least one incoming connection. However, in a network, there may also be states Y without any incoming connection; for example, such states can serve as external input. Their dynamics can be specified in an independent manner by any mathematical function f:$[0, \infty) \rightarrow [0, 1]$ over time t:

$$Y(t) = f(t) \text{ for all } t$$

Special cases of this are states Y with constant values over time, where for some constant $c \in [0, 1]$ it holds $f(t) = c$ for all t. For such constant states, still the general format can be used as well, as long as the speed factor $\boldsymbol{\eta}_Y$ is set at 0 and the combination function is well defined for zero arguments: then the general format reduces to $Y(t + \Delta t) = Y(t)$, and therefore the initial value is kept over time. But there are also other possible types of external input, for example, a repeated alternation of values 0 and 1 for some time intervals to model episodes in which a stimulus occurs and episodes in which it does not.

Examples of often used combination functions (see also Treur, 2016a, Ch 2, Table 2.10) are the following:

- the *identity* function **id**(.) for states with only one impact

$$\mathbf{id}(V) = V$$

- the *scaled sum* function $\mathbf{ssum}_{\boldsymbol{\lambda}}(..)$ with scaling factor $\boldsymbol{\lambda}$

$$\mathbf{ssum}_{\boldsymbol{\lambda}}(V_1,\ldots,V_k) = \frac{V_1 + \cdots + V_k}{\boldsymbol{\lambda}}$$

- the scaled minimum function $\mathbf{smin}_{\boldsymbol{\lambda}}(..)$ with scaling factor $\boldsymbol{\lambda}$

$$\mathbf{smin}_{\boldsymbol{\lambda}}(V_1,\ldots,V_k) = \frac{\min(V_1,\ldots,V_k)}{\boldsymbol{\lambda}}$$

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- the scaled maximum function **smax(..)** with scaling factor $\boldsymbol{\lambda}$

$$\mathbf{smax}_{\boldsymbol{\lambda}}(V_1, \ldots, V_k) = \frac{\max(V_1, \ldots, V_k)}{\boldsymbol{\lambda}}$$

- the *simple logistic sum* combination function $\mathbf{slogistic}_{\boldsymbol{\sigma},\boldsymbol{\tau}}(..)$ with steepness $\boldsymbol{\sigma}$ and threshold $\boldsymbol{\tau}$, defined by

$$\mathbf{slogistic}_{\boldsymbol{\sigma},\boldsymbol{\tau}}(V_1, \ldots, V_k) = \frac{1}{1 + \mathrm{e}^{-\boldsymbol{\sigma}(V_1 + \cdots + V_k - \boldsymbol{\tau})}}$$

- the *advanced logistic sum* combination function $\mathbf{alogistic}_{\boldsymbol{\sigma},\boldsymbol{\tau}}(..)$ with steepness $\boldsymbol{\sigma}$ and threshold $\boldsymbol{\tau}$, defined by

$$\mathbf{alogistic}_{\boldsymbol{\sigma},\boldsymbol{\tau}}(V_1, \ldots, V_k) = \left[\frac{1}{1 + \mathrm{e}^{-\boldsymbol{\sigma}(V_1 + \cdots + V_k - \boldsymbol{\tau})}} - \frac{1}{1 + \mathrm{e}^{\boldsymbol{\sigma}\boldsymbol{\tau}}}\right](1 + \mathrm{e}^{-\boldsymbol{\sigma}\boldsymbol{\tau}})$$

- the *Euclidean combination function* of *n*th order with scaling factor $\boldsymbol{\lambda}$ (generalizing the scaled sum $\mathbf{ssum}_{\boldsymbol{\lambda}}(...)$ for $\boldsymbol{n} = 1$) defined by

$$\mathbf{eucl}_{\mathbf{n},\boldsymbol{\lambda}}(V_1, \ldots, V_k) = \sqrt[n]{\frac{V_1{}^{\boldsymbol{n}} + \cdots + V_k{}^{\boldsymbol{n}}}{\boldsymbol{\lambda}}}$$

  Here, *n* can be any positive integer or even any positive real number.
- the *scaled geometric mean combination function* with scaling factor $\boldsymbol{\lambda}$

$$\mathbf{sgeomean}_{\boldsymbol{\lambda}}(V_1, \ldots, V_k) = \sqrt[k]{\frac{V_1 * \cdots * V_k}{\boldsymbol{\lambda}}}$$

For example, scaled minimum and maximum functions are often used in fuzzy logic-inspired modeling and modeling uncertainty in AI, and the logistic functions are often used in neural network-inspired modeling. The scaled sum functions, which are a special (linear) case of Euclidean functions, are often used in modeling of social networks. Geometric mean combination functions relate to product-based combination rules often used for probability-based approaches.

3. Asymptotic network behavior

Asymptotic behavior will be explored by analyzing possible equilibria. Stationary points and equilibria are defined as follows.

Definition 1 (Stationary point and equilibrium)**.** *A state Y has a stationary point at t if*

$$\frac{\mathbf{d}Y(t)}{\mathbf{d}t} = 0.$$

The network is in equilibrium at t if every state Y of the model has a stationary point at t.

Given the specific differential equation format for a temporal–causal network model, the following criterion can be found.

Lemma 1 (Criterion for a stationary point in a temporal–causal network)**.** *Let Y be a state and* $X_1, \ldots, X_k$ *be the states from which state Y gets incoming connections. Then Y has a stationary point at t if and only if*

$$\boldsymbol{\eta}_Y = 0$$

or

$$\mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y} X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y} X_k(t)\right) = Y(t)$$

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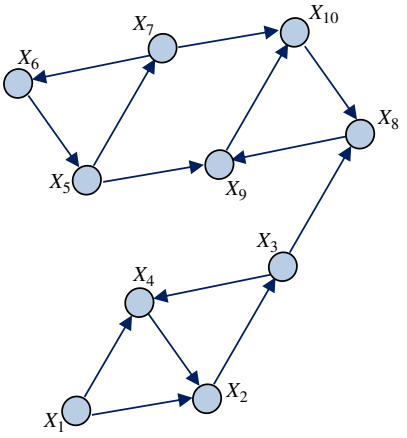


Figure 1. Example network.

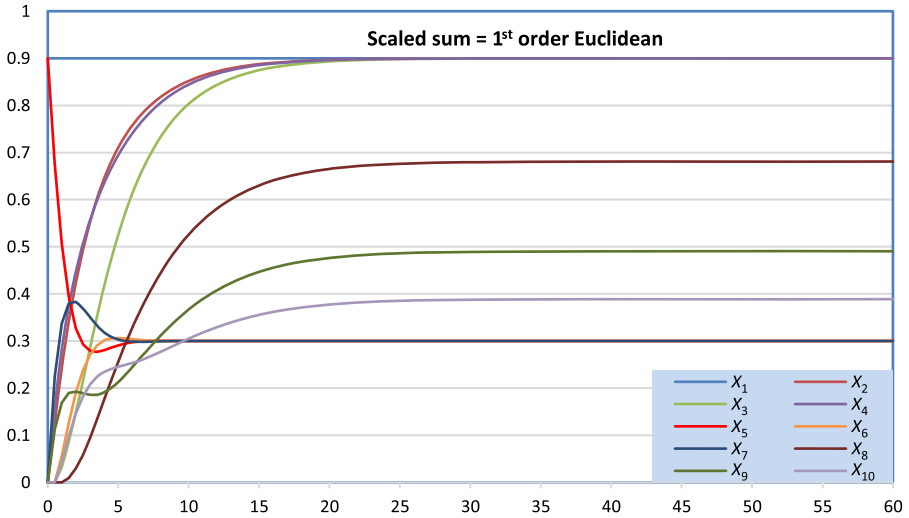


Figure 2. Example simulation for linear scaled sum combination functions.

As illustration, the example network shown in Figure 1 is used. The connection weights, speed factors, and initial values used are shown in Table 2. The simulation for $\Delta t = 0.5$ is shown in Figure 2.

Note that state X_1 has no incoming connections; in the simulation, it has initial value 0.9 and this stays constant at this level due to having speed factor 0. Also, X_5 has 0.9 as initial value. The other states have initial value 0. Note that in Section 6 theorems are presented from which it follows that the initial values of states X_2 to X_4 and X_8 to X_{10} are irrelevant for the asymptotic behavior as they do not have any effect on the asymptotic behavior; therefore, they were initially set at 0 here.

The speed factor of the states X_2 to X_{10} is 0.5. The combination function used is a normalized scaled sum function (see Section 5 for more details on normalization). In the simulation shown in Figure 2, states X_1 to X_4 all end up at value 0.9, states X_5 to X_7 all at value 0.3, and states X_8 to X_{10} at different individual values 0.681, 0.490, and 0.389, respectively. Overall, there is some clustering, but also some states get their own unique value. It can be observed that these unique values are in between the cluster values.

The combination function used for the simulation in Figure 2 is the scaled sum function, which is linear. It might be believed that this pattern depends on the combination function being linear.

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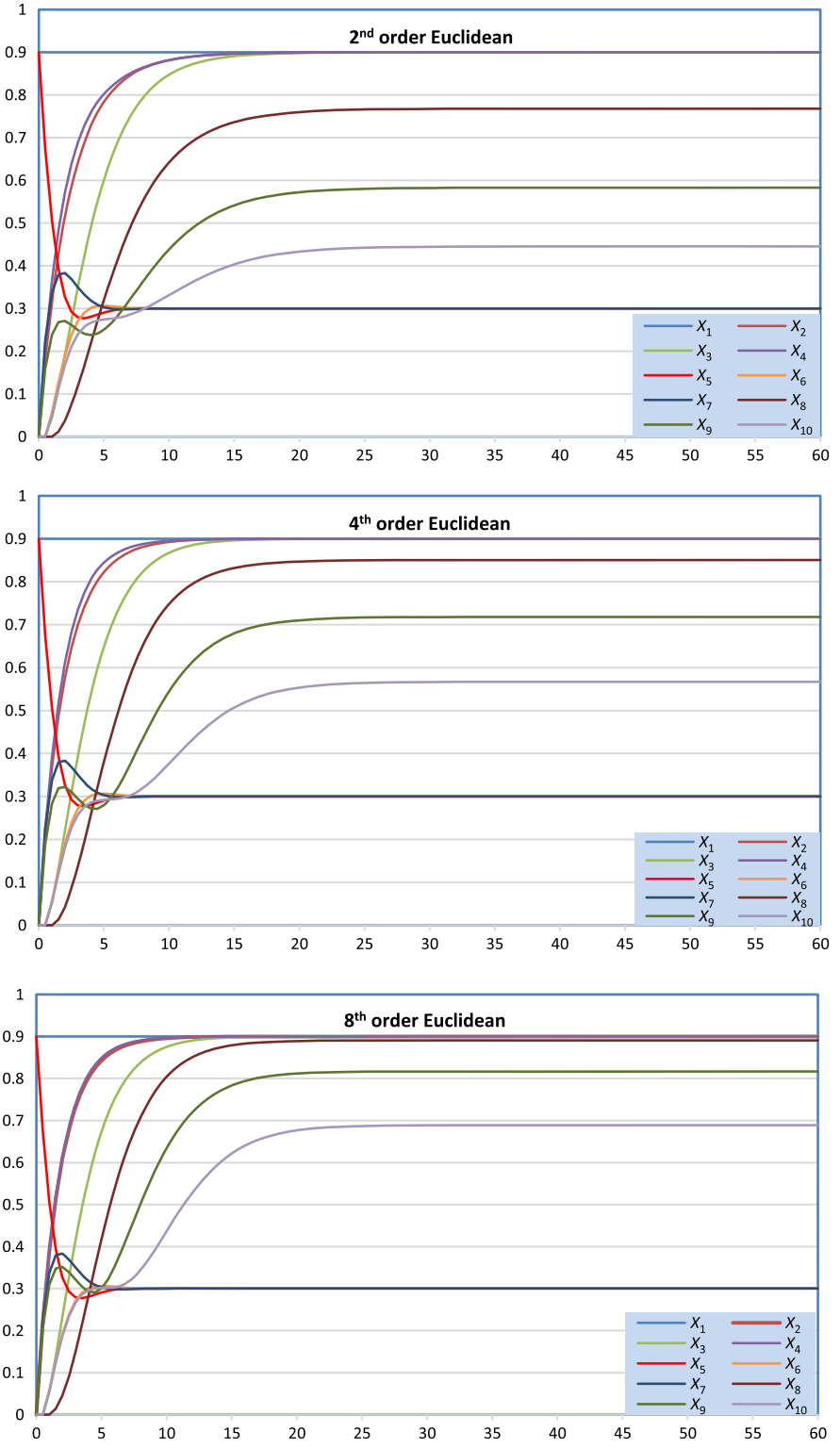


Figure 3. Simulations for nonlinear higher order Euclidean combination functions of order 2, 4, and 8.

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However, this is not the case. In Figure 3, three simulations are shown for nonlinear combination functions, namely, higher order Euclidean combination functions of order 2, 4, and 8, respectively. It is shown that the overall pattern is very similar with the same two groups going for 0.3 and 0.9, and the remaining three states X_8 to X_{10} getting each at different values but between these two values 0.3 and 0.9. The only difference is that the latter three values differ for the four considered combination functions, although they are in the same order. Note that in the graph for the 8th order Euclidean combination function state X_8 in the end gets a value very close but not equal to 0.9.

The question how such emerging asymptotic patterns can be explained will be addressed in the next three sections. It will be analyzed how the pattern depends on the network's characteristics, in particular on the connectivity within the network and the characteristics of the combination functions. Each of these two factors will be discussed first in Sections 4 and 5, respectively, after which in Section 6 they will be related to the emerging asymptotic patterns.

4. Connectivity and strongly connected components

When broadening the scope of analysis for a wider class of networks concerning connectivity, analysis based on the network's strongly connected components is useful. This is known from Graph Theory. Most of the following definitions can be found, for example, in Harary et al. (1965, Ch. 3) or in Kuich (1970, Section 6). Note that here only nonnegative connection weights are considered.

Definition 2 (Reachability and strongly connected components)**.**

a. *State Y is* forward reachable *from state X if there is a directed path from X to Y with nonzero connection weights and speed factors.*
b. *A network N is* strongly connected *if every two states are mutually forward reachable within N.*
c. *A state is called* independent *if it is not forward reachable by any other state.*
d. *A* subnetwork *of network N is a network whose states and connections are states and connections of N.*
e. *A strongly* connected component *C of a network N is a strongly connected subnetwork of N such that no larger strongly connected subnetwork of N contains it as a subnetwork.*

Strongly connected components C can be determined by choosing any node X of N and adding all nodes that are on any cycle through X. When a node X is not on any cycle, then it will form a singleton strongly connected component C by itself; this applies to all nodes of N with indegree or outdegree zero. Efficient algorithms have been developed to determine the strongly connected components of a graph (see, for example, Bloem et al., 2006; Fleischer et al., 2000; Gentilini et al., 2003; Li et al., 2014; Tarjan, 1972; Wijs et al., 2016). The strongly connected components of the example network from Figure 1 are shown in Figure 4.

Based on the strongly connected components, a form of abstracted picture of the network can be made, called the condensation graph (see Figure 5).

Definition 3 (Condensation graph)**.** *The* condensation *C(N) of a network N with respect to its strongly connected components is a graph whose nodes are the strongly connected components of N and whose connections are determined as follows: there is a connection from node C_i to node C_j in C(N) if and only if in N there is at least one connection from a node in the strongly connected component C_i to a node in the strongly connected component C_j.*

A condensation graph $C(N)$ is always an acyclic graph. The following theorem summarizes this (see also Harary et al., 1965, Ch. 3, Theorems 3.6 and 3.8 or Kuich, 1970, Section 6).

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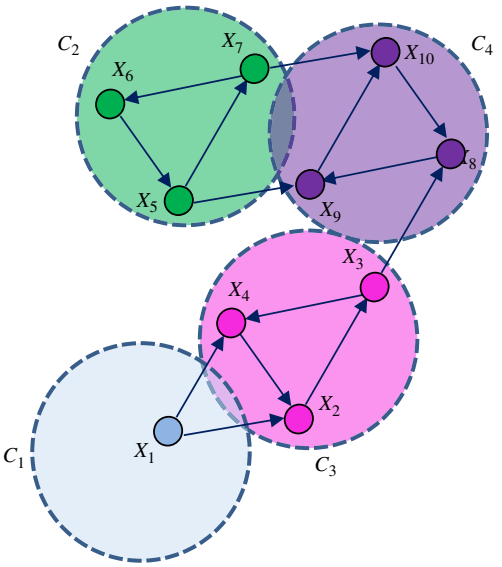


Figure 4. The strongly connected components within the example network.

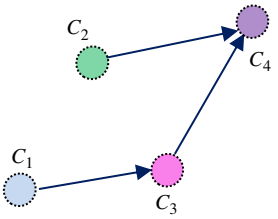


Figure 5. Condensation of the example network by its strongly connected components: the directed acyclic condensation graph $C(N)$.

Theorem 1 (Acyclic condensation graph)**.**

a. For any network N, its condensation graph C(N) is acyclic and has at least one state of outdegree zero and at least one state of indegree zero.
b. The network N is acyclic itself if and only if it is graph-isomorphic to C(N). In this case, the nodes in C(N) all are singleton sets {X} containing one state X from N.
c. The network N is strongly connected itself if and only if C(N) only has one node; this node is the set of all states of N.

The structure of an acyclic graph is much simpler than the structure of a cyclic graph. For example, for any acyclic directed graph, a stratification structure is defined (e.g., Chen, 2009). Here, such a construction is applied in particular to the condensation graph C(N), thus obtaining a stratified condensation graph SC(N) which will turn out very useful in Section 6 (see Figure 6).

Definition 4 (Stratified condensation graph)**.** *The stratified condensation graph for network N, denoted by SC(N), is the condensation graph C(N) together with a leveled partition* $S_0, \ldots, S_{h-1}$ *in strata* S_i *such that* $S_0 \cup \cdots \cup S_{h-1}$ *is the set of all nodes of C(N) and the* S_i *are mutually disjoint, which is defined inductively as follows. Here, h is the height of C(N), that is, the length of the longest path in C(N).*

(i) The stratum S_0 *is the set of nodes in C(N) without incoming connections in C(N).*
(ii) For each $i > 0$*, the stratum* S_i *is the set of nodes in C(N) for which all incoming connections in C(N) come only from nodes in* $S_0, \ldots, S_{i-1}$.

If node X is in stratum S_i, its *level* is i.

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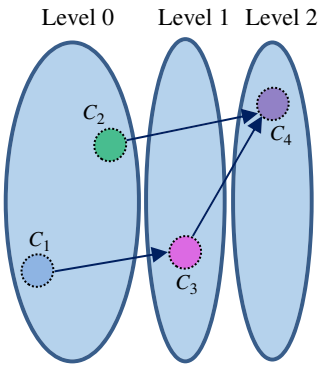


Figure 6. Stratified condensation graph SC(*N*) for the example network.

Table 3. Characteristics of Definition 5 for the example combination functions.

	(a)	(b)	(c)	(d)	(e)
id(..)	+	+	+	+	+
ssum$_\lambda$(..) (=**eucl**$_{n,\lambda}$(..), for $n=1$)	+	+	+	+	+
eucl$_{n,\lambda}$(..) for $n\neq 1$	+	+	+	−	−
sgeomean$_\lambda$(..) for nonzero values	+	+	+	−	−
smin$_\lambda$(..)	+	−	+	−	−
smax$_\lambda$(..)	+	−	+	−	−
slogistic$_{\sigma,\tau}$(..)	+	+	−	−	−
alogistic$_{\sigma,\tau}$(..)	+	+	−	−	−

5. Characteristics of combination functions

The following characteristics of combination functions have been found to relate to asymptotic behavior as discussed in Section 3. Note that for combination functions it is (silently) assumed that $c(V_1, \ldots, V_k) = 0$ iff $V_k = 0$ for all i.

Definition 5 (Monotonic, scalar-free, and additive for a combination function)**.**

a. A function c(..) *is called* monotonically increasing *if for all values* U_i, V_i *it holds*

$$U_i \leq V_i \quad \textit{for all} \quad i \Rightarrow c\,(U_1, \ldots, U_k) \leq c(V_1, \ldots, V_k)$$

b. A function c(..) *is called* strictly monotonically increasing *if*

$$U_i \leq V_i \textit{ for all } i, \textit{ and } U_j < V_j \textit{ for at least one } j \Rightarrow c\,(U_1, \ldots, U_k) < c(V_1, \ldots, V_k).$$

c. A function c(..) *is called* scalar-free *if for all* $\alpha > 0$ *and all* $V_1, \ldots, V_k$ *it holds*

$$c(\alpha V_1, \ldots, \alpha V_k) = \alpha c(V_1, \ldots, V_k).$$

d. A function c(..) *is called* additive *if for all* $U_1, \ldots, U_k$ *and* $V_1, \ldots, V_k$ *it holds*

$$c(U_1 + V_1, \ldots, U_k + V_k) = c\,(U_1, \ldots, U_k) + c(V_1, \ldots, V_k).$$

e. A function c(..) *is called* linear *if it is both scalar-free and additive.*

Note that these characteristics vary over the different examples of combination functions. Table 3 shows which of these characteristics apply to which combination functions. In general, the theorems that follow in Section 6 have the characteristics (a), (b), and (c) as conditions, so as can be seen in Table 3 they apply to **id**(..), **ssum**$_\lambda$(..), **eucl**$_{n,\lambda}$(..), and **sgeomean**$_\lambda$(..) (of which only the first two are linear and the last two are nonlinear, assuming $n \neq 1$ for the third one and

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nonzero values for the fourth one). The theorems do not apply to $\mathbf{smin}_{\boldsymbol{\lambda}}(..)$ and $\mathbf{smax}_{\boldsymbol{\lambda}}(..)$ (not strictly monotonous) and to $\mathbf{slogistic}_{\sigma,\tau}(..)$ and $\mathbf{alogistic}_{\sigma,\tau}(..)$ (not scalar-free). Note that different functions satisfying (a), (b), and (c) can also be combined to get more complex functions by using linear combinations with positive coefficients and function composition.

Definition 6 (Normalized)**.** *A network is normalized if for each state Y it holds* $\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}) = 1$, *where* $X_1, \ldots, X_k$ *are the states from which state Y gets incoming connections.*

As an example, for a Euclidean combination function $\mathbf{eucl}_{n,\boldsymbol{\lambda}}(..)$ of nth order, the scaling parameter choice

$$\boldsymbol{\lambda}_Y = \boldsymbol{\omega}_{X_1,Y}{}^{\boldsymbol{n}} + \cdots + \boldsymbol{\omega}_{X_k,Y}{}^{\boldsymbol{n}}$$

will provide a normalized network. Assuming $\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}) > 0$ for $\boldsymbol{\omega}_{X_i,Y} > 0$, this can be done more in general as follows:

(1) Normalizing a combination function

If any combination function $\mathbf{c}_Y(..)$ is replaced by $\mathbf{c}'_Y(..)$ defined as

$$\mathbf{c}'_Y(V_1, \ldots, V_k) = \frac{\mathbf{c}_Y(V_1, \ldots, V_k)}{\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})}$$

(note $\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}) > 0 > 0$ since $\boldsymbol{\omega}_{X_i,Y} > 0$), then the network becomes normalized.

(2) Normalizing the connection weights (for scalar-free combination functions)

For scalar-free combination functions also, normalization is possible by adapting the connection weights; define

$$\boldsymbol{\omega}'_{X_i,Y} = \frac{\boldsymbol{\omega}_{X_i,Y}}{\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})},$$

(assuming $\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}) > 0$ for $\boldsymbol{\omega}_{X_i,Y} > 0$), then indeed it holds:

$$\begin{aligned}\mathbf{c}_Y(\boldsymbol{\omega}'_{X_1,Y}, \ldots, \boldsymbol{\omega}'{}_{X_k,Y}) &= \mathbf{c}_Y\left(\frac{\boldsymbol{\omega}_{X_1,Y}}{\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})}, \ldots, \frac{\boldsymbol{\omega}_{X_k,Y}}{\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})}\right)\\ &= \frac{1}{\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})}\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})\\ &= 1\end{aligned}$$

Normalization is a necessary condition for applying the theorems developed in Section 6. Simulation is still possible when the network is not normalized. But the effect then usually is that activation is lost in an artificial manner (if the function values are lower than normalized) so that all values go to 0, or that activation is amplified in an artificial manner (if the function values are higher than normalized) so that all values go to 1. That makes less interesting behavior for practical applications and also less interesting analysis.

For different example functions, following normalization step (1), their normalized variants are given by Table 4.

Some of the implications of the above-defined characteristics are illustrated in the following proposition. This will be used in Section 6.

Proposition 1. *Suppose the network is normalized.*

a. If the combination functions are scalar-free and $X_1, \ldots, X_k$ *are the states from which state Y gets incoming connections, and* $X_1(t) = \cdots = X_k(t) = V$ *for some common value V, then also*

$$\mathbf{c}_Y\big(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)\big) = V.$$

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Table 4. Normalization of the different examples of combination functions.

Combination function	Notation	Normalizing scaling factor	Normalized combination function
Identity function	$\mathbf{id}(V)$	$\boldsymbol{\omega}_{X,Y}$	$\frac{V}{\boldsymbol{\omega}_{X,Y}}$
Scaled sum	$\mathbf{ssum}_{\boldsymbol{\lambda}}(V_1,\ldots,V_k)$	$\boldsymbol{\omega}_{X_1,Y}+\cdots+\boldsymbol{\omega}_{X_k,Y}$	$\frac{V_1+\cdots+V_k}{\boldsymbol{\omega}_{X_1,Y}+\cdots+\boldsymbol{\omega}_{X_k,Y}}$
Scaled maximun	$\mathbf{smax}_{\boldsymbol{\lambda}}(V_1,\ldots,V_k)$	$\max\left(\boldsymbol{\omega}_{X_1,Y},\ldots,\boldsymbol{\omega}_{X_k,Y}\right)$	$\frac{\max(V_1,\ldots,V_k)}{\max\left(\boldsymbol{\omega}_{X_1,Y},\ldots,\boldsymbol{\omega}_{X_k,Y}\right)}$
Scaled minimum	$\mathbf{smin}_{\boldsymbol{\lambda}}(V_1,\ldots,V_k)$	$\min\left(\boldsymbol{\omega}_{X_1,Y},\ldots,\boldsymbol{\omega}_{X_k,Y}\right)$	$\frac{\min(V_1,\ldots,V_k)}{\min\left(\boldsymbol{\omega}_{X_1,Y},\ldots,\boldsymbol{\omega}_{X_k,Y}\right)}$
Euclidean	$\mathbf{eucl}_{\boldsymbol{n},\boldsymbol{\lambda}}(V_1,\ldots,V_k)$	$\boldsymbol{\omega}_{X_1,Y}{}^{\boldsymbol{n}}+\cdots+\boldsymbol{\omega}_{X_k,Y}{}^{\boldsymbol{n}}$	$\sqrt[\boldsymbol{n}]{\frac{V_1{}^{\boldsymbol{n}}+\cdots+V_k{}^{\boldsymbol{n}}}{\boldsymbol{\omega}_{X_1,Y}{}^{\boldsymbol{n}}+\cdots+\boldsymbol{\omega}_{X_k,Y}{}^{\boldsymbol{n}}}}$
Simple logistic	$\mathbf{slogistic}_{\boldsymbol{\sigma},\boldsymbol{\tau}}(V_1,\ldots,V_k)$	$\mathbf{slogistic}_{\boldsymbol{\sigma},\boldsymbol{\tau}}\left(\boldsymbol{\omega}_{X_1,Y},\ldots,\boldsymbol{\omega}_{X_k,Y}\right)$	$\frac{1+e^{-\sigma(\boldsymbol{\omega}_{X_1,Y}+\cdots+\boldsymbol{\omega}_{X_k,Y}-\tau)}}{1+e^{-\sigma(V_1+\cdots+V_k-\tau)}}$
Advanced logistic	$\mathbf{alogistic}_{\boldsymbol{\sigma},\boldsymbol{\tau}}(V_1,\ldots,V_k)$	$\mathbf{alogistic}_{\boldsymbol{\sigma},\boldsymbol{\tau}}\left(\boldsymbol{\omega}_{X_1,Y},\ldots,\boldsymbol{\omega}_{X_k,Y}\right)$	$\frac{\frac{1}{1+e^{-\sigma(V_1+\cdots+V_k-\tau)}}-\frac{1}{1+e^{\sigma\tau}}}{\frac{1}{1+e^{-\sigma(\boldsymbol{\omega}_{X_1,Y}+\cdots+\boldsymbol{\omega}_{X_k,Y}-\tau)}}-\frac{1}{1+e^{\sigma\tau}}}$

b. If the combination functions are scalar-free and $X_1,\ldots,X_k$ are the states from which state Y gets incoming connections, and for $U_1,\ldots,U_k$, $V_1,\ldots,V_k$ and $\alpha\geq 0$ it holds $V_i=\alpha U_i$, then

$$\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V_1,\ldots,\boldsymbol{\omega}_{X_k,Y}V_k)=\alpha\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}U_1,\ldots,\boldsymbol{\omega}_{X_k,Y}U_k).$$

If in this situation in two different simulations, state values $X_i(t)$ and $X_i'(t)$ are generated then

$$X_i'(t)=\alpha X_i(t)\Rightarrow X_i'(t+\Delta t)=\alpha X_i(t+\Delta t).$$

c. If the combination functions are additive and $X_1,\ldots,X_k$ are the states from which state Y gets incoming connections, then for values $U_1,\ldots,U_k$, $V_1,\ldots,V_k$ it holds

$$\begin{aligned}\mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y}(U_1+V_1),\ldots,\boldsymbol{\omega}_{X_k,Y}(U_k+V_k)\right)&=\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}U_1,\ldots,\boldsymbol{\omega}_{X_k,Y}U_k)\\&\quad+\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V_1,\ldots,\boldsymbol{\omega}_{X_k,Y}V_k).\end{aligned}$$

If in this situation in three different simulations, state values $X_i(t)$, $X_i'(t)$, and $X_i'(t)$ are generated then

$$X_i''(t)=X_i(t)+X_i'(t)\Rightarrow X_i''(t+\Delta t)=X_i(t+\Delta t)+X_i'(t+\Delta t).$$

d. If the combination functions are scalar-free and monotonically increasing, and $X_1,\ldots,X_k$ are the states from which state Y gets incoming connections, and $V_1\leq X_1(t),\ldots,X_k(t)\leq V_2$ for some values V_1 and V_2, then also

$$V_1\leq\mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y}X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}X_k(t)\right)\leq V_2$$

and if $\boldsymbol{\eta}_Y\Delta t\leq 1$ and $V_1\leq Y(t)\leq V_2$ then $V_1\leq Y(t+\Delta t)\leq V_2$.

Proof.

a) This works as follows:

$$\mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y}X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}X_k(t)\right)=\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V,\ldots,\boldsymbol{\omega}_{X_k,Y}V)=V\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y},\ldots,\boldsymbol{\omega}_{X_k,Y})=V$$

b) and c) can easily be verified.

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d) This follows from

$$\begin{aligned} V_1 &= V_1 \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}) = \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} V_1, \ldots, \boldsymbol{\omega}_{X_k,Y} V_1) \\ &\leq \mathbf{c}_Y\big(\boldsymbol{\omega}_{X_1,Y} X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y} X_k(t)\big) \leq \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} V_2, \ldots, \boldsymbol{\omega}_{X_k,Y} V_2) \\ &= V_2 \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}) \\ &= V_2 \end{aligned}$$

and the second part from

$$\begin{aligned} Y(t+\Delta t) &= Y(t) + \boldsymbol{\eta}_Y\big[\mathbf{c}_Y\big(\boldsymbol{\omega}_{X_1,Y} X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y} X_k(t)\big) - Y(t)\big]\Delta t \\ &= \mathbf{c}_Y\left(\boldsymbol{\omega}_{X_1,Y} X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y} X_k(t)\right) \boldsymbol{\eta}_Y \Delta t + Y(t)(1 - \boldsymbol{\eta}_Y \Delta t) \\ &\leq V_2 \boldsymbol{\eta}_Y \Delta t + V_2(1 - \boldsymbol{\eta}_Y \Delta t) = V_2 \end{aligned}$$

and similarly for V_1

$$\begin{aligned} Y(t+\Delta t) &= \mathbf{c}_Y\big(\boldsymbol{\omega}_{X_1,Y} X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y} X_k(t)\big)\boldsymbol{\eta}_Y \Delta t + Y(t)(1 - \boldsymbol{\eta}_Y \Delta t) \\ &\geq V_1 \boldsymbol{\eta}_Y \Delta t + V_1(1 - \boldsymbol{\eta}_Y \Delta t) = V_1 \end{aligned}$$

□

6. Asymptotic network behavior and network characteristics

How the network structure characteristics concerning connectivity and combination function characteristics as discussed in Sections 4 and 5 relate to emerging network behavior is discussed in this section. As a first case, a network without cycles is considered. The following theorem has been proven using Lemma 1 from Section 3 and Proposition 1 (see Treur, 2018a).

Theorem 2 (Common state values provide equilibria)**.** *Suppose a network with nonnegative connections is based on normalized and scalar-free combination functions, and the states without any incoming connection have a constant value. Then the following hold.*

a. Whenever all states have the same value V, the network is in an equilibrium state.
b. If for every state for its initial value V it holds $V_1 \leq V \leq V_2$, then for all t for every state Y it holds $V_1 \leq Y(t) \leq V_2$. In an achieved equilibrium for every state for its equilibrium value V it holds $V_1 \leq V \leq V_2$.

Theorem 3 (Common equilibrium state values: acyclic case)**.** *Suppose an acyclic network with nonnegative connections is based on normalized and scalar-free combination functions.*

a. If in an equilibrium state the independent states all have the same value V, then all states have the same value V.
b. If, moreover, the combination functions are monotonically increasing, and in an equilibrium state the independent states all have values V with $V_1 \leq V \leq V_2$, then all states have values V with $V_1 \leq V \leq V_2$.

The following is a useful basic lemma for dynamics of normalized networks with combination functions that are (strictly) monotonically increasing and scalar-free (see Treur, 2018a).

Lemma 2. *Let a normalized network with nonnegative connections be given and its combination functions are monotonically increasing and scalar-free; then the following hold:*

a. (i) If for some node Y at time t for all nodes X with $\boldsymbol{\omega}_{X,Y} > 0$ it holds $X(t) \leq Y(t)$, then Y(t) is decreasing at t: $\frac{\mathbf{d}Y(t)}{\mathbf{d}t} \leq 0$.

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(ii) *If the combination functions are strictly increasing and at time t for all nodes X with* $\boldsymbol{\omega}_{X,Y} > 0$ *it holds* $X(t) \leq Y(t)$*, and a node X exists with* $X(t) < Y(t)$ *and* $\boldsymbol{\omega}_{X,Y} > 0$*, and the speed factor of Y is nonzero, then Y(t) is strictly decreasing at t:* $\frac{\mathbf{d}Y(t)}{\mathbf{d}t} < 0$.

b. (i) *If for some node Y at time t for all nodes X with* $\boldsymbol{\omega}_{X,Y} > 0$ *it holds* $X(t) \geq Y(t)$*, then Y(t) is increasing at t:* $\frac{\mathbf{d}Y(t)}{\mathbf{d}t} \geq 0$.

(ii) *If the combination function is strictly increasing and at time t for all nodes X with* $\boldsymbol{\omega}_{X,Y} > 0$ *it holds* $X(t) \geq Y(t)$*, and a node X exists with* $X(t) > Y(t)$ *and* $\boldsymbol{\omega}_{X,Y} > 0$*, and the speed factor of Y is nonzero, then Y(t) is strictly increasing at t:* $\frac{\mathbf{d}Y(t)}{\mathbf{d}t} > 0$.

The following theorem has been proven for strongly connected networks with cycles using Lemma 1 and 2 (see Treur, 2018a).

Theorem 4 (Common equilibrium state values: strongly connected cyclic case)**.** *Suppose the combination functions of the normalized network N are scalar-free and strictly monotonically increasing. Then the following hold.*

a. *If the network is strongly connected itself, then in an equilibrium state all states have the same value.*
b. *Suppose the network has one or more independent states and the subnetwork without these independent states is strongly connected. If in an equilibrium state all independent states have values V with* $V_1 \leq V \leq V_2$*, then all states have values V with* $V_1 \leq V \leq V_2$*. In particular, when all independent states have the same value V, then all states have this same value V.*

The first main, general theorem is formulated by Theorems 5 and 6.

Theorem 5 (Main theorem on equilibrium state values, part I)**.** *Suppose the network N is normalized and its combination functions are scalar-free and strictly monotonic. Let* SC*(N) be the stratified condensation graph of N. Then in an equilibrium state of N, the following hold.*

a. *Suppose* $C \in \mathrm{SC}(N)$ *is a strongly connected component of N of level 0, and in case it consists of a single state without any incoming connection, this state has a constant value. Then the following hold:*
   (i) *All states in N belonging to C have the same equilibrium value V.*
   (ii) *If for the initial values V of all states in N belonging to C, it holds* $V_1 \leq V \leq V_2$*, then also for the equilibrium values V of all states in C it holds* $V_1 \leq V \leq V_2$.
   (iii) *In particular, when all initial values of states in N belonging to C are equal to one value V, then the equilibrium value of all states in C is also V.*
b. *Let* $C \in \mathrm{SC}(N)$ *be a strongly connected component of N of level* $i > 0$*. Let* $C_1, \ldots, C_k \in \mathrm{SC}(N)$ *be the strongly connected components of N with an outgoing connection to C within the condensation graph* SC*(N). Then the following hold.*
   (i) *If for the equilibrium values V of all states in N belonging to* $C_1 \cup \cdots \cup C_k$ *it holds* $V_1 \leq V \leq V_2$*, then for all states in N belonging to C for their equilibrium value V it holds* $V_1 \leq V \leq V_2$.
   (ii) *In particular, when all equilibrium values of all states in N belonging to* $C_1 \cup \cdots \cup C_k$ *are equal to one value V, then also the equilibrium values of all states in N belonging to C are equal to the same V.*

Proof.

a. (i) follows from Theorem 3a).
   (ii) follows from Proposition 1b).
   (iii) This follows from (ii) with $V_1 = V_2 = V$.

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b. (i) This follows from Theorem 3b applied to C augmented with (as independent states) the states in $C_1 \cup \cdots \cup C_k$ with outgoing connections to states in C, with their values and these connections.
(ii) follows from (i) with $V_1 = V_2 = V$. □

Theorem 6 (Main theorem on equilibrium state values, part II)**.** *Suppose the network N is normalized and its combination functions are scalar-free and strictly monotonic. Let* SC*(N) be the stratified condensation graph of N. Then in an equilibrium state of N the following hold.*

a. If the equilibrium values of all states in every strongly connected component of level 0 in SC*(N) are equal to one value V, then the equilibrium state values of all states in N are equal to the same value V.*
b. If for the equilibrium values V of all states in every strongly connected component of level 0 in SC*(N) it holds $V_1 \leq V \leq V_2$, then for the equilibrium state values V of all states in N it holds $V_1 \leq V \leq V_2$.*
c. Suppose the states without any incoming connection have a constant value. If the initial values of all states in every strongly connected component of level 0 in SC*(N) are equal to one value V, then for the equilibrium state values of all states in N are equal to the same value V.*
d. Suppose the states without any incoming connection have a constant value. If for the initial values V of all states in every strongly connected component of level 0 in SC*(N) it holds $V_1 \leq V \leq V_2$, then for the equilibrium state values V of all states in N it holds $V_1 \leq V \leq V_2$.*

Proof. This follows by using induction over the number of strata in SC(N) and applying Theorem 4a) for the level 0 stratum and Theorem 4b) for the induction step from the strata of level $j < i$ to the stratum of level $i > 0$. □

As an illustration, for the example simulation, the following implications of these theorems can be found.

- Level 0 components
  The strongly connected components of level 0 are the subnetworks based on $\{X_1\}$ and $\{X_5, X_6, X_7\}$ (see Figures 3 and 5). As shown in Table 2, the initial values of X_1 and X_5 are 0.9, and the initial values for all other states are 0. From Theorem 5a)(i) and 4a)(ii), it follows that the equilibrium value of X_1 is 0.9, which indeed is the case, and those of X_5, X_6, X_7 are the same and ≤ 0.9; this is indeed confirmed in Figure 2, as these three equilibrium values of X_5, X_6, X_7 are all 0.3. This value 0.3 depends on the initial values of the states and the connection weights, which are not taken into account in the theorems; however, see also Theorem 7 below.
- Level 1 component
  For the level 1 component C_3, based on $\{X_2, X_3, X_4\}$, it goes as follows. The only incoming connection for C_3 is from equilibrium value 0.9 (implied by Theorem 5a)(ii)). By Theorem 5b)(ii), it follows that X_2, X_3, X_4 all have the same equilibrium value 0.9; this is indeed confirmed in Figure 2.
- Level 2 component
  The level 2 component C_4 is based on $\{X_8, X_9, X_{10}\}$. It has two incoming connections, one from X_3 in C_3 and one from X_5 in C_2. Their equilibrium values are 0.9 and 0.3, respectively, so they are not equal. Therefore, the above theorems do not imply that the equilibrium values of X_8, X_9, X_{10} are the same; indeed in Figure 2 they are different: 0.681, 0.490, and 0.389, respectively. But there is still an implication from Theorem 5b)(i), namely, that these equilibrium values should be ≥ 0.3 and ≤ 0.9. This is indeed confirmed in Figure 2.

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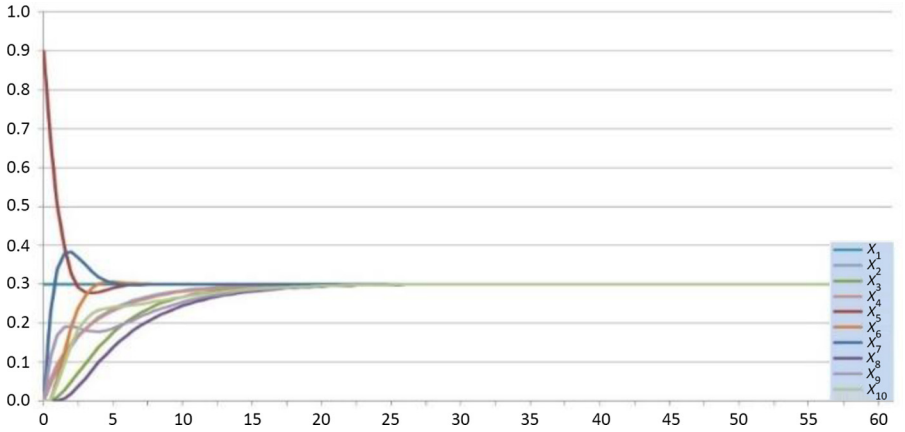


Figure 7. Variation of the example simulation for initial value 0.3 of X_1.

This illustrates how the above theorems have implications for simulations. Note that the specific equilibrium values 0.681, 0.490, and 0.389 are not predicted here. They also depend on the connection weights for the states X_8, X_9, X_{10} within component C_4, and these are not taken into account in the theorems; however, see also below, in the last part of this section.

Consider a variation, by setting the initial value of X_1 at 0.3 instead of 0.9. Then all equilibrium values turn out to become the same 0.3 (see Figure 7). Now the values of all states in the level 0 components C_1 and C_2 have the same value 0.3. As above, also the states in C_3 have the equilibrium value 0.3 because they are only affected by X_1 which has value 0.3. But now the equilibrium values of both X_3 in C_3 and X_5 in C_2 are the same 0.3, so this time Theorem 5b)(ii) can be applied to derive that all states in C_4 also have that same equilibrium value 0.3.

This predicts that all states of the network have value 0.3 in the equilibrium. Alternatively, Theorem 6a) can be applied for this case. By that theorem from the equal equilibrium values in the level 0 components C_1 and C_2 it immediately follows that all states in all components in the network have that same equilibrium value.

As seen above, in the theorems, the level 0 components play a central role, as initial nodes in the stratified condensation graph SC(N). Therefore, it can be useful to know more about them, for example, how their initial values determine all equilibrium values in the network. This is addressed for the case of a linear combination function in the following theorem.

Theorem 7 (Equilibrium state values in relation to level 0 components in the linear case)**.** *Suppose the network N is normalized and the combination functions are strictly monotonically increasing and linear. Assume that the states at level 0 that form a singleton component on their own are constant.*

Then the following hold:

a. *For each state Y, its equilibrium value is independent of the initial values of all states at some level $i > 0$. It is only dependent on the initial values for the states at level 0.*
b. *More specifically, let $B_1, \ldots, B_p$ be the states in level 0 components. Then for each state Y, its equilibrium value eq_Y is described by a linear function of the initial values $V_1, \ldots, V_p$ for $B_1, \ldots, B_p$, according to the following weighted average:*

$$\mathrm{eq}_Y(V_1, \ldots, V_k) = d_{B_1,Y} V_1 + \cdots + d_{B_p,Y}$$

Here, the $d_{B_i,Y}$ are real numbers between 0 and 1 and the sum of them is 1:

$$d_{B_1,Y} V_1 + \cdots + d_{B_p,Y} V_p = 1.$$

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c. *Each $d_{B_i,Y}$ is the equilibrium value for Y when the following initial values are used: $V_i = 1$ and all other initial values are 0:*
$d_{B_i,Y} = \text{eq}_Y(0, \ldots 0, 1, 0, ..0)$ *with 1 as ith argument.*

Proof. From Proposition 1, it follows that he equilibrium value of Y is a linear function of the initial values of all states of N. Therefore, the function is a linear combination of $e_i = \text{eq}_Y\ (0, \ldots 0, 1, 0, ..0)$ where only one state has initial value 1 and all other 0. An alternative, more theoretical linear algebra argument uses that the set of functions over time generated by the difference equations for different initial values forms an n-dimensional linear space with as basis the functions $d_i(t)$ generated for initial value 1 for state X_i and 0 for all other states. Therefore, each generated function is a linear combination of such functions. By substituting $t = 0$ in them, it is shown that the coefficients are the initial values and substituting t for an equilibrium shows that these initial values are the coefficients at that time point.

Now consider the different stratification levels. When all level 0 states have initial value 0, then by Theorem 5a)(iii), they will have equilibrium value 0 as well. Then, from Theorem 5b)(ii), it follows that all states will have equilibrium value 0. In particular, this holds for cases that only one of the states at a level $i > 0$ have value 1 and all other states have initial value 0. This shows that from the linear combination the coefficients of these terms are 0. Therefore, $\text{eq}_Y(..)$ is a function of $V_1, \ldots, V_p$ only. □

Note that Theorem 7c can be used to determine the values of the numbers $d_{B_i,Y}$ by simulation for each of these p initial value settings. However, in Section 7, it will also be shown how they can be determined by symbolically solving the equilibrium equations. Based on Theorem 7, for the case of linear combination functions, for level 0 components after each value $d_{B_i,Y}$ is determined, any equilibrium value can be predicted from the initial values by the identified linear expression.

Note that for the case of linear combination functions the equilibrium equations are linear and could be solved algebraically. But this does not provide additional information for nonsingleton level 0 components. They have an infinite number of solutions as every common value V is a solution; apparently, the linear equations always have a mutual dependency in this case. However, for components of level $i > 0$, solving the linear equations can provide specific values, due to the specific input values they get from one or more lower level components. In Section 7, such implications of the theorems for some example networks are shown. The next theorems show some variations on Theorem 7.

Theorem 8 (Equilibrium state values for level 0 components)**.** *Suppose the network N with states $X_1, \ldots, X_n$ is normalized and strongly connected. Then the following hold.*

a. *If the combination functions of the network N are scalar-free, then for given connection weights and speed factors, for any value $V \in [0, 1]$, there are initial values such that V is the common state value in an equilibrium achieved from these initial values.*
b. *For given connection weights and speed factors, let* $\text{eq}: [0, 1]^n \rightarrow [0, 1]$ *be the function such that* $\text{eq}(V_1, \ldots, V_k)$ *is the common state value for an equilibrium achieved from initial values $X_i(0) = V_i$ for all i. Then* $\text{eq}(0, \ldots, 0) = 0$, $\text{eq}(1, \ldots, 1) = 1$, *and the following hold:*
   (i) *If the combination functions of the network are scalar-free, then eq is scalar-free.*
   (ii) *If the combination functions of the network are additive, then eq is additive.*
c. *Suppose the combination functions of the network N are linear. For given connection weights and speed factors for each i, let e_i be the achieved common equilibrium value for initial values $X_i(0) = 1$ and $X_j(0) = 0$ for all $j \neq i$, that is, $e_i = eq(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 as ith argument. Then the sum of the e_i is 1, that is, $e_1 + \cdots + e_n = 1$ and in the general case for these given connection weights and speed factors, the common equilibrium value eq(…) is a*

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linear, monotonically increasing, continuous, and differentiable function of the initial values $V_1, \ldots, V_n$ *satisfying the following linear relation:*

$$\mathrm{eq}(V_1, \ldots, V_k) = e_1 V_1 + \cdots + e_n V_n$$

If the combination functions of N are strictly increasing, then $e_i > 0$ *for all i, and eq is also strictly increasing.*

Proof.

a) This follows from Proposition 1a) or d) with $V_1 = V_2 = V$.
b) and c). This follows from Propositions 1b) and 1c), and Lemma 2. □

Theorem 9 (Equilibrium state values for components of level $i > 0$)**.** *Suppose the network is normalized and consists of a strongly connected component plus a number of independent states* $A_1, \ldots, A_p$ *with outgoing connections to this strongly connected component. Then the following hold:*

a. Suppose the combination functions are scalar-free and $X_1, \ldots, X_k$ *are the states from which state Y gets incoming connections. If for* $U_1, \ldots, U_k$, $V_1, \ldots, V_k$ *and* $\alpha \geq 0$ *it holds* $V_i = \alpha\, U_i$ *for all i, then*

$$\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} V_1, \ldots, \boldsymbol{\omega}_{X_k,Y} V_k) = \alpha\, \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} U_1, \ldots, \boldsymbol{\omega}_{X_k,Y} U_k).$$

b. Suppose the combination functions are additive and $X_1, \ldots, X_k$ *are the states from which state Y gets incoming connections. Then if for values* $U_1, \ldots, U_k$, $V_1, \ldots, V_k$, $W_1, \ldots, W_k$, *it holds* $W_i = U_i + V_i$ *for all i, then*

$$\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} W_1, \ldots, \boldsymbol{\omega}_{X_k,Y} W_k) = \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} U_1, \ldots, \boldsymbol{\omega}_{X_k,Y} U_k) + \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} V_1, \ldots, \boldsymbol{\omega}_{X_k,Y} V_k).$$

c. Suppose all combination functions of the network N are linear. Then for given connection weights and speed factors, for each state Y, the achieved equilibrium value for Y only depends on the equilibrium values $V_1, \ldots, V_p$ *of states* $A_1, \ldots, A_p$*; the function* $\mathrm{eq}_Y(V_1, \ldots, V_p)$ *denotes this achieved equilibrium value for Y.*

d. Suppose the combination functions of the network N are linear. For the given connection weights and speed factors for each i, let $d_{i,Y}$ *be the achieved equilibrium value for state Y in a situation with equilibrium values* $A_i = 1$ *and* $A_j = 0$ *for all* $j \neq i$*, that is,*

$$d_{i,Y} = \mathrm{eq}_Y(0, \ldots 0, 1, 0, ..0)$$

with 1 as ith argument. Then in the general case for these given connection weights and speed factors, for each Y in the strongly connected component its equilibrium value is a linear, monotonically increasing, continuous, and differentiable function $\mathrm{eq}_Y(\ldots)$ *of the equilibrium values* $V_1, \ldots, V_p$ *of* $A_1, \ldots, A_p$ *satisfying the following linear relation:*

$$\mathrm{eq}_Y(V_1, \ldots, V_p) = d_{1,Y} V_1 + \cdots + d_{p,Y} V_p.$$

Here, the sum of the

$$d_{1,Y} + \cdots + d_{p,Y} = 1.$$

In particular, the equilibrium values are independent of the initial values for all states Y different from $A_1, \ldots, A_p$*. If the combination functions of N are strictly increasing, then* $d_{i,Y} > 0$ *for all i, and* $\mathrm{eq}_Y(..)$ *is also strictly increasing.*

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Table 5. Coefficients of the linear relations between limit values and initial values.

	X_1	X_5	X_6	X_7
X_8	0.634921	0.121693	0.121693	0.121693
X_9	0.31746	0.227513	0.227513	0.227513
X_{10}	0.148148	0.283951	0.283951	0.283951
	$d_{X_1,Y}$	$d_{X_5,Y}$	$d_{X_6,Y}$	$d_{X_7,Y}$

Proof.

a) and b) follow from Proposition 1

c) From a) and b), it follows that the equilibrium value of Y is a linear function of the initial values of all states of N. Therefore, the function is a linear combination of $e_i = \mathrm{eq}_Y(0, \ldots, 0, 1, 0, \ldots, 0)$, where only one state has initial value 1 and all other 0. However, when all independent states have (constant) value 0, from Theorem 5b)(ii), it follows that all states will have equilibrium value 0. In particular, this holds for cases that only one of the states that are not independent have initial value 1 and all other states have initial value 0. This shows that from the linear combination the coefficient e_i of these terms are 0. Therefore, $\mathrm{eq}_Y(\ldots)$ is a function of $V_1, \ldots, V_p$ only. From a) and b), it follows that $\mathrm{eq}_Y(V_1, \ldots, V_p)$ is linear, as indicated above. Therefore,

$$\begin{aligned}\mathrm{eq}_Y(V_1, \ldots, V_p) &= \mathrm{eq}_Y(V_1, 0, \ldots, 0) + .. + \mathrm{eq}_Y(0, .., 0, V_i, 0, .., 0) + .. + \mathrm{eq}_Y(0, \ldots, 0, V_p) \\ &= \mathrm{eq}_Y(1, 0, \ldots, 0)V_1 + .. + \mathrm{eq}_Y(0, .., 0, 1, 0, .., 0)V_i + .. + \mathrm{eq}_Y(0, \ldots, 0, 1)V_p \\ &= d_{1,Y}V_1 + \cdots + d_{p,Y}V_p\end{aligned}$$

□

Note that by using Theorem 3 instead of Theorem 5b)(ii) in the above proof a similar theorem is obtained for the case of an acyclic network; then, the equilibrium values of all states are linear combinations of the values of the initial states.

7. Further implications for example networks

In this section, it is shown what further conclusions can be drawn from the theorems presented in Section 6 for the example described in Section 3 and for an example Mental Network described in Schoenmakeret al. (2018). This shows that the applicability goes beyond only social networks. First, the earlier example described in Section 3 is analyzed; after that the new example will be addressed.

7.1 Further analysis of the example network shown in Figure 1

Theorems 7 to 9 are illustrated by the example network shown in Figure 1 as follows. Here there is only one independent constant state X_1 with singleton component. Moreover, the states in the other level 0 component C_2 are X_5, X_6, X_7, respectively (see Figure 5). So, from Theorem 7, it follows that the equilibrium value of any state Y is

$$\mathrm{eq}_Y(V_1, V_2, V_3, V_4) = d_{X_1,Y}V_1 + d_{X_5,Y}V_2 + d_{X_6,Y}V_3 + d_{X_7,Y}V_4$$

where $V_1 V_2$, V_3, and V_4 are the initial values of the states X_1, X_5, X_6, and X_7 in the level 0 components C_1 and C_2. For the example states $Y \in \{X_8, X_9, X_{10}\}$, the coefficients $d_{X_1,Y}$, $d_{X_5,Y}$, $d_{X_6,Y}$, $d_{X_7,Y}$ have been determined by simulation for the connection weights shown in Table 2 (and using speed factors 0.5), with these results shown in Table 5.

So, for example, for $Y = X_8$, the four coefficients are

$$\begin{aligned}d_{X_1,X_8} &= 0.634921 \\ d_{X_5,X_8} &= 0.121693\end{aligned}$$

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$$d_{X_6,X_8} = 0.121693$$

$$d_{X_7,X_8} = 0.121693.$$

Therefore, in a sense the equilibrium value of X_8 can be considered to be determined for 63.5% by the constant value of X_1 and for 12.2% by each of the initial values of X_5, X_6, X_7. These four values indeed sum up to 1% or 100%. Note that in this case the last three coefficients happen to be equal, as for the sake of simplicity this component is just one cycle and is therefore highly symmetric; this is not always the case. More specifically, given the above values, for the considered case the equilibrium value for X_8 is given by

$$\mathrm{eq}_{X_8}(V_1, V_2, V_3, V_4) = 0.634921\ V_1 + 0.121693\ V_2 + 0.121693\ V_3 + 0.121693\ V_4$$

with V_1 the constant value of X_1 and V_2, V_3, and V_4 the initial values of X_5, X_6, X_7, respectively. This is indeed confirmed in simulations.

For the example, as the scaled sum used is linear, solving the linear equations can also provide specific values. In this way, in line with Theorems 7 to 9, the specific equilibrium values of the states X_8, X_9, and X_{10} in C_4 can be determined algebraically from the equilibrium values $\boldsymbol{X_3}, \boldsymbol{X_5}$, and $\boldsymbol{X_7}$ of the states X_3, X_5, and X_7 in the lower level components C_2 and C_3. Using a symbolic solver (the online WIMS Linear Solver tool[1] was used), this can be done more in general. The linear equilibrium equations for X_8, X_9, and X_{10} are:

$$(\omega_{X_3,X_8} + \omega_{X_{10},X_8})X_8 = \omega_{X_3,X_8}\boldsymbol{X_3} + \omega_{X_{10},X_8}\ X_{10}$$

$$(\omega_{X_5,X_9} + \omega_{X_8,X_9})X_9 = \omega_{X_5,X_9}\boldsymbol{X_5} + \omega_{X_8,X_9}\ X_8$$

$$(\omega_{X_7,X_{10}} + \omega_{X_9,X_{10}})X_{10} = \omega_{X_7,X_{10}}\boldsymbol{X_7} + \omega_{X_9,X_{10}}\ X_9$$

These general equations have the following unique symbolic solution (displayed by the Linear Solver) when $\boldsymbol{X_3}, \boldsymbol{X_5}, \boldsymbol{X_7}$ are assumed given from the lower level components, and from this the values of the coefficients d_{X_i,X_j} of the linear relation from Theorem 8 can be determined: see Box 1.

$$X_8 = (\omega_{X10,X8}(\omega_{X7X10}\ \omega_{X8,X9}\ X_7 + \omega_{X5,X9}\ \omega_{X7,X10}\ X_7 + \omega_{X5,X9}\ \omega_{X9,X10}\ X_5) + \omega_{X3,X8}(\omega_{X8,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10}))\ X_3)$$
$$/(\omega_{X3,X8}(\omega_{X8,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X5,X9}(\omega_{X9X10}+\omega_{X7,X10})) + \omega_{X10,X8}(\omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X7,X10}\ \omega_{X8,X9}))$$

$$d_{X3,X8} = \omega_{X3,X8}(\omega_{X8,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10}))$$
$$/(\omega_{X3,X8}(\omega_{X8,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X5,X9}(\omega_{X9X10}+\omega_{X7,X10})) + \omega_{X10,X8}(\omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X7,X10}\ \omega_{X8,X9}))$$

$$d_{X5,X8} = \omega_{X10,X8}\ \omega_{X5,X9}\ \omega_{X9,X10}$$
$$/(\omega_{X3,X8}(\omega_{X8,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X5,X9}(\omega_{X9X10}+\omega_{X7,X10})) + \omega_{X10,X8}(\omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X7,X10}\ \omega_{X8,X9}))$$

$$d_{X7,X8} = \omega_{X10,X8}\ (\omega_{X7X10}\ \omega_{X8,X9} + \omega_{X5,X9}\ \omega_{X7,X10})$$
$$/(\omega_{X3,X8}(\omega_{X8,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X5,X9}(\omega_{X9X10}+\omega_{X7,X10})) + \omega_{X10,X8}(\omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X7,X10}\ \omega_{X8,X9}))$$

$$X_9 = (\omega_{X10,X8}(\omega_{X7,X10}\ \omega_{X8,X9}\ X_7 + \omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10})\ X_5) + \omega_{X3,X8}\ \omega_{X5X9}(\omega_{X9,X10}+\omega_{X7,X10})\ X_5 + \omega_{X3,X8}\ \omega_{X8,X9}(\omega_{X9,X10}+\omega_{X7,X10})\ X_3)$$
$$/(\omega_{X3,X8}(\omega_{X8,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10})) + \omega_{X10,X8}(\omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X7,X10}\ \omega_{X8,X9}))$$

$$d_{X3,X9} = \omega_{X3,X8}\ \omega_{X8,X9}\ (\omega_{X9,X10}+\omega_{X7,X10})$$
$$/(\omega_{X3,X8}(\omega_{X8,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10})) + \omega_{X10,X8}(\omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X7,X10}\ \omega_{X8,X9}))$$

$$d_{X5,X9} = \omega_{X5X9}\ (\omega_{X10,X8} + \omega_{X3,X8}\ )(\omega_{X9,X10}+\omega_{X7,X10})$$
$$/(\omega_{X3,X8}(\omega_{X8,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10})) + \omega_{X10,X8}(\omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X7,X10}\ \omega_{X8,X9}))$$

$$d_{X7,X9} = \omega_{X10,X8}\ \omega_{X7,X10}\ \omega_{X8,X9}$$

$$X_{10} = (\omega_{X3,X8}(\omega_{X7,X10}\ \omega_{X8,X9}\ X_7 + \omega_{X5,X9}\ \omega_{X7,X10}\ X_7 + \omega_{X5,X9}\ \omega_{X9,X10}\ X_5)$$
$$+ \omega_{X10,X8}(\omega_{X7,X10}\ \omega_{X8,X9}\ X_7 + \omega_{X5,X9}\ \omega_{X7,X10}\ X_7 + \omega_{X5,X9}\ \omega_{X9,X10}\ X_5) + \omega_{X3,X8}\ \omega_{X8,X9}\ \omega_{X9,X10}\ X_3)$$
$$/(\omega_{X3X8}(\omega_{X8X9}(\omega_{X9X10}+\omega_{X7X10}) + \omega_{X5X9}(\omega_{X9X10}+\omega_{X7X10})) + \omega_{X10,X8}(\omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X7,X10}\ \omega_{X8,X9}))$$

$$d_{X3,X10} = \omega_{X3,X8}\ \omega_{X8,X9}\ \omega_{X9,X10}$$
$$/(\omega_{X3X8}(\omega_{X8X9}(\omega_{X9X10}+\omega_{X7X10}) + \omega_{X5X9}(\omega_{X9X10}+\omega_{X7X10})) + \omega_{X10,X8}(\omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X7,X10}\ \omega_{X8,X9}))$$

$$d_{X5,X10} = \omega_{X5,X9}\ \omega_{X9,X10}\ (\omega_{X3,X8} + \omega_{X10,X8})$$
$$/(\omega_{X3X8}(\omega_{X8X9}(\omega_{X9X10}+\omega_{X7X10}) + \omega_{X5X9}(\omega_{X9X10}+\omega_{X7X10})) + \omega_{X10,X8}(\omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X7,X10}\ \omega_{X8,X9}))$$

$$d_{X7,X10} = \omega_{X7,X10}\ (\omega_{X3,X8} + \omega_{X10,X8})\ (\omega_{X8,X9} + \omega_{X5,X9})$$
$$/(\omega_{X3X8}(\omega_{X8X9}(\omega_{X9X10}+\omega_{X7X10}) + \omega_{X5X9}(\omega_{X9X10}+\omega_{X7X10})) + \omega_{X10,X8}(\omega_{X5,X9}(\omega_{X9,X10}+\omega_{X7,X10}) + \omega_{X7,X10}\ \omega_{X8,X9}))$$

Box 1. Explicit expressions for the equilibrium values of X_8, X_9 and X_{10} as displayed by the WIMS Linear Solver and of the coefficients d_{X_i,X_j} for them

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Figure 8. The strongly connected components within the second example network.

As a special case, if all occurring ω_{X_i, X_j} are set equal to one value ω, then the denominator becomes $7\boldsymbol{\omega}^3$ and the following values are obtained:

$$d_{X_3,X_8} = 4/7 \quad d_{X_3,X_9} = 2/7 \quad d_{X_3,X_{10}} = 1/7$$
$$d_{X_5,X_8} = 1/7 \quad d_{X_5,X_9} = 4/7 \quad d_{X_5,X_{10}} = 2/7$$
$$d_{X_7,X_8} = 2/7 \quad d_{X_7,X_9} = 1/7 \quad d_{X_7,X_{10}} = 4/7$$

Then the linear relations for the equilibrium values become

$$X_8 = 4/7\, \boldsymbol{X_3} + 1/7\, \boldsymbol{X_5} + 2/7\, \boldsymbol{X_7}$$
$$X_9 = 2/7\, \boldsymbol{X_3} + 4/7\, \boldsymbol{X_5} + 1/7\, \boldsymbol{X_7}$$
$$X_{10} = 1/7\, \boldsymbol{X_3} + 2/7\, \boldsymbol{X_5} + 4/7\, \boldsymbol{X_7}$$

7.2 Analysis of an example mental network

In this section, applicability is illustrated for a type of network which is not a social network. In general, Theorems 7 to 9 can be applied for many cases of networks that receive external input. This varies from Mental Networks that get input from external stimuli to Social Networks that are affected by context factors such as broadcasts from external sources that are received by members of the network. As an example of this, for the mental area, the Mental Network model from Schoenmaker et al. (2018) has been analyzed. The strongly connected components are as shown in Figure 8, with stratified condensation graph as in Figure 8; for the connection weights see Table 4). The model describes how the emotional charge of a received tweet affects the decision to retweet it. It can be explained by the following scenario considering Mark sending a tweet to Tim in which he expresses that he cannot wait to sing in the Christmas choir next week.

> 'This tweet contains both information and emotional charge: there is a choir performance next week, and secondly, Mark makes clear that he cannot wait for this event to happen. Tim's interpretation of this message is positively influenced by the

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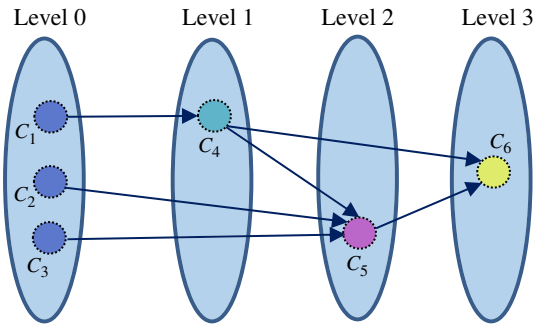


Figure 9. Stratified condensation graph SC(*N*) for the second example network.

> fact that Mark and Tim are friends. Tim does like to visit choir performances; therefore, he already has a positive association on the information that this event will take place. Reading about this Christmas performance, Tim gets slightly aroused and is focusing on the message. Mark's enthusiasm amplifies Tim's attention and arousal, which in turn lead to a positive interpretation of the tweet. Tim's positive interpretation of the message coupled with the fact that he is good friends with Mark and is excited about this performance leads to Tim's decision to retweet Mark's original Tweet.' (Schoenmaker et al., 2018, p. 138)

The states within the box Agent 1 all have a scaled sum combination function. The final state Sharing has $\mathbf{alogistic}_{\boldsymbol{\sigma},\boldsymbol{\tau}}(..)$ as combination function. For the analysis, the above theorems can be applied to the network when the state Sharing is left out of consideration.

The stratified condensation graph for this network is shown in Figure 9.

From this stratified condensation graph, a number of conclusions can be drawn:

- The level 0 states are the states Person, Information known, and Emotional charge in C_1, C_2, and C_3, respectively; therefore, these three states are the determining factors for the whole network.
- The level 1 state Relation with person will have the same equilibrium value as the level 0 state Person in C_1.
- When all level 0 states have the same equilibrium value V, then also all level 1 and level 2 states Relation with person, Opinion, Attention, Arousal, and Interpretation will have that same equilibrium value V. For example, when all level 0 states are constant 1, then all states as mentioned will end up in equilibrium value 1.
- When the level 0 states have different equilibrium values, then the level 2 states Opinion, Attention, Arousal, and Interpretation are expected to have different equilibrium values too, these values lay between the maximal and minimal values at level 0.

More specifically, in numbers, the following can be concluded. Suppose any given constant values $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ for the level 0 components in C_1, C_2, C_3, respectively. Then:

- at level 1, the equilibrium value in C_4 is $\mathbf{A_1}$
- at level 2, the equilibrium values of all four states in C_5 are between $\min(\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3})$ and $\max(\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3})$
- these equilibrium values of the four states in C_5 are linear functions in the form of weighted sums of $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$
- when all $\mathbf{A_i} = \mathbf{A}$ for one value, then at level 2, the equilibrium values of the states in C_5 are $\mathbf{A}$ as well.

The linear equilibrium equations for the states other than Sharing can be solved in a symbolic manner to obtain explicit algebraic expressions for their equilibrium values (again the online

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$$\mathbf{d}\text{Relation}/\mathbf{d}t = \eta_{\text{Relation}}[\omega_{\text{Person,Relation}}\ \text{Person} - \text{Relation}]$$

$$\mathbf{d}\text{Opinion}/\mathbf{d}t = \eta_{\text{Opinion}}\ [(\omega_{\text{Information,Opinion}}\ \text{Information} + \omega_{\text{Interpretation,Opinion}}\ \text{Interpretation})/\lambda_{\text{Opinion}} - \text{Opinion}]$$

$$\mathbf{d}\text{Interpretation}/\mathbf{d}t = \eta_{\text{Interpretation}}\ [\ (\omega_{\text{Relation,Interpretation}}\ \text{Relation} + \omega_{\text{Opinion,\ Interpretation}}\ \text{Opinion} + \omega_{\text{Attention,Interpretation}}\ \text{Attention} + \omega_{\text{Arousal,Interpretation}}\ \text{Arousal})/\lambda_{\text{Interpretation}} - \text{Interpretation}]$$

$$\mathbf{d}\text{Attention}/\mathbf{d}t = \eta_{\text{Attention}}\ [(\omega_{\text{Emotion,Attention}}\ \text{Emotion} + \omega_{\text{Relation,Attention}}\ \text{Relation} + \omega_{\text{Opinion,Attention}}\ \text{Opinion} + \omega_{\text{Interpretation,Attention}}\ \text{Interpretation})/\lambda_{\text{Attention}} -\ \text{Attention}]$$

$$\mathbf{d}\text{Arousal}\ /\mathbf{d}t = \eta_{\text{Arousal}}\ [(\omega_{\text{Emotion,Arousal}}\ \text{Emotion} + \omega_{\text{Relation,Arousal}}\ \text{Relation} + \omega_{\text{Opinion,Arousal}}\ \text{Opinion} + \omega_{\text{Interpretation,Arousal}}\ \text{Interpretation})/\lambda_{\text{Arousal}}\ - \text{Arousal}]$$

$$\mathbf{d}\text{Sharing}/\mathbf{d}t = \eta_{\text{Sharing}}\ [(\omega_{\text{Relation,Sharing}}\ \text{Relation} + \omega_{\text{Interpretation,Sharing}}\ \text{Interpretation} + \omega_{\text{Arousal,Sharing}}\ \text{Arousal})/\lambda_{\text{Sharing}} - \text{Sharing}]$$

Box 2. Overview of the differential equations of the second example network model.

$$\text{Relation} = \omega_{\text{Person,Relation}}\ \text{Person}$$

$$\text{Opinion} = (\omega_{\text{Information,Opinion}}\ \text{Information} + \omega_{\text{Interpretation,Opinion}}\ \text{Interpretation})/\lambda_{\text{Opinion}}$$

$$\text{Interpretation} = (\omega_{\text{Relation,Interpretation}}\ \text{Relation} + \omega_{\text{Opinion,\ Interpretation}}\ \text{Opinion} + \omega_{\text{Attention,Interpretation}}\ \text{Attention} + \omega_{\text{Arousal,Interpretation}}\ \text{Arousal})/\lambda_{\text{Interpretation}}$$

$$\text{Attention} = (\omega_{\text{Emotion,Attention}}\ \text{Emotion} + \omega_{\text{Relation,Attention}}\ \text{Relation} + \omega_{\text{Opinion,Attention}}\ \text{Opinion} + \omega_{\text{Interpretation,Attention}}\ \text{Interpretation})/\lambda_{\text{Attention}}$$

$$\text{Arousal} = (\omega_{\text{Emotion,Arousal}}\ \text{Emotion} + \omega_{\text{Relation,Arousal}}\ \text{Relation} + \omega_{\text{Opinion,Arousal}}\ \text{Opinion} + \omega_{\text{Interpretation,Arousal}}\ \text{Interpretation})/\lambda_{\text{Arousal}}$$

$$\text{Sharing} = (\omega_{\text{Relation,Sharing}}\ \text{Relation} + \omega_{\text{Interpretation,Sharing}}\ \text{Interpretation} + \omega_{\text{Arousal,Sharing}}\ \text{Arousal})/\lambda_{\text{Sharing}}$$

Box 3. Overview of the equilibrium equations of the second example network model.

Person $= X_1 = \mathbf{A_1}$ **Information** $= X_2 = \mathbf{A_2}$ **Emotion** $= X_3 = \mathbf{A_3}$

Relation $= X_4 = \omega_{\text{P,R}}\ \mathbf{A_1}$

Opinion $= X_5 =$

$$-[\mathbf{A_1}\ \omega_{\text{Int,O}}\ \omega_{\text{P,R}}(\lambda_{\text{Ar}}\ \lambda_{\text{At}}\ \omega_{\text{R,In}}t + \lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{RAt}} + \lambda_{\text{At}}\ \omega_{\text{Ar,Int}}\ \omega_{\text{R,Ar}})$$
$$+\mathbf{A_3}(\lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{E,At}} + \lambda_{\text{At}}\ \omega_{\text{Ar,Int}}\ \omega_{\text{E,Ar}})\ \omega_{\text{Int,O}}$$
$$+\mathbf{A_2}\ \omega_{\text{Inf,O}}((-\lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{Int,At}}) - \lambda_{\text{At}}\omega_{\text{Ar,Int}}\ \omega_{\text{Int,Ar}} + \lambda_{\text{Ar}}\ \lambda_{\text{At}}\ \lambda_{\text{I}})]$$
$$/\ [\omega_{\text{Int,O}}(\lambda_{\text{Ar}}\ \lambda_{\text{At}}\ \omega_{\text{O,Int}} + \lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{O,At}} + \lambda_{\text{At}}\ \omega_{\text{Ar,Int}}\ \omega_{\text{O,Ar}}) + \lambda_{\text{O}}(\lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{Int,At}} + \lambda_{\text{At}}\ \omega_{\text{Ar,Int}}\ \omega_{\text{Int,Ar}} - \lambda_{\text{Ar}}\ \lambda_{\text{At}}\ \lambda_{\text{I}})]$$

Interpretation $= X_6 =$

$$-[\mathbf{A_1}\ \lambda_{\text{O}}\ \omega_{\text{P,R}}(\lambda_{\text{Ar}}\ \lambda_{\text{At}}\ \omega_{\text{R,Int}} + \lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{R,At}} + \lambda_{\text{At}}\ \omega_{\text{Ar,Int}}\ \omega_{\text{R,Ar}})$$
$$+\mathbf{A_2}\ \omega_{\text{Inf,O}}(\lambda_{\text{Ar}}\ \lambda_{\text{At}}\ \omega_{\text{O,Int}} + \lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{O,At}} + \lambda_{\text{At}}\ \omega_{\text{Ar,Int}}\ \omega_{\text{O,Ar}})$$
$$+\mathbf{A_3}\ \lambda_{\text{O}}(\lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{E,At}} + \lambda_{\text{At}}\ \omega_{\text{Ar,Int}}\ \omega_{\text{E,Ar}})]$$
$$/\ [\omega_{\text{Int,O}}(\lambda_{\text{Ar}}\ \lambda_{\text{At}}\ \omega_{\text{O,Int}} + \lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{O,At}} + \lambda_{\text{At}}\ \omega_{\text{Ar,Int}}\ \omega_{\text{O,Ar}}) + \lambda_{\text{O}}(\lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{Int,At}} + \lambda_{\text{At}}\omega_{\text{Ar,Int}}\ \omega_{\text{Int,Ar}} - \lambda_{\text{Ar}}\ \lambda_{\text{At}}\ \lambda_{\text{I}})]$$

Attention $= X_7 =$

$$-[\mathbf{A_1}\ \omega_{\text{P,R}}\ (\omega_{\text{Int,O}}(\lambda_{\text{Ar}}\ \omega_{\text{O,At}}\ \omega_{\text{R,Int}} + \omega_{\text{Ar,Int}}(\omega_{\text{O,At}}\ \omega_{\text{R,Ar}} - \omega_{\text{O,Ar}}\ \omega_{\text{R,At}}) - \lambda_{\text{Ar}}\ \omega_{\text{O,Int}}\ \omega_{\text{R,At}})$$
$$+\lambda_{\text{O}}(\lambda_{\text{Ar}}\ \omega_{\text{Int,At}}\ \omega_{\text{R,Int}} + \omega_{\text{Ar,Int}}(\omega_{\text{Int,At}}\ \omega_{\text{R,Ar}} - \omega_{\text{Int,Ar}}\ \omega_{\text{R,At}}) + \lambda_{\text{Ar}}\ \lambda_{\text{I}}\ \omega_{\text{R,At}}))$$
$$+\mathbf{A_3}(\omega_{\text{Int,O}}(\omega_{\text{Ar,Int}}(\omega_{\text{E,Ar}}\ \omega_{\text{O,At}} - \omega_{\text{E,At}}\ \omega_{\text{O,Ar}}) - \lambda_{\text{Ar}}\ \omega_{\text{E,At}}\ \omega_{\text{O,Int}}) + \lambda_{\text{O}}(\omega_{\text{Ar,Int}}(\omega_{\text{E,Ar}}\ \omega_{\text{Int,At}} - \omega_{\text{E,At}}\ \omega_{\text{Int,Ar}}) + \lambda_{\text{Ar}}\ \lambda_{\text{I}}\ \omega_{\text{E,At}}))$$
$$+\mathbf{A_2}\ \omega_{\text{Inf,O}}\ (\lambda_{\text{Ar}}\ \omega_{\text{Int,At}}\ \omega_{\text{O,Int}} + \omega_{\text{Ar,Int}}(\omega_{\text{Int,At}}\ \omega_{\text{O,Ar}} - \omega_{\text{Int,Ar}}\ \omega_{\text{O,At}}) + \lambda_{\text{Ar}}\ \lambda_{\text{I}}\ \omega_{\text{O,At}})]$$
$$/[\omega_{\text{Int,O}}(\lambda_{\text{Ar}}\ \lambda_{\text{At}}\ \omega_{\text{O,Int}} + \lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{O,At}} + \lambda_{\text{At}}\ \omega_{\text{Ar,Int}}\ \omega_{\text{O,Ar}}) + \lambda_{\text{O}}(\lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{Int,At}} + \lambda_{\text{At}}\ \omega_{\text{Ar,Int}}\ \omega_{\text{Int,Ar}} - \lambda_{\text{Ar}}\ \lambda_{\text{At}}\ \lambda_{\text{I}})]$$

Arousal $= X_8 =$

$$-[\mathbf{A_1}\ \omega_{\text{P,R}}\ (\omega_{\text{Int,O}}(\lambda_{\text{At}}\ \omega_{\text{O,Ar}}\ \omega_{\text{R,Int}} + \omega_{\text{At,Int}}(\omega_{\text{O,Ar}}\ \omega_{\text{R,At}} - \omega_{\text{O,At}}\ \omega_{\text{R,Ar}}) - \lambda_{\text{At}}\ \omega_{\text{O,Int}}\ \omega_{\text{R,Ar}})$$
$$+\lambda_{\text{O}}(\lambda_{\text{At}}\ \omega_{\text{Int,Ar}}\ \omega_{\text{R,Int}} + \omega_{\text{At,Int}}(\omega_{\text{Int,Ar}}\ \omega_{\text{R,At}} - \omega_{\text{Int,At}}\ \omega_{\text{R,Ar}}) + \lambda_{\text{At}}\ \lambda_{\text{I}}\ \omega_{\text{R,Ar}}))$$
$$+\mathbf{A_3}(\omega_{\text{Int,O}}(\omega_{\text{At,Int}}(\omega_{\text{E,At}}\ \omega_{\text{O,Ar}} - \omega_{\text{E,Ar}}\ \omega_{\text{O,At}}) - \lambda_{\text{At}}\ \omega_{\text{E,Ar}}\ \omega_{\text{O,Int}}) + \lambda_{\text{O}}(\omega_{\text{At,Int}}(\omega_{\text{E,At}}\ \omega_{\text{Int,Ar}} - \omega_{\text{E,Ar}}\ \omega_{\text{Int,At}}) + \lambda_{\text{At}}\ \lambda_{\text{I}}\ \omega_{\text{E,Ar}}))$$
$$+\mathbf{A_2}\ \omega_{\text{Inf,O}}\ (\lambda_{\text{At}}\ \omega_{\text{Int,Ar}}\ \omega_{\text{O,Int}} + \omega_{\text{At,Int}}(\omega_{\text{Int,Ar}}\ \omega_{\text{O,At}} - \omega_{\text{Int,At}}\ \omega_{\text{O,Ar}}) + \lambda_{\text{At}}\ \lambda_{\text{I}}\ \omega_{\text{O,Ar}})\ ]$$
$$/\ [\omega_{\text{Int,O}}(\lambda_{\text{Ar}}\ \lambda_{\text{At}}\ \omega_{\text{O,Int}} + \lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{O,At}} + \lambda_{\text{At}}\ \omega_{\text{Ar,Int}}\ \omega_{\text{OAr}}) + \lambda_{\text{O}}(\lambda_{\text{Ar}}\ \omega_{\text{At,Int}}\ \omega_{\text{Int,At}} + \lambda_{\text{At}}\ \omega_{\text{Ar,Int}}\ \omega_{\text{Int,Ar}} - \lambda_{\text{Ar}}\ \lambda_{\text{At}}\ \lambda_{\text{I}})]$$

Box 4. Explicit algebraic solutions of the equilibrium equations of the second example network model as displayed by the WIMS Linear Solver; adopted from Schoenmaker et al. (2018).

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Table 6. Example values for the connection weights.

	State and connection	X_1 Person	X_2 Inf Known	X_3 Emo.	X_4 Relation	X_5 Opinion	X_6 Interpret	X_7 Attention	X_8 Arousal	X_9 Sharing b.
X_1	Person	0	0	0	1	0	0	0	0	0
X_2	Information	0	0	0	0	1	0	0	0	0
X_3	Emotional charge	0	0	0	0	0	0	1	1	0
X_4	Relationship with person	0	0	0	0	0	0.5	0.25	0.25	0.5
X_5	Opinion	0	0	0	0	0	0.75	0.25	0.25	0
X_6	Interpretation	0	0	0	0	0.75	0	0.25	0.25	1
X_7	Attention	0	0	0	0	0	0.75	0	0	0
X_8	Arousal	0	0	0	0	0	0.75	0	0	1
X_9	Sharing behavior	0	0	0	0	0	0	0	0	0

WIMS Linear Solver tool was used; see Box 3). Here, subscripts are abbreviated for the sake of briefness.

As can be seen, each of the equilibrium values is a linear combination of the three values $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ (as predicted by Theorem 8), where the coefficients are expressed in terms of specific connection weights and scaling factors. For example, this means that if all of these values $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are reduced by 20%, all equilibrium values will be reduced by 20%. This indeed is the case in simulation examples. If the values of the parameters for connection weights are assigned as in Table 6 and scaling factors $\boldsymbol{\lambda}_{\mathrm{Ar}} = 1.75$, $\boldsymbol{\lambda}_{\mathrm{Ar}} = 1.75$, $\boldsymbol{\lambda}_{\mathrm{O}} = 1.75$ and $\boldsymbol{\lambda}_{\mathrm{I}} = 2.75$, then the outcomes of the equilibrium values are:

$$\text{Person} = X_1 = \mathbf{A}_1; \text{Information} = X_2 = \mathbf{A}_2; \text{Emotion} = X_3 = \mathbf{A}_3; \text{Relation} = X_4 = \mathbf{A}_1;$$

$$\text{Opinion} = X_5 = 0.17307692\ \mathbf{A}_3 + 0.682692307\ \mathbf{A}_2 + 0.1442307692\ \mathbf{A}_1$$

$$\text{Interpretation} = X_6 = 0.40384615\ \mathbf{A}_3 + 0.259615384\ \mathbf{A}_2 + 0.336538461\ \mathbf{A}_1$$

$$\text{Attention} = X_7 = 0.65384615\ \mathbf{A}_3 + 0.13461538\ \mathbf{A}_2 + 0.21153846\ \mathbf{A}_1$$

$$\text{Arousal} = X_8 = 0.65384615\ \mathbf{A}_3 + 0.13461538\ \mathbf{A}_2 + 0.21153846\ \mathbf{A}_1$$

It can be seen that each of these equilibrium state values is a weighted average of $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ (for each the sum of these weights is 1, as predicted by Theorem 8). Therefore, in particular when all $\mathbf{A}_i$ are 1, all of these outcomes are 1. If only $\mathbf{A}_1$ and $\mathbf{A}_2$ are 1, then the outcomes depend just on the emotional charge $\mathbf{A}_3$:

$$\text{Person} = X_1 = 1.0; \text{Information} = X_2 = 1.0; \text{Emotion} = X_3 = \mathbf{A}_3; \text{Relation} = X_4 = 1.0;$$

$$\text{Opinion} = X_5 = 0.17307692\ \mathbf{A}_3 + 0.82692307;$$

$$\text{Interpretation} = X_6 = 0.40384615\ \mathbf{A}_3 + 0.59615384;$$

$$\text{Attention} = X_7 = 0.6538461\ \mathbf{A}_3 + 0.3461538;$$

$$\text{Arousal} = X_8 = 0.6538461\ \mathbf{A}_3 + 0.3461538;$$

$$\text{Sharing} = X_9 = \mathbf{alogistic}_{\sigma,\tau}(0.5,\ 0.40384615\ \mathbf{A}_3 + 0.59615384,\ 0.6538461\mathbf{A}_3 + 0.3461538).$$

It can be seen from this analysis that the equilibrium values of Attention and Arousal depend on about 65% on the emotional charge level, and as a consequence the impact of the emotional charge on the equilibrium value of Interpretation is about 40%. The effect of emotional charge on Sharing works through two causal pathways: via Interpretation and via Arousal. This leads to the function

$$\text{Sharing} = \mathbf{alogistic}_{\sigma,\tau}(0.5,\ 0.40384615\ \mathbf{A_3} + 0.59615384,\ 0.6538461\ \mathbf{A_3} + 0.3461538)$$

of $\mathbf{A}_3$, which is a monotonically increasing function of $\mathbf{A}_3$

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8. Discussion

To analyze and predict from its structure what asymptotic behavior a given network model will show is in general a challenging issue. For example, do all states in the network converge to the same value? Some results are available for the case of acyclic, fully connected or strongly connected networks and for linear combination functions only (e.g., Bosse et al., 2015). It is often believed that when nonlinear functions are used, such results become impossible. Also, networks that are not strongly connected are often not addressed as they are more difficult to handle. This paper shows what is still possible beyond the case of linear combination functions and also beyond the case of strongly connected networks.

In this paper, general theorems were presented that relate asymptotic network behavior to the network characteristics. The relevant network characteristics concern

- connectivity in terms of the network's strongly connected components and their mutual connections as shown in the network's condensation graph
- characteristics of the combination functions used to aggregate the effects of multiple incoming connections (in particular, monotonicity, scalar-freeness, and normalization).

The first item makes the approach applicable to any type of network connectivity, thus going beyond the limitation to strongly connected networks. The second item makes the approach applicable to a wider class of combination functions (most of which are nonlinear) going beyond the limitation to linear functions. However, there are also nonlinear functions that are not covered by this class. Some examples not covered are logistic functions, discrete threshold functions, and boolean functions, for example, as used in Karlsen & Moschoyiannis (2018) and Watts (2002). The current paper provides a first step to cover certain types of nonlinear functions. Nonlinear functions not covered yet form a next challenge that has been left open for now. In future research also, other types of nonlinear functions will be explored further. Note that the notion of temporal–causal network itself is not a limitation as it is a very general notion that covers all types of discrete or smooth dynamical systems, and all systems of first-order differential equations. For these results, see Treur (2017), building further, among others, on Ashby (1960) and Port & van Gelder (1995).

The presented theorems subsume and generalize existing theorems for specific cases such as similar theorems for acyclic networks, fully connected networks and strongly connected networks (e.g., Theorems 3 and 4 in Section 6), and theorems addressing only linear combination functions as one fixed type of combination function (e.g., Theorem 3 at p. 120 of Bosse et al., 2015).

The theorems can be applied to predict behavior of a given network, or to determine initial values in order to get some expected behavior. In particular, they can be used as a method of verification to check correctness of the implementation of a network. If simulation outcomes contradict the implications of the theorems, then some debugging of the implementation may be needed.

As already indicated in the Introduction section, after having developed the theorems presented here, it has turned out that these contributions also have some relations to research conducted from a different angle, namely, on control of networks (e.g., Liu et al., 2011, 2012; Moschoyiannis et al., 2016; Haghighi & Namazi, 2015; Karlsen & Moschoyiannis, 2018). In that area (e.g., Liu et al., 2011, 2012) usually a system of linear differential equations is used for the dynamics of the considered network with N nodes $x_1, \ldots, x_N$ represented over time t by states $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))$. The dynamics is based on the connections with weights a_{ij} from x_j to x_i, overall represented by a matrix $\mathbf{A} = (a_{ij})$. For the control M, additional nodes $u_1, \ldots, u_M$ are added, which are numerically represented over time t by states $\mathbf{u}(t) = (u_1(t), \ldots, u_M(t))$. These are meant to provide input at all time points in order to affect some of the network states (called drivers) over time. The latter nodes have connections to these driver nodes represented by an

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$N \times M$ input matrix $\mathbf{B} = (bij)$, where b_{ij} represents the weight of the connection from node u_j to node x_i. Then overall the dynamics of the extended network can be represented as

$$\frac{\mathbf{dx}(t)}{\mathbf{d}t} = \mathbf{Ax}(t) + \mathbf{Bu}(t)$$

Reachability within the network relates to the powers of matrix **A** and controllability of the network from the states $u_1, \ldots, u_M$ relate to the combined $N \times NM$ matrix $\mathbf{C} = (\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{N-1}\mathbf{B}) \in R^{N \times NM}$. Although precise mathematical criteria (e.g., Kalman, 1963) exist for this matrix **C** characterizing controllability of the network, such criteria often cannot be applied in practice as they depend on the precise values of the connection weights a_{ij} and in practical contexts usually these are not known. Therefore, in literature such as Liu et al. (2011, 2012), these weights are considered parameters, which introduces some complications: criteria for certain slightly different forms of (called structural) controllability are expressed in relation to these parameters (e.g., Lin, 1974); however, such criteria apply to (by far) most but not exactly all of such linear systems.

In contrast to the network control approach sketched in the previous paragraph, in the approach presented in the current paper, the lack of knowledge of specific weight values is not an issue, as these specific values are not used. Moreover, the theorems and their proofs do not make use of linearity assumptions, but instead of identified properties of a wider class of functions also including (a subset of the class of) nonlinear functions. Another difference is that the angle of controlling a network was not addressed in the current paper, as the focus was on an angle of verification of a network model. However, some of the theorems still can be used for controlling a network. For example, Theorem 6 can be applied when the states $u_1, \ldots, u_M$ of the vector **u** in the above formalization get outgoing connections (represented in matrix **B**) to the states within the level 0 components in the original network. Then the states within the level 0 components in the original network are used as drivers. More specifically, this theorem provides the following results for the considered class of nonlinear functions extending the class of linear functions.

- Theorem 6a) and b) show that the whole network can be controlled by only controlling the final equilibrium values of the states within the level 0 components of the network. This actually can be done by extending the network by nodes $\mathbf{u}_i$ that are connected to the states in level 0 components of the original network. In the extended network, this leads to singleton level 0 components $\{u_i\}$ and the other levels are increased by 1, for example, the level 0 components in the original network now become level 1 components in the extended network. Then, from Theorem 6a) and b), it follows that the equilibrium values of all states in the network depend on the equilibrium values of the states u_i in the level 0 components $\{u_i\}$, and these equilibrium values are $\lim_{t \to \infty} u_i(t)$, $i = 1, \ldots, M$.
- Note that if the u_i are kept constant over time, these limit values of the u_i are just the initial values $u_i(0)$; in this case, Theorem 6c) and d) apply. For example, for this case, Theorem 6c) shows that if these initial values $u_i(0)$ are all set at 1, then after some time all states of the network will get equilibrium value 1.

This illustrates how all states of the network can be controlled by only controlling the states within the level 0 components. Note that this has a partial overlap with what is found in Liu et al. (2012) for the linear case where also a decomposition based on the network's strongly connected components is used. In Theorems 7 to 9, it is described that some more can be said about how exactly the equilibrium value of each of the network's nodes depends on the initial or final values of the states in the level 0 components. In particular for the linear case, this equilibrium value of each state of the network is a linear function of the initial or equilibrium values of the states in the level 0 components.

This paper was invited as an extended (by more than 45%) version of Treur (2018b). Compared to Treur (2018b), the following were added in the current paper:

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- In Section 2, more combination functions are covered (scaled sum, scaled minimum, scaled maximum, simple logistic).
- In Section 3, three more simulations are included.
- In Section 5, Table 3 was added with an overview of the characteristics for the different combination functions.
- In Section 6, Theorem 7 was added.
- Section 7 was added as a whole; here in more detail the implications of the theorems are explored for two example networks.

Note

1 http://wims.unice.fr/wims/wims.cgi?session=DH1DFC9A6E.3&+lang=en&+module=tool%2Flinear%2Flinsolver.en

Conflict of interest. None

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