Tethering in pastures new

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Introduction

In a recent and illuminating article that provided much food for thought [1], the problem of tethering a goat at the edge of a circular pasture so as to restrict its attentions to only one half of the grazing supply was elegantly addressed and developed further to embrace the corresponding three-dimensional scenario involving a bird. The exercises resulted in mathematical formulations that required the use of numerical methods to extract practical results. Following the article, various questions and different scenarios sprang to my mind. The following poser perhaps best illustrates one of these, and it is the purpose of this Article to address this particular conundrum:

> A grazier has three troublesome beasts that are water averse, eat grass and who will, given half a chance, eat one another also in some fashion. The first will eat the other two and the second will eat only the third, which eats just grass. Having stabled and fed them in separate stalls during the winter months he plans to release them in the spring to an arbitrarily elliptic shaped pasture up to the water's edge in the middle of a lake. He has at his disposal:
>
> (1) A drum of tethering rope from which he can cut *just once* any required length, TBD (To Be Determined).
>
> (2) Slip rings and two tethering pegs that can be positioned only on the pasture boundary (i.e. at the water's edge).

Question 1: How might he best partition the ground using the shortest possible length of rope so that each creature can safely access equal amounts of pasture?

Question 2: Initially, he can make just one cut from the drum but if that restriction is lifted so that he is allowed to make two cuts, how might he otherwise economically configure the partition?

For the first question, it will be assumed without proof that perhaps a likely suitable scenario is one wherein the most ferocious of the beasts is collared to a slip ring that can slide along a rope tethered to pegs at each end of the minor axis of the elliptic pasture, the length of rope being sufficient to allow for access to just one third of the available grazing area. This allows for the most aggressive beast to pasture within an area common to two orthogonal ellipses, the major axis of the tethering ellipse lying along the minor axis of the elliptic island. The other two beasts can then be put to pasture safely on either side of this first beast's tethering area. This limiting case scenario is illustrated in Figure 1 below, but it is possible to envisage other scenarios. In the other limiting case where the major axis of the

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tethering ellipse lies along the major axis of the elliptic island pasture, the length of required tethering rope would be unnecessarily longer and, ergo, it will not be given further consideration. Also, for example, one might consider the case where the ellipses share a common origin but are oblique to one another, that is to say their major axes do not intersect at right angles, as in the figure. As likely as not, this arrangement would also result in a longer tether.

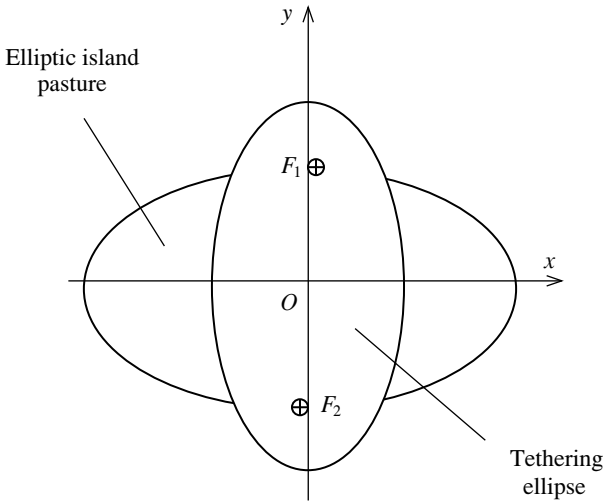


FIGURE 1: The single rope tethering scenario – showing the elliptic island with tethering pegs on its boundary at the water's edge, located at the focal points of the first beast's pasture

With respect to Figure 1, the tethering ellipse is shown extended into the water where there is no pasture; this is a no-go area for the beast because of its aversion to water. Also shown in the figure is a set of Cartesian axes whose origin is located at the centre of the island. The task then is to formulate the problem and develop first a mathematical expression for the area that is common to the two ellipses, after which it must be equated to one third of the area of the island pasture to provide an equation that can be solved numerically, following specified inputs (i.e. inputs related to the island size), so as to determine ultimately a required length for the tethering rope.

For the second question, the farmer must consider the possibility of locating two tethering pegs on the boundary, but this time each one can be located independently to serve as separate tethering points for two of the beasts. It is conceivable that they could be located almost anywhere about the boundary, depending on the size of the island. However, there is an obvious one, which is shown in Figure 2 and that will be the focus of attention here although a passing reference will be made to other possibilities.

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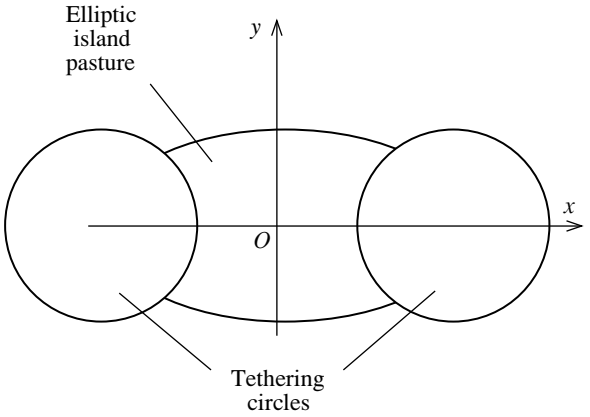


FIGURE 2: The two ropes tethering scenario

The two scenarios, the 'single rope' one and the 'two ropes' one will be considered separately hereunder.

Single rope scenario

There is symmetry in the geometry of the problem and it will suffice to determine the area that is common to both ellipses in the first quadrant, say, and multiply it by four to obtain the pasture available to the most aggressive of the beasts. Accordingly, it is necessary to determine first the coordinates of the point of intersection of both ellipses in that quadrant. With reference to the Cartesian system (Oxy) the tethering ellipse, E_1 say, can be described by the equation

$$\left(\frac{x}{b_1}\right)^2 + \left(\frac{y}{a_1}\right)^2 = 1. \tag{1}$$

In (1), b_1 denotes the semi-minor axis of the ellipse and a_1 denotes the semi-major axis. The eccentricity of the ellipse will be denoted by e_1 and it will be convenient to introduce at this stage an eccentricity angle, ψ_1 say, defined by $e_1 = \sin\psi_1$ so that $\dfrac{b_1}{a_1} = \cos\psi_1$. Similarly, the elliptic island pasture, E_2 say, will be described by the equation

$$\left(\frac{x}{a_2}\right)^2 + \left(\frac{y}{b_2}\right)^2 = 1. \tag{2}$$

Here the semi-major and semi-minor axes are denoted by a_2 and b_2, respectively, and the eccentricity will be denoted by $e_2 = \sin\psi_2$; again, the corresponding eccentricity angle, ψ_2, is such $\dfrac{b_2}{a_2} = \cos\psi_2$. Without loss of generality in what follows, the value for the parameter a_2 will be taken as unity when considering numerical aspects of the problem. Since the tethering ellipse has its foci at the ends of the minor axis of the pasturing

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ellipse it follows that $a_1e_1 = b_2$ and that the length of the tethering rope is simply $2a_1$. It is possible to establish also the following relationships that can be of use:

$$\frac{b_2}{a_1} = \sin\psi_1;\ \frac{a_1}{a_2} = \frac{\cos\psi_2}{\sin\psi_1};\ \frac{b_1}{b_2} = \frac{1}{\tan\psi_1};\ \frac{b_1}{a_2} = \frac{\cos\psi_1\cos\psi_2}{\sin\psi_1}$$
$$\frac{b_1b_2}{a_1a_2} = \cos\psi_1\cos\psi_2;\ \frac{a_1b_1}{a_2b_2} = \frac{\cos\psi_2}{\sin\psi_1\tan\psi_1}. \tag{3}$$

It will be seen that when the ellipse E_2 is fixed, the common area requirement fully determines both the eccentricity e_1 and the semi-major axis a_1 of the tethering ellipse E_1. In order to determine this area using standard integration methods it will be necessary first to determine the points of intersection of E_1 and E_2. This requires a simultaneous solution to equations (1) and (2). Specifically, it can be shown that for the x-coordinate,

$$x = a_2b_1\sqrt{\frac{a_1^2 - b_2^2}{a_1^2a_2^2 - b_1^2b_2^2}},$$

from which it follows in particular that

$$\frac{x^2}{b_1^2} = \frac{1 - \dfrac{b^2}{a_1^2}}{1 - \dfrac{b_1^2b_2^2}{a_1^2a_2^2}}$$

whence, using the appropriate results in (3),

$$\frac{x}{b_1} = \frac{\cos\psi_1}{\sqrt{1 - \cos^2\psi_1\cos^2\psi_2}}. \tag{4}$$

Similarly, it can be shown that

$$\frac{x}{a_2} = \left(\frac{b_1}{a_2}\right)\left(\frac{x}{b_1}\right) = \frac{\cos\psi_1\cos\psi_2}{\tan\psi_1\sqrt{1 - \cos^2\psi_1\cos^2\psi_2}}. \tag{5}$$

These representations will be employed in the equation that describes the common area for the ellipses E_1 and E_2. The area in question, which is the first beast's pasture, can be evaluated using expressions drawn from the integral calculus. The matter is addressed in greater detail in the appendix. Specifically, the common area will be written in the form $A = S(x) + T(x)$, where $S(x)$ is an integral representing the area of a central slice of the elliptic island pasture and $T(x)$ similarly comprises two equal area segments of the tethering ellipse. The evaluations are established in the appendix and are given there by equations (A1) and (A2) respectively; they are added to determine an expression for the common area A, which must be equated to $\frac{1}{3}\pi(a_2b_2)$. Using an appropriate result from (3), the ensuing equation can be reduced to the form

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$$\sin^{-1}\left(\frac{x}{a_2}\right)+\frac{x}{a_2}\sqrt{1-\frac{x^2}{a_2^2}}+\frac{\cos\psi_2}{\sin\psi_1\tan\psi_1}\left(\cos^{-1}\left(\frac{x}{b_1}\right)-\frac{x}{b_1}\sqrt{1-\frac{x^2}{b_1^2}}\right)-\frac{\pi}{6}=0. \quad (6)$$

The parameters $\frac{x}{b_1}$ and $\frac{x}{a_2}$ in this equation are expressed in terms of eccentricity angles in equations (4) and (5). So essentially the equation involves only these angles and it must be solved for the one due to the tethering ellipse when the other one, due to the island ellipse, is specified. It has to be solved numerically. In principle, this can be achieved by implementing an interactive search routine on a spreadsheet. For example, given an input value for ψ_2 (or e_2) it is possible to tabulate at a relatively coarse interval the function on the left-hand side of equation (6) for various values of the parameter ψ_1 (or e_1). It will be seen that the tabulated function changes sign at some particular parameter value (which is the root of interest) and at this point the process is repeated with a less coarse tabulation interval, and so on. Ultimately the process converges to an acceptable value (in terms of accuracy) for ψ_1, and hence e_1, and this is the value that meets the area requirement for the tethering ellipse. It is a straightforward process that does not involve the use of sophisticated numerical methods, such as a Newton-Raphson procedure; it is particularly well suited to manipulations on a spreadsheet. When the user is satisfied with the accuracy of the solution, it is easy enough to determine the required length of tethering rope from $2a_1 = (2a_2)\frac{\cos\psi_2}{\sin\psi_1}$.

Having set up the spreadsheet, it is possible to examine how the required rope length of the tethering ellipse changes with the various eccentricities. This exercise produced the results shown in Table 1 below.

e_2	e_1	*Length*
0.999	0.165716	0.539600
0.900	0.822502	1.059913
0.800	0.888220	1.351017
0.700	0.915098	1.560801
0.600	0.929579	1.721210
0.500	0.938405	1.845739
0.400	0.944112	1.941539
0.300	0.947851	2.012828
0.200	0.950239	2.062210
0.100	0.951564	2.091268
0.000	0.951990	2.100863

TABLE 1: Rope length variations with elliptic eccentricities

It can be appreciated from the results in the table that, not surprisingly, the length of required tethering rope increases as the eccentricity of the island diminishes.

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Having considered the aspect of just a single rope to meet the demands, attention will now be turned to the problem when the grazier is allowed to cut the rope into two lengths (TBD) and tether the beasts, as appropriate, so as to allow each one to access, as before, just one third of the available pasture. Clearly the most ferocious beast will be tethered to one of these pegs and the next most ferocious one will be tethered to the other peg, leaving the one who eats only grass to roam safely and graze between their respective areas. The grazier appreciates too that there are a number of ways in which he can do this, as discussed below.

Two ropes scenario

In this scenario (depicted in Figure 2 above) there are two lengths of rope, two boundary pegs and two slip rings that are available to the grazier who can position the pegs anywhere on the boundary. He understands that if the two pegs are positioned at opposite ends of the minor axis of the island there might be a clash of beasts if the required tether for each meets the objective regarding length but is long enough to provoke conflict, depending on the island eccentricity. Thus he settles to position the two pegs at the opposite ends of the major axis of the island (although he appreciates of course that it might be possible to position them either at the ends of any elliptic diameter, or arbitrarily around the boundary but with an increased complexity in the associated sums). This requires the two rope lengths to be equal. Each beast on the tether is then constrained to move within a circular arc, much like the goat in [1], and graze within an area formed by the intersection of this circle with the elliptic island pasture. In order to appreciate the sums for this scenario, an associated geometry has to be established. The elliptic island is as described by equation (2), but it will suffice now to drop the suffixes so that the semi-major and semi-minor axes of the elliptic island pasture are simply a and b, and the eccentricity is simply $e\,(=\,\sin\psi)$. It will suffice also to describe the circular tether at one end of the island's major axis and determine, initially, the intersection point with the elliptic island in one quadrant only, because of symmetry. Thus, the circular tether point at the positive end of the semi-major axis (see Figure 2) can be described by the equation

$$(x\,-\,a)^2\,+\,y^2\,=\,r^2. \tag{7}$$

Here, r denotes the tethering radius that must be determined to ensure that only one third of the elliptic island pasture is available. In order to evaluate this area (by integration) it is necessary first to determine the point of intersection of this tethering arc with the elliptic island pasture. Using the notation established earlier, subject to the proviso regarding suffixes, it is a straightforward matter to show, by eliminating y^2 from equations (2) and (7), that the x-point of intersection in the first quadrant is determined as a solution to the quadratic equation $(x\,-\,a)^2\,-\,\left(\frac{xb}{a}\right)^2\,=\,r^2\,-\,b^2$, which can be rearranged to the usual quadratic form, provided $a\,\neq\,b$, with the solution

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of interest given by

$$\frac{x}{a} = \frac{1}{e^2} - \frac{1}{e}\sqrt{\left(\frac{r}{a}\right)^2 + \frac{1}{e^2}\left(\frac{b}{a}\right)^4} = \frac{1}{e^2} - \frac{1-e^2}{e^2}\sqrt{1 + \left(\frac{e}{1-e^2}\right)^2\left(\frac{r}{a}\right)^2}. \quad (8)$$

The other solution to the quadratic involves a different sign for the square root term and it can be eschewed because it gives a point of intersection outside the elliptic island.

As with the previous scenario, the area that is available to the beast in this tethering arrangement, A' say, can be viewed as a sum of two areas in the form $A' = S'(x) + T'(x)$, which are described more fully in the appendix. $S'(x)$ denotes an integral representation for an area segment of the island ellipse and $T'(x)$ similarly denotes an area segment of the tethering circle. Their evaluations are given in (A3) and (A4) respectively and the corresponding sum is equated to one third of the elliptic island pasture, to yield after some algebraic manipulation the equation

$$\cos^{-1}\left(\frac{x}{a}\right) - \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}} + \frac{\left(\frac{r}{a}\right)^2}{\frac{b}{a}}\cos^{-1}\left(\frac{1-\frac{x}{a}}{\frac{r}{a}}\right) - \frac{\frac{r}{a}\left(1-\frac{x}{a}\right)}{\frac{b}{a}}\sqrt{1-\frac{\left(1-\left(\frac{x}{a}\right)\right)^2}{\left(\frac{r}{a}\right)^2}} - \frac{\pi}{3} = 0. \quad (9)$$

Here, $\frac{b}{a} = \sqrt{1-e^2}$, and the factor $\frac{x}{a}$ in this equation can be expressed in terms of $\frac{r}{a}$ via equation (8), where it can be seen that the eccentricity factor again figures. The equation is thus one that must be solved numerically to determine $\frac{r}{a}$ once an input value for e has been specified.

Having established the analysis, it is a straightforward matter to implement routines on a spreadsheet, as before, and obtain numerical results for given inputs. Specifically, for a given eccentricity and for an island semi-major axis of length unity, it is possible to vary interactively, as prescribed above, the value of the parameter $\frac{r}{a}$ until equation (9) is satisfied. In this manner, the following table showing how rope length varies with input can be constructed:

e	*Length*
0.999	1.470978
0.900	1.547893
0.800	1.613246
0.700	1.667470
0.600	1.711877
0.500	1.747643
0.400	1.775727
0.300	1.796870
0.200	1.811611
0.100	1.820316
0.000	1.823195

TABLE 2: Overall rope length required for two separate tethering lengths

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As a check, the interactive spreadsheet procedure was re-addressed and modified to accommodate the tethered goat scenario of [1] that triggered the author's initial interest. The numerical result so obtained was identical to that given in [1]. This is not entirely surprising since, for sufficiently small values of the eccentricity (i.e. when $e \to 0$), the elliptic shaped island degenerates to a circular one and it is possible to show then from equation (8) that $x \approx 1 - \dfrac{r^2}{2a^2}$. After employing the relationship

$$\cos^{-1}\left(1 - \tfrac{1}{2}u^2\right) = 2\sin^{-1}\left(\tfrac{1}{2}u\right),$$

equation (9) reduces to the form

$$\left(\frac{r}{a}\right)^2 \cos^{-1}\left(\frac{r}{2a}\right) + 2\sin^{-1}\left(\frac{r}{2a}\right) - \frac{r}{a}\sqrt{1 - \frac{r^2}{4a^2}} = \frac{\pi}{3}.$$

The terms on the left-hand side of this equation are recognizable as those shown in [1] for the goat problem.

Having established rope lengths for the two different scenarios, it remains to compare the respective merits of each.

Discussion

In order to appreciate the merits of the different scenarios it will be fruitful to compare, by way of a graph, rope length data shown in Table 2 with that shown in Table 1, as a function of the island eccentricity. The results of this exercise, that included more data points than those exhibited in the tables, are shown in Figure 3 as smooth curves.

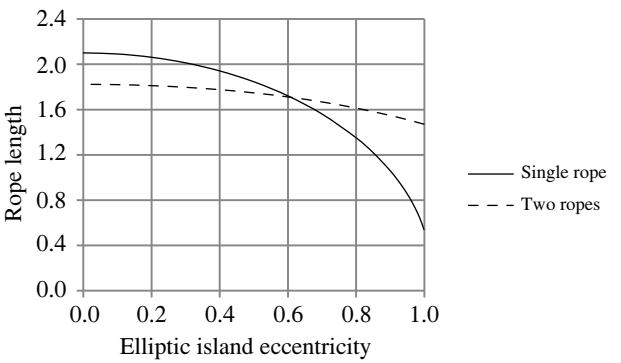


FIGURE 3: A comparison of rope lengths required for the different tethering arrangements

It is evident from the results shown in Figure 3 that if rope is a commodity to be used sparingly, the grazier can economise by opting for two separate tethering ropes (one pegged at each end of the semi-major axis of the elliptic island pasture) when the eccentricity of the island is less than

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its value at the crossover point, which is about 0.61. Beyond that, he is better off to opt for a single rope tether pegged at each end of the semi-minor axis of the island to provide an elliptic rather than circular tethering arrangement.

The interested reader might care to note that it is possible to determine more precisely the eccentricities and associated rope length at the crossover point shown in the figure. To do so, it will be necessary to solve equations (6) and (9) simultaneously when the overall rope lengths L and $2\frac{r}{a}$ for the two scenarios are equal. To this end, the suppression of suffixes relating to the parameters of the island ellipse will be lifted or not, depending on ease of description, and the rope lengths thus required for the two different scenarios will be the same when

$$\frac{r}{a} = \frac{\cos\psi}{\sin\psi_1}; \qquad (a = a_2,\ \psi = \psi_2,\ e = e_2).$$

A predictor-corrector method can be employed in the process as follows: first, take the approximate value for e_2 (= 0.61) and determine the corresponding value for the eccentricity angle ψ_2 from the relationship $e_2 = \sin\psi_2$. Substitute this value for the eccentricity angle into equation (6) and solve it interactively on the spreadsheet in the prescribed manner to obtain a predicted value for ψ_1 (or e_1). This allows for the determination of a value for $\frac{r}{a}$ from the above 'equal-length' tether relationship. This value for $\frac{r}{a}$ is then used in equation (9) to determine, again interactively, a corrected value for ψ_2 (or e_2). If the corrected value is the same as the starting value, the process terminates and the tether length requirement has been met. If not, the corrected value is re-substituted back into equation (6) to obtain an improved prediction for ψ_1, and so on. Ultimately the process converges to eccentricity values at the crossover point of $e_2 = 0.60912703$ and $e_1 = 0.92854842$; the subsequent rope lengths were found to be $\frac{r}{a} = 0.854099$ and $L = 2\frac{r}{a} = 1.708199$.

Conclusion

Two different tethering scenarios have been described to accommodate the pasturing of three troublesome beasts on an elliptic island. These scenarios are not unique and it is conceivable that other descriptions might meet also the objective. For example, if the farmer is allowed to peg the ground inside the boundary, not just on the boundary, then a multitude of additional scenarios arise and the ensuing exercise to determine required tether lengths will, in general, end up in numerical endeavours. These can be addressed comfortably using a spreadsheet in an interactive fashion, often circumventing the need to use more sophisticated number crunching techniques. In this respect, the process would be well suited to students who are keen to see numerical results without recourse to more esoteric methods and practices. It is left as an exercise for the reader to take such matters further.

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Acknowledgement

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References

1. Graham Jameson and Nicholas Jameson, Goats and birds, *Math. Gaz.* **101** (July 2017) pp. 296-300.

Appendix

The integrals required in the two different tethering scenarios are discussed here. The associated coordinate geometries are as described in the main body of text.

1. *Single rope scenario*

The schematic for this scenario is depicted below in Figure A1.

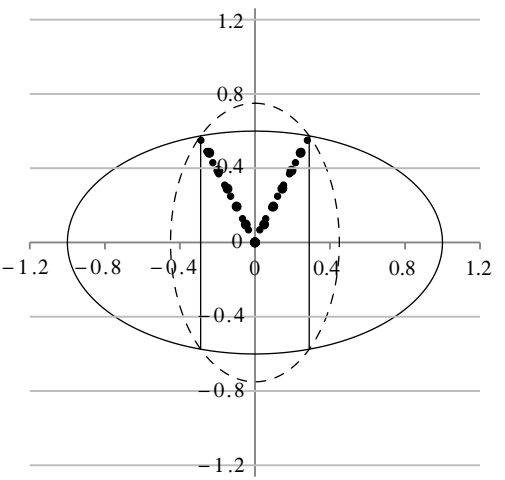


FIGURE A1: Schematic showing two orthogonal ellipses (E_1 vertical and E_2 horizontal) and the area in common

It may be appreciated from the figure that the area that is common to the two orthogonal ellipses comprises a central, vertical slice of the ellipse E_2 and two identical, vertical segments of the ellipse E_1. The area of the central slice will be denoted by $S(x)$ and it is given by

$$S(x) = 4\int_0^x y\,dx; \qquad y = b_2\sqrt{1 - \frac{x^2}{a_2^2}}.$$

The upper limit of integration is the x-value of the point of intersection of the two ellipses in the first quadrant. The substitution $x = a\sin t$ can be

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employed in the integral to reduce $S(x)$ to

$$\begin{aligned} S(x) &= 4a_2b_2 \int_0^{\sin^{-1}(x/a_2)} \cos^2 t \; dt \\ &= 2a_2b_2 \left[t + \frac{\sin 2t}{2} \right]_0^{\sin^{-1}(x/a_2)} \\ &= 2\left((a_2b_2) \sin^{-1} \frac{x}{a_2} + (a_2b_2) \frac{x}{a_2} \sqrt{1 - \frac{x^2}{a_2^2}} \right). \end{aligned} \tag{A1}$$

With respect to the vertical slice shown in the upper half space of Figure A1, it is of interest to note that the first term in parenthesis represents the area of a symmetric sector of the ellipse E_2. The second term in parenthesis represents the sum of triangular areas adjacent to this sector. The multiplicative factor 2 outside the parenthesis accounts for symmetry in the area the vertical slice (with respect to the major axis of E_2).

The sum of the remaining area segments that are common to the two ellipses will be denoted by $T(x)$, where

$$T(x) = 4 \int_x^{b_1} y \, dx; \qquad y = a_1 \sqrt{1 - \frac{x^2}{b_1^2}}.$$

This integral can also be evaluated in a straightforward manner by the change of variable method. Alternatively, the area sum can be viewed also as the complements of an area slice in the vertical ellipse E_1. Accordingly, the above result for $S(x)$ can be employed, *mutatis mutandis*, to write down the result without more ado. Specifically, replace a_2 by b_1, b_2 by a_1 and deduct the result from $\pi a_1 b_1$, to obtain

$$\begin{aligned} T(x) &= \pi a_1 b_1 - 2b_1 a_1 \left(\sin^{-1} \frac{x}{b_1} + \frac{x}{b_1} \sqrt{1 - \frac{x^2}{b_1^2}} \right) \\ &= 2a_1 b_1 \left(\cos^{-1} \frac{x}{b_1} - \frac{x}{b_1} \sqrt{1 - \frac{x^2}{b_1^2}} \right). \end{aligned} \tag{A2}$$

Use has been made of the relationship $\cos^{-1} u + \sin^{-1} u = \frac{1}{2}\pi$.

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2. *Two ropes scenario*

The schematic for this scenario is depicted in Figure A2.

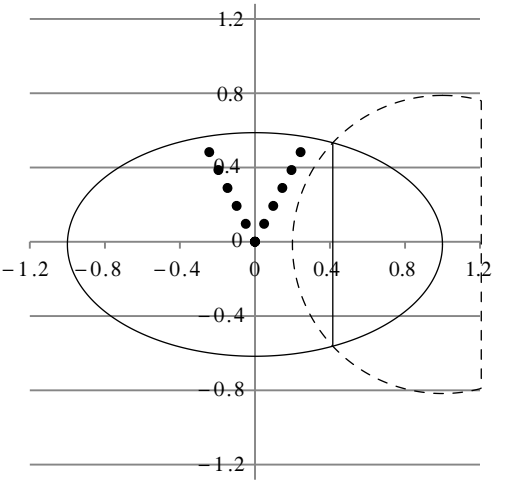


FIGURE A2: Schematic showing the horizontal ellipse E_2 and a tethering circle at one end of the major axis

From the figure, it can be seen that in this scenario the area in common comprises a circular segment and an elliptic one. The elliptic segment area will be denoted by $S'(x)$ and the circular one will be denoted by $T'(x)$. These are given by the integrals

$$S'(x) = 2\int_x^a y\,dx; \qquad y = b\sqrt{1 - \frac{x^2}{a^2}}$$

and

$$T'(x) = 2\int_{a-r}^{x} y\,dx; \qquad y = r\sqrt{1 - \frac{(a-x)^2}{r^2}}.$$

Here, x denotes the x-coordinate of the point of intersection of the ellipse and the circle in the first quadrant. Again, these integrals could be evaluated in the usual way by a suitable change of variable but it is possible also to write them down immediately. For example, the format of the integral for $S'(x)$ is the same as that for $T(x)$ above. Hence it will suffice to use result (A2) with appropriate changes to the parameters (replace a_1 by b and replace b_1 by a). Then

$$S'(x) = ab\left(\cos^{-1}\frac{x}{a} - \frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}}\right). \tag{A3}$$

For $T'(x)$, the area can be viewed as the complement of a strip, or slice, in a circular area. Thus (A1) can be employed with x replaced by $(a - x)$ and with a_2 and b_2 both replaced by r. The result must be deducted from πr^2.

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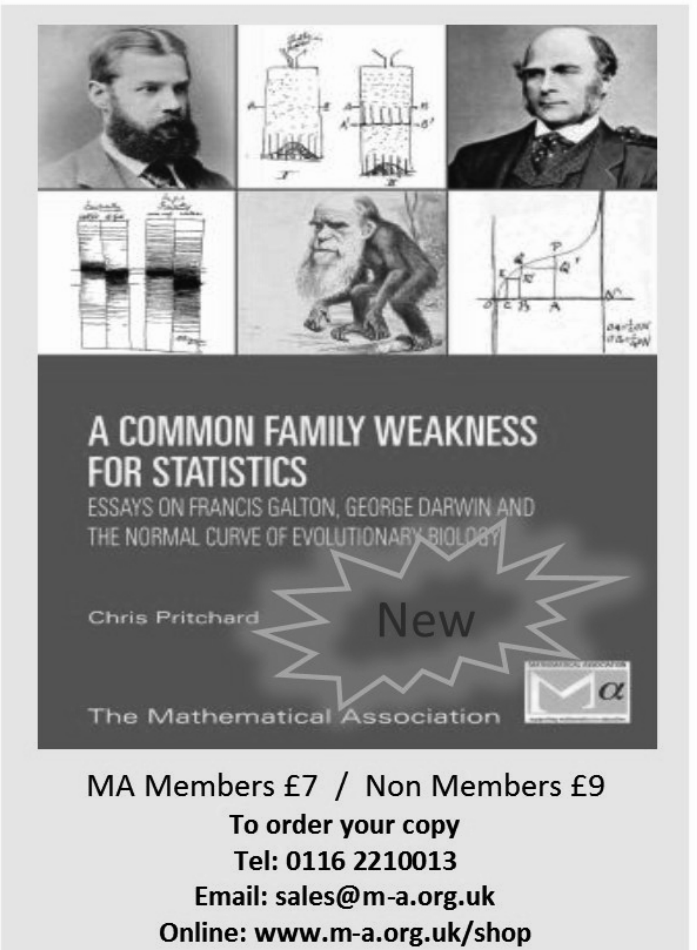
Since there is only one such segment, not two of them, the result must be halved to yield

$$T'(x) = \frac{1}{2}\left(\pi r^2 - 2r^2\left(\sin^{-1}\frac{a-x}{r} + \frac{a-x}{r}\sqrt{1-\frac{(a-x)^2}{r^2}}\right)\right)$$

$$= r^2\left(\cos^{-1}\frac{a-x}{r} - \frac{a-x}{r}\sqrt{1-\frac{(a-x)^2}{r^2}}\right). \quad \text{(A4)}$$

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