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Dynamics of subgroups of mapping class groups

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Abstract. Let $\mathrm{Mod}(S)$ be the mapping class group of a closed orientable surface S of genus $g \geq 2$. Let G be a non-elementary subgroup of $\mathrm{Mod}(S)$ such that the associated Bowen–Margulis measure is finite. In this paper, we give an asymptotic growth formula for G with respect to the Teichmüller metric.

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1. *Introduction*

Given a countable group G acting properly discontinuously by isometries on a proper metric space X, a natural quantity to study is the asymptotic orbit growth of the action, that is, the number $N(R, x)$ of elements displacing a basepoint x by at most a given number R. When X is a negatively curved contractible manifold and $M = X/\Gamma$ is compact, Margulis [22] proved in his 1969 thesis that $N(R, x)/e^{\delta R} \to_{R\to\infty} C_x$, where $C_x > 0$ is independent of R and δ is the topological entropy of the geodesic flow on M. Roblin [33] proved the same asymptotics in the more general setting where X is a $CAT(-1)$ space and with the cocompactness assumption replaced by the assumption that the so-called *Bowen–Margulis* measure on the unit tangent bundle of M is finite, a condition which covers, for instance, finite volume and geometrically finite rank one locally symmetric spaces. In recent years, a fruitful line of research has attempted to extend the results of Margulis and Roblin beyond the CAT(-1) setting. For instance, Link [19] proved similar counting results in rank one CAT(0) spaces. Zhu [38] and Blayac and Zhu [4] did so in the setting of properly convex Hilbert geometries.

Another such setting, which will be the object of this paper, is the action of the mapping class group of a closed surface on the corresponding Teichmüller space. The study of the dynamics of this action has long been influenced by analogy with the actions of discrete

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isometry groups of manifolds of negative curvature. Although the Teichmüller space of higher genus surfaces is not negatively curved in any meaningful global sense (e.g. it is not CAT(0) and not Gromov hyperbolic), in many ways it behaves 'asymptotically' or 'on average' like a negatively curved space, and the mapping class group as a lattice in such a space. It is interesting to exploit these analogies to extend the asymptotic orbit growth results of Margulis and Roblin to the setting of the mapping class group and its subgroups acting on Teichmüller space; this is the goal of this paper.

We now state the main result of this paper. Let $S = S_g$ be a connected closed surface of genus $g \geq 2$. Our results also hold for surfaces with punctures, but we only state them for closed surfaces. The mapping class group $\mathrm{Mod}(S)$ of S is the group of isotopy classes of orientation-preserving homeomorphisms of S. A subgroup $G < \mathrm{Mod}(S)$ of the mapping class group is called non-elementary if it contains two independent pseudo-Anosov mapping classes. For a non-elementary subgroup G of $\mathrm{Mod}(S)$ such that the Teichmüller geodesic flow on the unit cotangent bundle $QD^1(S)/G$ of $\mathrm{Teich}(S)/G$ has finite Bowen–Margulis measure (see §3 for the precise definition), we prove the following theorem.

THEOREM 1.1. (Theorem 5.19) *Let S be a closed surface of genus g and G be a non-elementary subgroup of the mapping class group* $\mathrm{Mod}(S)$ *so that the Bowen–Margulis measure μ_G on $QD^1(S)/G$ is finite. Let x, y be two points in Teichmüller space* $\mathrm{Teich}(S)$ *and denote the critical exponent of G by δ_G. We then have*

$$\lim_{R\to\infty} \frac{\sharp\{g \in G : d(x, gy) \leq R\}}{e^{\delta_G R}} = c,$$

where $c = 1/(\mu_G(QD^1(S)/G))$ and d is the Teichmüller metric.

In two special cases, the results of this paper were established earlier. One is when G is the full mapping class group $\mathrm{Mod}(S)$; in this case, the results were established by Athreya *et al* [1]. The other, when G is a convex-cocompact subgroup with at least one axis in the principal stratum, was studied by the first named author in his thesis and unpublished preprint [11]. The result of [1] uses in an essential way the $SL_2\mathbb{R}$ homogeneity of the measure of maximal entropy (Masur–Veech measure) of the geodesic flow, whereas the result in [11] uses in an essential way the Gromov hyperbolicity of the acting group G, neither of which occurs in our setting. Furthermore, the results of [1, 11] both use in an essential way some CAT(-1)-like properties of the Teichmüller metric, namely that two geodesics in Teichmüller space that fellow travel for a long time over compact parts of moduli space must get very close in the middle. This requires the use of some complicated analytic objects, such as the modified Hodge norm. This is in fact the reason for the assumption in [11] that some pseudo-Anosov axis must lie in the principal stratum. In the present paper, we unify, extend, and simplify the proofs of the results of [1, 11]. In addition, our results will apply to a much larger class of groups, including *statistically convex cocompact* subgroups of mapping class groups considered by Yang [37], such as free products of the form $H \star \langle p \rangle$, where H is free abelian generated by Dehn twists and p is a pseudo-Anosov element. We hope that they will also be applicable to mapping class groups of non-orientable surfaces studied by Khan [16], which can be realized as

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subgroups of mapping class groups of orientable surfaces preserving totally geodesic submanifolds of Teichmüller space. Inspired by and following the techniques of Link [19] in the rank one CAT(0) setting, we do not use any CAT(−1)-like homogeneity properties of the Teichmüller metric and, therefore, we expect our techniques to be applicable in the more general setting of actions with contracting elements (see e.g. [37] for details). To avoid using CAT(−1) properties of the Teichmüller metric, as well as to deal with global non-hyperbolicity of Teichmüller space, we had to make use of the horofunction compactification of Teich(S), the so-called Gardiner–Masur boundary $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, instead of Thurston's space of projective measured foliations, which is better understood geometrically but less natural from a dynamical point of view. Some limit points of G in $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ are non-Busemann, that is, cannot be approximated by geodesic rays. To deal with these points, we make use of so-called optimal geodesics, introduced by Azemar in [2], which can be thought of as approximating these points most efficiently. Various boundary 'shadows' illuminated by subsets of Teichmüller space with which we will work shall be defined in terms of these optimal geodesics.

2. *Background on Teichmüller spaces*

In this section, we introduce some background on Teichmüller spaces that will be used in the following. See, for instance [8] for more details on mapping class groups.

Let $S = S_g$ be a genus $g \geq 2$, closed, connected, orientable surface. All the arguments presented in this paper also apply to hyperbolic surfaces with punctures. The *mapping class group* $\mathrm{Mod}(S)$ of S is the group of isotopy classes of orientation-preserving homeomorphisms of S. Namely, if the group of orientation-preserving homeomorphisms of S is denoted by $\mathrm{Homeo}^+(S)$ and the group of homeomorphisms of S that are isotopic to the identity is denoted by $\mathrm{Homeo}_0(S)$, then

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S).$$

The *Teichmüller space* $\mathrm{Teich}(S)$ of S is the space of homotopy classes of hyperbolic structures $[(X, \phi)]$ on S. The mapping class group $\mathrm{Mod}(S)$ acts on $\mathrm{Teich}(S)$ isometrically with respect to the Teichmüller distance d. The cotangent bundle of $\mathrm{Teich}(S)$ can be canonically identified with the bundle $QD(S)$ of holomorphic quadratic differentials on S. Each element $q \in QD(S)$ defines an area on S. The unit cotangent bundle of $\mathrm{Teich}(S)$ is then identified with the subset $QD^1(S)$ of $QD(S)$ consisting of area one elements in $QD(S)$.

2.1. *Teichmüller compactification.*

Let $\mathcal{MF}(S)$ be the space of measured foliations on S and let $\mathcal{PMF}(S)$ be the space of projective measured foliations. The space $\mathcal{MF}(S)$ is a powerful tool for understanding $\mathrm{Teich}(S)$. In fact, to each $q \in QD(S)$ on a Riemann surface X, one can associate two measured foliations: the vertical measured foliation $\mathcal{V}(q)$ and the horizontal measured foliation $\mathcal{H}(q)$. This assignment gives a homeomorphism from $QD(S)$ onto its image in $\mathcal{MF}(S) \times \mathcal{MF}(S)$ (cf. [12]). Fixing a base point, the compactification of $\mathrm{Teich}(S)$ obtained by considering Teichmüller geodesic rays emanating from the base point is called the *Teichmüller compactification* of $\mathrm{Teich}(S)$ and is denoted

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by $\overline{\mathrm{Teich}(S)}^T$. The above assignment then gives a homeomorphism between the boundary $\overline{\mathrm{Teich}(S)}^T \setminus \mathrm{Teich}(S)$ and $\mathcal{PMF}(S)$.

The Teichmüller compactification is useful for many purposes. However, we require more than merely a compactification. The Teichmüller compactification depends on the choice of a base point X and, therefore, the action of $\mathrm{Mod}(S)$ on $\overline{\mathrm{Teich}(S)}^T$ is not continuous [15]. Nevertheless, this action remains well behaved in the following sense. Let $\mathcal{UE}(S) \subset \mathcal{PMF}(S)$ denote the set of projective classes of uniquely ergodic measured foliations on S. Since no ambiguity will arise, we write $\mathcal{UE}(S)$ simply as $\mathcal{UE}$. Let $\mathrm{Teich}(S) \bigcup_T \mathcal{UE}$ denote the image of $\mathrm{Teich}(S) \cup \mathcal{UE}$ in $\overline{\mathrm{Teich}(S)}^T$, equipped with the induced topology. We then have the following theorem.

THEOREM 2.1. (Masur [24]) *The action of the mapping class group* $\mathrm{Mod}(S)$ *on the Teichmüller space* $\mathrm{Teich}(S)$ *extends continuously to* $\mathrm{Teich}(S) \bigcup_T \mathcal{UE}$.

2.2. *Gardiner–Masur compactification.* For the study of counting problems for mapping class groups, another compactification of $\mathrm{Teich}(S)$, namely the Gardiner–Masur compactification, is often more suitable. We give a brief introduction to it and collect certain properties that will be used later. The reader is referred to [10, 20] for a detailed treatment.

Denote by $\mathcal{C}$ the set of (isotopy classes of) simple closed curves on S and $\mathrm{P}(\mathbb{R}_+^{\mathcal{C}})$ the projective space of $\mathbb{R}_+^{\mathcal{C}}$, where $\mathbb{R}_+^{\mathcal{C}}$ is the set of maps from $\mathcal{C}$ to $\mathbb{R}_+$. Denote the natural quotient by

$$\mathfrak{p} : \mathbb{R}_+^{\mathcal{C}} \to \mathrm{P}(\mathbb{R}_+^{\mathcal{C}}).$$

The map

$$\iota : X \mapsto (\mathrm{Ext}_X(\alpha)^{1/2})_{\alpha \in \mathcal{C}},$$

where $\mathrm{Ext}_X(\alpha)$ denotes the extremal length of α on X, defines an embedding of $\mathrm{Teich}(S)$ into $\mathbb{R}_+^{\mathcal{C}}$. Thus, the composition $\mathfrak{p} \circ \iota$ defines a map from $\mathrm{Teich}(S)$ to $\mathrm{P}(\mathbb{R}_+^{\mathcal{C}})$. Gardiner and Masur showed that the map $\mathfrak{p} \circ \iota$ is also an embedding and has a compact closure in $\mathrm{P}(\mathbb{R}_+^{\mathcal{C}})$. We identify $\mathrm{Teich}(S)$ with $\mathfrak{p} \circ \iota(\mathrm{Teich}(S))$. The closure $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ in $\mathrm{P}(\mathbb{R}_+^{\mathcal{C}})$ is called the *Gardiner–Masur compactification*. The boundary $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} = \overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \setminus \mathrm{Teich}(S)$ is called the *Gardiner–Masur boundary* of $\mathrm{Teich}(S)$.

The Gardiner–Masur compactification of $\mathrm{Teich}(S)$ has the following properties.

(A1) The Gardiner–Masur compactification of $\mathrm{Teich}(S)$ is homeomorphic to the *horofunction compactification* of $\mathrm{Teich}(S)$ endowed with the Teichmüller metric [20, 35]. The latter can be defined for any proper metric space and plays an important role in metric geometry.

(A2) The action of $\mathrm{Mod}(S)$ on $\mathrm{Teich}(S)$ extends continuously to an action on $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ [20]. Therefore, this compactification is different from the Teichmüller compactification $\overline{\mathrm{Teich}(S)}^T$.

(A3) The topology of $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ is complicated; in contrast, $\overline{\mathrm{Teich}(S)}^T$ is a $(6g-6)$-dimensional ball.

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(A4) Points in the Gardiner–Masur boundary $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ can be described by homogeneous functions on $\mathcal{MF}(S)$ [28]. Notice that [10, 29]

$$\mathcal{UE} \subsetneqq \mathcal{PMF}(S) \subsetneqq \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}.$$

The inclusion $\mathcal{PMF}(S) \subsetneqq \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ is only set-theoretic, whereas the inclusion $\mathcal{UE} \subsetneqq \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ is a topological embedding when $\mathcal{UE}$ is equipped with the topology induced from $\mathcal{PMF}(S)$ [29]. This also implies that the Gardiner–Masur compactification is distinct from the Thurston compactification.

(A5) The identity map on $\mathrm{Teich}(S)$ extends to an $\mathrm{Mod}(S)$-equivariant homeomorphism [24, 29]:

$$\mathrm{Teich}(S) \bigcup_{T} \mathcal{UE} \cong \mathrm{Teich}(S) \bigcup_{\mathrm{GM}} \mathcal{UE}.$$

Remark 2.2. In view of property (A5), the reader is encouraged to interpret the following arguments using the compactification $\overline{\mathrm{Teich}(S)}^{T}$ whenever possible.

2.3. *Teichmüller geodesics.* Since Teichmüller geodesics play an important role in our work, we require more precise information about the homeomorphism $\tilde{\Psi}$ from $QD(S)$ to its image in $\mathcal{MF}(S) \times \mathcal{MF}(S)$ [18]. The homeomorphism $\tilde{\Psi}$ is defined as follows. Let $\mathcal{C}$ be the set of isotopy classes of simple closed curves on S, $i : \mathcal{MF}(S) \times \mathcal{MF}(S) \to \mathbb{R}_+$ be the intersection function on $\mathcal{MF}(S)$, and

$$\tilde{\Delta} = \{(\lambda, \eta) \in (\mathcal{MF}(S))^2 : \exists \gamma \in \mathcal{C},\ i(\lambda, \gamma) = i(\eta, \gamma) = 0\}.$$

Then, $\tilde{\Psi} : QD(S) \to \mathcal{MF}(S) \times \mathcal{MF}(S) \setminus \tilde{\Delta}$ is given by setting $\tilde{\Psi}(q) = (\mathcal{H}(q), \mathcal{V}(q))$. Let

$$\Delta = \{(\lambda, [\eta]) \in \mathcal{MF}(S) \times \mathcal{PMF}(S) : \exists \gamma \in \mathcal{C},\ i(\lambda, \gamma) = i(\eta, \gamma) = 0\},$$

then this map further gives a homeomorphism

$$\begin{aligned} \Psi : QD^1(S) &\to \mathcal{MF}(S) \times \mathcal{PMF}(S) \setminus \Delta, \\ q &\mapsto (\mathcal{H}(q), [\mathcal{V}(q)]). \end{aligned} \tag{2.1}$$

The fact that the map Ψ is a homeomorphism implies that, for $([\lambda], [\eta]) \in \mathcal{PMF}(S) \times \mathcal{PMF}(S) \setminus \Delta$, there is a unique oriented unparameterized Teichmüller geodesic with projective horizontal measured foliation $[\lambda]$ and projective vertical measured foliation $[\eta]$. Since every oriented Teichmüller geodesic line determines a unique pair $([\xi], [\eta]) \in \mathcal{PMF}(S) \times \mathcal{PMF}(S)$, the map Ψ yields a parameterization of the space $\mathfrak{G}$ of all Teichmüller geodesic lines in $\mathrm{Teich}(S)$ by a subset of $\mathcal{PMF}(S) \times \mathcal{PMF}(S)$. This parameterization gives a topological structure, and hence, a Borel structure, on the set of geodesic lines.

The parameterization of $\mathfrak{G}$ given by Ψ is useful for many purposes; however, it is not compatible with the Teichmüller compactification in the sense that geodesic rays may have many accumulation points on the boundary. Since our approach relies heavily on geodesics, the Gardiner–Masur compactification (and hence, by property (A1), the

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horofunction compactification) provides, in a certain sense, a more compatible (though not one-to-one) parameterization that suffices for our purposes.

It is convenient to use the horofunction compactification, which we now recall, to describe this new parameterization of $\mathfrak{G}$. Fix a base point $o \in \mathrm{Teich}(S)$ and consider the set $C(\mathrm{Teich}(S), o)$ of all continuous functions on $\mathrm{Teich}(S)$ that vanish at o, equipped with the topology of uniform convergence on compact subsets. For any point $x \in \mathrm{Teich}(S)$, denote by d_x the function $y \mapsto d(x, y) - d(x, o)$. Then, the assignment

$$x \mapsto d_x$$

gives an embedding of $\mathrm{Teich}(S)$ into $C(\mathrm{Teich}(S), o)$ with a relatively compact closure $\overline{\mathrm{Teich}(S)}^{\mathrm{horo}}$, which is called the *horofunction compactification* of $\mathrm{Teich}(S)$. The complement $\partial_h \mathrm{Teich}(S)$ of $\mathrm{Teich}(S)$ in $\overline{\mathrm{Teich}(S)}^{\mathrm{horo}}$ is called the *horofunction boundary*. As noted in property (A1), it is known (cf. [20, 35]) that $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ can be identified with $\overline{\mathrm{Teich}(S)}^{\mathrm{horo}}$, under which $\partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ corresponds to $\partial_h \overline{\mathrm{Teich}(S)}^{\mathrm{horo}}$. Accordingly, we shall work exclusively with $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ and its points will be regarded as functions on $\mathrm{Teich}(S)$.

Given any geodesic ray $\boldsymbol{r} \subset \mathrm{Teich}(S)$, it is proved in [35] that $\boldsymbol{r}$ converges in $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ to a unique point in $\partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$. In particular, if $\boldsymbol{r}$ is determined by $q \in QD^1(S)$ whose vertical measured foliation $\mathcal{V}(q)$ is uniquely ergodic, then its unique limit point is given by the image of $\mathcal{V}(q)$ in $\partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ under the inclusion $\mathcal{UE} \subset \partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$. Points in $\partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ that are limit points of geodesic rays are called *Busemann points*. Thus, points in $\mathcal{UE} \subset \partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ are Busemann points. However, not every point in $\partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ is a Busemann point. In fact, there are non-Busemann points in $\mathcal{PMF}(S) \subset \partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ [28].

In the case of non-Busemann points, we employ optimal geodesics as asymptotic rays. Recall that, for a horofunction $\xi \in \partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, a geodesic $\ell \in \mathfrak{G}$ is called an *optimal geodesic for* ξ if, for all $t \in \mathbb{R}$, one has $\xi(\ell(t)) - \xi(\ell(0)) = -t$. An optimal geodesic for ξ passing through x may be regarded as the most efficient path from x to ξ. One can check that if $\ell : t \mapsto \ell(t)$ is a parameterized optimal geodesic, then so is $\ell(t + t_0)$ for all $t_0 \in \mathbb{R}$. It is proved in [2, Proposition 3.4] that given $\xi \in \partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ and $x \in \mathrm{Teich}(S)$, there exists a unique optimal geodesic for ξ passing through x. Moreover, when ξ is a Busemann point, this unique optimal geodesic is precisely the geodesic ray defining ξ.

To give a slightly more complete picture, we need to extend geodesic rays to geodesic lines. It is shown in [30, Theorem 1] that the intersection number function i on $\mathcal{MF}(S)$ can be extended to the preimage of $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ in $\mathbb{R}_+^{\mathcal{C}}$. Hence, for any two boundary points $\xi, \eta \in \partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, one can discuss whether $i(\xi, \eta)$ vanishes or not. A pair $(\xi, \eta) \in (\partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}})^2$ is called a *filling pair* if for all $\zeta \in \mathcal{MF}(S) \setminus \{0\}$,

$$i(\xi, \zeta) + i(\eta, \zeta) > 0.$$

A geodesic line ℓ is called *optimal for* (ξ, η) if $\ell(t)$ is optimal for ξ and $\ell(-t)$ is optimal for η.

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LEMMA 2.3. [21] *If* $(\xi, \eta) \in \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \times \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ *is a filling pair, then there exists a unique optimal geodesic for* (ξ, η). *If, moreover,* ξ *and* η *are Busemann points, this geodesic line converges in the positive direction to* $\xi \in \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ *and in the negative direction to* η.

Notice that if $\xi \in \mathcal{PMF}(S)$ is uniquely ergodic, then for any $\zeta \in \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \setminus \{\xi\}$, (ξ, ζ) is a filling pair [30, Lemma 7].

If $y \in \mathrm{Teich}(S)$ and $\xi \in \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, we denote by $\mathcal{L}(y, \xi)$ the unique optimal geodesic for ξ passing through y. If $(\xi, \eta) \in (\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}})^2$ is a filling pair, we denote the unique optimal geodesic for (ξ, η) by $\mathcal{L}(\xi, \eta)$. Denote by $\mathcal{N}$ the set of non-filling pairs. Then, there is a map

$$\begin{aligned} \Upsilon : (\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}})^2 \setminus \mathcal{N} &\to \mathfrak{G}, \\ (\xi, \eta) &\mapsto \mathcal{L}(\xi, \eta). \end{aligned} \tag{2.2}$$

Moreover, the restriction

$$\Upsilon|_{\mathcal{UE}\times\mathcal{UE}\setminus\mathrm{Diag}}$$

is the same as the parameterization given by (2.1). The following lemma shows that, compared with other compactifications, $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ is more compatible with the topological structure of $\mathfrak{G}$ when the latter is equipped with the topology of uniform convergence on compact sets.

LEMMA 2.4. [21] *Let* $\xi \in \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ *be uniquely ergodic and let* η *be an element of* $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \setminus \{\xi\}$. *Let* $\{x_n\}, \{y_n\}$ *be sequences in* $\mathrm{Teich}(S)$ *such that* $x_n \to \xi$ *and* $y_n \to \eta$. *Let* ℓ_n *be the geodesic segment joining* x_n *and* y_n. *Then, the sequence of geodesic segments* ℓ_n *converges to* $\mathcal{L}(\xi, \eta)$ *uniformly on compact sets. Moreover, if* $x_n = \xi$ *and* ℓ_n *is the geodesic ray* $\mathcal{L}(y_n, \xi)$, *then the sequence* $\{\ell_n\}$ *converges to* $\mathcal{L}(\eta, \xi)$.

Proof. The first assertion follows from [21, Theorem 1.5] together with the assumption that ξ is uniquely ergodic. We now prove the second assertion. Let $\{z_n \in \ell_n\}$ be a sequence that converges to ξ. By the first assertion, the sequence of geodesic segments $[z_n, y_n]$ converges to $\mathcal{L}(\xi, \eta)$. Choose a sequence of points $p_n \in [z_n, y_n]$ such that it converges to $p \in \mathcal{L}(\xi, \eta)$. Thus, $[z_n, p_n] \to [\xi, p]$ and $[p_n, y_n] \to [p, \eta]$. We parameterize ℓ_n and $\ell = \mathcal{L}(\xi, \eta)$ so that $\ell_n(0) = p_n$ and $\ell(0) = p$. The second assertion now follows from the definition of uniform convergence on compact sets. □

Convention. When $g = 2$, $\mathrm{Mod}(S_2)$ does not act faithfully on $\mathrm{Teich}(S_2)$. Instead, if we denote the center of $\mathrm{Mod}(S_2)$ by Z, then the quotient group $\mathrm{PMod(S_2)} = \mathrm{Mod}(S_2)/Z$ acts faithfully on $\mathrm{Teich}(S_2)$. Since we are concerned only with orbits of the action of $\mathrm{Mod}(S_2)$ on $\mathrm{Teich}(S_2)$, we adopt the following convention: when we refer to a non-elementary subgroup of $\mathrm{Mod}(S_2)$, we mean a non-elementary subgroup of $\mathrm{PMod(S_2)}$.

3. *The Bowen–Margulis measure*

3.1. *Conformal densities.* Let G be a subgroup of $\mathrm{Mod}(S)$. The subgroup G is called *non-elementary* if it contains two independent pseudo-Anosov mapping classes, that is,

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two pseudo-Anosov mapping classes with disjoint fixed point sets in $\mathcal{UE} = \mathcal{UE}(S)$, the subset of uniquely ergodic points in $\mathcal{PMF}(S)$. Such a subgroup is also called *sufficiently large* in [25]. The *limit set*, denoted by ΛG, is defined to be the closure in $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ of the set of fixed points in $\mathcal{UE} \subset \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ of pseudo-Anosov mapping classes contained in G. So, if G is non-elementary, then $|\Lambda G| = \infty$. Throughout this paper, the subgroup G is always assumed to be non-elementary.

One can define a G-conformal density in a very general situation; see, for instance, [7, 36]. Here, we use only a simplified version, which is sufficient for our purposes. We first recall the Busemann cocycles with respect to the Teichmüller metric. Let $x, y \in \mathrm{Teich}(S)$ and $\xi \in \mathcal{UE}$. The *Busemann cocycle* $\beta_\xi(x, y)$ *at* ξ is defined to be

$$\beta_\xi(x, y) = \lim_{z_n \to \xi} (d(x, z_n) - d(y, z_n)),$$

where $z_n \in \mathrm{Teich}(S)$ and $z_n \to \xi$ in $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$. By [28, 35], for $\xi \in \mathcal{UE}$,

$$\beta_\xi(x, y) = \frac{1}{2} \ln \frac{\mathrm{Ext}_x(\xi)}{\mathrm{Ext}_y(\xi)}. \tag{3.1}$$

Hence, $\beta_\cdot(x, y)$ is a continuous function on $\mathcal{UE}$. We note that the Busemann cocycle $\beta_\xi(x, y)$ considered here is defined only for $\xi \in \mathcal{UE}$.

Definition 3.1. [7, 36] Let $\mathcal{A} = \{\mu_x : x \in \mathrm{Teich}(S)\}$ be a family of finite Borel measures on $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ and $0 \le \alpha_G \le \infty$ be a real number. The family $\mathcal{A}$ is called an α_G-dimensional conformal density for a subgroup $G \le \mathrm{Mod}(S)$ if the following conditions hold.

- For any $x \in \mathrm{Teich}(S)$, μ_x is supported on ΛG.
- For any $x \in \mathrm{Teich}(S)$ and $g \in G$, $g_*\mu_x = \mu_{gx}$.
- For any $\xi \in \mathcal{UE} \cap \Lambda G$ and $x, y \in \mathrm{Teich}(S)$,

$$\frac{d\mu_x}{d\mu_y}(\xi) = \exp(-\alpha_G \beta_\xi(x, y)).$$

We now recall some general results on conformal densities for subgroups of mapping class groups from [36]. Notice that $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ is the horofunction compactification of $\mathrm{Teich}(S)$. Let $s \ge 0$ and $x, y \in \mathrm{Teich}(S)$. The Poincaré series associated to $G \curvearrowright \mathrm{Teich}(S)$ is defined to be

$$P_G(s, x, y) = \sum_{g \in G} \exp(-sd(gx, y)).$$

THEOREM 3.2. [36] *Let G be a non-elementary subgroup of* $\mathrm{Mod}(S)$ *and $\mathcal{A} = \{\mu_x\}$ be a positive α_G-conformal density for G. Then, the following three are equivalent:*

(1) *the Poincaré series $P_G(s, o, o)$ diverges at α_G for any (hence, for all) $o \in \mathrm{Teich}(S)$;*
(2) *the set of conical points ($\subset \mathcal{UE}$) has full μ_x-measure;*
(3) *the diagonal action of G on $(\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}})^2$ is ergodic with respect to $\mu_x \otimes \mu_x$.*

Notice that since G is non-elementary, any conformal density has positive dimension.

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3.2. *The Patterson–Sullivan construction.* The Patterson–Sullivan construction is a recipe for constructing conformal densities. We now briefly review the Patterson–Sullivan construction and the Bowen–Margulis measure in the setting of Teichmüller space. We refer the reader to [31] for an excellent exposition of the Patterson–Sullivan theory in the setting of negatively curved manifolds.

Let δ_G be the critical exponent of G with respect to the Teichmüller metric d. That is, let $o \in \mathrm{Teich}(S)$,

$$\delta_G = \inf\left\{s : \sum_{g\in G} \exp(-sd(o, go)) < \infty\right\}.$$

Notice that δ_G is independent of the choice of o and $\delta_G > 0$ as G is non-elementary. Recall that G is said to be *of divergence type* if its Poincaré series $P_G(s, o, o)$ of G diverges at the critical exponent $s = \delta_G$. Since G is non-elementary, it has contracting elements [27]. According to [37], δ_G is the same as the growth rate of G with respect to the Teichmüller metric.

We review the Patterson–Sullivan construction by assuming that G is of divergence type. For subgroups of convergence type, one can modify the construction as explained in [31]. Let $x \in \mathrm{Teich}(S)$ and $s > \delta_G$. Denote the Dirac measure at $y \in \mathrm{Teich}(S)$ by $\mathcal{D}_y$. Consider the family of probability measures $\{\nu_{x,s}\}$ on $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ given by

$$\nu_{x,s} = \frac{1}{P_G(s, x, x)} \sum_{g\in G} \exp(-sd(x, gx))\mathcal{D}_{gx}.$$

Since $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ is compact, the sequence of measures $\{\nu_{x,s} : s > \delta_G\}$ weak-* converges to some probability measure ν_x on $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ as $s \to \delta_G$. We now claim that the support of ν_x is a subset of $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$. Indeed, since $P_G(s, x, x)$ diverges at $s = \delta_G$, by assumption, and the action of G on $\mathrm{Teich}(S)$ is proper, $\nu_x(K) = 0$ holds for any compact subset K of $\mathrm{Teich}(S)$. Hence, it is supported on a subset of $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$. We now define ν_y for every $y \in \mathrm{Teich}(S)$ by

$$d\nu_y(\xi) = \exp(-\delta_G\beta_\xi(y, x))d\nu_x(\xi) \quad \text{for all } \xi \in \mathcal{UE}.$$

LEMMA 3.3. *If G is of divergence type, then the family of probability measures $\{\nu_x : x \in \mathrm{Teich}(S)\}$ is a δ_G-conformal density for G which has no atoms. Moreover, each measure ν_x is fully supported on $\Lambda G \cap \mathcal{UE}$.*

Proof. Since G is non-elementary, G contains pseudo-Anosov mapping classes. By [27], pseudo-Anosov mapping classes are contracting elements for the action of $\mathrm{Mod}(S)$ on $\mathrm{Teich}(S)$. However, [36] shows that the Gardiner–Masur boundary, equipped with the finite difference relation, gives a non-trivial convergence boundary. One can then apply [36] to conclude that $\{\nu_x\}$ is a δ_G-conformal density for G. Note that $\{\nu_x\}$ is a conformal density rather than a quasi-conformal density.

By [36] and Theorem 3.2, it follows that ν_x is fully supported on $\Lambda G \cap \mathcal{UE}$. To see that ν_x has no atoms, suppose that ν_x has an atom $\eta \in \Lambda G$, say of mass r. By Theorem 3.2 again, η must be a conical point, that is, there exists $D > 0$ such that the D-neighborhood of

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the geodesic $[x, \eta)$ intersects the orbit Gx infinitely often. Let $\gamma_n \in G$ be such a sequence. Then, by the triangle inequality, $\beta_\eta(\gamma_n x, x) \to -\infty$. So,

$$\nu_x(\gamma_n^{-1}\eta) = \nu_{\gamma_n x}(\eta) = \exp(-\delta_G \beta_\eta(\gamma_n x, x))\nu_x(\eta) \to \infty,$$

which contradicts the finiteness of ν_x. □

COROLLARY 3.4. *If G is a non-elementary subgroup of* $\mathrm{Mod}(S)$ *of divergence type, then there exists a δ_G-conformal density $\{\nu_x : x \in \mathrm{Teich}(S)\}$ without atoms for G which is fully supported on the set of conical points in $\mathcal{UE}$.*

Since each ν_x admits a unique completion, we henceforth assume that all such measures are complete. We now turn to the definition of the Bowen–Margulis measure on $QD^1(S)/G$. For $x \in \mathrm{Teich}(S)$ and $\xi, \eta \in \mathcal{UE}$ with $\xi \neq \eta$, set

$$\rho_x(\xi, \eta) = \lim_{z \to [\xi], w \to [\eta]} (d(x, z) + d(x, w) - d(z, w)) = \beta_\xi(x, u) + \beta_\eta(x, u), \tag{3.2}$$

where $u \in \mathrm{Teich}(S)$ is any point in the geodesic $\mathcal{L}(\xi, \eta)$. Now, we define a measure $\hat{\mu}_G$ on $\mathcal{PMF}(S) \times \mathcal{PMF}(S)$ by setting

$$d\hat{\mu}_G(\xi, \eta) = \exp(\delta_G \rho_x(\xi, \eta)) d\nu_x(\xi)\, d\nu_x(\eta) \quad \text{for all } \xi \neq \eta \in \mathcal{UE}.$$

Notice that since ν_x is supported on $\mathcal{UE}$, $\hat{\mu}_G$ extends to $\mathcal{PMF}(S) \times \mathcal{PMF}(S)$. We claim that $\hat{\mu}_G$ is independent of x and it is G-invariant for the diagonal action of G on $\mathcal{PMF}(S) \times \mathcal{PMF}(S)$. In fact, this claim follows from the definition of the conformal density and the definition of ρ_x.

The homeomorphism Ψ defined in (2.1) provides a *Hopf parameterization* of the unit cotangent bundle $QD^1(S)$ of $\mathrm{Teich}(S)$ as $(\mathcal{PMF}(S) \times \mathcal{PMF}(S) \setminus \Delta) \times \mathbb{R}$, where

$$\Delta = \{([\lambda], [\eta]) \in \mathcal{PMF}(S) \times \mathcal{PMF}(S) : \exists \gamma \in \mathcal{C},\ i(\lambda, \gamma) = i(\eta, \gamma) = 0\}.$$

Consequently, each point in $QD^1(S)$ can be parameterized as (ξ, η, s). We define a measure $\tilde{\mu}_G$ on $QD^1(S)$ by setting

$$d\tilde{\mu}_G = dt\, d\hat{\mu}_G.$$

Notice that since ν_x is fully supported on $\mathcal{UE}$, $\tilde{\mu}_G$ is fully supported on geodesic lines whose horizontal and vertical measured foliations are both uniquely ergodic. The measure μ_G on the quotient $QD^1(S)/G$ that descends from $\tilde{\mu}_G$ is called the *Bowen–Margulis measure associated to* G. Note that, *a priori*, the Bowen–Margulis measure μ_G need not be finite.

4. *Mixing of the Bowen–Margulis measure*

In this section, we prove that the geodesic flow $\{g_t\}$ on $QD^1(S)/G$ is mixing for a subgroup G with finite Bowen–Margulis measure. Our argument is closely related to that of Babillot [3] for rank one compact manifolds.

To prove the mixing result, we will need to make use of a result that G has *non-arithmetic pseudo-Anosov length spectrum*. More precisely, denote the spectrum of G by

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$$\mathrm{Spec}(G) = \{\log(\lambda_g) : g \in G \text{ is pseudo-Anosov and } \lambda_g \text{ is the dilatation of } g\}.$$

The following result has been recently proved by Choi and Kim in [6].

THEOREM 4.1. *Let G be a non-elementary subgroup of* $\mathrm{Mod}(S)$. *Then, the spectrum* $\mathrm{Spec}(G)$ *generates a dense subgroup of* $\mathbb{R}$.

We note that a proof of this result (which was purported to work even for *semigroups* of mapping class groups) was claimed in an earlier version of this paper as well as the PhD thesis [11, § 9] of the first named author. However, the authors of [6] correctly pointed out to us an essential gap in the argument, subsequently obtaining a correct proof.

We are now in a position to show the mixing of the geodesic flow with respect to the Bowen–Margulis measure μ_G. Notice that by the Poincaré recurrence theorem and Theorem 3.2, the finiteness of μ_G implies the divergence of G.

THEOREM 4.2. *Let G be a non-elementary subgroup of* $\mathrm{Mod}(S)$ *with finite Bowen–Margulis measure μ_G. Then, the geodesic flow g_t on $QD^1(S)/G$ is mixing with respect to μ_G. More precisely, let $A, B \subset QD^1(S)/G$ be Borel sets with finite μ_G-measure. If $\mu_G(QD^1(S)/G) < \infty$, then*

$$\lim_{t\to\pm\infty} \mu_G(A \cap g_{-t}B) = \frac{\mu_G(A)\mu_G(B)}{\mu_G(QD^1(S)/G)}.$$

Otherwise,

$$\lim_{t\to\pm\infty} \mu_G(A \cap g_{-t}B) = 0.$$

Since the proof is lengthy, we first state the following consequence.

COROLLARY 4.3. *Let G be a non-elementary subgroup of* $\mathrm{Mod}(S)$ *with finite Bowen–Margulis measure μ_G. Let $A, B \subset QD^1(S)/G$ be Borel subsets with finite μ_G-measure. Suppose that the quotient map $QD^1(S) \to QD^1(S)/G$ restricts to injective maps $\tilde{A} \to A$ and $\tilde{B} \to B$ on subsets $\tilde{A}, \tilde{B} \subset QD^1(S)$. If $\mu_G(QD^1(S)/G) < \infty$, then*

$$\lim_{t\to\pm\infty} \sum_{\gamma\in G} \tilde{\mu}_G(\tilde{A} \cap g_{-t}\gamma\tilde{B}) = \frac{\tilde{\mu}_G(A)\tilde{\mu}_G(B)}{\mu_G(QD^1(S)/G)}.$$

Otherwise,

$$\lim_{t\to\pm\infty} \sum_{\gamma\in G} \tilde{\mu}_G(\tilde{A} \cap g_{-t}\gamma\tilde{B}) = 0.$$

It is easy to see that if Corollary 4.3 holds for a finite index subgroup of G, then it also holds for G. Since $\mathrm{Mod}(S)$ is virtually torsion-free, we may therefore assume that G is torsion-free. The proof of Corollary 4.3 follows the same argument as [19, Corollary 5.6]. The only non-trivial ingredient in that proof, after unwrapping the integral, is the existence of a measurable fundamental domain with boundary of measure zero [19, Lemma 4.4]. Therefore, we provide the following analog.

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Recall that an open subset F of $Y \subset QD^1(S)$ is called a *fundamental domain* for the action of G on Y if $Y = \bigcup_g g\bar{F}$ and $gF \cap F = \emptyset$ for all $g \neq \mathrm{id}$ in G. Furthermore, it is called *locally finite* if the collection $\{gF : g \in G\}$ is locally finite. We continue to interpret $\mathrm{Mod}(S_g)$ as $\mathrm{PMod}(\mathrm{S_g})$ for $g = 2$.

LEMMA 4.4. *Let G be a torsion-free non-elementary subgroup of* $\mathrm{Mod}(S)$ *and $\tilde{\mu}_G$ be the measure on $QD^1(S)$ that descends to the Bowen–Margulis measure on $QD^1(S)/G$. Then, there is a G-invariant measurable subset $Y \subset QD^1(S)$ with full $\tilde{\mu}_G$-measure. Moreover, there exists a locally finite fundamental domain F for the action of G on Y so that $\tilde{\mu}_G(\partial F) = 0$.*

Sketch of the proof. The proof essentially follows from [19, Lemma 4.4]; however, we add some comments here, since for subgroups of $\mathrm{Mod}(S)$, the argument is more concrete. The idea is to use the fundamental domain constructed in [26] for the action of G on $\mathrm{Teich}(S)$ and then apply the trick in [33] to modify the boundary so that it has measure zero, as explained in [19, Lemma 4.4].

Consider

$$\mathcal{G} = \mathrm{Fix}(G) = \{q \in QD^1(S) : \exists \gamma \in G \setminus \{\mathrm{id}\}, \gamma q = q\}.$$

Then, $\mathcal{G}$ is closed, G-invariant, and g_t-invariant. Notice that if $\gamma q = q$, then $\gamma(\pi(q)) = \pi(q)$, where $\pi : QD^1(S) \to \mathrm{Teich}(S)$ is the projection. Such an element γ must have finite order. Since G is torsion-free, $\mathcal{G}$ is empty.

Choose a point $m_0 \in \mathrm{Teich}(S)$ whose stabilizer in G is trivial and consider the Dirichlet domain $\mathbb{D}_G$ associated to the action of G on $\mathrm{Teich}(S)$, namely,

$$\mathbb{D}_G = \{y \in \mathrm{Teich}(S) : d(y, m_0) < d(y, \gamma m_0)\ \forall \gamma \in G \setminus \mathrm{id}\}.$$

By [26, Theorem 5.9], $\mathbb{D}_G$ is a locally finite fundamental domain for the action of G on $\mathrm{Teich}(S)$. In fact, the boundary $\partial\mathbb{D}_G$ is a properly embedded submanifold of codimension one of $\mathrm{Teich}(S)$. Now, consider the preimage $\mathcal{D}_G$ of $\mathbb{D}_G$ in $QD^1(S)$ under the projection $\pi : QD^1(S) \to \mathrm{Teich}(S)$. That is,

$$\mathcal{D}_G = \{q \in QD^1(S) : d(\pi(q), m_0) < d(\pi(q), \gamma m_0)\ \forall \gamma \in G \setminus \mathrm{id}\}.$$

Then, $\mathcal{D}_G$ is a fundamental domain for the action of G on $QD^1(S)$ and $\pi(\mathcal{G}) \subset \partial\mathbb{D}_G$. Note that in our construction, $\bar{\mathcal{D}}_G = \pi^{-1}(\bar{\mathbb{D}}_G)$. Hence, unlike in [19], it is unnecessary to consider the so-called enlarged boundary. Nevertheless, it may happen that $\partial\mathcal{D}_G$ has positive $\tilde{\mu}_G$-measure, so we follow the same argument as in [19, Lemma 4.4] to conclude the existence of the required fundamental domain. We remark that the metric used in [19, Lemma 4.4] can be replaced by the $\mathrm{Mod}(S)$-invariant distance on $QD^1(S)$ defined by (see [36])

$$\tilde{d}(q_1, q_2) = \int_{-\infty}^{+\infty} \frac{d(\pi(g_t q_1), \pi(g_t q_2))}{2e^t}\, dt. \qquad \square$$

The rest of this section is devoted to the proof of Theorem 4.2. The proof of Theorem 4.2 is modeled on Babillot's argument [3] (see also [19, Theorem 5.4]). First, we recall a lemma (cf. [3, Lemma 1]) from the unitary representation theory.

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LEMMA 4.5. *Let $(X, \mathcal{B}, m, \{T_t\}_{t\in A})$ be a measure preserving dynamical system, where $(X, \mathcal{B})$ is a standard Borel space, m is a possibly unbounded Borel measure on $(X, \mathcal{B})$, and $\{T_t\}_{t\in A}$ is an action of a locally compact, second countable abelian group A on X by measure preserving transformations. Let $\phi \in L^2(X, m)$ be a real-valued function on X such that $\int \phi dm = 0$ if m is finite. If there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ going to infinity in A such that $\phi \circ T_{t_n}$ does not converge to 0 in the weak L^2 topology, then there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ going to infinity in A and a non-constant function χ in $L^2(X, m)$ such that, in the weak L^2 topology,*

$$\phi \circ T_{s_n} \to \chi, \quad \phi \circ T_{-s_n} \to \chi.$$

Proof of Theorem 4.2. Set $X = QD^1(S)/G$, $\mu = \mu_G$, and

$$\|\mu\| = \mu(QD^1(S)/G).$$

Suppose for contradiction that the geodesic flow is not mixing. Then, there exist $f \in C_c(X)$ (with $\int f\, d\mu = 0$ if m is finite) and a sequence $t_n \to \infty$ such that $f \circ g_{t_n}$ does not converge weakly to 0 in $L^2(X, \mu)$. By Lemma 4.5, there exist a non-constant function $\chi \in L^2(X, \mu)$ and a sequence $s_n \to \infty$ such that, weakly in $L^2(X, \mu)$,

$$f \circ g_{s_n} \to \chi, \quad f \circ g_{-s_n} \to \chi \ \ (n \to \infty).$$

By [3, Fact], χ is also the almost sure limit of the Cesàro averages of f for both positive and negative times.

Modify χ so that χ is defined on X and denote by Ξ the lift of χ to the unit tangent bundle $Y = QD^1(S)$ over Teich(S). For each $\epsilon > 0$, define the ϵ-smooth Ξ_ϵ to be

$$\Xi_\epsilon : q \in Y \mapsto \int_0^\epsilon \Xi(g_s(q))\, ds,$$

where g_t is the geodesic flow on Y. Since χ is not constant, by taking ϵ small enough, one can ensure that Ξ_ϵ is not constant and it is the limit of the corresponding Cesàro averages of the smoothing of f. Now, there is a full $\hat{\mu}_G$-measure subset $E_0 \subset (\Lambda G \times \Lambda G) \cap (\mathcal{UE} \times \mathcal{UE})$ such that for any $q \in Y$ with $([\mathcal{H}(q)], [\mathcal{V}(q)]) \in E_0$, the function

$$h_q : t \mapsto \Xi_\epsilon(g_t q)$$

is well defined and continuous for any real number t. Here, since ν_x is supported on $\mathcal{UE} \cap \Lambda G$, after modifying a null measure set, we regard $\Lambda G \times \Lambda G$ as a subset of $\mathcal{PMF}(S) \times \mathcal{PMF}(S)$ by considering the Thurston compactification $\overline{\text{Teich}(S)}^{Th}$. For each $q \in Y$, consider the period Per(q) of the map h_q. We note that Per(q) might be trivial and it only depends on the geodesic determined by q. Thus, it gives us a measurable map

$$\text{Per} : E_0 \to \text{Sub},$$

where Sub is the set of closed subgroups of $\mathbb{R}$. Note that Per is also G-invariant since Ξ_ϵ is G-invariant. By the double ergodicity of the diagonal action of G with respect to $\nu_x \otimes \nu_x$ (Theorem 3.2), Per is essentially constant.

Suppose that this constant is $\mathbb{R}$. Then, Ξ_ϵ is g_t-invariant and, hence, it defines a G-invariant measurable function on $\Lambda G \times \Lambda G \setminus \text{Diag}$. By the ergodicity of $\nu_x \otimes \nu_x$, Ξ_ϵ is

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constant, which contradicts our choice of ϵ. So, there exists $k \geq 0$ such that, almost surely, Per is $k\mathbb{Z}$. Denote this full measure subset of E_0 by E_1.

Our goal is to show that, in this case, $\mathrm{Spec}(G)$ is contained in $\frac{1}{2}k\mathbb{Z}$, which contradicts Theorem 4.1.

We first choose a full measure subset of E_1 with nice properties. Consider the lift $\tilde{f}$ of f to Y and define

$$\tilde{f}_\epsilon : Y \to \mathcal{R}, \quad q \mapsto \int_0^\epsilon \tilde{f}(g_s q)\, ds.$$

By the G-invariance of $\tilde{f}$, one obtains a function f_ϵ on X. By Lemma 4.5 and [3, Fact], one has a non-constant function χ_ϵ as a weak limit in $L^2(X, \mu)$ that is the push forward function of Ξ_ϵ. It is also the almost sure limit of the Cesàro averages for both positive and negative times

$$\frac{1}{K^2} \sum_{k=1}^{K^2} f_\epsilon \circ g_{\pm s_{n_k}}$$

for some sequence $\{n_k\}_k$. Denote by $\Xi_\epsilon^\pm$ the lifts of the limit superior of the Cesàro averages defined above. By Fubini's theorem, there is a set $E_2 \subset E_1$ of full measure such that for every $q \in Y$ with $([\mathcal{H}(q)], [\mathcal{V}(q)]) \in E_2$,

$$\Xi_\epsilon^+(q) = \Xi_\epsilon^-(q) = \Xi_\epsilon(q).$$

Furthermore, define E^- to be the set of points $[\zeta] \in \Lambda G \cap \mathcal{UE}$ such that for ν_x-almost every $[\eta]$, $([\zeta], [\eta]) \in E_2$ and E^+ to be the set of points $[\zeta] \in \Lambda G \cap \mathcal{UE}$ such that for ν_x-almost every $[\eta]$, $([\eta], [\zeta]) \in E_2$. By Fubini's theorem, $E := E_2 \bigcap (E^+ \times E^-)$ has full measure.

We now introduce the *cross ratio* of four points in E and relate it to $\log(\lambda_g)$ for a pseudo-Anosov mapping class $g \in G$.

Let $([\zeta_1], [\eta_1]) \in E$. Since E has full measure, one can choose $([\zeta_2], [\eta_2]) \in E_2$ so that $([\zeta_1], [\eta_2]) \in E_2$ and $([\zeta_2], [\eta_1]) \in E_2$. Now, take $q_0 \in QD^1(S)$ so that $([\mathcal{V}(q_0)], [\mathcal{H}(q_0)]) = ([\zeta_1], [\eta_1])$ and define $p_0 = \pi(q_0)$, where $\pi : QD^1(S) \to \mathrm{Teich}(S)$ is the projection. The pair (q_0, p_0) is said to be in the geodesic $\mathcal{L}([\zeta_1], [\eta_1])$. Once (q_0, p_0) has been fixed, there is a unique pair (q_1, p_1) in the geodesic $\mathcal{L}([\zeta_1], [\eta_2])$ such that $\mathrm{Ext}_{p_0}(\mathcal{V}(q_0)) = \mathrm{Ext}_{p_1}(\mathcal{V}(q_0))$. Consider $\mathcal{H}(q_1)$. There exists a unique pair (q_2, p_2) on the geodesic $\mathcal{L}([\eta_2], [\zeta_2])$ such that $\mathrm{Ext}_{p_1}(\mathcal{H}(q_1)) = \mathrm{Ext}_{p_2}(\mathcal{H}(q_1))$. Continuing this process, one finds a pair (q_3, p_3) in $\mathcal{L}([\zeta_2], [\eta_1])$ so that $\mathrm{Ext}_{p_2}(\mathcal{V}(q_2)) = \mathrm{Ext}_{p_3}(\mathcal{V}(q_2))$ and a pair (q_4, p_4) in $\mathcal{L}([\zeta_1], [\eta_1])$ with $\mathrm{Ext}_{p_3}(\mathcal{H}(q_3)) = \mathrm{Ext}_{p_4}(\mathcal{H}(q_3))$. Now, the two pairs (q_0, p_0) and (q_4, p_4) are in the same geodesic $\mathcal{L}([\zeta_1], [\eta_1])$; hence, there is a unique real number $\iota > 0$ such that

$$g_{\pm\iota} q_0 = q_4.$$

We call this number ι the *cross ratio* of $([\zeta_1], [\eta_1], [\zeta_2], [\eta_2])$ and denote it by

$$\iota = ([\zeta_1], [\eta_1]; [\zeta_2], [\eta_2]).$$

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By a result of Gardiner and Masur [10], if $q \in QD^1(X)$ with $(\zeta, \eta) = (\mathcal{V}(q), \mathcal{H}(q))$, then

$$i^2(\zeta, \eta) = \mathrm{Ext}_X(\zeta)\,\mathrm{Ext}_X(\eta).$$

This gives us that

$$([\zeta_1], [\eta_1]; [\zeta_2], [\eta_2]) = \left|\ln \frac{i(\eta_1, \zeta_1)i(\eta_2, \zeta_2)}{i(\eta_1, \zeta_2)i(\eta_2, \zeta_1)}\right|.$$

Hence, the cross ratio depends only on the choice of $([\zeta_1], [\eta_1], [\zeta_2], [\eta_2]) \in \mathcal{UE}^4$. This formula extends continuously to all 4-tuples of pairwise distinct points in $\mathcal{UE}$. Moreover, for any pseudo-Anosov mapping class $g \in \mathrm{Mod}(S)$, by considering $\mathcal{F}^u(g)$, $\mathcal{F}^s(g)$ and $[\beta]$, $g[\beta]$ for any $[\beta] \in \mathcal{UE}$, one has that the cross ratio

$$([\mathcal{F}^u(g)], [\beta]; [g\beta], [\mathcal{F}^u(g)]) = 2\ln(\lambda_g).$$

We now claim that if we start from $([\zeta_1], [\eta_1]) \in E$ and choose $[\zeta_2], [\eta_2]$ as above, then $([\zeta_1], [\eta_1]; [\zeta_2], [\eta_2])$ is in $k\mathbb{Z}$, the closed subgroup of $\mathbb{R}$ mentioned before. Combining this claim with the previous observation on the relationship between the cross ratio and the spectrum of G, the proof is complete.

For the claim, indeed, by [35, Theorem 1.1], one has

$$\lim_{t\to+\infty} d_T(\pi(g_t q_0), \pi(g_t q_1)) = 0, \quad \lim_{t\to-\infty} d_T(\pi(g_t q_1), \pi(g_t q_2)) = 0,$$

$$\lim_{t\to+\infty} d_T(\pi(g_t q_2), \pi(g_t q_3)) = 0, \quad \lim_{t\to-\infty} d_T(\pi(g_t q_3), \pi(g_t q_4)) = 0.$$

Hence, by the definition of E,

$$\begin{aligned}\Xi_\epsilon(q_0) = \Xi^+_\epsilon(q_0) = \Xi^+_\epsilon(q_1) = \Xi^-_\epsilon(q_1) = \Xi^-_\epsilon(q_2) = \Xi^+_\epsilon(q_2)\\ = \Xi^+_\epsilon(q_3) = \Xi^-_\epsilon(q_3) = \Xi^-_\epsilon(q_4) = \Xi^+_\epsilon(q_4) = \Xi_\epsilon(q_4).\end{aligned} \tag{4.1}$$

It follows that the cross ratio $([\zeta_1], [\eta_1]; [\zeta_2], [\eta_2])$ lies in $\mathrm{Per}(E_0)$, which completes the proof of the claim. □

5. *Asymptotics of orbit growth*

5.1. *Shadows, cones, and corridors.* For the reader's convenience, we introduce the notation, following [19, 33], that will be used in the following. Let $QD^1(S)$ be the space of unit-area quadratic differentials, which can be identified with the cotangent bundle of $\mathrm{Teich}(S)$, and let $\mathcal{MF}(S)$ be the space of measured foliations on S. We denote by $\mathcal{UE}$ the set of uniquely ergodic points in $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$. The natural projections involved here are given by

$$\begin{aligned}&\pi : QD^1(S) \to \mathrm{Teich}(S),\\ &\mathcal{V} : QD^1(S) \to \mathcal{MF}(S), \quad q \mapsto \mathcal{V}(q),\\ &\mathcal{H} : QD^1(S) \to \mathcal{MF}(S), \quad q \mapsto \mathcal{H}(q),\\ &[\cdot] : \mathcal{MF}(S) \to \mathcal{PMF}(S), \quad \xi \mapsto [\xi].\end{aligned} \tag{5.1}$$

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We use $\xi, \eta, \ldots$ to denote boundary points in $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$. When $\xi \in \mathcal{PMF}(S)$, we sometimes also use ξ to refer to one of its preimages under $[\cdot]$, when no confusion arises.

Set $\mathfrak{F} = (\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}})^2 \setminus \Delta$, where

$$\Delta = \{([\lambda], [\eta]) \in (\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}})^2 : \exists\, \gamma \in \mathcal{MF}(S) \setminus \{0\}, i(\lambda, \gamma) = i(\eta, \gamma) = 0\}.$$

Note that every pair $(\zeta, \psi) \in \mathfrak{F}$ determines a unique unparameterized Teichmüller geodesic $\mathcal{L}(\zeta, \psi)$, called the *Teichmüller geodesic determined by the pair* (ζ, ψ), by taking the optimal geodesic line for (ζ, ψ). When naturally oriented, it is referred to as the *oriented geodesic from* ζ *to* ψ. Note that if ζ and ψ belong to $\mathcal{UE}$, then $\mathcal{L}(\zeta, \psi)$ is exactly

$$\mathcal{L}(\zeta, \psi) = \{\pi(q) : q \in \mathcal{Q}D^1(S), [\mathcal{H}(q)] = \zeta, [\mathcal{V}(q)] = \psi\}.$$

For any point $x \in \mathrm{Teich}(S)$, the unique unit holomorphic quadratic differential q on x with $([\mathcal{H}(q)], [\mathcal{V}(q)]) = (\xi, \eta)$ will be denoted by $q(x; \xi, \eta)$.

For any subset $A \subset \mathrm{Teich}(S)$ and $r > 0$, the *r-neighborhood* $\mathcal{N}_r(A)$ of A is the subset consisting of points in $\mathrm{Teich}(S)$ whose distance to A is at most r. If A is a single point, we denote by $B(x, r)$ the open ball of radius r centered at x.

5.1.1. *Shadows.* For $r > 0$, define the *(measurable) shadow* of the ball $B(y, r)$ seen from x in $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ by

$$\mathcal{O}_r(x, y) = \{\beta \in \mathcal{UE} : \mathcal{L}(x, \beta) \cap B(y, r) \neq \emptyset\},$$

where $\mathcal{L}(x, \beta)$ is the unique optimal geodesic for β passing through x. Since $\beta \in \mathcal{UE}$, the geodesic $\mathcal{L}(x, \beta)$ is the geodesic ray starting from x whose associated quadratic differential has projectivized vertical measured foliation β. Although one could define the shadow as the set of boundary points satisfying the same condition, we adopt the above definition. Let $\zeta \in \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ and $y \in \mathrm{Teich}(S)$. Define

$$\mathcal{O}_r(\zeta, y) = \{\eta \in \mathcal{UE} : \mathcal{L}(\zeta, \eta) \cap B(y, r) \neq \emptyset\}.$$

For $r > 0$ and $x, y \in \mathrm{Teich}(S)$ with $d(x, y) > 2r$, define

$$\begin{aligned} \mathcal{O}_r^+(x, y) &= \{\beta \in \mathcal{UE} : \exists\, z \in B(x, r) \text{ such that } \mathcal{L}(z, \beta) \cap B(y, r) \neq \emptyset\}, \\ \mathcal{O}_r^-(x, y) &= \{\beta \in \mathcal{UE} : \forall\, z \in B(x, r),\ \mathcal{L}(z, \beta) \cap B(y, r) \neq \emptyset\}. \end{aligned} \tag{5.2}$$

The following lemma should be compared with [19, Lemma 2.1].

LEMMA 5.1. *Let $\zeta, \kappa \in \mathcal{UE}$ be two distinct projective measured foliations determining a Teichmüller geodesic and $o \in \mathrm{Teich}(S)$ be any point. Let $r > 0$, $\epsilon > 0$, and the Teichmüller geodesic line determined by ζ, κ be $\mathcal{L}(\zeta, \kappa)$. Suppose that $d(o, \mathcal{L}(\zeta, \kappa)) = r$. Then, there exists an open neighborhood $O_1 \times O_2 \subset (\mathcal{UE})^2$ of (ζ, κ) such that*

$$|d(o, \mathcal{L}(\mathcal{F}_1, \mathcal{F}_2)) - r| \leq \epsilon$$

for every $(\mathcal{F}_1, \mathcal{F}_2) \in O_1 \times O_2$. Moreover, every pair $(a, b) \in O_1 \times O_2$ is a filling pair and, hence, determines a Teichmüller geodesic.

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Proof. By [13, Lemma 1.4.3], the distance function on

$$\mathcal{PMF}(S) \times \mathcal{PMF}(S) \setminus \Delta$$

is continuous, which proves the first assertion of the lemma. Here, the set $\mathcal{PMF}(S)$ is equipped with its natural topology instead of the induced topology from $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$. For the second assertion, if not, then there are measured foliations $\xi_n \to \zeta$, $\eta_n \to \kappa$, and simple closed curves $\gamma_n \in \mathcal{C}$ such that $i(\xi_n, \gamma_n) = i(\eta_n, \gamma_n) = 0$. We may assume that $\gamma_n \to \iota$ with $\iota \in \mathcal{MF}(S)$. Then, $i(\iota, \zeta) = i(\iota, \kappa) = 0$. However, ι cannot be a minimal foliation (that is, a foliation without simple closed leaves); otherwise, ζ is topologically equivalent to κ [32] and, hence, has zero intersection with κ, which contradicts the assumption. So, there is a simple closed curve $\gamma \in \mathcal{C}$ such that $i(\gamma, \zeta) = i(\gamma, \kappa) = 0$, which also contradicts the assumption. □

COROLLARY 5.2. *Let $\mathcal{O}_r(\zeta, y)$ be as defined above. Then, $\mathcal{O}_r(\zeta, y)$ is open in $\mathcal{UE}$ and, hence, it is ν_x-measurable for all $x \in \mathrm{Teich}(S)$.*

Proof. Since the measure ν_x is complete by assumption, $\mathcal{UE}$ is ν_x-measurable. So, if $\zeta \in \mathcal{UE}$, the result follows from Lemma 5.1. Otherwise, we apply Lemma 2.4. □

Remark 5.3.

(1) The set $\mathcal{O}_r(x, y)$ is open in $\mathcal{UE}$ and, hence, ν_x-measurable.
(2) In general, it might not be true that if x_n converges to ξ in $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, then $\lim_{n\to\infty} \mathcal{O}_r^{\pm}(x_n, y) = \mathcal{O}_r(\xi, y)$.
(3) For $\xi \in \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, we define $\mathcal{O}_r^{\pm}(\xi, y) = \mathcal{O}_r(\xi, y)$.

We will also use the following modified boundary of $\mathcal{O}_r(\xi, x)$, as in [19],

$$\tilde{\partial}\mathcal{O}_r(\xi, x) = \{\eta \in \mathcal{UE} \setminus \{\xi\} : d(x, \mathcal{L}(\xi, \eta)) = r\}.$$

Notice that by Corollary 5.2, $\tilde{\partial}\mathcal{O}_r(\xi, x)$ is also ν_x-measurable.

5.1.2. *Cones.* Let $x \in \mathrm{Teich}(S)$ and $U \subset \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$. We denote by $\mathrm{Sect}_x(U)$ the set of all points y in $\mathrm{Teich}(S)$ such that there exists $\xi \in U$ with $y \in \mathcal{L}(x, \xi)$.

For $r > 0$, following [19], define the *cones*

$$C_r^+(x, U) = \{y \in \mathrm{Teich}(S) : \mathcal{O}_r^+(x, y) \cap U \neq \emptyset\}$$

and

$$C_r^-(x, U) = \{y \in \mathrm{Teich}(S) : \mathcal{O}_r^+(x, y) \subset U\}.$$

5.1.3. *Certain useful Borel sets.* The following sets will be useful. For $r > 0$ and $x, y \in \mathrm{Teich}(S)$ with $d(x, y) > 6r$, the *corridor* $L_r(x, y)$ is, intuitively, the set of pairs $(\xi, \eta) \in \mathcal{UE} \times \mathcal{UE}$ with $\xi \neq \eta$ such that the unique (oriented) Teichmüller geodesic $\mathcal{L}(\xi, \eta)$ passes first through $B(x, r)$ and then through $B(y, r)$.

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To define $L_r(x, y)$ rigorously, we use the nearest point projection. Let $x \in \mathrm{Teich}(S)$ and $\ell = \mathcal{L}(\xi, \eta)$ be a Teichmüller geodesic line. Denote

$$\Pi_\ell(x) = \{z \in \ell : d(x, z) = d(x, \ell)\}.$$

Then, Π_ℓ is called the *nearest point projection to* ℓ. Notice that in [19, 33], $\Pi_\ell(x)$ is always a single point because the distance function on CAT(0)-space is convex. However, this is no longer true since the Teichmüller distance is, in general, not convex (see [5]). Therefore, $\Pi_\ell(x)$ is a non-empty closed subset that may contain more than one point. Nevertheless, the diameter of $\Pi(x)$ is bounded by $2d(x, \ell)$. Recall that each ordered pair $(\xi, \eta) \in \mathcal{UE} \times \mathcal{UE}$ defines not only a unique unparameterized geodesic line $\mathcal{L}(\xi, \eta)$, but also an orientation $\overrightarrow{\xi\eta}$ on it pointing to η. For any two distinct points p, q in $\mathcal{L}(\xi, \eta)$, we say p is on the left of q if the oriented geodesic line $\overrightarrow{pq}$ is positive with respect to the orientation. For $x \in \mathrm{Teich}(S)$ and an oriented geodesic $\mathcal{L}(\xi, \eta)$, we define the *leftmost nearest projection* (to $\mathcal{L}(\xi, \eta)$) $w(x; \xi, \eta)$ to be the unique unit holomorphic quadratic differential on z such that $[\mathcal{H}(w(x; \xi, \eta))] = \xi$, $[\mathcal{V}(w(x; \xi, \eta))] = \eta$, $z = \pi(w(x; \xi, \eta)) \in \Pi_{\mathcal{L}(\xi,\eta)}(x)$, and z is leftmost in $\Pi_{\mathcal{L}(\xi,\eta)}(x)$. Note that we use the fact that $\Pi_{\mathcal{L}(\xi,\eta)}(x)$ is a closed set, which ensures that the notion of the leftmost point is well defined.

The *corridor* $L_r(x, y)$ for $d(x, y) > 6r$ is then defined to be the subset of $\mathcal{UE} \times \mathcal{UE}$ consisting of (ξ, η) with $d(x, \mathcal{L}(\xi, \eta)) < r$, $d(y, \mathcal{L}(\xi, \eta)) < r$ and so that $\Pi_{\mathcal{L}(\xi,\eta)}(x)$ is on the left of $\Pi_{\mathcal{L}(\xi,\eta)}(y)$ with $md(\Pi_{\mathcal{L}(\xi,\eta)}(x), \Pi_{\mathcal{L}(\xi,\eta)}(y)) > r$, where $md(A, B)$ stands for the minimal distance (rather than the Hausdorff distance) between two sets A, B. By Lemma 5.1, $L_r(x, y)$ is an open subset of $\mathcal{UE} \times \mathcal{UE}$; hence, $\nu_x \otimes \nu_x$-measurable.

Let g_t be the Teichmüller geodesic flow on $QD^1(S)$. Let $x \in \mathrm{Teich}(S)$ and $A, B \subset \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$. Set $\widehat{\mathfrak{F}} = \mathcal{UE} \times \mathcal{UE} \setminus \Delta$ and define

$$\begin{aligned} K_r(x) &= \left\{ g_s w(z; \xi, \eta) : (\xi, \eta) \in \widehat{\mathfrak{F}}, d(x, \mathcal{L}(\xi, \eta)) \le r, s \in \left(-\frac{r}{2}, \frac{r}{2}\right) \right\}, \\ K_A^+ &= K_r^+(x, A) = \{g_t w(z; \xi, \eta) \in K_r(x) : \eta \in A\}, \\ K_B^- &= K_r^-(y, B) = \{g_t w(z; \xi, \eta) \in K_r(y) : \xi \in B\}. \end{aligned} \tag{5.3}$$

It is straightforward to show the following lemma.

LEMMA 5.4. *If A is ν_x-measurable and B is ν_y-measurable, the sets $K_r(x)$, K_A^+, and K_B^- defined above are $\tilde{\mu}_G$-measurable.*

Proof. This follows from the continuity of the distance function at boundary points in $\mathcal{UE}$ (cf. [13, Lemma 1.4.3]) and from our choice of the leftmost nearest projection. □

Remark 5.5. Since $\Pi_{\mathcal{L}(\xi,\eta)}(x)$ has bounded diameter depending on $d(x, \mathcal{L}(\xi, \eta))$, the reader can check that whenever we use $w(x; \xi, \eta)$, the only relevant property is that $w(x, \xi, \eta) \in \Pi_{\mathcal{L}(\xi,\eta)}(x)$. Thus, one may define $w(x; \xi, \eta)$ in another way. Namely, define

$$K_r'(x) = \{g_s q(z; \xi, \eta) : (\xi, \eta) \in \widehat{\mathfrak{F}}, d(x, \mathcal{L}(\xi, \eta)) \le r, s \in \left(-\frac{r}{2}, \frac{r}{2}\right), z \in \Pi_{\mathcal{L}(\xi,\eta)}(x)\}.$$

One can check that this set is measurable. Since for any $(\xi, \eta) \in \mathcal{UE} \times \mathcal{UE}$, the set $K_r'(x) \cap \mathcal{L}(\xi, \eta)$ is a compact set and the projection map $\phi : K_r'(x) \to \mathcal{UE} \times \mathcal{UE}$ is continuous, one

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can find a measurable section of ϕ. We define $w(x; \xi, \eta)$ to be the value of the measurable section of ϕ at (ξ, η) and define $K_r(x)$ as before. In particular, $K_r(x)$ is measurable.

Remark 5.6. As pointed out in [5], the work of J. H. C. Whitehead implies that for each $x \in \mathrm{Teich}(S)$, one can choose r_x, depending on x, such that for each geodesic $\mathcal{L}(\xi, \eta)$ passing through $B(x, r_x)$, $\Pi_{\mathcal{L}(\xi,\eta)}(x)$ is, in fact, a connected segment or a point. However, this is more than we need, since in the following, we will only use the boundedness of $\Pi_{\mathcal{L}(\xi,\eta)}(x)$.

5.2. *Properties of shadows and cones.* We now list important properties of shadows and cones, following [19]. We defer to the appendix those arguments that carry over to our setting with only minor modifications.

Concerning $\mathcal{O}_r(\xi, x)$ and $\mathcal{O}_r^{\pm}(x, y)$ defined in §5.1, we have the following lemma.

LEMMA 5.7. *Let* $\xi \in \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ *and* $o \in \mathrm{Teich}(S)$. *Then, the set*

$$\{r > 0 : \nu_o(\tilde{\partial}\mathcal{O}_r(\xi, o)) > 0\}$$

is at most countable.

Proof. The proof is the same as that of [19, Lemma 6.2]. See also Note A.1 in Appendix A for details. □

LEMMA 5.8. *Let* $\xi \in \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, $o \in \mathrm{Teich}(S)$, *and* $r > 0$. *Suppose that* $y_n \in \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ *converges to* ξ. *Then, we have:*

(a)

$$\limsup_{n\to\infty}(\mathcal{O}_r^{\pm}(y_n, o)) \subset (\mathcal{O}_r(\xi, o) \cup \tilde{\partial}\mathcal{O}_r(\xi, o));$$

(b)

$$\mathcal{O}_r(\xi, o) \subset \liminf_{n\to\infty}(\mathcal{O}_r^{\pm}(y_n, o)).$$

Proof. We first prove part (b). Choose $\eta \in \mathcal{O}_r(\xi, o) \subset \mathcal{UE}$. Consider the unique geodesic $\mathcal{L}(\xi, \eta)$ determined by ξ, η, and the leftmost nearest projection $w_{\xi,\eta}$ of o to $\mathcal{L}(\xi, \eta)$. It follows that $d(o, w_{\xi,\eta}) < r$. Let $\epsilon > 0$ be such that $B(w_{\xi,\eta}, \epsilon) \subset B(o, r)$. By Lemma 2.4, there exists an open neighborhood U of ξ in $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ such that, for any points $z \in U$, there exists a geodesic $\mathcal{L}(z, \eta)$ with

$$d(w_{\xi,\eta}, \mathcal{L}(z, \eta)) < \epsilon.$$

Notice that z may lie either in $\mathrm{Teich}(S)$ or in $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$. Now, choose N such that for all $n \geq N$, we have $B(y_n, r) \in U$ when $y_n \in \mathrm{Teich}(S)$ and $y_n \in U$ when $y_n \in \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$. It follows that $\eta \in \mathcal{O}_r^-(y_n, o) \subset \mathcal{O}_r^+(y_n, o)$ for $n \geq N$.

We now prove part (a). It suffices to show that

$$\limsup_{n\to\infty}(\mathcal{O}_r^+(y_n, o)) \subset \mathcal{O}_r(\xi, o) \cup \tilde{\partial}\mathcal{O}_r(\xi, o).$$

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Let $\iota \in \limsup_{n\to\infty}(\mathcal{O}_r^+(y_n, o))$. Then, there exists an infinite subsequence, still denoted by $\{y_n\}$, such that $\iota \in \mathcal{O}_r^+(y_n, o)$ for all n. Denote by $\ell = \mathcal{L}(\xi, \iota)$ the geodesic determined by ξ and ι. We may assume that $y_n \in \mathrm{Teich}(S)$. Indeed, if $y_n \in \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, then we can take a sequence $y_n' \in \mathrm{Teich}(S)$ converging to y. Now, consider oriented geodesics ℓ_n constructed as follows. For $y_n \in \mathrm{Teich}(S)$, let ℓ_n be the geodesic extending the oriented geodesic ray $-\mathcal{L}(y_n, \iota)$. Observe that $d(o, \ell_n) < r$. Furthermore, reparameterize ℓ_n so that $\beta_\iota(o, \ell(o)) = \beta_\iota(o, \ell_n(0)) = 0$. Let $s_n \in \mathbb{R}$ be such that $\ell_n(s_n)$ is the leftmost point in $\Pi_{\ell_n}(o)$. Computations as in [19, (26)] show that $s_n \in (-r, r)$. On the one hand, $y_n \to \xi$. On the other hand, passing to a subsequence if necessary, one may assume that $s_n \to s \in [-r, r]$. By Lemma 2.4, we have $d(\ell_n(s_n), \ell(s_n)) \to 0$ as $n \to \infty$. Therefore,

$$\begin{aligned} d(o, \ell(\mathbb{R}) &\le d(o, \ell(s)) \\ &\le \lim_{n\to\infty} (d(o, \ell_n(s_n)) + d(\ell_n(s_n), \ell(s_n)) + d(\ell(s_n), \ell(s))) \\ &\le r. \end{aligned} \tag{5.4}$$

□

LEMMA 5.9. *Let $\xi \in \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, $o \in \mathrm{Teich}(S)$, and $r > 0$ be such that $\nu_o(\mathcal{O}_r(\xi, o)) > 0$ and $\nu_o(\tilde{\partial}\mathcal{O}_r(\xi, o)) = 0$. Then, for every $\epsilon > 0$, there exists an open neighborhood U_ξ of ξ in $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ such that for every $a \in U_\xi$,*

$$e^{-\epsilon}\nu_o(\mathcal{O}_r(\xi, o)) \le \nu_o(\mathcal{O}_r^{\pm}(a, o)) \le e^{\epsilon}\nu_o(\mathcal{O}_r(\xi, o)).$$

Proof. With Lemma 5.8 at hand, the proof proceeds as in [19, Corollary 6.4], noting that Theorem 3.2 implies $\nu_o(\mathcal{UE}) = 1$. See also Note A.2 for details. □

Concerning $C_r^{\pm}(x, U)$ defined in §5.1, it is straightforward to check the following lemma.

LEMMA 5.10.

(1) *We have $C_r^-(x, U) \subset C_r^+(x, U)$.*
(2) *For any $\rho > 0$ and $x_i, y_i \in \mathrm{Teich}(S)(i = 1, 2)$ with $d(x_i, y_i) < \rho$, $\mathcal{O}_r^+(x_1, y_1) \subset \mathcal{O}_{r+\rho}^+(x_2, y_2)$.*
(3) *For any $\rho > 0$ and $x_i, y_i \in \mathrm{Teich}(S)(i = 1, 2)$ with $d(x_i, y_i) < \rho$,*
$$y_1 \in C_r^+(x_1, U) \Rightarrow y_2 \in C_{r+\rho}^+(x_2, U).$$
(4) *For any $\rho > 0$ and $x_i, y_i \in \mathrm{Teich}(S)(i = 1, 2)$ with $d(x_i, y_i) < \rho$,*
$$y_1 \in C_{r+\rho}^-(x_1, U) \Rightarrow y_2 \in C_r^-(x_2, U).$$
(5) *If $r_1 \le r_2$, then*
$$C_{r_1}^+(x, U) \subset C_{r_2}^+(x, U), C_{r_2}^-(x, U) \subset C_{r_1}^-(x, U).$$

Proof. See [19, Lemma 6.5]. □

LEMMA 5.11. *Let $r > 0$, $\gamma \in G$, $x, y \in \mathrm{Teich}(S)$, and $A, B \subset \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ be Borel subsets.*

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(1) *If* $(\zeta, \eta) \in L_r(x, \gamma y) \cap (\gamma B \times A)$, *then*

$$(\gamma y, \gamma^{-1} x) \in C_r^+(x, A) \times C_r^+(y, B)$$

and

$$(\zeta, \eta) \in \mathcal{O}_r^+(\gamma y, x) \times \mathcal{O}_r^+(x, \gamma y).$$

(2) *Suppose that* $d(x, \gamma y) > 12r$. *If* $(\gamma y, \gamma^{-1} x) \in C_r^-(x, A) \times C_r^-(y, B)$, *then*

$$\{(\zeta, \eta) \in (\mathcal{UE})^2 : \eta \in \mathcal{O}_r^-(x, \gamma y), \zeta \in \mathcal{O}_r(\eta, x)\} \subset L_r(x, \gamma y) \cap (\gamma B \times A). \quad (5.5)$$

Proof. For items (1) and (2), the proofs are the same as in [19, Lemmas 6.7 and 6.8], since the arguments there rely only on the definitions of $C_r^{\pm}$, $\mathcal{O}_r^{\pm}$. See Note A.3 for details. □

We record the following properties of $K_A^+ = K_r^+(x, A)$, $K_B^- = K_r^-(y, B)$. Let

$$\Psi' : QD^1(S) \to \mathcal{PMF}(S) \times \mathcal{PMF}(S), \quad q \mapsto ([\mathcal{H}(q)], [\mathcal{V}(q)])$$

be the map obtained by composing Ψ in (2.1) with the natural projection $\mathcal{MF}(S) \to \mathcal{PMF}(S)$ in the first factor.

LEMMA 5.12. *For all* $\gamma \in G$ *such that* $d(x, \gamma y) \geq 12r$, *we have*

$$\Psi'\left(\bigcup_{t>0} \{K_A^+ \cap g_{-t} \gamma K_B^-\} \right) = L_r(x, \gamma y) \cap (\gamma B \times A).$$

Proof. The proof follows that of [19, Lemma 6.9]. The condition $d(x, \gamma y) \geq 12r$ guarantees that $md(\Pi_{\mathcal{L}(\xi,\eta)}(x), \Pi_{\mathcal{L}(\xi,\eta)}(\gamma y)) > r$, as required in the definition of $L_r(x, \gamma y)$. See Note A.4 for details. □

We will also need estimates for the Bowen–Margulis measures of K_A^+ and K_B^-. First, recall that (see (3.2)) for $\xi \neq \eta \in \mathcal{UE}$, $x \in \mathrm{Teich}(S)$, and $u \in \mathcal{L}(\xi, \eta)$,

$$\rho_x(\xi, \eta) = \beta_\xi(x, u) + \beta_\eta(x, u),$$

and (see (3.1))

$$\beta_\xi(x, y) = \frac{1}{2} \ln \frac{\mathrm{Ext}_x(\xi)}{\mathrm{Ext}_y(\xi)}.$$

By Kerckhoff's formula [15],

$$\beta_\xi(x, y) \leq d(x, y).$$

Hence,

$$\begin{aligned} \eta \in \mathcal{O}_r(\xi, x) &\Rightarrow \rho_x(\xi, \eta) \leq 2r, \\ (\xi, \eta) \in L_r(x, \gamma y) &\Rightarrow \rho_x(\xi, \eta) \leq 2r. \end{aligned} \quad (5.6)$$

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By the definition of $\tilde{\mu}_G$ and the fact that ν_x is fully supported on $\mathcal{UE}$, we have

$$\begin{aligned}
\tilde{\mu}_G(K_A^+) &= \int_A \left(\int_{\mathcal{O}_r(\xi,x)} e^{\delta_G \rho_x(\xi,\eta)} d\nu_x(\eta) \right) d\nu_x(\xi) \int \mathbb{1}_{K_A^+}(g_t w(x;\xi,\eta))\, dt \\
&= r \int_A \left(\int_{\mathcal{O}_r(\xi,x)} e^{\delta_G \rho_x(\xi,\eta)} d\nu_x(\eta) \right) d\nu_x(\xi), \\
\tilde{\mu}_G(K_B^-) &= \int_B \left(\int_{\mathcal{O}_r(\eta,y)} e^{\delta_G \rho_x(\xi,\eta)}\, d\nu_y(\eta) \right) d\nu_y(\xi) \int \mathbb{1}_{K_B^-}(g_t w(x;\xi,\eta))\, dt \\
&= r \int_B \left(\int_{\mathcal{O}_r(\eta,y)} e^{\delta_G \rho_x(\xi,\eta)} d\nu_y(\xi) \right) d\nu_y(\eta).
\end{aligned} \tag{5.7}$$

From (5.7) and (5.6), it follows that

$$\begin{aligned}
r \int_A \nu_x(\mathcal{O}_r(\xi,x))\, d\nu_x(\xi) \le \tilde{\mu}_G(K_A^+) \le r e^{2r\delta_G} \int_A \nu_x(\mathcal{O}_r(\xi,x))\, d\nu_x(\xi), \\
r \int_B \nu_y(\mathcal{O}_r(\eta,y))\, d\nu_y(\eta) \le \tilde{\mu}_G(K_B^-) \le r e^{2r\delta_G} \int_B \nu_y(\mathcal{O}_r(\eta,y))\, d\nu_y(\eta).
\end{aligned} \tag{5.8}$$

For the following lemma, recall that for $x \in \mathrm{Teich}(S)$ and a geodesic line $\mathcal{L}(\xi,\eta)$, we have defined the nearest point projection $\Pi_{\mathcal{L}(\xi,\eta)}(x)$ and the leftmost nearest projection $w(x;\xi,\eta) \in QD^1(S)$.

LEMMA 5.13. *Let $T_0 > 12r$, $\gamma \in G$ with $d(x,\gamma y) \ge 12r$, $T > T_0 + 6r$, $s \in (-r/2, r/2)$, and $(\xi,\eta) \in L_r(x,\gamma y)$.*

(1) *If $d(x,\gamma y) \in (T_0 + 3r, T]$, then*

$$\int_{T_0}^{T+3r} e^{\delta_G t} \mathbb{1}_{K_r(\gamma y)}(g_{t+s} w(x;\xi,\eta))\, dt \ge r e^{-3r\delta_G} e^{\delta_G d(x,\gamma y)}. \tag{5.9}$$

(2) *We have*

$$\int_{T_0}^{T-6r} e^{\delta_G t} \mathbb{1}_{K_r(\gamma y)}(g_{t+s} w(x;\xi,\eta))\, dt \le r e^{3r\delta_G} e^{\delta_G d(x,\gamma y)}. \tag{5.10}$$

(3) *If $d(x,\gamma y) \le T_0 - 3r$ or $d(x,\gamma y) > T$, then*

$$\int_{T_0}^{T-6r} e^{\delta_G t} \mathbb{1}_{K_r(\gamma y)}(g_{t+s} w(x;\xi,\eta))\, dt = 0. \tag{5.11}$$

Proof. Consider $w(x;\xi,\eta)$ and $w(\gamma y;\xi,\eta)$. There is $\tau > 0$ such that $w(\gamma y;\xi,\eta) = g_\tau w(x;\xi,\eta)$. Since $(\xi,\eta) \in L_r(x,\gamma y)$, we have $|d(x,\gamma y) - \tau| < 2r$. Note that

$$K_r(\gamma y) \cap \mathcal{L}(\xi,\eta) = \left\{ g_s w(\gamma y;\xi,\eta) : s \in \left(-\frac{r}{2}, \frac{r}{2} \right) \right\},$$

and $g_{t+s} w(x;\xi,\eta) \in K_r(\gamma y)$ if and only if $t + s - \tau \in (-r/2, r/2)$. Hence, if $\tau - s - (r/2) \ge T_0$ and $\tau - s + (r/2) \le T + 3r$, then

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$$\begin{aligned}\int_{T_0}^{T+3r} & e^{\delta_G t}\mathbb{1}_{K_r(\gamma y;\xi,\eta)}(g_{t+s}w(x;\xi,\eta))\,dt\\ &\geq \int_{\tau-s-r/2}^{\tau-s+r/2} e^{\delta_G t}\,dt\\ &\geq re^{-3r\delta_G}e^{\delta_G d(x,\gamma y)}.\end{aligned} \tag{5.12}$$

However, one checks that $d(x,\gamma y)\in(T_0+3r,T]$ implies that $\tau-s-(r/2)\geq T_0$ and $\tau-s+(r/2)\leq T+3r$, and hence, item (1) holds. For item (2), if $g_{t+s}w(x;\xi,\eta)\in K_r(\gamma y)$, then

$$t+s-\tau\leq\frac{r}{2},\ \tau-(t+s)\leq\frac{r}{2}.$$

It follows that

$$\begin{aligned}\int_{T_0}^{T-6r} & e^{\delta_G t}\mathbb{1}_{K_r(\gamma y)}(g_{t+s}w(x;\xi,\eta))\,dt\\ &\leq \int_{\tau-s-r/2}^{\tau-s+r/2} e^{\delta_G t}\,dt\\ &\leq re^{\delta_G(\tau-s+(r/2))}\\ &\leq re^{3r\delta_G}e^{\delta_G d(x,\gamma y)}.\end{aligned} \tag{5.13}$$

For item (3), it suffices to note that if $d(x,\gamma y)\leq T_0-3r$, then $\tau-s+(r/2)\leq T_0$, and if $d(x,\gamma y)\geq T$, then $\tau-s-3r\geq T-6r$. □

Finally, we record the following consequence of Lemma 5.12.

LEMMA 5.14. *Let $\gamma\in G$ and $d(x,\gamma y)\geq 12r$. Let $r>0$, then*

$$\begin{aligned}&\tilde{\mu}_G(K_A^+\cap g_{-t}\gamma K_B^-)\\ &\quad=\int_{L_r(x,\gamma y)\cap(\gamma B\times A)} e^{\delta_G\rho_x(\xi,\eta)}\,d\nu_x(\xi)\,d\nu_y(\eta)\int_{-(r/2)}^{r/2}\mathbb{1}_{K_r(\gamma y)}(g_{t+s}w(x;\xi,\eta))\,ds.\end{aligned} \tag{5.14}$$

5.3. *Subexponential growth of orbits in cones.* In this section, we prove a technical counting lemma that will serve as a substitute for [19, Lemma 6.6].

PROPOSITION 5.15. *Let G be a non-elementary subgroup of divergence type of* $\mathrm{Mod}(S)$ *with the critical exponent δ_G. Let $x,y\in\mathrm{Teich}(S)$ and $r>0$. Given two open subsets $\widehat{W}\subset\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ and $U\subset\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$:*

(1) *if A is a subset of $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ whose closure $\bar{A}$ in $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ is contained in $\widehat{W}\cap\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, then*

$$\sharp\{g\in G: gy\in(C_r^{\pm}(x,A)\setminus\widehat{W})\cap B(R,x)\}=o(e^{\delta_G R});$$

(2) *if $B\subset\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ and its closure $\overline{B}$ satisfies $\overline{B}\cap\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}\subset U$, then*

$$\sharp\{g\in G: gy\in(B\setminus C_r^{\pm}(x,U))\cap B(R,x)\}=o(e^{\delta_G R}).$$

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We remark that in the CAT(0) setting of [19, Lemma 6.6], the corresponding quantities were proved to be *finite* and do not merely have slow exponential growth rates. Due to various intricacies involved in Teichmüller geometry, we were not able to prove this in our setting. Nevertheless, the exponential estimates we were able to establish turn out to be sufficient.

The proof of Proposition 5.15 will rely on following lemma.

LEMMA 5.16. *Let $x \in \mathrm{Teich}(S)$ and let $z_n \in \mathrm{Teich}(S)$ be a sequence of points with $d(z_n, x) \to \infty$ lying on geodesic rays starting at points x_n within distance r of x with uniquely ergodic vertical foliations $\zeta_n \in \mathcal{UE}$. Assume that ζ_n converges in the Gardiner–Masur compactification to $\zeta \in \mathcal{UE} \subset \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$. Then, we have $z_n \to \zeta$.*

Proof. We first claim that any Gromov–Hausdorff accumulation point of the geodesic segments $[x_n, \zeta_n)$ is $[y, \zeta)$ for some $y \in B_r(x)$. This follows from Klarreich [17, Proposition 5.1], Miyachi [29, Corollary 1 in § 6.1], Masur's two boundaries theorem [24], and a diagonal argument. Now, since $[x_n, z_n] \subset [x_n, \zeta_n)$, any Gromov–Hausdorff limit of $[x_n, z_n]$ is $[y, \zeta)$. Since ζ is the only accumulation point in $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ of $[y, \zeta)$, the result follows. □

Lemma 5.16 immediately implies the following corollary.

COROLLARY 5.17. *For any $A \subset \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ contained in the closure of uniquely ergodic points, we have $\overline{C^{\pm}(x, A)} \cap \mathcal{UE} = \overline{A} \cap \mathcal{UE}$.*

We will also need the following lemma.

LEMMA 5.18. *Assume that the action $G \curvearrowright \mathrm{Teich}(S)$ has purely exponential growth. For $x \in \mathrm{Teich}(S)$, let $A_{K,R}(x)$ be the set of $y \in \mathrm{Teich}(S)$ with $R - K \le d(x, y) < R$. Then, for any $x, y \in \mathrm{Teich}(S)$, for large enough $K > 0$, there is a $C > 0$ such that for any Borel set $A \subset \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, we have*

$$\lim \sup_{R\to\infty} \frac{|A \cap A_{K,R}(x) \cap Gy|}{|A_{K,R}(x) \cap Gy|} \le C\nu_x(\overline{A}).$$

Proof. Let $K > 0$ be large enough so that $\nu_x(\mathcal{O}_K(x, gy)) \asymp_\times e^{-\delta d(x,gy)}$ [36] and $|A_{K,R}(x) \cap Gy| \asymp_\times e^{\delta R}$. Here, the notation $A(R) \asymp_\times B(R)$ is $A(R) \asymp_{x,y,K} B(R)$, which means $\lim_{R\to\infty}(A(R)/B(R)) = c \neq 0$, where c depends only on the choice of x, y, and K. Since the number of orbit points in any ball of finite radius is finite, a point of $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ is in at most D shadows $\mathcal{O}_K(x, gy)$, $gy \in A_{K,R}(x)$.

Thus,

$$\sum_{\{g:gy\in A_{K,R}(x)\cap A\}} \nu_x(\mathcal{O}_K(x, gy)) \le D\nu_x\Bigg(\bigcup_{\{g:gy\in A_{K,R}(x)\cap A\}} \mathcal{O}_K(x, gy)\Bigg).$$

Moreover, by Lemma 2.4, any uniquely ergodic point of

$$\bigcap_{n\in\mathbb{N}z} \bigcup_{\{g:gy\in A_{K,R}(x)\cap A\}} \mathcal{O}_K(x, gy)$$

lies in $\overline{A}$.

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Thus, for large enough n, we have

$$\nu_x\Bigg(\bigcup_{\{g:gy\in A_{K,R}(x)\cap A\}}\mathcal{O}_K(x,gy)\Bigg)\le 2\nu_x(\overline{A}),$$

so

$$\begin{aligned}\frac{|A\cap A_{K,R}(x)\cap Gy|}{|A_{K,R}(x)\cap Gy|}&\simeq e^{-hR}|A\cap A_{K,R}(x)\cap Gy|\simeq\sum_{\{g:gy\in A_{K,R}(x)\cap A\}}\nu_x(\mathcal{O}_K(x,gy))\\&\le D\nu_x\Bigg(\bigcup_{\{g:gy\in A_{K,R}(x)\cap A\}}\mathcal{O}_K(x,gy)\Bigg)\le 2D\nu_x(\overline{A}),\end{aligned}$$

where the implied constants depend on x, y, and K but not on R. □

Proof of Proposition 5.15. If the action has purely exponential growth, the result follows from Lemma 5.18 and Corollary 5.17. Otherwise, [**37**, Theorem B] implies that $e^{-\delta R}|B_R(x)\cap Gy|\to 0$ and the result follows trivially. □

5.4. *Asymptotic formulas.*

THEOREM 5.19. *Let S be a closed surface of genus g and let G be a non-elementary subgroup of the mapping class group* Mod(S). *Let x, y be two points in Teichmüller space* Teich(S). *Denote the associated Bowen–Margulis measure on $QD^1(S)/G$ by μ_G and the critical exponent of G by δ_G.*

(1) *If G is convergent, then*

$$\lim_{R\to\infty}\frac{\sharp\{g\in G:d(x,gy)\le R\}}{e^{\delta_G R}}=0.$$

(2) *If G is divergent and μ_G is infinite, then*

$$\lim_{R\to\infty}\frac{\sharp\{g\in G:d(x,gy)\le R\}}{e^{\delta_G R}}=0.$$

(3) *If μ_G is finite, then*

$$\lim_{R\to\infty}\frac{\sharp\{g\in G:d(x,gy)\le R\}}{e^{\delta_G R}}=c,$$

where $c=1/(\mu_G(QD^1(S)/G))$.

Having laid out the necessary technical results in the previous sections, our proof follows the strategy of [**19**, **33**]. A key observation is that the argument in [**19**] remains valid even when the number of orbit points in a sector that are far from the boundary is not finite, but merely has slower exponential growth rate than G itself, a property we established in Proposition 5.15. This observation allows us to use the estimates established in §5.3 to replace [**19**, Lemma 6.6].

Proof. We treat only item (3) in the next subsection. The proof of item (2) is the same as that of item (3), except that the first conclusion of Corollary 4.3 is replaced by the second. Item (1) follows from the dissipativity of convergent groups. □

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5.4.1. *Proof of item (3).* Item (3) in Theorem 5.19 follows from the following more general theorem by taking f to be $\mathbb{1}_{\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \times \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}}$. For $x \in \mathrm{Teich}(S)$, the Dirac measure at x is denoted by $\mathcal{D}_x$.

THEOREM 5.20. *Let G be a non-elementary subgroup of* $\mathrm{Mod}(S)$ *with finite Bowen–Margulis measure μ_G. Let $x, y \in \mathrm{Teich}(S)$, and denote the Patterson–Sullivan measures by ν_x and ν_y, respectively. Set $\|\mu_G\| = \mu_G(QD^1(S)/G)$. Then, the following weak convergence in $C(\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \times \overline{\mathrm{Teich}(S)}^{\mathrm{GM}})^*$ holds as $R \to \infty$,*

$$\delta_G e^{-\delta_G R} \sum_{\substack{g \in G \\ d(x,gy) \le R}} \mathcal{D}_{gy} \otimes \mathcal{D}_{g^{-1}x} \rightharpoonup \frac{1}{\|\mu_G\|} \nu_x \otimes \nu_y.$$

The remainder of this section is devoted to the proof of Theorem 5.20. We follow the arguments in [**19**, § 8], with the necessary technical modifications. The proof is therefore not self-contained; we indicate only the required modifications and refer the reader to [**19**, §8] for the remaining details.

Throughout this subsection, we assume that G is a non-elementary subgroup of $\mathrm{Mod}(S)$ with finite Bowen–Margulis measure μ_G. To simplify notation in the following, for $x, y \in \mathrm{Teich}(S)$, we set

$$\mu_{x,y}^R = \delta_G e^{-\delta_G R} \sum_{\substack{g \in G \\ d(x,gy) \le R}} \mathcal{D}_{gy} \otimes \mathcal{D}_{g^{-1}x},$$

and, whenever r, A, B are fixed, we write (see §5.1)

$$K^+ = K_r^+(x, A), \quad K^- = K_r^-(y, B).$$

Recall that conical points are contained in $\mathcal{UE}$ and, hence, for any $\xi \in \partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, the pair (ξ, η) is a filling pair for every conical point $\eta \neq \xi$. In particular, $\mathcal{L}(\xi, \eta)$ is non-empty.

PROPOSITION 5.21. *Let $\epsilon > 0$ and $(\xi_0, \eta_0) \in (\partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}})^2$. Let $x, y \in \mathrm{Teich}(S)$ have trivial stabilizers in G and satisfy $x \in \mathcal{L}(\xi_0, \xi^+)$, $y \in \mathcal{L}(\eta_0, \eta^+)$ for some conical points $\xi^+ \in \mathrm{supp}(\nu_x)$ and $\eta^+ \in \mathrm{supp}(\nu_y)$. Then, there exist open neighborhoods $U, W \subset \partial_{\mathrm{GM}} \overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ of ξ_0 and η_0, respectively, such that for all Borel subsets $A \subset U$, $B \subset W$,*

$$\limsup_{R \to \infty} \mu_{x,y}^R(C_1^-(x, A) \times C_1^-(y, B)) \le e^{\epsilon} \nu_x(A) \nu_y(B) / \|\mu_G\|,$$
$$\liminf_{R \to \infty} \mu_{x,y}^R(C_1^+(x, A) \times C_1^+(y, B)) \ge e^{-\epsilon} \nu_x(A) \nu_y(B) / \|\mu_G\|.$$

Proof. Throughout the proof, we adopt the notation of §5.1. Let $\epsilon > 0$ and $\rho = \min\{d(x, \gamma x), d(y, \gamma y) : \gamma \in G \setminus \{\mathrm{id}\}\} > 0$. Choose

$$r \in \left(0, \min\left\{\frac{1}{10}, \frac{\rho}{6}, \frac{\epsilon}{60\delta_G}\right\}\right)$$

such that

$$\nu_x(\tilde{\partial} \mathcal{O}_r(\xi_0, x)) = \nu_y(\tilde{\partial} \mathcal{O}_r(\eta_0, y)) = 0.$$

The existence of such an r is guaranteed by Lemma 5.7.

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CLAIM 1. *We have*

$$\nu_x(\mathcal{O}_r(\xi_0, x))\nu_y(\mathcal{O}_r(\eta_0, y)) > 0.$$

Proof of Claim 1. Indeed, by assumption, $\xi^+ \in \mathcal{O}_r(\xi_0, x)$, $\eta^+ \in \mathcal{O}_r(\eta_0, y)$, and ξ^+, η^+ lie in the support of measures ν_x, ν_y. It suffices to show that there exist open neighborhoods O_1, O_2 of ξ_0 and η_0 contained in the corresponding sets $\mathcal{O}_r(\xi_0, x)$ and $\mathcal{O}_r(\eta_0, y)$. This follows from Corollary 5.2. □

Thanks to Lemma 5.9, there exist open neighborhoods $\widehat{U}$, $\widehat{W}$ of ξ_0 and η_0, respectively, in the compactification, such that whenever $(a, b) \in \widehat{U} \times \widehat{W}$, one has

$$\begin{aligned} e^{-(\epsilon/30)}\nu_x(\mathcal{O}_r(\xi_0, x)) \le \nu_x(\mathcal{O}_r^{\pm}(a, x)) \le e^{\epsilon/30}\nu_x(\mathcal{O}_r(\xi_0, x)), \\ e^{-(\epsilon/30)}\nu_y(\mathcal{O}_r(\eta_0, y)) \le \nu_y(\mathcal{O}_r^{\pm}(b, y)) \le e^{\epsilon/30}\nu_x(\mathcal{O}_r(\eta_0, y)). \end{aligned} \tag{5.15}$$

Let U, W be open neighborhoods of ξ_0, η_0 in $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ such that their closures satisfy

$$\bar{U} \subset \widehat{U} \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}, \quad \bar{W} \subset \widehat{W} \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}.$$

Let $A \subset U$ and $B \subset W$ be arbitrary Borel subsets, and set

$$M = r^2\nu_x(\mathcal{O}_r(\xi_0, x))\nu_y(\mathcal{O}_r(\xi_0, y)).$$

We apply Roblin's method, which consists of providing bounds for

$$\int_{T_0}^{T+3r} e^{\delta_G t} \sum_{\gamma \in G} \tilde{\mu}_G(K^+ \cap g_{-t}\gamma K^-)\, dt$$

and

$$\int_{T_0}^{T-6r} e^{\delta_G t} \sum_{\gamma \in G} \tilde{\mu}_G(K^+ \cap g_{-t}\gamma K^-)\, dt$$

using two approaches: the mixing properties of the geodesic flow and direct computations.

CLAIM 2. *Let $T_0 > 60r$, $T - 6r > T_0$. Then, for $t \gg T_0$, we have*

$$\begin{aligned} &(e^{\delta_G(T-6r)} - e^{\delta_G T_0})M\nu_x(A)\nu_x(B) \\ &\quad \le e^{(2\epsilon/5)}\|\mu_G\|\delta_G \int_{T_0}^{T-6r} e^{\delta_G t} \sum_{\gamma \in G} \tilde{\mu}_G(K^+ \cap g_{-t}\gamma K^-)\, dt, \\ &(e^{\delta_G(T+3r)} - e^{\delta_G T_0})M\nu_x(A)\nu_x(B) \\ &\quad \ge e^{-(8\epsilon/15)}\|\mu_G\|\delta_G \int_{T_0}^{T+3r} e^{\delta_G t} \sum_{\gamma \in G} \tilde{\mu}_G(K^+ \cap g_{-t}\gamma K^-)\, dt. \end{aligned} \tag{5.16}$$

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Proof of Claim 2. By the choice of r and ρ, one has

$$\text{for all } \gamma \in G \setminus \{\mathrm{id}\}, \quad K_r(x) \cap \gamma K_r(x) = \emptyset, \quad K_r(y) \cap \gamma K_r(y) = \emptyset.$$

Thus, under the projection $QD^1(S) \to QD^1(S)/G$, the subsets $K^{\pm}$ inject into $QD^1(S)/G$. It follows from Corollary 4.3 that

$$\lim_{t\to\infty} \sum_{\gamma\in G} \tilde{\mu}_G(K^+ \cap g_{-t}\gamma K^-) = \frac{\tilde{\mu}_G(K^+)\tilde{\mu}_G(K^-)}{\mu_G(QD^1(S)/G)}.$$

Let $T_0 > 20r$. For $t \gg T_0$, the computations in the proof of [**19**, Proposition 8.2, pp. 827–828], together with (5.8) and (5.15), conclude the proof of the claim (see also Note A.5 for details of the computation). □

CLAIM 3. *We have*

$$\liminf_{R\to\infty} \mu^R_{x,y}(C_1^+(x, A) \times C_1^+(y, B)) \geq e^{-\epsilon} \nu_x(A)\nu_y(B)/\|\mu_G\|. \tag{5.17}$$

Proof of Claim 3. By (3.2),

$$\rho_x(\xi, \eta) = \lim_{z\to[\xi],w\to[\eta]} (d(x, z) + d(x, w) - d(z, w)).$$

It follows that $\rho_x(\xi, \eta) \leq 2r$ whenever $(\xi, \eta) \in L_r(x, \gamma y)$. Furthermore, as $T_0 > 60r$, $K^+ \cap g_{-t}\gamma K^- \neq \emptyset$ for some $t \geq T_0$ implies that $d(x, \gamma y) \geq 12r$. By the conformality of the measure ν_x, we have

$$\int_{\mathcal{O}_r^+(x,\gamma y)} d\nu_x(\eta) e^{\delta_G d(x,\gamma y)} \leq e^{4r\delta_G} \nu_y(\mathcal{O}_r^+(\gamma^{-1}x, y)).$$

Here, we use the fact, which follows from the triangle inequality, that if $\eta \in \mathcal{O}_r^+(x, \gamma y)$, then the Busemann cocycle satisfies $\beta_\eta(x, y) \geq d(x, \gamma y) - 4r$. With these remarks in mind, since $r < 1$ and $T_0 > 60r$, it follows from Lemma 5.14, Lemma 5.13 (2), Lemma 5.11, Lemma 5.10 (5), and the fact that ν_x and ν_y are fully supported on $\mathcal{UE}$ that

$$\begin{aligned}
&\int_{T_0}^{T-6r} e^{\delta_G t} \sum_{\gamma\in G} \tilde{\mu}_G(K^+ \cap g_{-t}\gamma K^-)\, dt \\
&\overset{(5.14)}{=} \sum_{\gamma\in G} \int_{T_0}^{T-6r} dt \left(e^{\delta_G t} \int_{L_r(x,\gamma y)\cap(\gamma B\times A)} e^{\delta_G \rho_x(\xi,\eta)}\, d\nu_x(\xi)\, d\nu_x(\eta) \right. \\
&\qquad \left. \times \int_{-r/2}^{r/2} \mathbb{1}_{K_r(\gamma y)}(g_{t+s} w(x; \xi, \eta))\, ds \right) \\
&\overset{(5.10),\,(5.11)}{\leq} r^2 e^{5r\delta_G} \sum_{\substack{\gamma\in G \\ d(x,\gamma y)\leq T}} e^{\delta_G d(x,\gamma y)} \int_{L_r(x,\gamma y)\cap(\gamma B\times A)} d\nu_x(\xi)\, d\nu_x(\eta) \\
&\overset{\text{Lemma 5.11(1)}}{\leq} r^2 e^{5r\delta_G} \sum_{\substack{\gamma\in G \\ d(x,\gamma y)\leq T \\ (\gamma y,\gamma^{-1}x)\in C_1^+(x,A)\times C_1^+(y,B)}} e^{\delta_G d(x,\gamma y)} \\
&\qquad \times \int_{\mathcal{O}_r^+(\gamma y,x)\times\mathcal{O}_r^+(x,\gamma y)} d\nu_x(\xi)\, d\nu_x(\eta)
\end{aligned}$$

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$$\leq r^2 e^{5r\delta_G} \sum_{\substack{\gamma\in G\\ d(x,\gamma y)\leq T\\ (\gamma y,\gamma^{-1}x)\in C_1^+(x,A)\times C_1^+(y,B)}} e^{\delta_G d(x,\gamma y)}\nu_x(\mathcal{O}_r^+(\gamma y,x))\nu_x(\mathcal{O}_r^+(x,\gamma y))$$

$$\leq r^2 e^{9r\delta_G} \sum_{\substack{\gamma\in G\\ d(x,\gamma y)\leq T\\ (\gamma y,\gamma^{-1}x)\in C_1^+(x,A)\times C_1^+(y,B)}} \nu_x(\mathcal{O}_r^+(\gamma y,x))\nu_y(\mathcal{O}_r^+(\gamma^{-1}x,y)). \tag{5.18}$$

Set

$$\mathcal{X} = (C_1^+(x,A)\times C_1^+(y,B))\cap(\widehat{U}\times\widehat{W})$$

and

$$\mathcal{Y} = (C_1^+(x,A)\times C_1^+(y,B))\setminus(\widehat{U}\times\widehat{W}).$$

Then,

$$\mathcal{X}\cup\mathcal{Y} = C_1^+(x,A)\times C_1^+(y,B).$$

It follows that

$$\begin{aligned}
&\int_{T_0}^{T-6r} e^{\delta_G t}\sum_{\gamma\in G}\tilde{\mu}_G(K^+\cap g_{-t}\gamma K^-)\,dt\\
&\leq r^2 e^{9r\delta_G}\underbrace{\sum_{\substack{\gamma\in G\\ d(x,\gamma y)\leq T\\ (\gamma y,\gamma^{-1}x)\in\mathcal{X}}}\nu_x(\mathcal{O}_r^+(\gamma y,x))\nu_y(\mathcal{O}_r^+(\gamma^{-1}x,y))}_{\mathrm{I}}\\
&\quad + r^2 e^{9r\delta_G}\underbrace{\sum_{\substack{\gamma\in G\\ d(x,\gamma y)\leq T\\ (\gamma y,\gamma^{-1}x)\in\mathcal{Y}}}\nu_x(\mathcal{O}_r^+(\gamma y,x))\nu_y(\mathcal{O}_r^+(\gamma^{-1}x,y))}_{\mathrm{II}}\\
&\leq \mathrm{I}+\mathrm{II}.
\end{aligned}\tag{5.19}$$

Note that, on the one hand, by (5.15) and the choice $r<\min\{1,\epsilon/60\delta_G\}$, we have

$$\begin{aligned}
\mathrm{I} &\leq r^2 e^{11\epsilon/30}\sum_{\substack{\gamma\in G\\ d(x,\gamma y)\leq T\\ (\gamma y,\gamma^{-1}x)\in\mathcal{X}}}\nu_x(\mathcal{O}_r(\xi_0,x))\nu_y(\mathcal{O}_r(\eta_0,y))\\
&\leq e^{11\epsilon/30}M\frac{e^{T\delta_G}}{\delta_G}\mu_{x,y}^T(C_1^+(x,A)\times C_1^+(y,B)).
\end{aligned}\tag{5.20}$$

On the other hand, by Proposition 5.15(1) and the fact that $\|\nu_x\|=\|\nu_y\|=1$, we have, as $T\to\infty$,

$$\mathrm{II} = o(e^{T\delta_G}). \tag{5.21}$$

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Hence,

$$\int_{T_0}^{T-6r} e^{\delta_G t} \sum_{\gamma\in G} \tilde{\mu}_G(K^+ \cap g_{-t}\gamma K^-)\, dt$$
$$\leq e^{11\epsilon/30} M \frac{e^{T\delta_G}}{\delta_G} \mu_{x,y}^T(C_1^+(x, A) \times C_1^+(y, B)) + e^{T\delta_G} o(1). \tag{5.22}$$

Combining Claim 2 with the above estimates, we complete the proof of Claim 3 after dividing by $Me^{\delta_G(T-6r)}\|\mu_G\|$. □

CLAIM 4. *We have*

$$\limsup_{R\to\infty} \mu_{x,y}^R(C_1^-(x, A) \times C_1^-(y, B)) \leq e^{\epsilon} \nu_x(A)\nu_y(B)/\|\mu_G\|. \tag{5.23}$$

Proof of Claim 4. The proof of Claim 4 follows the same line as that of Claim 3. We start by making several remarks. Observe that if $(\xi, \eta) \in (\mathcal{UE})^2$, then $\rho_x(\xi, \eta) \geq 0$. By Lemma 5.11(2), Lemma 5.10, and the assumption that $r < 1$, whenever $(\gamma y, \gamma^{-1}x) \in C_1^-(x, A) \times C_1^-(y, B) \subset C_r^-(x, A) \times C_r^-(y, B)$,

$$\{(\zeta, \eta) : \eta \in \mathcal{O}_r^-(x, \gamma y),\ \zeta \in \mathcal{O}_r(\eta, x)\} \subset L_r(x, \gamma y) \cap (\gamma B \times A).$$

Moreover, since $\gamma y \in C_1^-(x, A) \subset C_r^-(x, A)$, we have

$$\mathcal{O}_r^-(x, \gamma y) \subset \mathcal{O}_r^+(x, \gamma y) \subset A \subset \widehat{U}.$$

As a consequence of (5.15) at the beginning of the proof, if $\xi \in \mathcal{O}_r^-(x, \gamma y)$, then

$$\nu_x(\mathcal{O}_r(\xi, x)) \geq e^{-(\epsilon/30)} \nu_x(\mathcal{O}_r(\xi_0, x)). \tag{5.24}$$

Additionally, if $\gamma^{-1}x \in \widehat{W}$, then by (5.15) again and conformality, we have

$$\int_{\mathcal{O}_r^-(x,\gamma y)} d\nu_x(\xi) e^{\delta_G d(x,\gamma y)} \geq \nu_y(\mathcal{O}_r^-(\gamma^{-1}x, y))$$
$$\geq e^{-(\epsilon/30)} \nu_y(\mathcal{O}_r(\eta_0, y)). \tag{5.25}$$

As in the proof of Claim 3, combining the above remarks with Lemma 5.14, Lemma 5.13 (1), and the fact that ν_x and ν_y are fully supported on $\mathcal{UE}$, we obtain

$$\int_{T_0}^{T+3r} e^{\delta_G t} \sum_{\gamma\in G} \tilde{\mu}_G(K^+ \cap g_{-t}\gamma K^-)\, dt$$
$$\overset{(5.14)}{\geq} \sum_{\gamma\in G} \int_{T_0}^{T+3r} dt \Bigg(e^0 \int_{L_r(x,\gamma y)\cap(\gamma B\times A)} d\nu_x(\xi)\, d\nu_y(\eta)$$
$$\times \int_{-(r/2)}^{r/2} e^{\delta_G t} \mathbb{1}_{K_r(\gamma y)}(g_{t+s} w(x; \xi, \eta))\, ds \Bigg)$$
$$\overset{(5.9)}{\geq} r^2 e^{-3r\delta_G} \sum_{\substack{\gamma\in G \\ T_0+3r<d(x,\gamma y)\leq T}} e^{\delta_G d(x,\gamma y)} \int_{L_r(x,\gamma y)\cap(\gamma B\times A)} d\nu_x(\xi)\, d\nu_x(\eta)$$

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$$\geq r^2 e^{-(\epsilon/20)} \sum_{\substack{\gamma\in G\\ T_0+3r<d(x,\gamma y)\leq T\\ (\gamma y,\gamma^{-1}x)\in C_1^-(x,A)\times C_1^-(y,B)}} e^{\delta_G d(x,\gamma y)} \int_{\mathcal{O}_r^-(x,\gamma y)} d\nu_x(\xi)\, d\nu_x(\mathcal{O}_r(\xi,x))$$

$$\geq r^2 e^{-(\epsilon/12)} \sum_{\substack{\gamma\in G\\ T_0+3r<d(x,\gamma y)\leq T\\ (\gamma y,\gamma^{-1}x)\in C_1^-(x,A)\times C_1^-(y,B)}} e^{\delta_G d(x,\gamma y)} \int_{\mathcal{O}_r^-(x,\gamma y)} d\nu_x(\xi)\, d\nu_x(\mathcal{O}_r(\xi_0,x))$$

$$\geq r^2 e^{-(\epsilon/12)} \sum_{\substack{\gamma\in G\\ T_0+3r<d(x,\gamma y)\leq T\\ (\gamma y,\gamma^{-1}x)\in C_1^-(x,A)\times C_1^-(y,B)}} \nu_y(\mathcal{O}_r^-(\gamma^{-1}x,y))\nu_x(\mathcal{O}_r(\xi_0,x)). \tag{5.26}$$

Set

$$\mathcal{X}' = (C_1^-(x,A)\times C_1^-(y,B))\cap(\widehat{U}\times\widehat{W})$$

and

$$\mathcal{Y}' = (C_1^-(x,A)\times C_1^-(y,B))\setminus(\widehat{U}\times\widehat{W}).$$

Then,

$$\begin{aligned}
&\int_{T_0}^{T+3r} e^{\delta_G t}\sum_{\gamma\in G}\tilde{\mu}_G(K^+\cap g_{-t}\gamma K^-)\,dt\\
&\quad\geq r^2 e^{-(\epsilon/12)} \underbrace{\sum_{\substack{\gamma\in G\\ T_0+3r<d(x,\gamma y)\leq T\\ (\gamma y,\gamma^{-1}x)\in\mathcal{X}'}} \nu_y(\mathcal{O}_r^-(\gamma^{-1}x,y))\nu_x(\mathcal{O}_r(\xi_0,x))}_{\text{III}}\\
&\quad+ r^2 e^{-(\epsilon/12)} \underbrace{\sum_{\substack{\gamma\in G\\ T_0+3r<d(x,\gamma y)\leq T\\ (\gamma y,\gamma^{-1}x)\in\mathcal{Y}'}} \nu_y(\mathcal{O}_r^-(\gamma^{-1}x,y))\nu_x(\mathcal{O}_r(\xi_0,x))}_{\text{IV}}.
\end{aligned} \tag{5.27}$$

It follows from the previous remarks that

$$\begin{aligned}
\text{III} &\geq r^2 e^{-(7\epsilon/60)} \sum_{\substack{\gamma\in G\\ T_0+3r<d(x,\gamma y)\leq T\\ (\gamma y,\gamma^{-1}x)\in\mathcal{X}'}} \nu_y(\mathcal{O}_r(\eta_0,y))\nu_x(\mathcal{O}_r(\xi_0,x))\\
&\geq e^{-(7\epsilon/60)} M\frac{e^{T\delta_G}}{\delta_G}\mu_{x,y}^T(\mathcal{X}') - C_1\\
&\geq e^{-(7\epsilon/60)} M\frac{e^{T\delta_G}}{\delta_G}\mu_{x,y}^T(C_1^-(x,A)\times C_1^-(y,B)) - Me^{-(7\epsilon/60)}\|\mathcal{Y}_T'\| - C_1,
\end{aligned} \tag{5.28}$$

where $\|\mathcal{Y}_T'\|$ denotes the cardinality of

$$\mathcal{Y}_T' = \{\gamma\in G : d(s,\gamma y)\leq T, (\gamma y,\gamma^{-1}x)\in\mathcal{Y}'\}$$

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and C arises from the fact that, for fixed x and y, there are only finitely many elements in $\gamma \in G$ with $d(x, \gamma y) \leq T_0 + 3r$. Note that by Proposition 5.15, we have

$$\|\mathcal{Y}'_T\| = o(e^{T\delta_G}).$$

Since IV ≥ 0, combining these estimates, we complete the proof of Claim 4 exactly as in the proof of Claim 3. □

The proof of the proposition follows from Claims 3 and 4. □

PROPOSITION 5.22. *Let $\epsilon > 0$ and $x, y \in \mathrm{Teich}(S)$. Then, for any two points $\xi_0, \eta_0 \in \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, there exist $r > 0$ and open neighborhoods*

$$U, W \subset \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$$

of ξ_0 and η_0, respectively, such that for all Borel subsets $A \subset U$, $B \subset W$,

$$\limsup_{R\to\infty} \mu^R_{x,y}(C_r^-(x, A) \times C_r^-(y, B)) \leq e^{\epsilon}\nu_x(A)\nu_y(B)/\|\mu_G\|,$$
$$\liminf_{R\to\infty} \mu^R_{x,y}(C_r^+(x, A) \times C_r^+(y, B)) \geq e^{-\epsilon}\nu_x(A)\nu_y(B)/\|\mu_G\|.$$

Proof. The proof follows the same line as that of [19, Proposition 8.3], combined with the strategy used in Proposition 5.21, where [19, Lemma 6.6(a)] is replaced by Proposition 5.15(1). We refer the reader to Note A.6 for details as well. □

Proof of Theorem 5.20. The proof follows the same line as that of [19, Theorem 8.1]. Let $x, y \in \mathrm{Teich}(S)$ and $\epsilon > 0$. For any two points $\xi_0, \eta_0 \in \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, fix $r > 0$ and open neighborhoods $U, W \subset \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ of ξ_0, η_0 ensured by Proposition 5.22. Let $\widehat{U}, \widehat{W}$ be two open sets of $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ such that $\widehat{U} \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} = U$ and $\widehat{W} \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} = W$. Let $\mathcal{A}, \mathcal{B}$ be Borel subsets of $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ with closures $\overline{\mathcal{A}} \subset \widehat{U}, \overline{\mathcal{B}} \subset \widehat{W}$ such that $\partial(\mathcal{A} \times \mathcal{B})$ has $\nu_x \otimes \nu_y$-measure zero.

Let $\alpha > 0$. Choose open subsets A_+, B_+ and compact subsets A_-, B_- of $\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ such that

$$\begin{aligned}
A_- \subset \mathcal{A}^\circ \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \subset \overline{\mathcal{A}} \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \subset A_+ \subset U,\\
B_- \subset \mathcal{B}^\circ \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \subset \overline{\mathcal{B}} \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \subset B_+ \subset W,\\
\nu_x(\mathcal{A}^\circ \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \setminus A_-) < \alpha,\ \nu_x(A_+ \setminus \overline{\mathcal{A}} \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}) < \alpha,\\
\nu_y(\mathcal{B}^\circ \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \setminus B_-) < \alpha,\ \nu_y(B_+ \setminus \overline{\mathcal{B}} \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}) < \alpha.
\end{aligned} \tag{5.29}$$

Note that

$$\begin{aligned}
&\overline{\mathcal{A}} \times \overline{\mathcal{B}}\\
&\quad = ((\overline{\mathcal{A}} \times \overline{\mathcal{B}}) \setminus (C_r^-(x, A_+) \times C_r^-(y, B_+))) \cup ((\overline{\mathcal{A}} \times \overline{\mathcal{B}}) \cap (C_r^-(x, A_+) \times C_r^-(y, B_+))),\\
&C_r^+(x, A_-) \times C_r^+(y, B_-)\\
&\quad = ((\mathcal{A} \times \mathcal{B}) \cap (C_r^+(x, A_-) \times C_r^+(y, B_-))) \cup (C_r^+(x, A_-) \times C_r^+(y, B_-) \setminus (\mathcal{A} \times \mathcal{B})).
\end{aligned} \tag{5.30}$$

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By Propositions 5.15, 5.22, and the fact that

$$\nu_x \otimes \nu_y(\partial(\mathcal{A} \times \mathcal{B})) = 0,$$

we have

$$\begin{aligned}\limsup_{R\to\infty} \mu^R_{x,y}(\mathcal{A} \times \mathcal{B}) &\le \limsup_{R\to\infty} \mu^R_{x,y}((C_r^-(x, A_+) \times C_r^-(y, B_+)) \\ &\le \frac{e^{\epsilon}\nu_x(A_+)\nu_y(B_+)}{\|\mu_G(G)\|} \\ &\le \frac{e^{\epsilon}(\nu_x(\overline{\mathcal{A}})\nu_y(\overline{\mathcal{B}}) + \alpha(\|\nu_x\| + \|\nu_y\|))}{\|\mu_G(G)\|} \\ &\le \frac{e^{\epsilon}(\nu_x(\mathcal{A})\nu_y(\mathcal{B}) + \alpha(\|\nu_x\| + \|\nu_y\|))}{\|\mu_G(G)\|}. \end{aligned} \tag{5.31}$$

Take α small enough. One has

$$\limsup_{R\to\infty} \mu^R_{x,y}(\mathcal{A} \times \mathcal{B}) \le \frac{e^{\epsilon}\nu_x(\mathcal{A})\nu_y(\mathcal{B})}{\|\mu_G(G)\|}.$$

Similarly, one obtains that

$$\limsup_{R\to\infty} \mu^R_{x,y}(\mathcal{A} \times \mathcal{B}) \ge \frac{e^{-\epsilon}\nu_x(\mathcal{A})\nu_y(\mathcal{B})}{\|\mu_G(G)\|}.$$

Hence, for a continuous function f compactly supported in $\widehat{U} \times \widehat{W}$,

$$\begin{aligned} e^{-\epsilon} &\int_{\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \times \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}} f\, d(\nu_x \otimes \nu_y) \\ &\le \liminf_{R\to\infty} \int_{\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \times \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}} f\, d(\mu^R_{x,y}) \\ &\le \limsup_{R\to\infty} \int_{\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \times \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}} f\, d(\mu^R_{x,y}) \\ &\le e^{\epsilon} \int_{\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \times \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}} f\, d(\nu_x \otimes \nu_y). \end{aligned} \tag{5.32}$$

The proof then follows by exactly the same argument as in the last paragraph of the proof of [**19**, Theorem 8.1]. □

6. *Applications*

We present several cases where Theorem 5.19 can be applied.

6.1. *Mapping class groups.* If G is the full mapping class group $\mathrm{Mod}(S)$, then one can use item (3) in Theorem 5.19 to get an asymptotic growth formula for the mapping class group with respect to the Teicmüller metric, since in this case, $\delta_G = 6g - 6$ is the entropy of the Techmüller geodesic flow g_t on the unit tangent bundle $QD^1(S)$ of $\mathrm{Teich}(S)$ and μ_G is, up to a scale, the Masur–Veech measure constructed by Masur [**23**] and Veech [**34**], which is finite. This is the main result of [**1**], which is inspired by the method in Margulis's

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thesis [22]. Our method here is, however, modeled on the ones given by Roblin [33] and Link [19], so we obtain another proof for the case $\mathrm{Mod}(S)$.

6.2. *Convex cocompact subgroups.* A subgroup $G \leq \mathrm{Mod}(S)$ is called *convex cocompact* if some G-orbit in $\mathrm{Teich}(S)$ is quasi-convex (see [9, 14]. Convex cocompact subgroups of $\mathrm{Mod}(S)$ are analogs of convex cocompact subgroups of Kleinian groups. It also has many interesting properties, e.g. ΛG consists of uniquely ergodic measured foliations and it is virtually purely pseudo-Anosov. An important feature for a convex cocompact subgroup G of $\mathrm{Mod}(S)$ is that there is a G-invariant subset W_G of $\overline{\mathrm{Teich}(S)}^{Th}$ such that G acts on W_G cocompactly. This shows that the corresponding Bowen–Margulis measure μ_G is finite, so one can use item (3) in Theorem 5.19 to get an asymptotic formula for orbit growth. In fact, under the assumption that at least one pseudo-Anosov element has an axis in the principal stratum, this formula is obtained in the first named author's thesis [11], where a large part of the proof is quite different from the current paper.

6.3. *Other examples.* Statistically convex cocompact (SCC) subgroups for general actions with contracting elements were introduced by Yang [37]. In the setting of mapping class group actions on Teichmüller space, they are defined as follows. Let $G < \mathrm{Mod}(S)$, $o \in \mathrm{Teich}(S)$, and a constant $M > 0$. Let O_M denote the set of $g \in G$ such that some geodesic from $B(o, M)$ to $B(go, M)$ lies outside of the M-neighborhood of the orbit Go. We say G is SCC if for some $M > 0$, the set O_M has strictly smaller exponential growth rate than G itself (this definition can be shown to be independent of the basepoint $o \in \mathrm{Teich}(S)$). Yang [37] showed that SCC actions have coarse purely exponential growth:

$$0 < \lim \inf_{R\to\infty} |Go \cap B(o, R)|/e^{\delta_G R} \leq \lim \inf_{R\to\infty} |Go \cap B(o, R)|/e^{\delta_G R} < \infty.$$

This immediately implies that G is divergent and item (2) of our Theorem 5.19 then implies that G admits finite Bowen–Margulis measure. Therefore, item (3) of Theorem 5.19 applies and gives precise multiplicative asymptotics for orbit growth of G.

Examples of non-convex-cocompact SCC subgroups of mapping class groups can be obtained as free products of a free abelian group H generated by Dehn twists and a cyclic group generated by a single pseudo-Anosov γ. See [37, Proposition 6.6].

A. *Appendix. Supplementary notes*

In this appendix, we include some notes for the convenience of the reader, adapting the corresponding arguments in [19].

A.1. *Proof of Lemma 5.7.* Denote

$$A_n = \left\{ r > 0 : \nu_o(\tilde{\partial}\mathcal{O}_r(\xi, o)) > \frac{1}{n} \right\}.$$

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Since ν_o is finite and different values for r give disjoint $\tilde{\partial}\mathcal{O}_r(\xi, o)$, the number $|A_n|$ is finite for any fixed number $n \in \mathbb{Z}_{>0}$. Hence,

$$\{r > 0 : \nu_o(\tilde{\partial}\mathcal{O}_r(\xi, o)) > 0\} = \bigcup_n A_n$$

is at most countable.

A.2. *Proof of Lemma 5.9.* By our construction, $\nu_o(\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}) = \nu_o(\mathcal{UE}) = 1$. According to Lemma 5.8, one has

$$\begin{aligned}
&\nu_o(\mathcal{O}_r(\xi, o)) \\
&\quad \le \nu_o(\liminf_{n\to\infty}(\mathcal{O}_r^{\pm}(y_n, o))) \\
&\quad \le \liminf_{n\to\infty} \nu_o(\mathcal{O}_r^{\pm}(y_n, o)) \\
&\quad \le \limsup_{n\to\infty} \nu_o(\mathcal{O}_r^{\pm}(y_n, o)) \\
&\quad \le \nu_o(\limsup_{n\to\infty}(\mathcal{O}_r^{\pm}(y_n, o))) \\
&\quad \le \nu_o((\mathcal{O}_r(\xi, o) \cup \tilde{\partial}\mathcal{O}_r(\xi, o))) \\
&\quad = \nu_o(\mathcal{O}_r(\xi, o)).
\end{aligned} \tag{A.1}$$

Thus, we have, as claimed,

$$\begin{aligned}
&\lim_{n\to\infty} \nu_o(\mathcal{O}_r^{\pm}(y_n, o)) \\
&\quad = \nu_o(\mathcal{O}_r(\xi, o)).
\end{aligned} \tag{A.2}$$

A.3. *Proof of Lemma 5.11.* The reader can easily verify item (1) from the definitions. We now turn to item (2).

Let $\eta \in \mathcal{O}_r^-(x, \gamma y)$ and $\zeta \in \mathcal{O}_r(\eta, x)$. By definition, the geodesic $\mathcal{L}(\eta, \zeta)$ has distance r with x. Reversing the direction of $\mathcal{L}(\eta, \zeta)$, one gets $\mathcal{L}(\zeta, \eta)$. One then checks directly that $(\zeta, \eta) \in L_r(x, \gamma y)$. To show that $\zeta \in \gamma B$ and $\eta \in A$, note that, by the definition of $C_r^-(x, A)$, we have $\gamma y \in C_r^-(x, A)$. It follows that $\eta \in \mathcal{O}_r^-(x, \gamma y) \subset A$. Since $(\zeta, \eta) \in L_r(x, \gamma y)$, we have $\zeta \in \mathcal{O}_r^+(\gamma y, x)$, which implies that $\gamma^{-1}\zeta \in \mathcal{O}_r^+(y, \gamma^{-1}x)$. Here, we use the fact that G acts continuously on $\mathrm{Teich}(S) \bigcup_{\mathrm{GM}} \mathcal{UE}$. Since $\gamma^{-1}x \in C_r^-(y, B)$, the definition of $C_r^-(y, B)$ implies that $\mathcal{O}_r^+(y, \gamma^{-1}x) \subset B$. Therefore, $\zeta \in \gamma B$.

A.4. *Proof of Lemma 5.12.* The proof follows from the following two statements.

CLAIM A.1. *We have* $\Psi'(\bigcup_{t>0}\{K_A^+ \cap g_{-t}\gamma K_B^-\}) \subset L_r(x, \gamma y) \cap (\gamma B \times A)$.

Proof of Claim A.1. Let $q \in K_A^+ \cap g_{-t}\gamma K_B^-$ for some $t > 0$. Then, by definition of $K^{\pm}$ (see (5.3)), we have $\Psi'(q) := (\xi, \eta) \in \mathcal{UE} \times \mathcal{UE}$ with $\xi \in \gamma B$ and $\eta \in A$. Write $q = w(x; \xi, \eta)$ and let $\tau \in \mathbb{R}$ be such that

$$w(\gamma y; \xi, \eta) = g_t q.$$

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On the one hand, by the definition of $K^{\pm}$ (see (5.3)), we have $|d(x, \gamma y) - \tau| \leq 2r$, which, together with the assumption $d(x, \gamma y) \geq 12r$, implies that $\tau > 0$. On the other hand, the inequality $d(x, \gamma y) \geq 12r$ also implies that $md(\Pi_{\mathcal{L}(\xi,\eta)}(x), \Pi_{\mathcal{L}(\xi,\eta)}(\gamma y)) > r$. Therefore, we conclude that $\Psi'(q) \subset L_r(x, \gamma y)$. □

CLAIM A.2. *We have* $\Psi'(\bigcup_{t>0}\{K_A^+ \cap g_{-t}\gamma K_B^-\}) \supset L_r(x, \gamma y) \cap (\gamma B \times A)$.

Proof of Claim A.2. Let

$$(\xi, \eta) \in L_r(x, \gamma y) \cap (\gamma B \times A).$$

Consider the geodesic $\ell = \mathcal{L}(\xi, \eta)$. By definition, there exist points $m, n \in \ell$ with $g_t m = n$ for some $t > 0$ such that

$$\Psi'(\ell) = (\xi, \eta), \quad d(x, m) < r, \quad d(ry, n) < r.$$

Set $q = w(x; \xi, \eta)$ and let $\tau \in \mathbb{R}$ be such that $g_\tau q = w(\gamma y; \xi, \eta)$. Then,

$$q \in K_A^+, \quad g_\tau q \in \gamma K_B^-, \quad \tau > 0. \qquad \square$$

A.5. *Details of the computation for Claim 2 in Proposition 5.21.* Fix $\epsilon > 0$. By the mixing property of the geodesic flow, there exists $T_0 > 12r$ such that for $t \geq T_0$, we have

$$\begin{aligned} e^{-(\epsilon/3)}\tilde{\mu}_G(K^+)\tilde{\mu}_G(K^-) &\leq \|\mu_G\| \sum_{\gamma\in G} \tilde{\mu}_G(K^+ \cap g_{-t}\gamma K^-) \\ &\leq e^{\epsilon/3}\tilde{\mu}_G(K^+)\tilde{\mu}_G(K^-). \end{aligned} \tag{A.3}$$

By (5.8) and (5.15), we have, for $A \subset \widehat{V}$ and $B \subset \widehat{W}$,

$$\begin{aligned} re^{-(\epsilon/30)}\nu_x(\mathcal{O}_r(\xi_0, x))\nu_x(A) \leq \tilde{\mu}_G(K^+) \leq re^{2r\delta_G}e^{\epsilon/30}\nu_x(\mathcal{O}_r(\xi_0, x))\nu_x(A), \\ re^{-(\epsilon/30)}\nu_y(\mathcal{O}_r(\eta_0, y))\nu_y(B) \leq \tilde{\mu}_G(K^-) \leq re^{2r\delta_G}e^{\epsilon/30}\nu_y(\mathcal{O}_r(\eta_0, y))\nu_y(B). \end{aligned} \tag{A.4}$$

Since $M = r^2\nu_x(\mathcal{O}_r(\xi_0, x))\nu_y(\mathcal{O}_r(\eta_0, y)) > 0$ by Claim 1 and $r\delta_G \leq (\epsilon/60)$,

$$\begin{aligned} e^{-(\epsilon/15)}M\nu_x(A)\nu_y(B) &\leq \tilde{\mu}_G(K^+)\tilde{\mu}_G(K^-) \\ &\leq e^{\epsilon/5}M\nu_x(A)\nu_y(B). \end{aligned}$$

By the inequalities in (A.3), we have, for all $t \geq T_0$,

$$\begin{aligned} M\nu_x(A)\nu_y(B) &\leq e^{2\epsilon/5}\|\mu_G\| \sum_{\gamma\in G} \tilde{\mu}_G(K^+ \cap g_{-t}\gamma K^-), \\ M\nu_x(A)\nu_y(B) &\geq e^{-(8\epsilon/15)}\|\mu_G\| \sum_{\gamma\in G} \tilde{\mu}_G(K^+ \cap g_{-t}\gamma K^-). \end{aligned}$$

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Thus,

$$\begin{aligned}(e^{\delta_G(T-6r)} - e^{\delta_G T_0})M\nu_x(A)\nu_x(B) &= \delta_G \int_{T_0}^{T-6r} e^{t\delta_G} M\nu_x(A)\nu_x(B)\, dt \\ &\le e^{2\epsilon/5}\|\mu_G\|\delta_G \int_{T_0}^{T-6r} e^{\delta_G t} \sum_{\gamma\in G} \tilde{\mu}_G(K^+ \cap g_{-t}\gamma K^-)\, dt, \\ (e^{\delta_G(T+3r)} - e^{\delta_G T_0})M\nu_x(A)\nu_x(B) &= \delta_G \int_{T_0}^{T+3r} e^{t\delta_G} M\nu_x(A)\nu_x(B)\, dt \\ &\ge e^{-(8\epsilon/15)}\|\mu_G\|\delta_G \int_{T_0}^{T+3r} e^{\delta_G t} \sum_{\gamma\in G} \tilde{\mu}_G(K^+ \cap g_{-t}\gamma K^-)\, dt.\end{aligned} \tag{A.5}$$

A.6. *Proof of Proposition 5.22.* Let $(\xi_0, \eta_0) \in (\partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}})^2$. Choose conical points ξ^+ and η^+ in the supports of the measures ν_x and ν_y, respectively, and points $x_0 \in \mathcal{L}(\xi_0, \xi^+)$, $y_0 \in \mathcal{L}(\eta_0, \eta^+)$ with trivial stabilizers in G. Apply Proposition 5.21 to $(x_0, y_0, \xi_0, \eta_0)$ to obtain the desired open neighborhoods $\widehat{U}_0$, $\widehat{W}_0$ of ξ_0 and η_0, where ϵ is taken to be $\epsilon/3$.

Choose an open neighborhood U_0 of ξ_0 in $\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ so that $U_0 \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}} \subset \widehat{U}_0$ and with the following property: if $a \in U_0 \cap \mathcal{UE}$, then $|\beta_a(x_0, x) - \beta_{\xi_0}(x_0, x)| \le (\epsilon/6\delta_G)$ and if $a \in \mathrm{Teich}(S) \cap U_0$, then

$$|d(x_0, a) - d(x, a) - \beta_{\xi_0}(x_0, x)| \le \frac{\epsilon}{6\delta_G}. \tag{A.6}$$

Choose an open neighborhood W_0 of η_0 in the same way.

Let $U \subset \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$, $W \subset \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}$ be open neighborhoods of ξ_0 and η_0, respectively, so that

$$\overline{U} \subset U_0 \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}, \quad \overline{W} \subset W_o \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}.$$

Let $A \subset U$, $B \subset W$ be arbitrary Borel sets.

Notice that if $\xi \in A \cap \mathcal{UE}$, then $\beta_{\xi_0}(x_0, x) < \beta_\xi(x_0, x) + (\epsilon/6\delta_G)$. Hence, by the conformality of the measure, we obtain

$$\begin{aligned}e^{\delta_G\beta_{\xi_0}(x_0,x)}\nu_{x_0}(A) &= \int_A e^{\delta_G\beta_{\xi_0}(x_0,x)}\, d\nu_{x_0}(\xi) \\ &\le e^{\epsilon/6}\int_A d\nu_x(\xi) = e^{\epsilon/6}\nu_x(A).\end{aligned} \tag{A.7}$$

One can deduce similarly that

$$e^{\delta_G\beta_{\xi_0}(y_0,y)}\nu_{y_0}(B) \le e^{\epsilon/6}\nu_y(B).$$

Set $r = 1 + \max\{d(x, x_0), d(y, y_0)\}$ and

$$V_{-r} = \{x \in \mathrm{Teich}(S) : \overline{B_r(z)} \subset U_0\} \cup (U_0 \cap \partial_{\mathrm{GM}}\overline{\mathrm{Teich}(S)}^{\mathrm{GM}}).$$

If $d(x, \gamma y) \le T$ and $(\gamma y, \gamma^{-1}x) \in V_{-r} \times W_0$, then

$$(\gamma y_0, \gamma^{-1}x) \in U_0 \times W_0.$$

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By the choice of r, one can then check, by using (A.6) and the one for y_0, that

$$\begin{aligned} d(x_0, \gamma y_0) &\leq d(y, \gamma^{-1}x) + \beta_{\eta_0}(y_0, y) + \beta_{\xi_0}(x_0, x) + \frac{\epsilon}{3\delta_G} \\ &\leq T + \beta_{\eta_0}(y_0, y) + \beta_{\xi_0}(x_0, x) + \frac{\epsilon}{3\delta_G}. \end{aligned} \tag{A.8}$$

Set

$$\begin{aligned} \mathcal{X}_1 &= (C_r^-(x, A) \times C_r^-(y, B)) \cap (V_{-r} \times W_0), \\ \mathcal{X}_2 &= (C_r^-(x, A) \times C_r^-(y, B)) \setminus (V_{-r} \times W_0), \end{aligned}$$

and

$$\mathcal{X}_3 = C_1^-(x, A) \times C_1^-(y, B) \cap V_0 \times W_0.$$

We then have, for $T \gg 1$,

$$\begin{aligned} &\{\gamma \in G : d(x, \gamma y) \leq T, (\gamma y, \gamma^{-1}x) \in \mathcal{X}_1\} \\ &\quad \subset \left\{\gamma \in G : d(x_0, \gamma y_0) \leq T + \beta_{\eta_0}(y_0, y) + \beta_{\xi_0}(x_0, x) + \frac{\epsilon}{3\delta_G}, (\gamma y_0, \gamma^{-1}x_0) \in \mathcal{X}_3\right\}. \end{aligned} \tag{A.9}$$

Note that by Proposition 5.15, the number of elements in $\mathcal{X}_2$ is $o(e^{T\delta})$. Therefore, by Proposition 5.21,

$$\begin{aligned} &\limsup_{T\to\infty} \mu_{x,y}^T(C_r^-(x, A) \times C_r^-(y, B)) \\ &\quad \leq e^{\epsilon/3} e^{\delta_G(\beta_{\xi_0}(x_0,x)+\beta_{\eta_0}(y_0,y))} \\ &\qquad \times \left(\limsup_{T\to\infty} \mu_{x_0,y_0}^{T+\beta_{\eta_0}(y_0,y)+\beta_{\xi_0}(x_0,x)+c\epsilon/3\delta_G}(C_1^-(x_0, A) \times C_1^-(y_0, B))\right) \\ &\quad \leq \frac{1}{\|\mu_G\|} e^{2\epsilon/3} e^{\delta_G(\beta_{\eta_0}(y_0,y)+\beta_{\xi_0}(x_0,x))} \nu_{x_0}(A)\nu_{y_0}(B). \end{aligned} \tag{A.10}$$

Combining this inequality with (A.7) and the corresponding formula for $\nu_y(B)$, we complete the proof of the upper limit. The argument for the lower limit is similar.

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