Euro. Jnl of Applied Mathematics (2001), *vol.* 12, *pp.* 645–664. Printed in the United Kingdom

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Asymptotic and numerical studies of the leader election algorithm

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(*Received 4 April 2000; revised 29 January 2001*)

A leader is to be elected from n people using the following algorithm. Each person flips a coin. Those people who wind up with tails (which occurs with probability p, $0 < p < 1$) move on to the next stage. Those with heads are eliminated. Let H_n denote the number of stages needed until there is a single winner. We analyze the moments and the probability distribution of H_n. In the symmetric model we have an unbiased coin with $p = 1/2$; in the asymmetric model $p \neq 1/2$. We analyze these models asymptotically, for $n \to \infty$, using a variety of analytical and numerical approaches. This leads to simple derivations of some existing results, as well as some new results for the asymmetric case. Our analysis makes some assumptions about the forms of various asymptotic expansions as well as their asymptotic matching.

1 Introduction

We consider the following random leader election algorithm. There are a total of n people and a single leader is to be elected from among them. Each person flips a coin and those who wind up with tails move on to the next stage, while those with heads do not. If at a particular stage all candidates throw heads, the toss is deemed inconclusive and all advance to the next stage. This procedure is repeated until there remains a single winner. We assume that the flips are all independent and that tails occurs with probability p with $0 < p < 1$. Then heads occurs with the remaining probability $q = 1 - p$. We let H_n denote the number of stages needed to elect a leader out of n people. This is a random variable whose probability distribution $\text{Prob}[H_n = k] = P(k;n)$ we seek to compute. We define $H_1 = 0$ and for $n \geqslant 2$, $P(k;n)$ is clearly nonzero if and only if $k \geqslant 1$ (in particular, at least one toss is needed).

This problem was posed and analyzed by Prodinger [6] for the case of a fair coin with $p = q = 1/2$. Recently, further analyses of this model were provided by Fill, Mahmoud & Szpankowski [2] and Janson & Szpankowski [3]. In Fill *et al.* [2], the authors also discuss applications of this model to computer and communications networks, including token ring networks. This model also has some interesting connections with digital trees called tries (see Fill *et al.* [2] for more discussion).

Here we shall obtain some old and some new results for this model. For the symmetric model with $p = q = 1/2$, we obtain a simple expression for $P(k;n)$ as a Bernoulli

https://doi.org/10.1017/S0956792501004508 Published online by Cambridge University Press

polynomial (cf. § 3 and also Fill *et al.* [2]). Then we obtain asymptotic results for the probability distribution, which are valid for k and/or $n \to \infty$. In particular, we recover the asymptotic distribution given in Fill *et al.* [2]. This corresponds to an asymptotic approximation to P valid for $k, n \to \infty$ with $\xi = n2^{-k} = O(1)$.

For the asymmetric model ($p \neq q$) we first analyze the mean $E(H_n)$ and variance $\mathrm{var}(H_n)$ in the limit as $n \to \infty$. We give the leading one or two terms in the asymptotic expansions of these quantities, and indicate how to obtain full asymptotic series, in powers of n^{-1}. The expression for $E(H_n)$ we obtain is much different from that in Janson & Szpankowski [3], but we show (cf. Appendix A) that they are equivalent. We give a more detailed description of the variance than the one given in Janson & Szpankowski [3]. Our approach is completely elementary, and does not involve the use of complex variables, Mellin transforms, etc.

We also study $P(k;n)$ asymptotically, again for k and/or $n \to \infty$, in the asymmetric case. For $n \to \infty$ the most important scale is $\xi = np^k = O(1)$, as most of the probability mass is concentrated in the range $k = \log_{1/p}(n) + O(1)$. In Janson & Szpankowski [3], an expression of the form $P(k;n) = Q(\xi) + O(n^{-1})$ was obtained for the asymptotic distribution. However, the characterization of the function $Q(\xi)$ is in terms of an abstract measure and it is difficult, e.g. to sketch a graph of $Q(\xi)$ from the results in Janson & Szpankowski [3]. Here we aim at a more concrete description of $Q(\xi)$, by giving asymptotic results as $\xi \to 0$ and $\xi \to \infty$, and numerical results that indicate the basic shape of this function for various values of $p = 1 - q$.

Our analysis combines classical methods such as saddle point approximations, together with asymptotic methods of applied mathematics, such as matched asymptotic expansions. The latter have not been used much in the analysis of algorithms. We have previously used similar methods to analyze the QUICKSORT algorithm [4] and the height of digital search trees [5]. We make certain assumptions about the forms of the asymptotic expansions and their asymptotic matching.

In § 2 we state the basic equations. In § 3 we briefly discuss the symmetric model. The asymmetric model is discussed in § 4–6, where we analyze the moments (§ 4), analytic properties of the distribution (§ 5) and numerical properties of the distribution (§ 6).

2 Problem statement

We let H_n be the number of coin tosses needed to elect a leader out of n candidates. We denote the probability distribution of this random variable as

$$P(k;n) = \mathrm{Prob}[H_n = k], \quad (n \geqslant 1, k \geqslant 0) \tag{2.1}$$

with

$$\widetilde{P}(k;n) = \mathrm{Prob}[H_n \leqslant k], \quad (n \geqslant 1, k \geqslant 0). \tag{2.2}$$

The generating function

$$G_n(u) = E[u^{H_n}] = \sum_{k=0}^{\infty} u^k P(k;n) \tag{2.3}$$

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satisfies [3] the recurrence relation

$$G_n(u) = uq^n G_n(u) + u\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}G_k(u) + (1-u)\delta_{n,1}, \tag{2.4}$$

and this holds for all $n \geqslant 0$. Following [2] and [3] we define $G_0(u) = 0$, which corresponds to setting $H_0 = \infty$. From (2.4) it follows that $G_1(u) = 1$ so that $P(k;1) = \delta_{k,0}$. This means that if there is only one candidate, then no coin tosses are needed. For $n \geqslant 2$, we find that $G_n(0) = 0$ so that $P(0;n) = 0$. The probabilities themselves satisfy

$$P(k;n) = q^n P(k-1;n) + \sum_{\ell=0}^{n}\binom{n}{\ell}p^\ell q^{n-\ell}P(k-1;\ell), \quad (k \geqslant 1, n \geqslant 2) \tag{2.5}$$

with the boundary conditions $P(k;0) = 0$ and $P(k;1) = \delta_{k,0}$.

The probabilities (2.2) satisfy

$$\widetilde{P}(k;n) = q^n\widetilde{P}(k-1;n) + \sum_{\ell=0}^{n}\binom{n}{\ell}p^\ell q^{n-\ell}\widetilde{P}(k-1;\ell) + \delta_{n,1}\delta_{k,0}, \ (k \geqslant 0, n \geqslant 1). \tag{2.6}$$

From (2.6) it follows that $\widetilde{P}(k;1) = 1$ for $k \geqslant 0$ and we have $\widetilde{P}(k;0) = 0$ for all k. When $n = 2$ we have

$$\widetilde{P}(k;2) = 1 - (p^2+q^2)^k, (k \geqslant 0); \quad P(k;2) = 2pq(p^2+q^2)^{k-1}, \ (k \geqslant 1). \tag{2.7}$$

We define the moments (for $n \geqslant 1$) by

$$x(n) = E[H_n] = \sum_{k=1}^{\infty} kP(k;n) \tag{2.8}$$

$$w(n) = E\big[H_n(H_n-1)\big] = \sum_{k=1}^{\infty} k(k-1)P(k;n) \tag{2.9}$$

and note that the variance is $\mathrm{var}[H_n] = w(n) + x(n) - x^2(n)$. We also set $x(0) = w(0) = 0$. By differentiating (2.4) with respect to u and setting $u = 1$, we find that the first two factorial moments satisfy

$$x(n) = 1 + q^n x(n) + \sum_{\ell=0}^{n}\binom{n}{\ell}p^\ell q^{n-\ell}x(\ell), \ (n \geqslant 2) \tag{2.10}$$

and

$$w(n) = 2\big[x(n) - 1\big] + q^n w(n) + \sum_{\ell=0}^{n}\binom{n}{\ell}p^\ell q^{n-\ell}w(\ell), \ (n \geqslant 2). \tag{2.11}$$

3 The symmetric case

We briefly discuss the symmetric case with $p = q = 1/2$. We shall obtain a simple derivation of some of the results in Fill *et al.* [2], and also analyze $P(k;n)$ asymptotically, for various ranges of n and k.

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We set $p = q = 1/2$ in (2.6), replace k by $k+1$, and let $\widetilde{P}(k;n) = 2^{-kn}R(k;n)$. We thus have

$$R(k+1;n) - R(k;n) = \sum_{\ell=0}^{n} \binom{n}{\ell} 2^{k(n-\ell)} R(k;\ell), \ (k \geqslant 0). \tag{3.1}$$

Next, we recall the addition theorem [4, p. 804] for the Bernoulli polynomials $B_n(x)$:

$$B_n(x+h) = \sum_{\ell=0}^{n} \binom{n}{\ell} h^{n-\ell} B_\ell(x). \tag{3.2}$$

Setting $x = h = 2^k$ in (3.2), then $x = 0$ and $h = 2^k$, and subtracting the two expressions yields

$$\begin{aligned} B_n(2^{k+1}) - B_n(2^k) &= \big[B_n(2^{k+1}) - B_n(0)\big] - \big[B_n(2^k) - B_n(0)\big] \\ &= \sum_{\ell=0}^{n} \binom{n}{\ell} 2^{k(n-\ell)} \big[B_\ell(2^k) - B_\ell(0)\big]. \end{aligned} \tag{3.3}$$

By comparing (3.1) and (3.3), we see that $R(k;n) = B_n(2^k) - B_n(0)$, and then

$$\widetilde{P}(k;n) = 2^{-kn}\big[B_n(2^k) - B_n(0)\big]. \tag{3.4}$$

Note that $B_0(x) = 1$ so that $\widetilde{P}(k;0) = 0$ for all k. Also, $\widetilde{P}(k;1) = 2^{-k}\big[2^k - 1/2 - (-1/2)\big] = 1$ for $k \geqslant 0$ and $\widetilde{P}(0;n) = B_n(1) - B_n(0) = \delta_{n,1}$. This verifies that (3.4) satisfies all of the boundary conditions. The expression (3.4) is equivalent to that in Theorem 1 in Fill *et al.* [2]. The probabilities (2.1) are obtained from $P(k;n) = \widetilde{P}(k;n) - \widetilde{P}(k-1;n)$.

By using the Taylor series definition of the Bernoulli polynomials [4, p. 804], we obtain the following integral representation for $\mathrm{Prob}[H_n \leqslant k]$:

$$\widetilde{P}(k;n) = \frac{n!}{2\pi i} \int_C \frac{2^{-k}(e^z - 1)}{z^n\big[\exp(z2^{-k}) - 1\big]} dz \tag{3.5}$$

where C is any closed loop about $z = 0$. This expression also follows from (14) in Fill *et al.* [2]. The integral in (3.5) is useful for the asymptotic evaluation of $\widetilde{P}$. We first fix n and let $k \to \infty$. Then we expand $1/(e^\eta - 1) = 1/\eta - 1/2 + \eta/12 + O(\eta^2)$ with $\eta = z2^{-k}$, and (3.5) becomes

$$\widetilde{P}(k;n) = 1 - \frac{1}{2}n2^{-k} + \frac{1}{12}n(n-1)4^{-k} + O(8^{-k}), \ k \to \infty, \ n \geqslant 3. \tag{3.6}$$

The above also follows from Theorem 1 in Fill *et al.* [2].

Next we let $n \to \infty$ with k fixed. Now we approximate the integrand in (3.5) using

$$\frac{1}{\exp(z2^{-k}) - 1} = \exp(-z2^{-k}) + O\big(\exp(-2z2^{-k})\big)$$

and evaluate (3.5) by the saddle point method. We have

$$\widetilde{P}(k;n) = \frac{n!}{2\pi i} \int_C 2^{-k} \exp\big[z(1-2^{-k}) - n\log z\big]\big\{1 + O\big(e^{-z}, \exp(-z2^{-k})\big)\big\} dz. \tag{3.7}$$

There is a saddle point where

$$\frac{d}{dz}\big[z(1-2^{-k}) - n\log z\big] = 0$$

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so that

$$z = z_0 \equiv \frac{n}{1-2^{-k}} = \frac{n2^k}{2^k-1}.$$

Then we obtain the standard estimate for (3.7) in the form

$$\begin{aligned}\widetilde{P}(k;n) &= \frac{n!2^{-k}}{2\pi i}\exp\left[z_0(1-2^{-k}) - n\log z_0\right]\sqrt{\frac{2z_0^2\pi}{n}}i\left[1+O\left(\frac{1}{n}\right)\right] \\ &= \frac{n!e^n}{n^n\sqrt{2\pi n}}\frac{n}{2^k-1}(1-2^{-k})^n\left[1+O\left(\frac{1}{n}\right)\right] \\ &\sim n2^{-k}(1-2^{-k})^{n-1},\ n\to\infty. \end{aligned} \tag{3.8}$$

Here we have used Stirling's formula to approximate $n!$.

Expression (3.6) can be used to compute the probability that many tosses are needed to decide an election with just a few candidates, while expression (3.8) gives the probability that only a few tosses will be needed to elect a leader from among many candidates. These are both rare events and the corresponding probabilities are exponentially small. For n large, most of the probability mass will be concentrated for large values of k. Thus, we consider the limit where n and k are simultaneously large. As observed in Fill *et al.* [2], the appropriate scaling is $n,k\to\infty$ with $\xi = n2^{-k} = O(1)$. We return to (3.5), set $z = nu$, and use the Taylor series

$$\frac{1}{e^{\xi u}-1} = \sum_{m=0}^{\infty}\frac{F_m(\xi)}{m!}(u-1)^m \tag{3.9}$$

so that

$$F_0 = \frac{1}{e^\xi - 1},\quad F_1 = -\frac{\xi e^\xi}{(e^\xi-1)^2},\quad F_2 = \frac{\xi^2 e^\xi(e^\xi+1)}{(e^\xi-1)^3}.$$

Then (3.5) becomes (with C first changed to nC)

$$\widetilde{P}(k;n) = \frac{\xi n!}{n^n}\frac{1}{2\pi i}\int_C \frac{e^{nu}}{u^n}\left[F_0 + F_1(u-1) + \frac{1}{2}F_2(u-1)^2 + O\left((u-1)^3, e^{-nu}\right)\right]du. \tag{3.10}$$

The integrals in (3.10) are easily evaluated using

$$\frac{1}{2\pi i}\int_C (u-1)^m\frac{e^{nu}}{u^n}du = \sum_{j=0}^{m}\binom{m}{j}(-1)^{m+j}\frac{n^{n-j-1}}{(n-j-1)!},\ n\geqslant j+1.$$

Using the above in (3.10) yields the asymptotic expansion of $\widetilde{P}$ for $n\to\infty$ with $\xi = O(1)$. Below we summarize the main results of this section.

Theorem 3.1 *When $p = q = 1/2$ we have*

$$\widetilde{P}(k;n) = 2^{-kn}\left[B_n(2^k) - B_n(0)\right],$$

$$P(k;n) = 2^{-kn}\left[B_n(2^k) - B_n(0)\right] - I\{k\geqslant 1\}2^{-(k-1)n}\left[B_n(2^{k-1}) - B_n(0)\right]$$

where $I\{\cdot\}$ is the indicator function and the above holds for all $k,n\geqslant 0$. Asymptotic expansions of $\widetilde{P}$ are

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(a) $\widetilde{P}(k;n) = 1 - \frac{1}{2}n2^{-k} + \frac{1}{12}n(n-1)4^{-k} + O(8^{-k})$, $k \to \infty$, $n \geqslant 3$

(b) $\widetilde{P}(k;n) \sim n2^{-k}(1-2^{-k})^{n-1}$, $n \to \infty$

(c) $\widetilde{P}(k;n) = \dfrac{\xi}{e^{\xi}-1} + \dfrac{1}{n}\xi^2\left[\dfrac{e^{\xi}}{(e^{\xi}-1)^2} - \dfrac{\xi e^{\xi}(e^{\xi}+1)}{2(e^{\xi}-1)^3}\right] + O(n^{-2})$;
$k, n \to \infty$, $\xi = n2^{-k} = O(1)$.

From Theorem 1 we easily obtain the leading term(s) in the expansion(s) of $P(k;n)$ as

$$P(k;n) \sim n2^{-k-1},\ k \to \infty,\ n \geqslant 2$$

$$P(k;n) \sim n2^{-k}(1-2^{-k})^{n-1},\ n \to \infty$$

$$P(k;n) = \frac{\xi}{e^{\xi}+1} + O(n^{-1});\ k, n \to \infty, \xi = O(1).$$

The result for n large and k fixed can be interpreted as follows. The probability that a single sequence of tosses of length k has all tails is 2^{-k}, while the probability the remaining $n-1$ sequences have at least one heads is $(1-2^{-k})^{n-1}$. The factor n accounts for the fact that the sequence of tails can occur in any of the n sequences. We can also easily interpret the result for $k \to \infty$ and $n = 2$. With two sequences the probability that the election process lasts exactly k rounds is exactly the probability that two sequences of heads and tails agree for the first $k-1$ tosses and differ on the k^{th} toss, and this is equal to $2^{-(k-1)} \cdot 2^{-1} = 2^{-k}$.

We observe that the asymptotic expansions match in various intermediate limits. For example, when (c) is expanded for $\xi \to 0$ it becomes $1 - \xi/2 + \ldots$, which agrees with (a). Expanding (c) for $\xi \to \infty$ yields $\xi e^{-\xi} + \ldots$, which agrees with the expansion of (b) as $k \to \infty$. This observation will be useful for analyzing the asymmetric model, for which the calculation of $\widetilde{P}$ seems much harder.

4 The asymmetric model: moments

We consider the moment equations (2.10) and (2.11). We view (2.10) as a discrete integral equation for $x(n)$. We shall convert it to another discrete integral equation, which will prove useful for evaluating $x(n)$ asymptotically, for $n \to \infty$. We define the operator $\mathscr{D}$ by

$$\mathscr{D}f(n) = \sum_{\ell=0}^{n} \binom{n}{\ell} p^{\ell} q^{n-\ell} f(\ell)$$

and consider equations of the form $(I - \mathscr{D})x(n) = y(n)$ where I is the identity operator. A straightforward calculation shows that

$$\mathscr{D}^2 f(n) = \sum_{m=0}^{n}\left[\sum_{\ell=0}^{m}\binom{m}{\ell}p^{\ell}q^{m-\ell}f(\ell)\right]\binom{n}{m}p^m q^{n-m}$$

$$= \sum_{\ell=0}^{n}\binom{n}{\ell}p^{2\ell}(1-p^2)^{n-\ell}f(\ell)$$

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and by induction on M, we obtain

$$\mathscr{D}^M f(n) = \sum_{\ell=0}^{n} \binom{n}{\ell} p^{M\ell}(1-p^M)^{n-\ell} f(\ell), M \geqslant 1.$$

If $(I-\mathscr{D})x(n) = y(n)$ we write

$$\begin{aligned} x(n) = (I-\mathscr{D})^{-1} y(n) &= (I+\mathscr{D}+\mathscr{D}^2+\cdots)y(n) \\ &= \sum_{\ell=0}^{\infty}\sum_{j=0}^{n} \binom{n}{j} p^{\ell j}(1-p^\ell)^{n-j} y(j). \end{aligned} \tag{4.1}$$

The inversion in (4.1) is clearly only valid for certain sequences $y(n)$. We need $y(0) = 0$ for the outer sum to converge when $j = 0$. We apply (4.1) to (2.10) by writing the latter as $(I-\mathscr{D})x(n) = y(n)$ where $y(n) = 1 + q^n x(n)$ for $n \geqslant 2$ with $y(0) = y(1) = 0$. Then (4.1) yields

$$\begin{aligned} x(n) = &\sum_{\ell=0}^{\infty}\sum_{j=2}^{n} \binom{n}{j} p^{\ell j}(1-p^\ell)^{n-j} \\ &+ \sum_{\ell=0}^{\infty}\sum_{j=2}^{n} \binom{n}{j} p^{\ell j}(1-p^\ell)^{n-j} q^j x(j). \end{aligned} \tag{4.2}$$

The above holds for all $n \geqslant 0$, as $x(0) = x(1) = 0$. While (4.2) does not explicitly determine $x(n)$, it is much more useful than (2.10) for evaluating $x(n)$ in the limit $n \to \infty$.

We set

$$\ell = \left\lfloor \frac{\log n}{\log(1/p)} \right\rfloor + k = \frac{\log n}{\log(1/p)} - \left\{ \frac{\log n}{\log(1/p)} \right\} + k \tag{4.3}$$

where $\lfloor z \rfloor$ is the greatest integer $\leqslant z$ and $\{z\}$ is the fractional part of z. The first sum in (4.2) then becomes

$$\begin{aligned} a(n) &= \sum_{\ell=0}^{\infty} \left[1 - (1-p^\ell)^n - np^\ell(1-p^\ell)^{n-1}\right] \\ &= \sum_{k=-\lfloor \log_{1/p}(n) \rfloor}^{\infty} \left[1 - \left(1 - \frac{p^{k-\beta}}{n}\right)^n - p^{k-\beta}\left(1-\frac{p^{k-\beta}}{n}\right)^{n-1}\right] \end{aligned} \tag{4.4}$$

where

$$\beta = \beta(n) = \left\{ \frac{\log n}{\log(1/p)} \right\} = \{\log_{1/p}(n)\}.$$

By splitting the sum in (4.4) into the ranges $k \leqslant -1$ and $k \geqslant 0$ and using $(1-\Delta/n)^n = e^{-\Delta} + O(n^{-1})$ we are led to

$$\begin{aligned} a(n) = \frac{\log n}{\log(1/p)} - \beta &+ \sum_{j=0}^{\infty} \left[1 - (1+p^{j-\beta})\exp(-p^{j-\beta})\right] \\ &- \sum_{j=-\infty}^{-1} (1+p^{j-\beta})\exp(-p^{j-\beta}) + O(n^{-1}). \end{aligned} \tag{4.5}$$

Note that the above is oscillatory in n, in view of the appearance of the fractional part

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β. Our approximation to the summand in (4.4) does not apply for all k in the range $-\lfloor \log_{1/p}(n)\rfloor \leqslant k < \infty$, but the major contribution to the sum as $n \to \infty$ comes from the range $k = O(1)$. The $O(n^{-1})$ correction term in (4.5) can easily be computed by refining the approximation to $(1 - \Delta/n)^n$ to an asymptotic series in powers of n^{-1}. Now consider the second sum in (4.2). We again use (4.3) and note that $p^\ell = n^{-1}p^{k-\beta}$ and

$$\binom{n}{j} n^{-j} = \frac{1}{j!}\left[1 + O(n^{-1})\right],\ n \to \infty$$
$$\left(1 - \frac{1}{n}p^{k-\beta}\right)^{n-j} = \exp(-p^{k-\beta}) + O(n^{-1}),\ n \to \infty.$$

Thus as $n \to \infty$ the second sum in (4.2) is $O(1)$ and behaves as

$$\sum_{j=2}^{\infty} \frac{q^j}{j!} x(j) \left[\sum_{k=-\infty}^{\infty} p^{(k-\beta)j} \exp(-p^{k-\beta})\right] + O(n^{-1}). \tag{4.6}$$

To see that the above series converges we consider the inner sum (with $\beta = 0$ for simplicity). In view of (A.13) in the appendix we have, for $j \to \infty$,

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} p^{kj} \exp(-p^k) &= \frac{1}{\log(1/p)} \sum_{\ell=-\infty}^{\infty} \Gamma(j + ai\ell), a = \frac{2\pi}{\log(1/p)} \\
&\sim \frac{1}{\log(1/p)} \sum_{\ell=-\infty}^{\infty} e^{(j+ai\ell)\log(j+ai\ell)} e^{-(j+ai\ell)} \sqrt{\frac{2\pi}{j + ai\ell}} \\
&\sim \frac{1}{\log(1/p)} \sqrt{\frac{2\pi}{j}} e^{-j} e^{j\log j} \sum_{\ell=-\infty}^{\infty} e^{ai\ell \log j} \exp\left(-\frac{a^2\ell^2}{2j}\right) \\
&= O\left(\Gamma(j)\sqrt{j}\right).
\end{aligned}$$

Here we have used the fact that the Stirling approximation for $\Gamma(z)$ applies for $z \to \infty$ and $|\arg z| < \pi$. Since $x(n)$ grows only logarithmically, it follows that the terms in the j-sum decay (roughly) geometrically as $j \to \infty$. Thus the series converges rapidly and is useful for numerical calculations. By combining (4.4)–(4.6) we obtain a two-term asymptotic representation for $x(n)$. The leading term is $\log_{1/p}(n)$ and the next term is $O(1)$ as $n \to \infty$.

A similar calculation can be used to evaluate $w(n)$ in (2.11). By writing this equation as $(I - \mathscr{D})w(n) = 2\left[x(n) - 1\right] + q^n w(n)$ we obtain (for $n \geqslant 2$)

$$\begin{aligned}
w(n) = &-2\sum_{\ell=0}^{n} \left[1 - (1-p^\ell)^n - np^\ell(1-p^\ell)^{n-1}\right] \\
&+2\sum_{\ell=0}^{\infty} \left[\sum_{j=2}^{n} \binom{n}{j} p^{\ell j}(1-p^\ell)^{n-j} x(j)\right] \\
&+\sum_{\ell=0}^{\infty} \sum_{j=2}^{n} \binom{n}{j} p^{\ell j}(1-p^\ell)^{n-j} q^j w(j) \\
\equiv\ &S_1 + S_2 + S_3
\end{aligned} \tag{4.7}$$

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where S_i are the three sums in (4.7). By using (4.2) we can show that

$$S_1 + S_2 = 2\sum_{k=0}^{\infty} k\left[1 - (1-p^k)^n - np^k(1-p^k)^{n-1}\right] + 2\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=2}^{n}\sum_{i=2}^{j}\binom{n}{j}\binom{j}{i}p^{\ell j}p^{mi}(1-p^{\ell})^{n-j}(1-p^m)^{j-i}q^i x(i). \tag{4.8}$$

Now consider the limit $n \to \infty$. Writing the first sum in the right side of (4.8) as $\tilde{a}(n)$ and setting $k = \log_{1/p}(n) + j - \beta$ we obtain

$$\tilde{a}(n) = \lfloor \log_{1/p}(n) \rfloor^2 - \lfloor \log_{1/p}(n) \rfloor + 2\lfloor \log_{1/p}(n) \rfloor S_0 + 2S_0' + O(n^{-1}) \tag{4.9}$$

where

$$S_0 = \sum_{j=-\infty}^{\infty}\left[I\{j \geqslant 0\} - (1+p^{j-\beta})\exp(-p^{j-\beta})\right]$$
$$S_0' = \sum_{j=-\infty}^{\infty} j\left[I\{j \geqslant 0\} - (1+p^{j-\beta})\exp(-p^{j-\beta})\right]. \tag{4.10}$$

After some calculation, we find that the four-fold sum in (4.8), for $n \to \infty$, is given by

$$2\lfloor \log_{1/p}(n) \rfloor \sum_{i=2}^{\infty}\frac{q^i}{i!}x(i)\left[\sum_{\ell=-\infty}^{\infty} p^{(\ell-\beta)i}\exp(-p^{\ell-\beta})\right]$$
$$+2\sum_{i=2}^{\infty}\frac{q^i}{i!}x(i)\left[\sum_{\ell=-\infty}^{\infty}(\ell+1)p^{(\ell-\beta)i}\exp(-p^{\ell-\beta})\right] + O(n^{-1}). \tag{4.11}$$

Finally, the leading term for S_3 in (4.7) is the same as (4.6), with $x(j)$ replaced by $w(j)$. By combining (4.9) with (4.11) we obtain a three-term approximation to $w(n)$ as $n \to \infty$. The leading term is $O(\log^2 n)$, the second term is $O(\log n)$ and the third term is $O(1)$. Then we can easily compute the asymptotic variance from $\mathrm{var}(H_n) = w(n) + x(n) - x^2(n)$, and we find that the variance is $O(1)$ for $n \to \infty$. We summarize the final results below.

Theorem 4.1 *The moments $x(n)$ and $w(n)$ satisfy* (4.2) *and* (4.7)*, respectively. For $n \to \infty$ we have*

$$E(H_n) = x(n) = \frac{\log n}{\log(1/p)} - \beta + S_0 + S_* + O(n^{-1})$$
$$var(H_n) = S_0 + S_* - (S_0 + S_*)^2 + 2S_0' + 2S_*' + T_* + O\left(\frac{\log n}{n}\right)$$

where

$$\beta = \left\{\frac{\log n}{\log(1/p)}\right\} = \textit{fractional part of } \log_{1/p}(n)$$

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and

$$\begin{bmatrix} S_0 \\ S_0' \end{bmatrix} = \sum_{j=-\infty}^{\infty} \begin{bmatrix} 1 \\ j \end{bmatrix} \left[I\{j \geqslant 0\} - (1+p^{j-\beta})\exp(-p^{j-\beta}) \right],$$

$$\begin{bmatrix} S_* \\ S_*' \end{bmatrix} = \sum_{j=2}^{\infty} \frac{q^j}{j!} x(j) \sum_{k=-\infty}^{\infty} \begin{bmatrix} 1 \\ k+1 \end{bmatrix} p^{(k-\beta)j} \exp(-p^{k-\beta}),$$

$$T_* = \sum_{j=2}^{\infty} \frac{q^j}{j!} w(j) \sum_{k=-\infty}^{\infty} p^{(k-\beta)j} \exp(-p^{k-\beta}).$$

While the relations in Theorem 4.1 are implicit rather than explicit, they are useful for numerical purposes, as all the sums that appear are rapidly convergent. We need only generate the first few $x(n)$ and $w(n)$ from (2.10) and (2.11), and use these to evaluate S_*, S_*' and T_*. Now we relate Theorem 4.1 to the results in Janson & Szpankowski [3]. We note that, in view of the appearance of β, the answers are periodic functions of $\log_{1/p}(n)$, with period equal to one. Such functions can be represented as Fourier series in the form

$$\alpha_0 + {\sum_{k=-\infty}^{\infty}}' \alpha_k e^{2\pi i k\beta} \tag{4.12}$$

where we have isolated the constant term α_0 in the Fourier series. In Janson & Szpankowski [3], a formula for $E(H_n)$ was obtained in the form (4.12). In Appendix A we show that our result is equivalent to that in Theorem 1(i) in Janson & Szpankowski [3]. The authors also show that $\mathrm{var}(H_n) = O(1)$ and identify the constant term (cf. Theorem 1(ii) in Janson & Szpankowski [3]) in the Fourier series for the variance. After a lengthy calculation, it is possible to show that this agrees with the result in our Theorem 4.1. The latter can also be used to identify the other Fourier coefficients. We comment that the present derivation uses only elementary methods and makes no use of complex variables, Mellin transforms, etc. In Janson & Szpankowski [3] the derivation relies heavily on Mellin transforms. Our final results do not involve any oscillatory sums and are well suited to numerical computations.

5 The asymmetric model: distribution

We consider the probability distribution $P(k;n)$, and also $\widetilde{P}(k;n)$. We have not been able to obtain this explicitly, so we discuss asymptotic and numerical methods. We first attempt to obtain analogs of the formulas in parts (a)–(c) of Theorem 3.1, for the asymmetric case.

First we consider (2.6) in the limit $n \to \infty$ with k fixed. We have $\widetilde{P}(0;n) = \delta_{n,1}$ and then

$$\widetilde{P}(1;n) = q^n \widetilde{P}(0;n) + npq^{n-1} \sim np(1-p)^{n-1}.$$

By computing $\widetilde{P}(k;n)$ for the first few k we conclude that

$$\widetilde{P}(k;n) \sim np^k \left[q(1+p+p^2+\cdots+p^{k-1}) \right]^{n-1} = np^k(1-p^k)^{n-1}, \ n \to \infty.$$

When $p = 1/2$, this reduces to Theorem 3.1(b). The interpretation of the above is similar to that given below Theorem 3.1 for the symmetric case. Note that now the probability

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of a single sequence of k tosses having all tails is p^k and that of the remaining $n-1$ sequences having at least one heads is $(1-p^k)^{n-1}$.

Next we let $k \to \infty$ with n fixed. We set $\widetilde{P}(k;n) = 1 - \overline{P}(k;n)$ to find that for $k \geqslant 1$

$$-\overline{P}(k;n) = q^n - q^n \overline{P}(k-1;n) - \sum_{\ell=0}^{n} \binom{n}{\ell} q^{n-\ell} p^\ell \overline{P}(k-1;\ell). \tag{5.1}$$

We recall that $\overline{P}(k;0) = 1$ and $\overline{P}(k;1) = 0$ so that (5.1) simplifies to

$$\overline{P}(k;n) = q^n \overline{P}(k-1;n) + \sum_{\ell=2}^{n} \binom{n}{\ell} q^{n-\ell} p^\ell \overline{P}(k-1;\ell). \tag{5.2}$$

Since (5.2) is 'constant-coefficient' in k, we assume that $\overline{P}(k;n) \sim a^k g(n)$ for $k \to \infty$. Using this form in (5.2) yields

$$ag(n) = (p^n + q^n) g(n) + \sum_{\ell=2}^{n-1} \binom{n}{\ell} q^{n-\ell} p^\ell g(\ell). \tag{5.3}$$

Setting $n = 2$ determines a as

$$a = p^2 + q^2. \tag{5.4}$$

In view of (2.7) we have $g(2) = 1$ and then (5.3) (with (5.4)) may be used to compute $g(n)$ recursively, for $n \geqslant 3$. Hence,

$$(p^2 + q^2 - p^n - q^n) g(n) = \sum_{\ell=2}^{n-1} \binom{n}{\ell} q^{n-\ell} p^\ell g(\ell), \quad n \geqslant 3. \tag{5.5}$$

When $p = q = 1/2$ we have $g(n) = n/2$.

Now consider k and n simultaneously large, with $\xi = np^k$ fixed. We set

$$\widetilde{P}(k;n) \equiv \widetilde{Q}(np^k;n) = \widetilde{Q}(\xi;n)$$

with which (2.6) becomes

$$\widetilde{Q}(\xi;n) = q^n \widetilde{Q}\left(\frac{\xi}{p};n\right) + \sum_{\ell=0}^{n} \binom{n}{\ell} p^\ell q^{n-\ell} \widetilde{Q}\left(\xi \frac{\ell}{np};\ell\right) + \delta_{n,1}\delta_{k,0}. \tag{5.6}$$

For n and k large, we may neglect the first and third terms in the right side of (5.6). Furthermore, the 'kernel' in the sum $\binom{n}{\ell} p^\ell q^{n-\ell}$ behaves as $n \to \infty$ as the delta function $\delta(\ell - np)$. We thus expand $\widetilde{Q}$ in the sum in (5.6) about $\ell = np$:

$$\widetilde{Q}\left(\xi \frac{\ell}{np};\ell\right) = \widetilde{Q}(\xi;np) + \left(\widetilde{Q}_1 \frac{\xi}{np} + \widetilde{Q}_2\right)\Big|_{\ell=np} (\ell - np)$$
$$+ \frac{1}{2}\left(\widetilde{Q}_{11}\left(\frac{\xi}{np}\right)^2 + 2\widetilde{Q}_{12}\frac{\xi}{np} + \widetilde{Q}_{22}\right)\Big|_{\ell=np} (\ell - np)^2 + \cdots .$$

Here $\widetilde{Q}_1$ (resp. $\widetilde{Q}_2$) is the derivative of $\widetilde{Q}$ with respect to its first (resp. second) argument. Then (5.6) becomes, as $n \to \infty$,

$$\widetilde{Q}(\xi;n) = \widetilde{Q}(\xi;np) + \frac{pq}{2}\left[\frac{\xi^2}{np^2}\widetilde{Q}_{\xi\xi}(\xi;np) + \frac{2\xi}{p}\frac{\partial}{\partial(np)}\widetilde{Q}_\xi(\xi;np) + n\frac{\partial^2}{\partial(np)^2}\widetilde{Q}(\xi;np)\right] + \cdots . \tag{5.7}$$

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We assume that as $n \to \infty$ $\widetilde{Q}$ has an expansion of the form $\widetilde{Q}(\xi;n) = A(\xi) + n^{-1}B(\xi) + O(n^{-2})$. Then $\partial_n \widetilde{Q}_\xi = O(n^{-2})$ and $n\partial_n^2 \widetilde{Q} = O(n^{-2})$, and hence (5.7) becomes

$$A(\xi) + \frac{1}{n}B(\xi) + O(n^{-2}) = A(\xi) + \frac{1}{np}B(\xi) + \frac{q}{2np}\xi^2 A''(\xi) + O(n^{-2}) \tag{5.8}$$

which yields

$$B(\xi) = -\frac{1}{2}\xi^2 A''(\xi). \tag{5.9}$$

This expresses the second term in the expansion of $\widetilde{Q}$ in terms of the leading term, but the latter remains undetermined. When $p = q = 1/2$ we have $A(\xi) = \xi/(e^\xi - 1)$ and then (5.9) is consistent with Theorem 3.1(c).

Some further information about $A(\xi)$ can be obtained by asymptotically matching this approximation to those valid in the other limits. For $k \to \infty$ we have

$$np^k(1-p^k)^{n-1} = \xi\left(1 - \frac{\xi}{n}\right)^{n-1} \sim \xi e^{-\xi}$$

so that we expect that $A(\xi) \sim \xi e^{-\xi}$ as $\xi \to \infty$. By matching $A(\xi)$ as $\xi \to 0^+$ to the approximation valid for n fixed and $k \to \infty$ we obtain

$$1 - A(\xi) \sim g(n)(p^2+q^2)^k;\ n \to \infty, \xi \to 0.$$

We need to know the behavior of $g(n)$ as $n \to \infty$. By using ideas similar to those in § 4, we convert (5.5) to the equivalent equation

$$g(n) = \sum_{m=0}^{\infty}\sum_{\ell=2}^{n} p^{m\ell}(1-p^m)^{n-\ell}\binom{n}{\ell}\frac{q^\ell}{(p^2+q^2)^{m+1}}g(\ell),\ n \geqslant 2 \tag{5.10}$$

with $g(2) = 1$. Note that when $n = 2$, (5.10) becomes simply $g(2) = g(2)$. For $n \to \infty$ we set $m = \lfloor \log_{1/p}(n) \rfloor + k$ in (5.10) and note that

$$p^{m\ell}\binom{n}{\ell} = p^{(k-\beta)\ell}\frac{1}{n^\ell}\binom{n}{\ell} \sim \frac{1}{\ell!}p^{(k-\beta)\ell}$$

and

$$(p^2+q^2)^{-m} = \exp\left(-\log(p^2+q^2)\left[\frac{\log n}{\log(1/p)} + k - \beta\right]\right) = n^\nu (p^2+q^2)^{\beta-k}$$

where

$$\nu = \frac{\log(p^2+q^2)}{\log p} > 0.$$

Then (5.10) can be replaced by the asymptotic relation

$$g(n) = n^\nu\left[\sum_{\ell=2}^{\infty}\frac{q^\ell}{\ell!}g(\ell)\overline{A}(\ell;n) + O(n^{-1})\right] \tag{5.11}$$

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where

$$\overline{A}(\ell;n)=\sum_{k=-\infty}^{\infty}p^{(k-\beta)\ell}(p^2+q^2)^{\beta-k-1}\exp(-p^{k-\beta}). \tag{5.12}$$

Thus $\overline{A}$ varies slowly and periodically with n. When $p=q=1/2$, (5.11) reduces to $g(n)=n/2$. We also note that $\nu\to 0$ as $p\downarrow 0$, $\nu\to 2$ as $p\uparrow 1$ and $\nu=1$ if $p=1/2$. We next write (5.11) and (5.12) as Fourier series. We use

$$\sum_{k=-\infty}^{\infty}\delta(\eta-k+\beta)=1+\sum_{j=-\infty}^{\infty}{}'\exp\bigl[2\pi ij(\eta+\beta)\bigr] \tag{5.13}$$

where the last sum excludes the term with $j=0$. Using (5.13) in (5.12) yields

$$\begin{aligned}\overline{A}(\ell;n)&=\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(\eta-k+\beta)\exp(-p^{\eta})\frac{p^{\ell\eta}}{(p^2+q^2)^{1+\eta}}d\eta\\&=\int_{-\infty}^{\infty}\frac{p^{\ell\eta}}{(p^2+q^2)^{1+\eta}}\exp(-p^{\eta})d\eta\\&\quad+\sum_{j=-\infty}^{\infty}{}'e^{2\pi ij\beta}\int_{-\infty}^{\infty}\frac{p^{\ell\eta}}{(p^2+q^2)^{1+\eta}}e^{2\pi ij\eta}\exp(-p^{\eta})d\eta\\&=\int_0^{\infty}\frac{z^{\ell-\nu-1}e^{-z}}{(p^2+q^2)\log(1/p)}dz\\&\quad+\sum_{j=-\infty}^{\infty}{}'e^{2\pi ij\beta}\int_0^{\infty}\frac{z^{\ell-\nu-1}e^{-z}}{(p^2+q^2)\log(1/p)}\exp\left(\frac{2\pi ij\log z}{\log p}\right)dz\\&=\frac{\Gamma(\ell-\nu)}{(p^2+q^2)\log(1/p)}+\sum_{j=-\infty}^{\infty}{}'e^{2\pi ij\beta}\frac{\Gamma(\ell-\nu+2\pi ij/\log p)}{(p^2+q^2)\log(1/p)}.\end{aligned} \tag{5.14}$$

In §6 we shall show that retaining only the constant term in the Fourier series, which yields the approximation

$$g(n)\approx n^{\nu}\frac{1}{(p^2+q^2)\log(1/p)}\sum_{\ell=2}^{\infty}\frac{\Gamma(\ell-\nu)}{\ell!}q^{\ell}g(\ell), \tag{5.15}$$

is often sufficient to obtain an accurate estimate of $g(n)$ for n large.

We next develop a numerical method for computing $A(\xi)$, which is the leading term for $\widetilde{P}(k;n)$ in the limit when n and k are large with $\xi=np^k=O(1)$. We write (2.4) in the form $(I-u\mathscr{D})G_n(u)=uq^nG_n(u)+(1-u)\delta_{n,1}$ so that $G_n(u)=[1+u\mathscr{D}+u^2\mathscr{D}^2+\cdots]\bigl[uq^nG_n(u)+(1-u)\delta_{n,1}\bigr]$. An explicit calculation shows that this is the same as

$$\begin{aligned}G_n(u)&=n(1-pu)\sum_{\ell=0}^{\infty}u^{\ell}p^{\ell}(1-p^{\ell})^{n-1}\\&\quad+\sum_{\ell=0}^{\infty}u^{\ell+1}\sum_{j=2}^{n}\binom{n}{j}q^jp^{\ell j}(1-p^{\ell})^{n-j}G_j(u).\end{aligned}$$

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Inverting the generating function (2.3), the above leads to

$$P(k;n) = n\left[p^k(1-p^k)^{n-1} - I\{k>0\}p^k(1-p^{k-1})^{n-1}\right]$$
$$+\sum_{\ell=0}^{k-1}\sum_{j=2}^{n}\binom{n}{j}q^j p^{\ell j}(1-p^\ell)^{n-j}P(k-1-\ell;j). \tag{5.16}$$

This is exact for all $k,n \geqslant 0$. For $n\to\infty$, (5.16) becomes

$$P(k;n) \equiv Q(\xi;n) = \xi(e^{-\xi} - e^{-\xi/p}) + O(n^{-1})$$
$$+\sum_{m=1}^{\infty}\sum_{j=2}^{\infty}\frac{q^j}{j!}\xi^j p^{-(m+1)j}\exp(-\xi p^{-m-1})P(m;j).$$

Here we have used $P(0;j)=0$ for $j\geqslant 2$. Thus, writing

$$P(k;n) = Q(\xi) + O(n^{-1})$$

we have

$$Q(\xi) = \xi(e^{-\xi} - e^{-\xi/p}) + \sum_{m=1}^{\infty}\sum_{j=2}^{\infty}\frac{q^j}{j!}\xi^j p^{-(m+1)j}\exp(-\xi p^{-m-1})P(m;j). \tag{5.17}$$

Again the double sum must be evaluated by first using (2.5) to compute $P(m;j)$. We also have

$$Q(\xi) = A(\xi) - A(\xi/p).$$

It follows from (5.17) that $Q(\xi)\sim A(\xi)\sim \xi e^{-\xi}$ as $\xi\to\infty$, and this is precisely the result we obtained by using asymptotic matching.

Now let $\xi\to 0^+$ in (5.17). The dominant contribution to the sum will come from large values of m, where we may use the approximation $P(m;j)\sim 2pq(p^2+q^2)^{m-1}g(j)$. Denoting by $S(\xi)$ the double sum in (5.17), we have

$$S(\xi)\sim\sum_{m=0}^{\infty}\sum_{j=2}^{\infty}\frac{q^j}{j!}\xi^j p^{-(m+1)j}2pq(p^2+q^2)^{m-1}g(j)\exp(-\xi p^{-m-1}). \tag{5.18}$$

We furthermore scale m to be large so that $\xi p^{-m} = O(1)$ as $\xi\to 0^+$. Letting $m = \log_p\xi + \ell - \Delta$ where $\Delta = \{\log\xi/\log p\} = \{\log_p\xi\}$ we obtain from (5.18)

$$S(\xi)\sim\frac{2pq}{p^2+q^2}\xi^\nu\sum_{\ell=-\infty}^{\infty}\sum_{j=2}^{\infty}\frac{q^j}{j!}g(j)\frac{p^{(\ell+\Delta-1)j}}{(p^2+q^2)^{\ell+\Delta}}\exp(-p^{\Delta+\ell-1}). \tag{5.19}$$

Using (5.19) in (5.17) and noting that $0<\nu<2$, we find that as $\xi\to 0^+$ $Q(\xi)$ is given by the right side of (5.19). But, for any non-integer z we have $\{z\}+\{-z\}=1$, so that $\Delta = 1-\beta$. By using (5.11) with (5.12) we can obtain the behavior of $1-g(n)(p^2+q^2)^k$ as $n\to\infty$. The approximation to $P(k;n)$ in the double limit where $k\to\infty$ and then $n\to\infty$ is thus $2pq(p^2+q^2)^{k-1}g(n)$ where we use (5.11) for $g(n)$ as $n\to\infty$. However, in view of (5.19), this is precisely the same as the expansion of $S(\xi)$ (and hence $Q(\xi)$) as $\xi\to 0^+$. This verifies that the expansion for $P(k;n)$ valid for $k\to\infty$, $n=O(1)$ and that for $k,n\to\infty$ with ξ fixed asymptotically match.

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Below we summarize the main results of this section. These made some assumptions about the forms of the various asymptotic expansions.

Result 5.1 For the asymmetric model we have the asymptotic expansions

(a) $1 - \widetilde{P}(k;n) \sim g(n)(p^2+q^2)^k,\ k \to \infty.$
Here $g(n)$ is computed for $n \geqslant 3$ from the recursion (5.5), with $g(2) = 1$. For $n \to \infty$, $g(n)$ is given by (5.11) with $\nu = \log(p^2+q^2)/\log p \in (0,2)$.

(b) $\widetilde{P}(k;n) \sim np^k(1-p^k)^{n-1},\ n \to \infty.$

(c) $\widetilde{P}(k;n) = A(\xi) - \dfrac{1}{2n}\xi^2 A''(\xi) + O(n^{-2});\ k,n \to \infty, \xi = np^k = O(1).$

(d) For $\xi = O(1)$ we have

$$P(k;n) = Q(\xi) + O(n^{-1})$$

where

$$Q(\xi) = \xi(e^{-\xi} - e^{-\xi/p}) + \sum_{m=1}^{\infty}\sum_{j=2}^{\infty} \frac{q^j}{j!}\xi^j p^{-(m+1)j}\exp(-\xi p^{-m-1})P(m;j)$$

and $Q(\xi) = A(\xi) - A(\xi/p)$, $A(0) = 1$, $A(\xi) = \sum_{m=0}^{\infty} Q(\xi p^{-m})$.

The asymptotic results for $P(k;n)$ are easily obtained from parts (a) and (b) as

$$P(k;n) \sim 2pq\, g(n)(p^2+q^2)^{k-1},\ k \to \infty$$
$$P(k;n) \sim np^k(1-p^k)^{n-1},\ n \to \infty.$$

When $p = q = 1/2$, Result 5.1 reduces to the asymptotic formulas in Theorem 3.1. In Appendix B we show that the expression for $Q(\xi)$ becomes explicit, and we get

$$Q(\xi) = \frac{\xi}{e^{\xi}+1}. \tag{5.20}$$

In the asymmetric case our results are not completely explicit, but they are useful for the numerical calculation of $Q(\xi)$ and $g(n)$ (and hence $P(k;n)$), as we show in § 6.

6 Numerical results

In Figure 1 we plot the function $Q(\xi)$ for ξ in the range $[0,10]$ and for $p = 0.2$, 0.4, 0.6 and 0.8. Our method for computing Q is simply to truncate both sums that appear in the definition of Q in Result 5.1 at some fairly large value, call it N. Then we compute $P(m;j)$ numerically using the recursion (2.5). We chose N large enough so that increasing N to $N+1$ and plotting both functions on the same graph leads to the two curves being indistinguishable. The exact result clearly satisfies $Q(0) = 0$ and we also have $Q_N(0) = 0$ for any N, where Q_N corresponds to truncation of the series in Theorem 4.1. For small ξ the approximation $Q(\xi) \approx Q_N(\xi)$ is not very good as the tails of the sum(s) become important in this limit. However, we have done an asymptotic analysis of $Q(\xi)$ as $\xi \to 0^+$ in the previous section.

From Figure 1 (and indeed from the formula in Result 5.1) we see that $Q(\xi)$ is unimodal. If $Q_{\max}(p)$ is the maximum value of $Q(\xi) = Q(\xi;p)$, then we see that $Q_{\max}$ decreases as p increases, with approximate values being $Q_{\max} = 0.36$, 0.315, 0.23, 0.116 for $p = 0.2$, 0.4,

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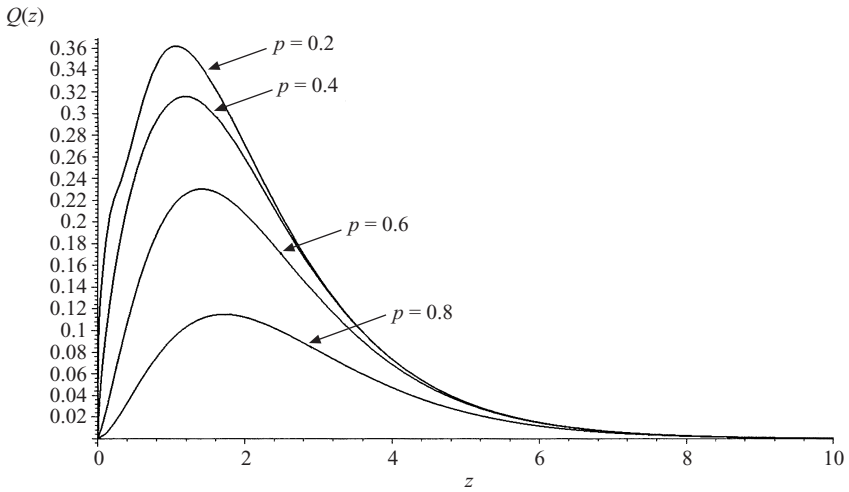


FIGURE 1. A sketch of the function Q for $p = 0.2$, 0.4, 0.6 and 0.8.

0.6, 0.8. The corresponding values of ξ where $Q_{\max}$ is achieved are (approximately) 1.0, 1.2, 1.4, 1.6.

In Table 1, we test the accuracy of the approximation (5.15). We tabulate $g(n)n^{-\nu}$ (computing $g(n)$ exactly using the recurrence (5.5)) for $n = 100$, 200, and 300 and compare the result to $C(p)$, which is defined as the right side of (5.15) (divided by n^{ν}). The series converges quite rapidly, except when $q \uparrow 1$ (i.e. $p \downarrow 0$). We computed enough terms in the series so that we get $C(p)$ to five decimal places. We see that for the tabulated points, which cover the interval $0.1 < p < 0.99$, $C(p)$ remains in the range $[0.375, 0.533]$. As p increases from zero, $C(p)$ increases until about $p = 0.3$, then decreases to a local minimum at about $p = 0.7$, then again increases to $C(p) = 0.5$ as $p \to 1$. The table also shows that the oscillations that correspond to the higher Fourier coefficients (cf. (5.11) and (5.14)) are important only for small values of p. When $p = 0.5$ (the symmetric case) these oscillations are absent completely. It is also easy to show that they disappear in the limit $p \to 1$. The data in Table 1 suggests that perhaps $g(n)n^{-\nu}$ converges to a limit as $n \to \infty$ for any p in the range [0.5,1]. This would mean that the higher Fourier coefficients obtained by using (5.14) in (5.11) all vanish. However, it is difficult to verify this directly, since these coefficients also involve $g(n)$.

An interesting open question is whether we can get an explicit formula for $Q(\xi)$ (and hence $g(n)$) in the non-symmetric case. Even if not, our numerical calculations suggest that it may be fruitful to do asymptotic analyses of the problem in the limits $p \to 0$, $p \to 1$ and $p \approx 1/2$.

Appendix A

We show the equivalence between our expression for $E(H_n)$ in Theorem 5.1, and the result in Theorem 1(i) in Janson & Szpankowski [3]. For simplicity, we take $\beta = \{\log_{1/p}(n)\} = 0$, but the exact same calculation applies to the case $0 < \beta < 1$.

First we compare the $O(1)$ terms in the expansion of $E(H_n)$ that do not involve $x(n)$,

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Table 1. *Accuracy of the approximation* (5.15)

p	$C(p)$	$g(n)n^{-v}$		
		$n = 100$	$n = 200$	$n = 300$
0.1	0.37585	0.36643	0.41732	0.39449
0.2	0.48669	0.48480	0.48570	0.49476
0.3	0.53273	0.53412	0.53225	0.53328
0.4	0.52867	0.52918	0.52892	0.52886
0.5	0.50000	0.5	0.5	0.5
0.6	0.47167	0.47082	0.47125	0.47139
0.7	0.45723	0.45539	0.45631	0.45662
0.8	0.45920	0.45631	0.45776	0.45824
0.9	0.47477	0.47082	0.47280	0.47346
0.99	0.49714	0.49225	0.49470	0.49551

to the corresponding terms in Janson & Szpankowski [3]. Setting $\beta = 0$, we must show that

$$\sum_{j=0}^{\infty}\left[1-(1+p^j)\exp(-p^j)\right]-\sum_{j=1}^{\infty}(1+p^{-j})\exp(-p^{-j})$$
$$=\frac{1}{2}-\frac{1-\gamma}{\log(1/p)}-\sum_{k\neq 0}\alpha_k \tag{A.1}$$

where

$$\alpha_k=\frac{1}{\log(1/p)}\left(1+\frac{2\pi ik}{\log(1/p)}\right)\Gamma\left(\frac{2\pi ik}{\log(1/p)}\right). \tag{A.2}$$

Here γ is Euler's constant and $\Gamma(\cdot)$ is the Gamma function.

We start from the Fourier series

$$\{z\}=\frac{1}{2}-\sum_{n=1}^{\infty}\frac{1}{n\pi}\sin(2n\pi z),\ z\neq\ \text{integer} \tag{A.3}$$

whose derivative yields

$$\frac{1}{2}+\sum_{n=1}^{\infty}\cos(2n\pi z)=\frac{1}{2}\sum_{m=-\infty}^{\infty}\delta(z-m). \tag{A.4}$$

For real a, we have the integral representations

$$\Gamma(aik)+\Gamma(-aik)=\frac{1}{aik}\int_0^{\infty}[t^{iak}-t^{-iak}]e^{-t}dt$$
$$=\frac{1}{aik}\int_0^{\infty}2i\sin(ak\log t)e^{-t}dt \tag{A.5}$$

and

$$aik\left[\Gamma(aik)-\Gamma(-aik)\right]=2\int_0^{\infty}\cos(ak\log t)e^{-t}dt. \tag{A.6}$$

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Using (A.5) and (A.6), the right side of (A.1) becomes

$$\frac{1}{2}+\frac{\gamma-1}{\log(1/p)}-(I_1+I_2)$$

where

$$I_1=I_1(a)=\frac{1}{\pi}\int_0^\infty\left[\sum_{k=1}^\infty\frac{\sin(ak\log t)}{k}\right]e^{-t}dt,\ a=\frac{2\pi}{\log(1/p)}$$

$$I_2=I_2(a)=\frac{a}{\pi}\int_0^\infty\left[\sum_{k=1}^\infty\cos(ak\log t)\right]e^{-t}dt. \tag{A.7}$$

In view of (A.4) we evaluate I_2 as

$$\begin{aligned}I_2&=\frac{a}{\pi}\int_0^\infty\left[-\frac{1}{2}+\frac{1}{2}\sum_{m=-\infty}^\infty\delta\left(\frac{\log t}{\log(1/p)}-m\right)\right]e^{-t}dt\\&=\frac{-1}{\log(1/p)}+\sum_{m=-\infty}^\infty\int_0^\infty p^{-\xi}\exp(-p^{-\xi})\delta(\xi-m)d\xi\\&=-\frac{1}{\log(1/p)}+\sum_{m=-\infty}^\infty p^{-m}\exp(-p^{-m}).\end{aligned} \tag{A.8}$$

Similarly, we have

$$\begin{aligned}I_1'(a)&=\frac{1}{\pi}\int_0^\infty\left[\sum_{k=1}^\infty\cos(ak\log t)\right]e^{-t}\log t\,dt\\&=\frac{1}{\pi}\int_0^\infty\left[-\frac{1}{2}+\frac{1}{2}\sum_{m=-\infty}^\infty\delta\left(\frac{\log t}{\log(1/p)}-m\right)\right]e^{-t}\log t\,dt\\&=\frac{\gamma}{2\pi}+\frac{1}{2\pi}\sum_{m=-\infty}^\infty\int_0^\infty\xi p^{-\xi}\log^2(1/p)\exp(-p^{-\xi})\delta(\xi-m)d\xi\\&=\frac{\gamma}{2\pi}+\frac{2\pi}{a^2}\sum_{m=-\infty}^\infty mp^{-m}\exp(-p^{-m})\\&=\frac{\gamma}{2\pi}+\frac{d}{da}\left[\sum_{m=-\infty}^{-1}\exp(-e^{2\pi m/a})+\sum_{m=1}^\infty\exp(-e^{2\pi m/a})\right].\end{aligned}$$

By integrating the above with respect to a we obtain

$$I_1(a)-I_1(0^+)=\frac{\gamma a}{2\pi}+\sum_{m=-\infty}^{-1}\left[\exp(-p^{-m})-1\right]+\sum_{m=1}^\infty\exp(-p^{-m}). \tag{A.9}$$

From (A.3) we obtain

$$\lim_{z\to0^\pm}\left[\sum_{n=1}^\infty\frac{\sin(2n\pi z)}{n}\right]=\pm\frac{\pi}{2}. \tag{A.10}$$

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We write the integral in (A.7) as $\int_0^\infty(\cdots)dt = \int_0^1(\cdots)dt + \int_1^\infty(\cdots)dt$ and use (A.10) to obtain

$$I_1(0^+) = \frac{1}{\pi}\int_0^1 -\frac{\pi}{2}e^{-t}dt + \frac{1}{\pi}\int_1^\infty \frac{\pi}{2}e^{-t}dt = -\frac{1}{2} + e^{-1}. \tag{A.11}$$

Using (A.11), (A.9) and (A.8) we obtain the right side of (A.1) as

$$\frac{1}{2} + \frac{\gamma - 1}{\log(1/p)} + \frac{1}{\log(1/p)} - \sum_{m=-\infty}^{\infty} p^{-m}\exp(-p^{-m})$$
$$-\left[\frac{\gamma}{\log(1/p)} + e^{-1} - \frac{1}{2} + \sum_{m=1}^{\infty}\left(\exp(-p^m) - 1\right) + \sum_{m=1}^{\infty}\exp(-p^{-m})\right].$$

After some simplification, we can easily show that the above agrees with the left side of (A.1).

Next we consider the part of the $O(1)$ term in the expansion of $E(H_n)$ that involves $x(n)$. Setting $\beta = 0$ we must show that

$$\sum_{j=2}^{\infty}\frac{q^j}{j!}x(j)\left[\sum_{k=-\infty}^{\infty}p^{kj}\exp(-p^k)\right] = \frac{T_1^*(0)}{\log(1/p)} + \frac{1}{\log(1/p)}\sum_{k\neq 0}T_1^*\left(\frac{2\pi ik}{\log(1/p)}\right) \tag{A.12}$$

where

$$T_1^*(s) = \sum_{n=2}^{\infty}\frac{q^n}{n!}x(n)\Gamma(s+n)$$

is as in Theorem 1(i) in Janson & Szpankowski [3]. By comparing coefficients in (A.12), it suffices to show that

$$\sum_{k=-\infty}^{\infty}p^{kn}\exp(-p^k) = \frac{1}{\log(1/p)}\left[\Gamma(n) + \sum_{k\neq 0}\Gamma(n + aik)\right],\ a = \frac{2\pi}{\log(1/p)}. \tag{A.13}$$

But, the left side of (A.13) is equal to

$$\sum_{m=-\infty}^{\infty}\int_0^\infty p^{-n\xi}\exp(-p^{-\xi})\delta(\xi - m)d\xi$$
$$= \frac{1}{\log(1/p)}\int_0^\infty t^{n-1}e^{-t}\left[\sum_{m=-\infty}^{\infty}\delta\left(\frac{\log t}{\log(1/p)} - m\right)\right]dt$$
$$= \frac{1}{\log(1/p)}\int_0^\infty t^{n-1}e^{-t}\left[1 + 2\sum_{m=1}^{\infty}\cos(am\log t)\right]dt$$
$$= \frac{1}{\log(1/p)}\left[\Gamma(n) + \sum_{m\neq 0}\int_0^\infty t^{n-1}t^{aim}e^{-t}dt\right]$$

which is the same as the right side. This establishes the equivalence between the expression for $E(H_n)$ in Janson & Szpankowski [3] and that in Theorem 5.1.

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Appendix B

We evaluate the expression for $Q(\xi)$ in Result 5.1, in the symmetric case. Setting $\xi \to \xi/2$ and using the expression for $P(m;j)$ in Theorem 3.1, we obtain

$$Q(\xi/2) = \frac{1}{2}\xi(e^{-\xi/2} - e^{-\xi}) + \sum_{m=0}^{\infty}\sum_{j=2}^{\infty}\frac{2^{-j}}{j!}\xi^j \exp(-2^m\xi) \times \big[B_j(2^m) - B_j(0) - 2^j B_j(2^{m-1})I\{m \geqslant 1\} + B_j(0)2^j I\{m \geqslant 1\}\big]. \quad \text{(B.1)}$$

Using (cf. [1])

$$\sum_{n=0}^{\infty}\frac{u^n}{n!}B_n(x) = \frac{ue^{ux}}{e^u - 1},\ B_0(x) = 1,\ B_1(x) = x - \frac{1}{2}$$

we evaluate the sum(s) over j in (B.1) to get

$$Q(\xi/2) = \frac{1}{2}\xi(e^{-\xi/2} - e^{-\xi}) + \sum_{m=0}^{\infty}\exp(-2^m\xi)\Big\{\frac{\xi/2}{e^{\xi/2}-1}\big(\exp(\xi 2^{m-1}) - 1\big) - \frac{\xi}{2}2^m - I\{m \geqslant 1\}\Big[\frac{\xi}{e^{\xi}-1}\big(\exp(\xi 2^{m-1}) - 1\big) - \xi 2^{m-1}\Big]\Big\}. \quad \text{(B.2)}$$

Now,

$$\sum_{m=0}^{\infty} 2^{m-1}\xi\exp(-2^m\xi)\big[I\{m \geqslant 1\} - 1\big] = -\frac{\xi}{2}e^{-\xi}$$

$$\sum_{m=0}^{\infty}\big[\exp(-\xi 2^{m-1}) - \exp(-\xi 2^m)\big] = e^{-\xi/2},$$

and

$$\sum_{m=0}^{\infty} I\{m \geqslant 1\}\big[\exp(-\xi 2^{m-1}) - \exp(-\xi 2^m)\big] = e^{-\xi}.$$

Using the above in (B.2) we obtain, after some simplification,

$$Q(\xi/2) = \frac{\xi/2}{e^{\xi/2}-1} - \frac{\xi}{e^{\xi}-1} = \frac{\xi/2}{e^{\xi/2}+1}$$

which establishes (5.20).

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