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Non-uniqueness for specifications in $\ell^{2+\epsilon}$

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Abstract. For every $p > 2$, we construct a regular and continuous specification (g-function), which has a variation sequence that is in ℓ^p and which admits multiple Gibbs measures. Combined with a result of Johansson and Öberg [Square summability of variations of g-functions and uniqueness in g-measures. *Math. Res. Lett.* **10**(5–6) (2003), 587–601], this determines the optimal modulus of continuity for a specification which admits multiple Gibbs measures.

1. *Introduction*

For a finite set A, let $P(A)$ be the set of probability distributions on A. A *specification* g (also commonly known as a g-function) is a measurable function from $A^{\mathbb{Z}^-}$ to $P(A)$. A specification g is *regular* if there exists $\varrho > 0$ such that for every sequence $\{b_n\}_{n\in\mathbb{Z}^+}$ and every $a \in A$, we have that $(g(\{b_n\}))(a) \geq \varrho$. We focus on specifications that are regular and continuous with respect to the product topology. An invariant measure compatible with a specification g is a shift-invariant probability measure μ on $A^{\mathbb{Z}}$ such that for every $f : A \to \mathbb{R}$,

$$\mathbf{E}_\mu(f(x_0)|x_{-1}, x_{-2}, \ldots) = \mathbf{E}_{g(x_{-1},x_{-2},\ldots)}(f) \quad \text{almost surely.}$$

With a slight abuse of language, we will refer to those measures from hereafter as *Gibbs measures*. It is easy to show that every continuous specification has a Gibbs measure.

Given a past, $x_{-1}, x_{-2}, \ldots$, the specification g tells us the probability distribution for x_0, the next state of the process. Thus, the specification and the past determine the stochastic evolution of the process. One common example of a specification is a finite state space Markov chain. The specification for a k-step Markov chain is determined

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by $x_{-1}, \ldots, x_{-k}$. For this reason Döblin and Fortet referred to specifications as 'chains with infinite connections' [**7**].

The question of whether a specification g uniquely determines the stationary process under a natural 'mixing' assumption has been prominent since the pioneering work of Döblin and Fortet [**7**]. However, in the last three decades reasonable progress was achieved only in providing sufficient conditions for the uniqueness of Gibbs measures (see [**21**] for detailed discussions and references). Harris [**12**] studied the behavior of lumped Markov chains and introduced important coupling ideas that were used by several later authors. Keane [**15**] introduced the notion of a continuous g-function and gave conditions under which a g-function has a unique measure. One natural way to express uniqueness conditions is in terms of the modulus of continuity of g. To quantify the modulus of continuity of a specification g, we define the *variation of g at distance k* to be

$$\mathrm{var}_k(g) = \sup\{\|g(b) - g(b')\|_1 \mid b_1 = b'_1, b_2 = b'_2, \ldots, b_k = b'_k\}.$$

The continuity of g is equivalent to the convergence of the sequence $\mathrm{var}_k(g)$ to zero.

In [**23**], Walters showed that if $\mathrm{var}(g) \in \ell^1$, then the Gibbs measure is unique. Ledrappier [**17**] established a strong connection between specifications and the thermodynamic formalism of statistical mechanics. In particular, they related specifications with symbolic dynamics as developed in works of Sinai [**20**], Ruelle [**19**], Bowen [**4**] and others.

Walters's work was sharpened by Lalley [**16**] and Berbee [**3**], who showed that the Gibbs measure is unique if

$$\sum_{n=1}^{\infty} \exp\left(-\sum_{m=1}^{n} \mathrm{var}_m(g)\right) = \infty.$$

On the other hand, by the early 1980s it was known that the equilibrium measures of appropriately chosen one-dimensional long-range Ising models are not unique (see [**8**] for the hierarchical-type models and [**11**] for the $1/r^2$ decaying cases). Existence of a phase transition was later established for the one-dimensional long-range percolation model (see [**2**, **18**]) and finally for the one-dimensional Fortuin–Kasteleyn (FK) random cluster model [**1**]. Nevertheless until the 1990s it was widely believed that continuous and regular specifications admit a unique Gibbs measure.

However, in 1993, Bramson and Kalikow [**5**] provided a remarkable example of a continuous and regular specification that admits multiple Gibbs measures. During the time when that paper was being revised, Hulse constructed a different example of a continuous and regular specification with multiple Gibbs measures [**13**]. The variation of the function g in Bramson and Kalikow's construction is not in ℓ^p for any p. In fact, in their example $\mathrm{var}_k(g) \geq C/(\log k)$ for some constant C. This gave rise to the following question: for which values of p does $\mathrm{var}(p) \in \ell^p$ imply uniqueness?

A few years ago Stenflo further sharpened Berbee's work [**22**]. However, the results of Berbee and Stenflo, while improving over Walters' result, are still in the realm of ℓ^1. Recently, Johansson and Öberg [**14**] showed that if g is regular and $\{\mathrm{var}_k(g)\}_{k=1}^{\infty}$ is in ℓ^2, then g admits a unique Gibbs measure. Our main result is the following.

THEOREM 1. *For every $p > 2$, there exists a regular specification g such that $\{\mathrm{var}_k(g)\} \in \ell^p$ and g admits multiple Gibbs measures.*

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This shows that the result of Öberg and Johansson is tight.

Remarks. In [**9**, **10**] Fernández and Maillard proved a Dobrushin-type uniqueness condition. This condition is not comparable with the variation conditions. Recently, Dias and Friedli [**6**] took a closer look at a class of chains inspired by the construction introduced in those papers.

2. *Construction*

We will use the alphabet of size four, $A = \{+1, -1\}^2$. We fix a parameter $\epsilon \in (0, 0.5)$. Given this, we pick a positive integer $K = K(\epsilon)$ such that the inequalities in equations (10), (12) and (13) below are true for all $k \geq K$. Note that all those inequalities are true for all large enough k. We now begin to define a regular continuous specification

$$g(x, y) = g^{\epsilon}(x, y) : A^{\mathbb{Z}^-} \to P(A).$$

The choice of (x_0, y_0) given $\{x_{-i}, y_{-i}\}_{i=1}^{\infty}$ consists of four steps.

(1) Choose y_0 independently of $\{x_{-i}, y_{-i}\}_{i=1}^{\infty}$, so that $y_0 = +1$ with probability 0.5 and $y_0 = -1$ with probability 0.5.
(2) Using the values of $\{y_{-i}\}_{i=0}^{\infty}$, choose a (deterministic) set of odd size $S \subset \mathbb{Z}^-$.
(3) Using the values of $\{y_{-i}\}_{i=0}^{\infty}$, choose a (deterministic) value $0 \leq \upsilon < 0.4$.
(4) Let z be the majority value of $\{x_t : t \in S\}$. Choose $x_0 = z$ with probability $0.5 + \upsilon$ and $x_0 = -z$ with probability $0.5 - \upsilon$.

In order to complete the first step, the second coordinate (y) must be independent and identically distributed with distribution $(\frac{1}{2}, \frac{1}{2})$. We ensure this if for all x and y,

$$g(x, y)(1, 1) + g(x, y)(-1, 1) = g(x, y)(1, -1) + g(x, y)(-1, -1) = \tfrac{1}{2}. \tag{1}$$

Because of (1) and the fact that our definitions of S and υ given below depend only on $\{y_i\}_{i=-\infty}^{0}$ and $\{x\}_{i=-\infty}^{-1}$, we can first choose $\{y_i\}_{i=-\infty}^{\infty}$ to be an independent and identically distributed sequence and then choose the x sequence. This will simplify our notation.

The most intricate part of the construction is the choice of the set $S \subset \mathbb{Z}^-$. Before we choose S, we need some notation. For every positive integer k, let I_k be the sequence of length

$$\ell_k = \lceil (1 + \epsilon)^k \rceil$$

such that the last element is -1 and all other elements are $+1$. We also define

$$\beta_k = 2^{\ell_k} \tag{2}$$

and

$$\nu_k = \frac{\beta_k}{\beta_{k-1}}, \tag{3}$$

which is within a constant factor of $2^{\epsilon \cdot \ell_{k-1}}$.

Now we define a block structure, which will allow us to choose the set $S \subset \mathbb{Z}^-$ and the parameter υ.

Definition 2. Let $k > K$. A *k-block for y* is an interval $[a, b)$ and the values $\{y_i\}_{i=a}^{b-1}$ such that:

(1) $(y_{a-\ell_k+1}, y_{a-\ell_k+2}, \ldots, y_a) = (y_{b-\ell_k+1}, y_{b-\ell_k+2}, \ldots, y_b) = I_k$;
(2) for no $c \in (a, b)$ does $(y_{c-\ell_k+1}, y_{c-\ell_k+2}, \ldots, y_c) = I_k$.

Note that $[a, b)$ being a k-block depends, in fact, also on values of y outside $[a, b)$. Note that this definition is invariant under a shift of y in the following sense: given y and

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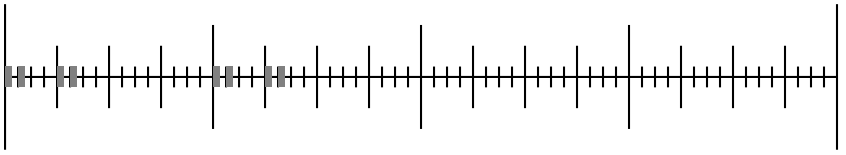


FIGURE 1. A block. The opening is marked in gray.

any integer t, define $\sigma^t(y)$ by $\sigma^t(y)_i = y_{i+t}$ for all i. Then if $[a, b)$ is a k-block for $\sigma^t(y)$, then $[a + t, b + t)$ is a k-block for y.

Also note that the length of a k-block is likely to be close to β_k, and that the number of $(k-1)$-blocks inside a k-block is likely to be close to ν_k. Precise statements of these two claims will be used extensively in the next section.

Definition 3. We say that the *beginning* of a K-block $B = [a, b)$ is the interval

$$[a, \min(b, a + \beta_K^{1-\epsilon})).$$

For $k > K$, we say that the *beginning* of a k-block $B = [a, b)$ is the intersection of:

- the union of the first $\nu_k^{1-\epsilon}$ $(k-1)$-blocks in B; and
- $[a, a + \beta_{k+1})$.

Definition 4. The *opening* $C(B)$ of a k-block B is the set of points $t \in B$ such that t is in the beginning of its j-block for every $K \le j \le k$.

See Figure 1 for a visual representation of a block and its opening.

Notation. We let $B_k(t)$ denote the k-block containing t, and we let $C(B_k(t))$ denote the opening of $B_k(t)$.

For a k-block B, we define the *signature* of B to be

$$X(B) = \operatorname{sign} \sum_{t \in C'(B)} x_t, \tag{4}$$

where

$$C'(B) = \begin{cases} C(B) & \text{if } |C(B)| \text{ is odd,} \\ C(B) \setminus \{\max(C(B))\} & \text{otherwise.} \end{cases}$$

The purpose of working with $C'(B)$ instead of $C(B)$ is that its cardinality is always odd, so $X(B)$ can only be $+1$ or -1.

By Definition 4, we see that the event that 0 is in the opening of its k-block is measurable with respect to

$$\sigma(y_i : i \in [-2\beta_{k+1}, 0]).$$

Indeed, if the block is shorter than $2\beta_{k+1}$, then the event is determined by elements of the block, and if it is longer (which is measurable with respect to this σ-algebra), then 0 is not in the opening.

This fact will be used to show that our specification is continuous and to show how quickly $\operatorname{var}_k(g)$ approaches 0.

Define k_0 to be the supremum of all j such that 0 is in the opening of its j-block if there is such a j. If there is no such value j, then we take $k_0 = K - 1$. We now define S to be

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the following set: if $k_0 = \infty$, then S is the empty set. If $k_0 < \infty$, then we define S to be $C'(B)$, where B is the $(k_0 + 1)$-block containing 0. Note that $P(k_0 = \infty) = 0$.

Next we choose the value of υ. If

$$\min(S) < -\beta_{k_0+2}, \quad \text{then } \upsilon = 0. \tag{5}$$

Otherwise, we take

$$\upsilon = \begin{cases} 0.4 & \text{if } k_0 = K - 1, \\ \beta_{k_0}^{-1/2+\epsilon} & \text{if } k_0 \geq K. \end{cases} \tag{6}$$

Finally, we set the probability that $x_0 = 1$ given $\{y\}_{i=-\infty}^{\infty}$ and $\{x\}_{i=-\infty}^{-1}$ to be

$$(g(x, y))(1, y_0) = 0.5 + \upsilon \cdot \operatorname{sign} \sum_{t \in S} x_t = 0.5 + \upsilon X(B_{k_0+1}(0)). \tag{7}$$

By the definition of a signature in equation (4), this completes the definition of our specification.

For ease of notation, for any two sequences $x = (x_{-k})_{k=1}^{\infty}$ and $y = (y_{-k})_{k=1}^{\infty}$, and x_0 and y_0 in $\{+1, -1\}$, we write $g_{x_0}(x, y, y_0)$ for $(g(x, y))(x_0, y_0)$.

The condition defined by (5) is important because when combined with (6) it implies that if $\max\{-s : s \in S\}$ is large, then υ is close to zero. This is essential for proving continuity and for estimating $\operatorname{var}_k(g)$. Also notice that g is symmetric in the x coordinate. This property will play a key role in showing that g supports multiple Gibbs measures. We write this in the form of a lemma.

LEMMA 1. *For all $x \in \{1, -1\}^{\mathbb{Z}^-}$, $y \in \{1, -1\}^{\mathbb{Z}^-}$ and $w, z \in \{1, -1\}$,*

$$g_{(w,z)}(x, y) = g_{(-w,z)}(-x, y).$$

2.1. *Continuity.* We now show that g is continuous with respect to the product topology. The two most important elements of the construction are that for all y:

(1) there exists N such that the set S is determined by $\{y_i\}_{i=-N}^{0}$;
(2) the larger N is, the closer υ is to zero.

In the next lemma we quantify these two statements and show that they imply that g is continuous.

LEMMA 2. *The specification g is regular and continuous. Moreover, the sequence $var_j(g) \in l^p$ for all*

$$p > \frac{2(1+\epsilon)^2}{1-2\epsilon}. \tag{8}$$

Note that as ϵ approaches 0, the bound in (8) goes to 2.

Proof. Let $k > K$. We want to estimate $\operatorname{var}_j(g)$ for $\beta_k < j \leq \beta_{k+1}$.

Let $\{x_i^{(1)}, y_i^{(1)}\}_{i \in \mathbb{Z}^-}$ and $\{x_i^{(2)}, y_i^{(2)}\}_{i \in \mathbb{Z}^-}$ be such that $x_i^{(1)} = x_i^{(2)}$ and $y_i^{(1)} = y_i^{(2)}$ for every $i > -j$. It is enough to estimate $|g_1(x^{(1)}, y^{(1)}, l) - g_1(x^{(2)}, y^{(2)}, l)|$ for $l = -1, +1$. Fix l and, for $h = 1, 2$ let

$$y_i^{(h)} = \begin{cases} y_i^{(h)} & \text{if } i < 0, \\ l & \text{if } i = 0. \end{cases}$$

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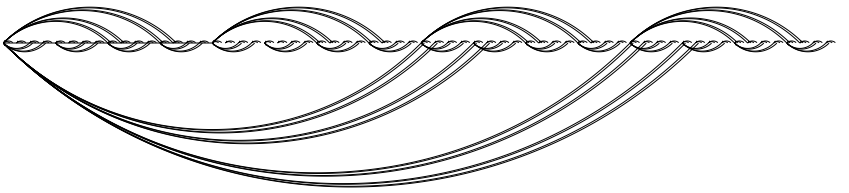


FIGURE 2. The arrows point from every point t to a point in $S(t)$, which is defined in an analogous manner as $S = S(0)$.

If $k_0(y^{(1)}) \le k-2$, then $k_0(y^{(1)}) = k_0(y^{(2)})$ and g_1 depends only on l, $\{x_i^{(h)}, y_i^{(h)}\}_{i>-\beta_k}$. Therefore, in this case $g_1(x^{(1)}, y^{(1)}, l) - g_1(x^{(2)}, y^{(2)}, l) = 0$.

If, on the other hand, $k_0(y^{(1)}) \ge k-1$, then $k_0(y^{(2)}) \ge k-1$ as well and therefore $|g_1(x^{(h)}, y^{(h)}, l) - 0.5| < \beta_{k-1}^{-1/2+\epsilon}$. Thus,

$$\mathrm{var}_j(g) < 2(\beta_{k-1})^{-1/2+\epsilon} < 8((\beta_{k+1})^{1/(1+\epsilon)^2})^{-1/2+\epsilon} < 8j^{-(1-2\epsilon)/(2(1+\epsilon)^2)}. \quad (9)$$

□

3. *Multiple measures*

The goal of this section is to prove the following lemma.

LEMMA 3. *For every* $\epsilon \in (0, 0.5)$*, the specification* g^ϵ *admits multiple measures.*

Theorem 1 is a direct consequence of Lemmas 2 and 3.

To see that the function g admits multiple Gibbs measures, we introduce the following notation. Choose an arbitrary t. We will prove that $X(B_k(t)) = X(B_{k+1}(t))$ with extremely high probability (remember that in (4) we defined $X(B_k(t))$ to be the majority value of $\{x_j\}_{j\in C'(B_k(t))}$). Next we note that for any t and t' and for all k large enough, $B_k(t) = B_k(t')$ and thus $X(B_k(t)) = X(B_k(t'))$. (See Figure 2.) Thus, by the symmetry (Lemma 1), there exist at least two invariant measures: one where $\lim_{k\to\infty} X(B_k(t)) = 1$ almost surely and one where $\lim_{k\to\infty} X(B_k(t)) = -1$ almost surely.

Assume that μ is a stationary and ergodic measure on $\{(\pm 1, \pm 1)\}^{\mathbb{Z}}$ which is a Gibbs measure for g.

In §§3.1 and 3.2, we prove results about the block structure. The block structure depends only on the y sequence, which is an independent and identically distributed (0.5, 0.5) sequence. In §3.3, we use the results obtained in §§3.1 and 3.2 to understand the (more complicated) structure of the x sequence and show the existence of two different Gibbs measures.

3.1. *Good blocks.*

In this subsection we define the notion of a good block. A good block is a block whose length is close to its expected length and the size of its opening is also close to its expected value. Our aim in this subsection is to show that a k-block is good with high probability. In the next subsections we show that good blocks exhibit behavior that yields multiple Gibbs measures.

Definition 5. A k-block B is *good* if:

(1) $|B| < \beta_k^{1+\epsilon}$; and

(2) $|C(B)| > \beta_k^{1-\epsilon} 2^{-k}$.

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In order to show that most blocks are good, we put a measure on k-blocks by conditioning on y such that $\{y_i\}_{i=-l_k+2}^{1} = I_k$. For a k-block $B = \{y_i\}_{i=1}^{n-1}$ such that:

(1) $y_1 = -1$;
(2) $y_{n-l_k} = \cdots y_{n-1} = 1$; and
(3) there is no occurrence of I_k in B,

we define the measure

$$m_k(B) = \mathbf{P}(B_k(1) = B | \{y_i\}_{i=-l_k+2}^{1} = I_k) = 2^{-|B|}.$$

Note that m_k is the distribution of a typical k-block seen as an unnumbered sequence in $\{-1, 1\}^{\mathbb{Z}}$. Thus, $(|B|/\beta_k)m_k(B)$ is the corresponding size biased distribution. With these definitions, we can check that for any μ which is a Gibbs measure for g and any k and B,

$$\mathbf{P}(B_k(0) \text{ is a translate of } B) = \frac{|B| m_k(B)}{\beta_k}.$$

LEMMA 4. *For all $k \geq K$,*

$$\sum_{B \text{ not good}} m_k(B) < 3 \cdot 2^{-3k}.$$

Proof. We prove the lemma by induction on k. For $k = K$, we need only to check the first condition. The probability that a k-block is longer than $c\ell_k$ is less than $(1 - 2^{-\ell_k})^c$. Thus, the probability that the first condition in the definition is not satisfied is at most

$$(1 - 2^{-\ell_k})^{\beta_k^{1+\epsilon}/\ell_{k-1}} < 2^{-3k}. \tag{10}$$

Note that the inequality in (10) holds for all k large enough because β_k is exponential in ℓ_k, which is exponential in k. Our choice of K guarantees that (10) holds for all $k \geq K$.

If $k > K$ and:

(1) $|B| < \beta_k^{1+\epsilon}$;
(2) there are at least $\nu_k^{1-\epsilon}$ $(k-1)$-blocks in B; and
(3) at least half of the $\nu_k^{1-\epsilon}$ $(k-1)$-blocks in the beginning of B are good,

then

$$|C(B)| \geq \tfrac{1}{2} \nu_k^{1-\epsilon} \beta_{k-1}^{1-\epsilon} 2^{-k+1} \geq \beta_k^{1-\epsilon} 2^{-k} \tag{11}$$

and B is good.

As the bound in (10) holds for any $k \geq K$, we have that the probability that the first condition is not satisfied is at most 2^{-3k}. The probability that conditions 3.1 and 3.1 are satisfied is the probability that if we select $\nu_k^{1-\epsilon}$ $(k-1)$-blocks independently according to m that none of them is a k-block and that at least half of them are good. The probability that a $(k-1)$-block is also a k-block is ν_k^{-1}. Thus, the probability of having a k-block among $\nu_k^{1-\epsilon}$ $(k-1)$-blocks is less than

$$\nu_k^{1-\epsilon} \nu_k^{-1} = \nu_k^{-\epsilon} \leq 2^{-3k}, \tag{12}$$

where (12) holds for k large enough because ν_k is super-exponential in k.

By the induction hypothesis, a $(k-1)$-block is not good with probability less than 2^{-k+1}. Thus, the probability that half of a sequence of $\nu_k^{1-\epsilon}$ $(k-1)$-blocks chosen independently are not good is at most

$$2^{\nu_k^{1-\epsilon}} (2^{-k})^{.5\nu_k^{1-\epsilon}} \leq 2^{-\nu_k^{1-\epsilon}} \leq 2^{-3k}, \tag{13}$$

where, again, the second inequality holds for k large enough because ν_k is super-exponential in k. □

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LEMMA 5. *For every $k \geq K$,*

$$\mathbf{P}(B_k(0) \text{ is good}) > 1 - 2^{-k}.$$

Proof. First, we want to estimate

$$\tilde{Y} = \sum_{|B| \leq \beta_k} m_k(B).$$

Here $\tilde{Y}$ is the probability that I_k appears somewhere in $\{y_j\}_{j=1}^{\beta_k}$. For $i = 1, \ldots, \ell_k$, let E_i be the event that there exists h with $h \equiv i \mod \ell_k$ such that $(y_h, y_{h+1}, \ldots, y_{h+\ell_k-1}) = I_k$. The $\{E_i\}$ are negatively associated and, for every i,

$$\mathbf{P}(E_i) = 1 - (1 - \beta_k^{-1})^{\beta_k/\ell_k}.$$

Therefore,

$$\tilde{Y} \geq 1 - (1 - \beta_k^{-1})^{\beta_k}. \tag{14}$$

On the other hand, for $i = 1, \ldots, 1000$, let F_i to be the event that I_k appears in

$$\{y_j : j = \beta_k i/1000 \ldots \beta_k(i+1)/1000 - 1\}.$$

Then the F_i are independent and, using a first-moment argument, we see that for every i, $\mathbf{P}(F_i) \leq 1/1000$. The event measured by $\tilde{Y}$ is the union of the F_i plus the event that I_k appears on the seams between the blocks. Therefore,

$$\tilde{Y} \leq 1 - (1 - 1/1000)^{1000} + 1000\ell_k 2^{-\ell_k}. \tag{15}$$

For any j, let

$$\tilde{Z} = \sum_{j\beta_k < |B| \leq (j+1)\beta_k} m_k(B)$$

be the probability that the first appearance of I_k is in the interval $[j\beta_k, (j+1)\beta_k)$. Then

$$\tilde{Z} = \tilde{Y}(1 - \tilde{Y})^j (1 - \ell_k 2^{-\ell_k})^j, \tag{16}$$

where the last term comes from the event that I_k appears on the seam between two consecutive intervals of length β_k.

From (14), (15) and (16), we get that for any j,

$$(5/2)^{-j} > \sum_{j\beta_k < |B| \leq (j+1)\beta_k} m_k(B) > 2^{-2j-2}.$$

From this, we get

$$\sum_{|B| > k\beta_k} m_k(B) > \sum_{j \geq k} 2^{-2j-2} > 3 \cdot 2^{-3k}$$

and

$$\sum_{|B| > k\beta_k} |B| m_k(B) < \sum_{j \geq k} (5/2)^{-j}(j+1)\beta_k < 2^{-k}\beta_k.$$

Thus, for any set S of k-blocks such that

$$\sum_{B \in S} m_k(B) < 3 \cdot 2^{-3k},$$

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we have that

$$\sum_{B\in S} |B| m_k(B) < 2^{-k}\beta_k.$$

Combining this last statement with Lemma 4,

$$\mathbf{P}(B_k(0) \text{ is good}) \geq 1 - \frac{1}{\beta_k}\left(\sum_{B \text{ not good}} |B| m_k(B)\right) > 1 - 2^{-k}. \qquad \square$$

3.2. *Beautiful points.* In this subsection we define the notion of a beautiful point. Our goal in this subsection will be to show that most points are beautiful.

Definition 6. We say that $t \in \mathbb{Z}$ is *k-beautiful* if for every $j \geq k$:
(1) $B_j(t)$ is good; and
(2) t is not in the beginning of $B_j(t)$. (The *beginning* of a block was defined in Definition 3.)

From Definition 6, it follows immediately that if t and s belong to the same k-block, and t is $(k+1)$-beautiful, then s is also $(k+1)$-beautiful.

LEMMA 6. *Almost surely, for every t there exists $\hat{k}(t)$ such that t is $\hat{k}(t)$-beautiful.*

Proof. First we show that

$$\mathbf{P}(0 \text{ is } k\text{-beautiful}) > 1 - 2^{-k+2}.$$

Lemma 5 tells us that the probability that the j-block containing 0 is good is greater than $1 - 2^{-j}$. As there are at most $\nu_j^{1-\epsilon}$ $(k-1)$-blocks in the beginning of $B_k(0)$ and the expected number of $(k-1)$-blocks in $B_k(0)$ is ν_j, the probability that 0 is in the beginning of $B_j(0)$ is less than

$$\nu_j^{1-\epsilon}/\nu_j = \nu_j^{-\epsilon}.$$

Therefore, 0 is k-beautiful with probability at least

$$1 - \sum_{j=k}^{\infty}(2^{-j} + \nu_j^{-\epsilon}) = 1 - \sum_{j=k}^{\infty} 2^{-j+1} = 1 - 2^{-k+2}.$$

Thus, by Borel–Cantelli, we get that $\hat{k}(0)$ exists almost surely. The lemma is true because μ is shift invariant. $\square$

3.3. *Proof of Lemma 3.* In the previous subsections we only discussed the structure induced by the y values. In this subsection we will shed some light on the structure of the x values.

We assume that μ is an ergodic Gibbs measure for g.

LEMMA 7. *For all t,*

$$X(t) = \lim_{k\to\infty} X(B_k(t))$$

exists almost surely and is equal to 1 *or* -1.

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Proof. By Lemma 6, we get $\hat{k}$ such that t is $\hat{k}$-beautiful. Using that $B_k(t) = [a, b)$ is good for every $k > \hat{k}$, that μ is a Gibbs measure with respect to g and equation (6) in the definition of g, we get that given y and $\{x_i\}_{i<a}$, the values of $x_s : s \in C'(B_k(t))$ are independent and identically distributed. By Lemma 1, it causes no loss of generality to assume that $X(B_{k+1}(t)) = 1$. For every s in $C'(B_k(t))$,

$$\mathbf{P}(x_s = 1 \mid y, \{x_i\}_{i<a}) = \tfrac{1}{2} + \beta_k^{-1/2+\epsilon}. \tag{17}$$

Also, the fact that $B_k(t)$ is good implies that

$$|C'(B_k(t))| \geq \beta_k^{1-\epsilon} 2^{-k} - 1.$$

Thus, we get

$$\mathbf{P}\left(X(B_k(t)) = X(B_{k+1}(t))\right) = \mathbf{P}\left(\sum_{s \in C'(B_k(t))} x_s > 0 \mid y, \{x_i\}_{i<a}\right).$$

We have that

$$\mathbf{E}\left(\sum_{s \in C'(B_k(t))} x_s \mid y, \{x_i\}_{i<a}\right) = 2|C'(B_k(t))|\beta_k^{-1/2+\epsilon}.$$

The standard deviation of the sum is less than $\frac{1}{2}\sqrt{|C'(B_k(t))|}$. Thus, by Chebyshev's inequality, we have that

$$\begin{aligned}
\mathbf{P}(X(B_k(t)) \neq X(B_{k+1}(t))) &\leq \mathbf{P}\left(\sum_{s \in C'(B_k(t))} x_s > 0 \mid y, \{x_i\}_{i<a}\right) \\
&\leq \left(\frac{2|C'(B_k(t))|\beta_k^{-1/2+\epsilon}}{\frac{1}{2}\sqrt{|C'(B_k(t))|}}\right)^{-2} \\
&\leq 16|C'(B_k(t))|^{-1}\beta_k^{1-2\epsilon} \\
&\leq 32\beta_k^{\epsilon-1}2^k\beta_k^{1-2\epsilon} \\
&\leq 32\beta_k^{-\epsilon}2^k \\
&\leq 2^{-k}.
\end{aligned}$$

By Borel–Cantelli, there are only finitely many values of k such that $X(B_k(t)) \neq X(B_{k+1}(t))$ and therefore $X(t)$ exists almost surely. As $|C'(B_k(t))|$ is odd for all k and t, the limit must be either 1 or -1. □

We are now ready to prove Lemma 3.

Proof of Lemma 3. For every t and s, $B_k(t) = B_k(s)$ for all k sufficiently large. Therefore, $X(t) = X(s)$ for all s and t. (See Figure 2.) Since μ is ergodic and $X(0)$ is shift invariant, $X(0)$ is a μ almost sure constant, which we denote by $X(\mu)$. Let

$$\tilde{\mu}(A) = \mu(\{(x, y) : (-x, y) \in A\}).$$

Then, by Lemma 1, $\tilde{\mu}$ is also a Gibbs measure for g. On the other hand, $X(\tilde{\mu}) = -X(\mu) \neq X(\mu)$ and therefore $\mu \neq \tilde{\mu}$. □

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4. *An open problem*

Our construction is not monotone—by changing a y value from a -1 to a $+1$ we may change the set S at which we look, and then reduce the probability that the function outputs $+1$ at the x coordinate. The construction of Bramson and Kalikow, on the other hand, is monotone. Therefore, we ask the following question.

Question 1. Are there a value of p and a continuous monotone regular specification g such that $\mathrm{var}(g) \in \ell^p$ and g admits multiple Gibbs measures?

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