Ergod. Th. & Dynam. Sys. (2013), **33**, 624–642 © *Cambridge University Press, 2012*
doi:10.1017/S0143385711001167

Invariant measures with bounded variation densities for piecewise area preserving maps

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(*Received* 10 *February* 2011 *and accepted in revised form* 13 *December* 2011)

Abstract. We investigate the properties of absolutely continuous invariant probability measures (ACIPs), especially those measures with bounded variation densities, for piecewise area preserving maps (PAPs) on $\mathbb{R}^d$. This class of maps unifies piecewise isometries (PWIs) and piecewise hyperbolic maps where Lebesgue measure is locally preserved. Using a functional analytic approach, we first explore the relationship between topological transitivity and uniqueness of ACIPs, and then give an approach to construct invariant measures with bounded variation densities for PWIs. Our results 'partially' answer one of the fundamental questions posed in [**13**]—to determine all invariant non-atomic probability Borel measures in piecewise rotations. When restricting PAPs to interval exchange transformations (IETs), our results imply that for non-uniquely ergodic IETs with two or more ACIPs, these ACIPs have very irregular densities, i.e. they have unbounded variation.

1. *Introduction*

Conservative systems are often used as models of the physical world, where conservative is usually understood as energy preserving (i.e. where energy is invariant under time evolution). In this article, we consider conservative systems that are governed by discrete-time dynamical systems. In particular, we focus on multidimensional *piecewise area preserving maps* (PAPs), which are a general extension of interval exchange transformations (IETs) into $\mathbb{R}^d$. Regarding IETs, Keane conjectured that minimality implies unique ergodicity in [**17**] and this conjecture holds for IETs with two or three intervals. However, counterexamples have been constructed; see [**18, 21**]. Thereafter, Masur [**23**] and Veech [**27**] have independently demonstrated that almost every topologically transitive IET (with respect to Lebesgue measure) is uniquely ergodic; simultaneously, Keane and Rauzy [**19**] revealed that unique ergodicity holds for a Baire residual subset of the space of IETs. To fully understand the densities of absolutely continuous invariant measures (ACIPs) for these non-uniquely ergodic counterexamples, it is natural to explore equivalent conditions to the topological transitivity in IETs in terms of ACIPs.

https://doi.org/10.1017/S0143385711001167 Published online by Cambridge University Press

When extending to the multidimensional PAPs, we are facing at least two technical obstacles: complicated topology in high dimensions and non-local preservation of distance. For the class of PAPs which preserve distance locally, a special case of interest is the class of piecewise isometries (PWIs). Establishing the properties of their ACIPs will partially contribute to answering a fundamental question posed in [**13**], i.e. to determine all invariant non-atomic probability Borel measures for piecewise rotations. This question is still open so far.

For the class of PAPs that do not preserve distance locally, a particular case is piecewise hyperbolic maps. For these maps, properties that have been studied include transitivity and possession of a unique physical measure (e.g. see works of Boyarsky and Góra [**3**] and Viana [**28**]). These studies use a functional analytic approach by choosing a 'reasonable' function space and applying a *transfer operator* on this space. They study statistical properties of the system by looking at the operator fixed point and determining if there is a spectral gap. In one-dimensional piecewise expanding maps, the space of bounded variation functions has been demonstrated to be such a 'reasonable' space [**3, 28**]. In higher dimensions, the space of multidimensional bounded variation functions can still be chosen under certain assumptions [**5, 20, 26**] and contains a classical anisotropic Sobolev space of Triebel–Lizorkin type [**2**].

In this article, our interest is to explore the structure of ACIPs and the relationship between the uniqueness of such measures and topological properties, e.g. the existence of dense orbits, topological transitivity and minimality for multidimensional PAPs (particularly for PWIs) by applying the functional analytic approach. Definitions of PAPs and PWIs are given below.

Let X be a compact subset of $\mathbb{R}^d$ and $(X, \mathfrak{B}, m)$ be a probability space. For convenience, m always denotes d-dimension normalized Lebesgue measure on X, and $\mathfrak{B}$ is the Borel σ-field. We say $\mathcal{P} = \{\omega_i\}_{i=0}^{r-1}$ is a *topological partition* of X if: (i) $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$; (ii) $\bigcup_{i=0}^{r-1} \omega_i = X$; and (iii) for each ω_i, $\mathrm{int}(\omega_i) \neq \emptyset$ and $m(\partial\omega_i) = 0$. Here each ω_i is called an atom; int A and ∂A are the interior and boundary of A, respectively.

Definition 1. A non-singular map $f : (X, \mathfrak{B}, m) \to (X, \mathfrak{B}, m)$ with a topological partition $\mathcal{P} = \{\omega_i\}_{i=0}^{r-1}$ is called a *piecewise area preserving map* (PAP) if $f|_{\mathrm{int}(\omega_i)} \in C^1$ for each ω_i and $|\det Df(x)| \equiv 1$ for $x \in \bigcup_{i=0}^{r-1} \mathrm{int}(\omega_i)$. Here non-singularity means that f is measurable (with respect to $\mathfrak{B}$) and $m(A) = 0$ implies $m(f^{-1}(A)) = m(f(A)) = 0$ for any $A \in \mathfrak{B}$; and Df refers to the Jacobian matrix. We say a PAP f is piecewise-invertible-area-preserving if $f|_{\omega_i}$ is invertible for each ω_i, and say f is an invertible PAP if f is globally invertible. In particular, if each $f|_{\mathrm{int}(\omega_i)}$ is an isometry (i.e. preserving Euclidean distance), then we say f is a *piecewise isometry* (PWI).

Our definition of PAPs includes piecewise hyperbolic maps with determinant ± 1, e.g. Baker's map, Arnold's cat map, area preserving Hénon map and standard map [**22**]. However, we will mainly concentrate on methods working on non-hyperbolic maps such as PWIs.

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For a PAP $f : X \to X$, the ACIPs are classified based on the density properties as†

$$\mathcal{M}_I(f) := \{\mu \text{ is an ACIP with respect to } f\},$$
$$\mathcal{M}_{IB}(f) := \left\{\mu \in \mathcal{M}_I(f) : \frac{d\mu}{dm} = \eta\Big|_X \text{ for some } \eta \in BV(\Omega), \text{ where } \Omega \supset X \text{ is an open ball}\right\},$$
$$\mathcal{M}_{IC}(f) := \left\{\mu \in \mathcal{M}_I(f) : \frac{d\mu}{dm} \text{ is } m\text{-almost everywhere continuous}\right\},$$

where $BV(\Omega)$ is the space of bounded variation functions (see Definition 2). We chose to work with $\mathcal{M}_{IB}$ and $\mathcal{M}_{IC}$ for the following reasons.

- These spaces are 'large enough' Banach subspaces of $L^1(m)$, i.e. they contain discontinuous functions [**25**].
- Functions in these spaces have 'good' geometric properties, e.g. $\chi_E \in BV(\Omega)$ implies that the measurable subset $E \subset \Omega$ has finite perimeter [**9**].
- These spaces coincide with those chosen in piecewise hyperbolic maps in [**2, 5, 26**].
- These spaces are invariant under the *transfer operator* (defined in §2.1) for PWIs.

It is clear that $\mathcal{M}_{IB} \subset \mathcal{M}_{IC} \subset \mathcal{M}_I$ for one-dimensional invertible PAPs, while, in higher dimensions, $\mathcal{M}_{IB} \cup \mathcal{M}_{IC} \subset \mathcal{M}_I$. Additionally, for non-invertible PAPs, the set $\mathcal{M}_I$ is possibly empty and conditions for which $\mathcal{M}_I \neq \emptyset$ are discussed in §3.2.

The novelty of this article is that we introduce multidimensional bounded variation functions to analyze ACIPs for PAPs, especially for PWIs. In Theorem 1, we explore the relationship between the set of nomadic points and the sets $\mathcal{M}_{IC}$, $\mathcal{M}_{IB}$ for invertible multidimensional PAPs. In particular, we demonstrate that when the set of nomadic points has a positive Lebesgue measure, both $\mathcal{M}_{IB}$ and $\mathcal{M}_{IC}$ are singletons. This can be applied to non-uniquely ergodic IETs constructed in [**18, 21**] to show the irregularity of densities of their ACIPs. For invertible PWIs, in Theorem 2 we give an approach to construct invariant measures with bounded variation densities. These results partially answer one of Goetz's questions in [**13**].

The paper is organized in the following way. Preliminaries and the main results are stated in §2, then applications along with discussions are in §3 and finally proofs are given in §4.

2. *Preliminaries and main results*

In this section, we give the formal definitions of *transfer operator* and *multidimensional bounded variation*, and then state the main results which are connected to one of the open questions in [**13**].

2.1. *Transfer operator.*

Let $(X, \mathfrak{B}, m)$ be a probability space, where m is normalized Lebesgue measure, and let $f : X \to X$ be a non-singular map. The *transfer operator* $\mathcal{L}_f : L^1(m) \to L^1(m)$ associated with f is defined up to m-almost everywhere equivalence

† $d\mu/dm \in L^1(m)$ is m-almost everywhere continuous means that its equivalence class contains a representative which is an m-almost everywhere continuous map.

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as follows [**3**]:

$$\int_A \mathcal{L}_f \varphi \, dm = \int_{f^{-1}(A)} \varphi \, dm, \quad \varphi \in L^1(m),\ A \in \mathfrak{B}.$$

This *transfer operator* possesses the following dual property [**28**]:

$$\int (\mathcal{L}_f \varphi)\psi \, dm = \int \varphi \cdot (\psi \circ f) \, dm, \quad \varphi \in L^1(m),\ \psi \in L^\infty(m).$$

For an invertible PAP $f : X \to X$ with a topological partition $\mathcal{P} := \{\omega_0, \ldots, \omega_{r-1}\}$, the *transfer operator* can be simplified to be

$$\mathcal{L}_f(\varphi) = \varphi \circ f^{-1} \quad \text{for all } \varphi \in L^1(m).$$

2.2. *Multidimensional bounded variation.* There are various definitions of multidimensional bounded variation functions, e.g. see Appendix A.1 and [**9, 29**]. These definitions can be reduced to the usual notion of bounded variation in one dimension; see Appendix A.1. We state one of these as follows.

Definition 2. [**9**] Let Ω be an open set of $\mathbb{R}^d$. A function $\eta \in L^1(\Omega)$ is a *bounded variation* function ($\eta \in BV(\Omega)$) if

$$\operatorname{var}(\eta) := \sup\left\{ \int_\Omega \eta \cdot \operatorname{div} \vec{\phi} \, dm : \vec{\phi} \in C_c^1(\Omega, \mathbb{R}^d),\ \| \vec{\phi} \|_\infty \le 1 \right\} < \infty. \tag{1}$$

Here

$$\vec{\phi} = (\phi_i)_{i=1}^d, \quad \operatorname{div} \vec{\phi} = \sum_{i=1}^d \frac{\partial \phi_i}{\partial x_i}, \quad \| \vec{\phi} \|_\infty := \sup_x | \vec{\phi}(x)|$$

and $C_c^1(\Omega, \mathbb{R}^d)$ is the set of $\vec{\phi} \in C^1(\Omega, \mathbb{R}^d)$ with compact support. We define a norm on $BV(\Omega)$ by $\|\eta\|_{BV} := \|\eta\|_1 + \operatorname{var}(\eta)$.

For functions of bounded variation, we state the corresponding Helly's theorem [**9**] below.

HELLY'S THEOREM. [**9**] *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain with Lipschitz boundary. Assume that $\{\eta_n\}_{n=1}^\infty$ is a sequence in $BV(\Omega)$ satisfying $\sup_n \|\eta_n\|_{BV} < \infty$; then there exist a subsequence $\{\eta_{n_k}\}_{k=1}^\infty$ and a function $\eta \in BV(\Omega)$ such that $\eta_{n_k} \to \eta$ in $L^1(\Omega)$ as $k \to \infty$.*

2.3. *Main results.* Let X be a compact subset of $\mathbb{R}^d$ and $f : X \to X$ be an invertible map. We say $x \in X$ is a *nomadic point* of f if $O_f(x) := \{f^i(x) \mid i \in \mathbb{Z}\}$ is dense in X. We denote $\operatorname{nom}(f)$ to be the set of all nomadic points of the map f. If $\operatorname{nom}(f) = X$, then f is called *minimal*.

THEOREM 1. *Let $(X, \mathfrak{B}, m)$ be a probability space, where m is the normalized Lebesgue measure, and $f : X \to X$ be an invertible* PAP *with a topological partition $\mathcal{P} = \{\omega_i\}_{i=0}^{r-1}$. Then the following hold:*

(i) *if $m(\operatorname{nom}(f)) > 0$, then $\mathcal{M}_{IB}(f) \cup \mathcal{M}_{IC}(f) = \{m\}$;*

(ii) *if $f|_{\omega_i}$ is a homeomorphism for each ω_i, then $\mathcal{M}_{IC}(f) = \{m\}$ implies $\operatorname{nom}(f) \neq \emptyset$.*

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We remark here that even for area preserving diffeomorphisms, $\mathrm{nom}(f) \neq \emptyset$ does not necessarily imply $m(\mathrm{nom}(f)) > 0$; see e.g. Fayad and Katok [**10**].

COROLLARY 1. *Suppose f is an invertible* PAP *with $m(\mathrm{nom}(f)) > 0$ and there exists a measure $m \neq \mu \in \mathcal{M}_I(f)$; then $\varphi := (d\mu/dm) \notin BV(\Omega)$ and the set of discontinuities of φ has positive Lebesgue measure.*

Theorem 2 below aims to construct ACIPs with bounded variation densities for invertible PWIs, say $f : X \to X$ with a topological partition $\mathcal{P} = \{\omega_0, \ldots, \omega_{r-1}\}$. As a bounded variation function is defined on an open set, we choose an open ball $\Omega \supset X$ and extend f to $\overline{f} : \Omega \to \Omega$ by

$$\overline{f}(x) = \begin{cases} f(x), & x \in X, \\ x, & x \in \Omega \backslash X. \end{cases} \tag{2}$$

Given any $\eta \in BV(\Omega)$, the sequence of variations $\{\mathrm{var}(\mathcal{L}^n_{\overline{f}}\eta)\}_{n=0}^{\infty}$ is not necessarily uniformly bounded [**20**]. Therefore, we alternatively work with functions η which lie in a plausible proper subset $BV^*(\Omega)$ (see below). This subset is associated with a Sobolev space.

Let $\omega_r := \Omega \backslash X$ and without ambiguity we still write $\mathcal{P} = \{\omega_0, \ldots, \omega_r\}$ as a topological partition of Ω. Moreover, we denote

$$\partial\mathcal{P}^{\infty} := \{x \in \Omega : \overline{f}^n(x) \in \partial\mathcal{P} \text{ for some } n \geq 0\},$$

where $\partial\mathcal{P} := \bigcup_{i=0}^{r} \partial\omega_i$, and define a δ-neighborhood of $\partial\mathcal{P}^{\infty}$ by

$$N_\delta := \{x \in \Omega, \mathrm{dist}(x, \partial\mathcal{P}^{\infty}) < \delta\}. \tag{3}$$

For a given invertible PWI f, the function subspace $BV^*(\Omega)$ that we consider is defined as

$$BV^*(\Omega) := \{\eta \in BV(\Omega) : \eta|_{N_\delta} \in W^{1,2} \text{ for some } \delta > 0\}, \tag{4}$$

where $W^{1,2}$ is a Sobolev space (see Appendix A.2). We remark that different invertible PWIs f determine different function subspaces $BV^*(\Omega)$ individually, but in all cases $W^{1,2}(\Omega) \subseteq BV^*(\Omega) \subseteq BV(\Omega)$.

THEOREM 2. *Suppose $f : X \to X$ is an invertible* PWI *and $\Omega \supset X$ is an open ball. Then, given any $\eta \in BV^*(\Omega)$ with $\eta|_X \geq 0$ and $\|\eta|_X\|_1 > 0$, there exists a subsequence of the Birkhoff averages of the transfer operator $\mathcal{L}_{\overline{f}}$, which converges to a function $\overline{\eta} \in BV(\Omega)$ in $L^1(m)$, i.e.*

$$\frac{1}{n_k} \sum_{i=0}^{n_k-1} \mathcal{L}^i_{\overline{f}}\eta \to \overline{\eta} \in BV(\Omega) \quad \text{as } k \to \infty$$

and by normalization $d\mu := \overline{\eta}|_X \, dm \in \mathcal{M}_{IB}(f)$.

Concerning the open question [**13**] in piecewise rotations (defined in Appendix A.3), the following corollaries give a universal approach to partially determine the ACIPs.

COROLLARY 2. *Suppose $f : X \to X$ is an invertible piecewise rotation; then:*

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(i) $\mathcal{M}_I(f) = \{\varphi\, dm : \varphi = \mathbb{E}(\varphi|\mathcal{I}),\ \varphi \in L^1(m)\}$, *where* $\mathcal{I} = \{B \in \mathcal{B} : f^{-1}(B) = B \bmod m\}$;

(ii) *given any* $\eta \in BV^*(\Omega)$ *satisfying* $\eta|_X \geq 0$ *and* $\|\eta|_X\|_1 > 0$, *any accumulation point of* $\{(1/n)\sum_{i=0}^{n-1} \mathcal{L}_f^i \eta\}_{n=1}^{\infty}$ *is an invariant density in* $\mathcal{M}_{IB}(f)$. *Furthermore, if* $m(\mathrm{nom}(f)) > 0$, *then* $\mathcal{M}_{IB}(f) = \{m\}$.

COROLLARY 3. *Suppose* $f : X \to X$ *is a non-invertible piecewise rotation. Let* $X^+ := \bigcap_{i=0}^{\infty} f^i(X)$ *and define* $f^+ : \overline{X^+} \to \overline{X^+}$ *as in equation (5) in §3.2. Then* f^+ *is* m*-almost everywhere invertible. Furthermore:*

(i) *if* $m(\overline{X^+}) > 0$, *then the statements in Corollary 2 hold for* f^+ *and*

$$\mathcal{M}_I(f) = \{\mu(\cdot) := \nu(\cdot \cap \overline{X^+}),\ \forall \nu \in \mathcal{M}_I(f^+)\};$$

(ii) *if* $m(\overline{X^+}) = 0$, *then* $\mathcal{M}_I(f) = \mathcal{M}_{IB}(f) = \mathcal{M}_{IC}(f) = \emptyset$.

The proof of Corollary 2 is based on Theorems 1 and 2 and Lemma 8, while the proof of Corollary 3 is based on Lemma 1 and Proposition 2. We remark that ACIPs only give a subset of non-atomic probability Borel measures. Therefore, to fully answer the question in piecewise rotations [**13**], we have to explore singular non-atomic probability invariant measures. For instance, when $m(\overline{X^+}) = 0$, it is natural to consider Hausdorff measure. This is discussed at the end of §3.2.

3. *Applications and discussions*

In this section, we consider two main applications. We first consider IETs (see Appendix A.4 for the definition) which are non-uniquely ergodic. We apply Theorem 1 to show that the densities of their ACIPs can be irregular. We then consider multidimensional piecewise invertible area preserving maps and apply Theorems 1 and 2 to study their invariant densities. At the end of this section, we give a short discussion of the open question posed in [**13**] for piecewise rotations.

3.1. *Interval exchange transformations.*

For an IET f, the set $\mathcal{M}_{IC}(f)$ can be refined to

$$\mathcal{M}'_{IC}(f) := \left\{\mu \in \mathcal{M}_I(f) : \frac{d\mu}{dm} := \varphi \text{ has at most countably many discontinuity points}\right\},$$

where m is the normalized Lebesgue measure. Observe that $\mathcal{M}_{IB}(f) \subset \mathcal{M}'_{IC}(f) \subset \mathcal{M}_{IC}(f)$. Moreover, it is known that topological transitivity† implies minimality for IETs (see e.g. [**16**, Corollary 14.5.11]). Hence, by applying Theorem 1, we have the following corollary that characterizes the minimality properties of IETs.

COROLLARY 4. *For any IET* $f : [0, 1) \to [0, 1)$,

$$f \text{ is minimal} \Leftrightarrow \mathcal{M}_{IC}(f) = \{m\} \Leftrightarrow \mathcal{M}'_{IC}(f) = \{m\}.$$

† Topological transitivity means that for any open sets U and V, there exists $n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$.

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This corollary can be used to investigate Keane's conjecture, namely that minimality implies unique ergodicity for IETs [**17**]. This conjecture was shown to be false and we review two well-known counterexamples here.

Keynes and Newton [**21**] considered the following map. Let $\widehat{T}_{\gamma\beta} : [0, 1+\beta) \to [0, 1+\beta)$ be given by

$$\widehat{T}_{\gamma\beta}(x) = \begin{cases} x+1 & \text{if } 0 \le x < \beta, \\ x+\gamma \pmod 1 & \text{if } \beta \le x < 1+\beta. \end{cases}$$

By choosing appropriate β and γ (see [**6, 21**]), the map $T_{\gamma\beta}(x) := \widehat{T}_{\gamma\beta}(x(1+\beta))/(1+\beta)$ is minimal and has an eigenvalue -1. This implies that $T^2_{\gamma\beta}$ is not uniquely ergodic and its ergodic measures belong to $\mathcal{M}_I$ (see [**21**] for details).

Keane [**18**] also constructed an IET with four intervals satisfying a strong irrationality condition that implies minimality. Under certain conditions, there exist two different ergodic measures μ_1 and μ_2. Moreover, such ergodic measures are either both in $\mathcal{M}_I$ or one is Lebesgue measure and the other is singular. For the measure that is singular, the Hausdorff dimension has been recently estimated [**7**]. Together with our results, we could obtain a better understanding of ergodic measures for non-uniquely ergodic IETs.

For the examples above, there are no explicit formulae for their densities (even if these densities belong to $\mathcal{M}_I$). One of the difficulties in constructing counterexamples can be seen from the fact that these ergodic measures are in $\mathcal{M}_I(f)\backslash\mathcal{M}_{IC}(f)$ from Corollary 4. In the following proposition, we provide a more explicit description of the invariant densities for non-uniquely ergodic IETs.

PROPOSITION 1. *Let f be any topologically transitive IET on* $[0, 1)$. *Suppose $m \neq \mu \in \mathcal{M}_I(f)$; then the following hold:*

(i) *the density of μ is a simple function (i.e. a linear combination of finitely many characteristic functions);*

(ii) *for any representative from the equivalence class $\varphi := d\mu/dm$, φ is discontinuous everywhere and* $\operatorname{supp}\mu = [0, 1)$ *(recall that if $x \in \operatorname{supp}\mu$, then, for any open ball B_x containing x, $\mu(B_x) > 0$ holds* [**16**, p. 141]*).*

Remark 1. For the two ergodic measures $\mu_1, \mu_2 \in \mathcal{M}_I$ (i.e. $\mu_1, \mu_2 \ll m$) in the examples of Keynes and Newton [**21**] and Keane [**18**], we can derive that $\mu_1 \perp \mu_2$ (i.e. there exists a subset $E \subset [0, 1)$ such that $\mu_1(E) = 1$ but $\mu_2(E) = 0$) and, moreover, $\operatorname{supp}\mu_1 = \operatorname{supp}\mu_2 = [0, 1)$ from Proposition 1. Hence, the measures μ_1 and μ_2 intermingle with each other in some sense.

3.2. *Piecewise invertible area preserving maps.* In this subsection, we aim to understand the structure of ACIPs for piecewise invertible area preserving maps $f : X \to X$. For such a map f, the set $X^+ := \bigcap_{i=0}^{\infty} f^i(X)$ is invariant under the map, i.e. $f(X^+) = X^+$ [**12**]. In particular, for PWIs, X^+ is almost closed, i.e. $m(X^+) = m(\overline{X^+})$ [**1**]. Here we show that such almost closedness of X^+ is valid for a broad range of piecewise invertible area preserving maps.

LEMMA 1. *Let $f : X \to X$ be a piecewise invertible area preserving map with a topological partition $\mathcal{P} = \{\omega_0, \ldots, \omega_{r-1}\}$. Suppose $f_i := f|_{\operatorname{int}\omega_i}$ is Lipschitz for each ω_i; then $m(X^+) = m(\overline{X^+})$ and $f|_{X^+}$ is m-almost everywhere invertible.*

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Under the conditions of Lemma 1, it is not necessary to have the property $f(\overline{X^+}) \subseteq \overline{X^+}$, but we can define a map f^+ that is m-almost everywhere equal to f and for which $f^+(\overline{X^+}) \subset \overline{X^+}$ (see below). Since each f_i is Lipschitz, there exists a continuous extension $\widehat{f_i} : \overline{\text{int}\,\omega_i} \to \overline{f_i(\text{int}\,\omega_i)}$. For any $x \in (\bigcup_{i=0}^{r-1} \partial\omega_i) \cap \overline{X^+}$, let $g(x) := \widehat{f_{i^*}}(x)$, where $i^* := \min\{i : x \in \partial\omega_i\}$. Then we can define $f^+ : \overline{X^+} \to \overline{X^+}$ to be

$$f^+(x) = \begin{cases} f(x), & x \in \text{int}(\omega_i) \cap X^+, \\ g(x) & \text{otherwise.} \end{cases} \tag{5}$$

Moreover, if f_i is bi-Lipschitz, the map f^+ can be shown to be non-singular and to be m-almost everywhere invertible. The non-singularity and m-almost everywhere invertibility of f^+ allow us to obtain Theorems 1 and 2 for $f^+ : \overline{X^+} \to \overline{X^+}$. The ACIPs of f^+ can further be used to determine the ACIPs of f, as follows.

PROPOSITION 2. *Let $f : X \to X$ be a piecewise invertible area preserving map with a topological partition $\mathcal{P} := \{\omega_0, \ldots, \omega_{r-1}\}$ and $f^+ : \overline{X^+} \to \overline{X^+}$ be defined as in equation (5). Suppose that $f_i := f|_{\text{int}\,\omega_i}$ is bi-Lipschitz continuous. Then f^+ is non-singular and m-almost everywhere invertible. Moreover, the following hold:*

(i) *if $m(\overline{X^+}) > 0$, then $\mathcal{M}_I(f) = \{\mu(\cdot) := \nu(\cdot \cap \overline{X^+}), \forall \nu \in \mathcal{M}_I(f^+)\}$;*

(ii) *if $m(\overline{X^+}) = 0$, then $\mathcal{M}_I(f) = \emptyset$.*

When $m(\overline{X^+}) = 0$, it is natural to consider invariant measures that are absolutely continuous with respect to Hausdorff measure. If we let $s = \dim_H \overline{X^+}$ and, furthermore, if f^+ satisfies the following conditions:

(1) $0 < \mathcal{H}^s(X^+) = \mathcal{H}^s(\overline{X^+}) < \infty$;

(2) $\mathcal{H}^s$ is an invariant measure for f^+;

(3) f^+ is non-singular with respect to $\mathcal{H}^s$, i.e. $\mathcal{H}^s((f^+)^{-1}(A)) = \mathcal{H}^s(f^+(A)) = 0$ whenever $\mathcal{H}^s(A) = 0$,

then, by arguments analogous to those used in Proposition 2, we can show that f^+ is $\mathcal{H}^s$-almost everywhere invertible.

The above three conditions can be achieved for some piecewise invertible area preserving maps. We take interval translation maps (see Appendix A.4) as examples. Condition (2) is demonstrated in [**4**], while condition (3) can be inferred by combining condition (2) and the definition of Hausdorff measure. The $\mathcal{H}^s$-almost everywhere closedness of X^+ can be shown by adapting the proof of Lemma 1. Moreover, by [**11**, Theorem 9.3, p. 130], condition (1) holds for particular interval translation maps where $\overline{X^+}$ are self-similar sets satisfying an open set condition and positive Hausdorff dimension [**4**].

Concerning the open question in [**13**], for non-invertible piecewise rotations in the case of $m(\overline{X^+}) = 0$, we consider absolutely continuous (with respect to $\mathcal{H}^s$) invariant probability measures. We denote

$$\mathcal{H}_I^s(f) := \{\nu \text{ probability invariant measure of f } : \nu \ll \mathcal{H}^s\}.$$

PROPOSITION 3. *Suppose $f : X \to X$ is a two-dimensional piecewise rotation with $m(\overline{X^+}) = 0$ and $s := \dim_H \overline{X^+} > 1$; then X^+ is $\mathcal{H}^s$-almost everywhere closed and f^+ is non-singular with respect to $\mathcal{H}^s$. Moreover, the following hold:*

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(i) *if* $0 < \mathcal{H}^s(\overline{X^+}) < \infty$, *then* $\mathcal{H}^s$ *is an invariant measure of* f^+ *and* f^+ *is* $\mathcal{H}^s$*-almost everywhere invertible; furthermore,* $\mathcal{H}^s_I(f) = \{\mu(\cdot) := \nu(\cdot \cap \overline{X^+}), \forall \nu \in \mathcal{H}^s_I(f^+)\}$;

(ii) *if* $\mathcal{H}^s(\overline{X^+}) = 0$, *then* $\mathcal{H}^s_I(f) = \emptyset$.

Under the condition $s := \dim_H \overline{X^+} > 1$, the proof of Proposition 3 is analogous to the proofs of Proposition 2 and Lemma 1. However, this condition does not always hold. For instance, the Cartesian product of interval translation maps (as defined in [**4**]) with themselves provides some examples of piecewise rotations with Hausdorff dimension s ranging over $0 \le s \le 1$. It might be interesting to explore conditions for $s > 1$.

Regarding the determination of all the invariant non-atomic probability measures for piecewise rotations, it might be necessary to consider the structure of $\mathcal{H}^s_I(f^+)$ in the case of $\mathcal{H}^s(\overline{X^+}) = \infty$. In this case, f^+ is not necessarily $\mathcal{H}^s$-almost everywhere invertible; however, we note that, by [**11**, Theorem 4.10, p. 68], there exists a compact subset $E \subset \overline{X^+}$ with $0 < \mathcal{H}^s(E) < \infty$. We suggest establishing a non-atomic probability invariant measure of f^+ induced by E as a reference measure and leave this for further studies.

4. *Proofs*

We first state a basic lemma regarding $\mathcal{M}_I(f)$, i.e. the set of all ACIPs with respect to a map f, with a standard proof.

LEMMA 2. *Let* $f : X \to X$ *be a non-singular map and* $\varphi \in L^1(m)$*; then* $\mathcal{L}_f \varphi = \varphi$ *if and only if* $d\mu := \varphi\, dm \in \mathcal{M}_I(f)$.

Proof. Suppose $\mathcal{L}_f \varphi = \varphi$. Let $d\mu := \varphi\, dm$; then, for each $A \subset X$,

$$\mu(A) = \int_A \varphi\, dm = \int_A \mathcal{L}_f \varphi\, dm = \int_{f^{-1}(A)} \varphi\, dm = \mu(f^{-1}(A)),$$

which implies $\mu \in \mathcal{M}_I(f)$.

On the other hand, if $d\mu := \varphi\, dm$ is an invariant measure of f, i.e. $\mu(f^{-1}(A)) = \mu(A)$ for any Borel set $A \subset X$, then

$$\int_A \mathcal{L}_f \varphi\, dm = \int_{f^{-1}(A)} \varphi\, dm = \mu(f^{-1}(A)) = \mu(A) = \int_A \varphi\, dm,$$

which implies $\mathcal{L}_f \varphi = \varphi$. □

4.1. *Proof of Theorem 1.*

To prove statement (i), we first show that $\mathcal{M}_{IC} = \{m\}$ followed by a proof of $\mathcal{M}_{IB} = \{m\}$. For statement (ii), we start with a lemma showing the equivalence between topological transitivity and the existence of a nomadic point.

Proof of the statement (i) in Theorem 1. Consider any $\mu \in \mathcal{M}_{IC}$. Take a representative $\varphi = d\mu/dm$ from the equivalence class such that φ is m-almost everywhere continuous. Furthermore, take any point $x' \in \bigcup_{i=0}^{r-1} \operatorname{int} \omega_i$, where φ is continuous. Since $m(\operatorname{nom}(f)) > 0$, we can choose a nomadic point x^* such that for any $n \in \mathbb{Z}$, the equality $\varphi \circ f^{-n}(x^*) = \varphi(x^*)$ holds. Then there exists a subsequence $\{f^{k_t}(x^*)\}$ such that as $|k_t| \to \infty$, $f^{k_t}(x^*) \to x'$ because x^* is a nomadic point. By the continuity of φ at x', we have

$$\varphi(x') = \varphi\Big(\lim_{|k_t| \to \infty} f^{k_t}(x^*)\Big) = \varphi(x^*).$$

This implies $\varphi \equiv 1$, i.e. $\mathcal{M}_{IC} = \{m\}$.

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Consider the case $\mu \in \mathcal{M}_{IB}(f)$, i.e. $\varphi = d\mu/dm$ with $\varphi = \eta|_X$ for some $\eta \in BV(\Omega)$. Hence, by [**29**, p. 178], m-almost every $x \in \Omega$ is a *regular point* of η. By a regular point x_0, we mean that there exists a unit vector $a \in \mathbb{R}^d$ such that the limits

$$\lim_{x \to x_0, \langle x - x_0, a \rangle > 0} \eta(x) \quad \text{and} \quad \lim_{x \to x_0, \langle x - x_0, a \rangle < 0} \eta(x)$$

exist, where $\langle \cdot, \cdot \rangle$ is the inner product. Therefore, by analogous arguments to those used in $\mathcal{M}_{IC}(f) = \{m\}$, it follows that for any regular point $x_0 \in \bigcup_{i=0}^{r-1} \operatorname{int}(\omega_i)$,

$$\lim_{x \to x_0, \langle x - x_0, a \rangle > 0} \varphi(x) = \lim_{x \to x_0, \langle x - x_0, a \rangle < 0} \varphi(x).$$

Hence, by [**29**, p. 168], we have $\lim_{x \to x_0} \varphi(x) = \varphi(x^*)$, where x^* is a nomadic point. In addition, $\varphi \in L^1(m)$ implies that m-almost every $x \in X$ is a Lebesgue point of φ, i.e.

$$\lim_{r \to 0^+} \frac{1}{m(B(x, r))} \int_{B(x,r)} |\varphi(y) - \varphi(x)| \, dm(y) = 0.$$

Therefore, if a regular point x_0 is also a Lebesgue point, then

$$\begin{aligned} 0 \le |\varphi(x_0) - \varphi(x^*)| &= \lim_{r \to 0^+} \frac{1}{m(B(x_0, r))} \int_{B(x_0,r)} |\varphi(x_0) - \varphi(x^*)| \, dm(y) \\ &\le \lim_{r \to 0^+} \frac{1}{m(B(x_0, r))} \int_{B(x_0,r)} |\varphi(y) - \varphi(x_0)| \, dm(y) \\ &\quad + \lim_{r \to 0^+} \frac{1}{m(B(x_0, r))} \int_{B(x_0,r)} |\varphi(y) - \varphi(x^*)| \, dm(y) = 0. \end{aligned}$$

This implies that $\varphi(x_0) = \varphi(x^*)$, meaning that $\varphi \equiv 1$. □

Before proving statement (ii), we first formulate an equivalent condition of topological transitivity.

LEMMA 3. *Let $f : X \to X$ be an invertible PAP with a topological partition $\mathcal{P} = \{\omega_i\}_{i=0}^{r-1}$. Suppose $f|_{\omega_i}$ is a homeomorphism for each ω_i; then the following are equivalent:*

(i) *f has a nomadic point;*

(ii) *f is strongly topologically transitive, that is, for any open sets U, V, there exists $n \in \mathbb{Z}$ such that $\operatorname{int}(f^n(U)) \cap V \neq \emptyset$;*

(iii) *f is topologically transitive.*

Since PAPs are not necessarily continuous at every point, the proof of Lemma 3 will not be standard (see [**30**] for the continuous version). Therefore, we provide the details of the proof here.

Proof of Lemma 3. '(ii) implies (iii)' is direct and we only need to prove '(i) implies (ii)' and '(iii) implies (i)'. For convenience, we denote $\mathcal{P}^{(n)} := \bigvee_{i=0}^{n} f^{-i}(\mathcal{P})$ for $n \ge 0$ and $\mathcal{P}^{(n)} := \bigvee_{i=-1}^{n} f^{-i}(\mathcal{P})$ for $n < 0$, and denote by $\omega^{(n)}$ one of the atoms in the topological partition $\mathcal{P}^{(n)}$ for any $n \in \mathbb{Z}$.

'(i) ⇒ (ii).' We prove this by contradiction. Suppose there exist open sets $U, V \neq \emptyset$ such that for any $n \in \mathbb{Z}$, $\operatorname{int}(f^n(U)) \cap V = \emptyset$. Given a nomadic point x^* of f, there exist $n_1, n_2 \in \mathbb{Z}$ such that $f^{n_1}(x^*) \in U$ and $f^{n_2}(x^*) \in V$. Let $t := n_2 - n_1$; then there exists an atom $\omega^{(t)} \in \mathcal{P}^{(t)}$ such that $f^{n_1}(x^*) \in \omega^{(t)}$.

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Suppose $f^{n_1}(x^*) \in \operatorname{int} \omega^{(t)}$. Since $f^t|_{\operatorname{int}\omega^{(t)}}$ is homeomorphic, we have $f^t(U \cap \operatorname{int}\omega^{(t)}) = \operatorname{int} f^t(U \cap \operatorname{int}\omega^{(t)})$. Therefore,

$$f^{n_2}(x^*) = f^t(f^{n_1})(x^*) \in f^t(U \cap \operatorname{int}\omega^{(t)}) \subset \operatorname{int} f^t(U),$$

which implies $f^{n_2}(x^*) \in \operatorname{int}(f^t(U)) \cap V$. This is a contradiction.

Suppose $f^{n_1}(x^*) \in \partial\omega^{(t)} \cap \omega^{(t)}$. Let $l := n_1 + n_2$; then there exists an atom $\omega^{(l)}$ such that $x^* \in \omega^{(l)}$. Since $f^{n_1}|_{\omega^{(l)}}$, $f^{n_2}|_{\omega^{(l)}}$ are continuous, there exists $x' \in \omega^{(l)}$ sufficiently close to x^* such that $f^{n_1}(x') \in \operatorname{int}\omega^{(t)} \cap U$ and $f^{n_2}(x') \in V$. Repeat the same process as the above case, using x' in place of x^*; this completes the proof.

'(iii) $\Rightarrow$ (i).' Suppose $\{U_i\}_{i=1}^{\infty}$ is a countable base for X. By (iii), there exists an $n_1 \in \mathbb{Z}$ such that $f^{n_1}(U_1) \cap U_2 \neq \emptyset$. Let $y := f^{n_1}(x) \in f^{n_1}(U_1) \cap U_2$, where $x \in \omega^{(n_1)} \in \mathcal{P}^{(n_1)}$.

If $x \in \operatorname{int}\omega^{(n_1)}$, then there exists an open ball $x \in B_x \subset \omega^{(n_1)}$ such that $f^{n_1}(B_x) \subset \operatorname{int} f^{n_1}(U_1)$. Therefore, $y \in \operatorname{int}(f^{n_1}(U_1)) \cap U_2$.

Otherwise, if $x \in \partial\omega^{(n_1)} \cap \omega^{(n_1)}$, by using an approach analogous to that used in '(i) $\Rightarrow$ (ii)', then there exists $x' \in \operatorname{int}\omega^{(n_1)} \cap U_1$ and $y' := f^{n_1}(x') \in \operatorname{int}(f^{n_1}(U_1)) \cap U_2$. Hence,

$$\operatorname{int}(f^{n_1}(U_1)) \cap U_2 \neq \emptyset.$$

Therefore, there exists a closed ball B_2 such that $B_2 \subset \operatorname{int}(f^{n_1}(U_1)) \cap U_2 \cap \operatorname{int}\omega^{(n_1)}$. Moreover, since $f^{n_1}|_{B_2}$ is a homeomorphism, $V_1 := f^{-n_1}(B_2)$ is closed. Analogously, for open sets $\operatorname{int} B_2$ and U_3, there exist $n_2 \in \mathbb{Z}$ and a closed ball $B_3 \subset \operatorname{int}(f^{n_2}(B_2)) \cap U_3$, with $f^{n_2}|_{B_3}$ being a homeomorphism. Let $V_2 := f^{-n_2}(B_3) \subset B_2$; then V_2 is closed and $f^{-n_1}(V_2) \subset V_1$.

If one continues this process, there will exist $\{n_i\}_{i=1}^{\infty}$ and a sequence of non-empty closed sets $\{V_i\}_{i=1}^{\infty}$ such that $f^{-n_i}(V_{i+1}) \subset V_i$ for each i. Therefore, $\bigcap_{i=1}^{\infty} f^N(V_{i+1}) \neq \emptyset$, where $N = -\sum_{j=1}^{i} n_j$. We note that $V_i \subset U_i$ for each i. Fix any $\bar{x} \in \bigcap_{i=1}^{\infty} f^N(V_{i+1})$; then $\bar{x}$ is nomadic. This completes the proof. □

Remark 2. There does exist a map f that is topologically transitive but has no nomadic points [**24**]. However, Lemma 3 does not apply to this case as f does not extend continuously to the boundary from its interior.

Proof of statement (ii) in Theorem 1. We argue by contradiction. Suppose f has no nomadic points; so, by Lemma 3, there will exist two open sets $U, V \subset X$ such that $\operatorname{int}(f^n(U)) \cap V = \emptyset$ for all $n \in \mathbb{Z}$. Therefore, $\operatorname{int}(f^{i+n}(U)) \cap \operatorname{int} f^i(V) = \emptyset \mod m$ for any $n, i \in \mathbb{Z}$. Let

$$U^* := \bigcup_{i=-\infty}^{\infty} \operatorname{int} f^i(U) \quad \text{and} \quad V^* := \bigcup_{i=-\infty}^{\infty} \operatorname{int} f^i(V).$$

Then $U^* \cap V^* = \emptyset \mod m$ and both U^* and V^* are invariant under f up to m-almost everywhere. Hence, both $m|_{U^*}$ and $m|_{V^*}$ are invariant measures and $m|_{U^*} \neq m|_{V^*}$. Since $\partial U^* \subset \bigcup_{i=-\infty}^{\infty} \partial f^i(U)$ and $\partial V^* \subset \bigcup_{i=-\infty}^{\infty} \partial f^i(V)$, this implies $m(\partial U^*) = m(\partial V^*) = 0$. That is, the discontinuity points of χ_{U^*} and χ_{V^*} lie in a set of zero Lebesgue measure. Therefore, $m|_{U^*}, m|_{V^*} \in \mathcal{M}_{IC}$, which contradicts the uniqueness of measures in $\mathcal{M}_{IC}$. □

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Remark 3. When restricting PAPs to IETs, each open set of X is a union of countably many open intervals; therefore, χ_{U^*} and χ_{V^*} have at most countably many discontinuities. This fact is used to prove Corollary 4.

4.2. *Proof of Theorem 2.* We prove Theorem 2 by the following lemmas. Recall that given an invertible PWI $f : X \to X$ and an open ball $\Omega \supset X$, we can extend f into $\overline{f} : \Omega \to \Omega$ as in (2) and define $BV^*(\Omega) \subset BV(\Omega)$ as in (4).

LEMMA 4. *Let Ω be an open subset of $\mathbb{R}^d$. Then $\eta \in BV^*(\Omega)$ if and only if $\eta = \eta^{(1)} + \eta^{(2)}$, where $\eta^{(1)} \in BV(\Omega)$ with $\eta^{(1)}|_{N_{\delta'}} = 0$ for some $\delta' > 0$ (recall that $N_{\delta'}$ is defined in equation (3)) and $\eta^{(2)} \in W^{1,2}(\Omega)$.*

Proof. We only prove the sufficiency. Suppose $\eta \in BV^*(\Omega)$; then there exists a $\delta > 0$ such that $\eta|_{N_\delta} \in W^{1,2}$. Let $\delta' = \delta/2$; then $\overline{N_{\delta'}} \subset N_\delta$. Hence, there exists a bump function $B(x) \in C_c^\infty(\Omega)$ such that $B|_{\overline{N_{\delta'}}} \equiv 1$ and $B|_{N_\delta^c} \equiv 0$ (here N_δ^c denotes the complement of N_δ). We define

$$\eta^{(2)} := \eta \cdot B, \quad \eta^{(1)} := \eta - \eta^{(2)}.$$

It is clear that $\eta^{(2)} \in W^{1,2}(\Omega)$ and $\eta^{(1)} \in BV(\Omega)$ with $\eta^{(1)}|_{N_{\delta'}} = 0$. □

For convenience, we denote the sequence of the Birkhoff averages of $\mathcal{L}_{\overline{f}}$ on an L^1 function η by $\{\eta_n\}_{n=1}^\infty$, i.e.

$$\eta_n := \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{L}_{\overline{f}}^i \eta.$$

LEMMA 5. *Suppose $f : X \to X$ is an invertible PWI and $\Omega \supset X$ is an open ball. Then, for any $\eta \in W^{1,2}(\Omega)$, there exists a subsequence $\{n_k\}_{k=1}^\infty$ such that $\eta_{n_k} \to \overline{\eta} \in W^{1,2}(\Omega)$ in L^1 as $k \to \infty$.*

Proof. Since $\overline{f} : \Omega \to \Omega$ is a PWI, it follows that $\overline{f}_i := \overline{f}|_{\operatorname{int} \omega_i} = A_i x + c_i$, where A_i is an orthogonal matrix and c_i is a translation vector. Hence, by using the orthogonality and the definition of $\|\cdot\|_{W^{1,2}}$, we have

$$\|\eta_2 \circ \overline{f}^{-i}\|_{W^{1,2}} = \|\eta_2\|_{W^{1,2}}.$$

Moreover, since $W^{1,2}(\Omega)$ is a Hilbert space, the lemma can be shown directly by using the Banach–Saks theorem [**15**] (see Appendix A.2). □

LEMMA 6. (Change of coordinates) [**8**, p. 535] *Let $\psi : W \to \Omega$ be a C^2-diffeomorphism, where W and Ω are open subsets of $\mathbb{R}^d$. Given $\overrightarrow{\phi} \in C^1(\Omega, \mathbb{R}^d)$, let*

$$\overrightarrow{\phi}^{\psi}(y) := D\psi^{-1}(y)\, \overrightarrow{\phi}(\psi(y));$$

then

$$\operatorname{div}(|\det D\psi|\, \overrightarrow{\phi}^{\psi}) = (\operatorname{div} \overrightarrow{\phi}) \circ \psi \cdot |\det D\psi|. \tag{6}$$

LEMMA 7. *Let $f : X \to X$ be an invertible piecewise isometry with a topological partition $\mathcal{P} = \{\omega_0, \omega_1, \ldots, \omega_{r-1}\}$. Suppose that $\eta \in BV(\Omega)$ with $\eta|_{N_\delta} = 0$ for some $\delta > 0$; then $\operatorname{var}(\mathcal{L}_{\overline{f}}\eta) \le \operatorname{var}(\eta) < \infty$.*

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Proof. Let $\omega_r := \Omega \backslash X$ and $\overline{f}_i := \overline{f}|_{\omega_i}$ for each ω_i, $i = 0, \ldots, r$. For any $\eta \in BV(\Omega)$, we have $\mathcal{L}_{\overline{f}}\eta \in BV(\Omega)$. Recall that

$$\operatorname{var}(\mathcal{L}_{\overline{f}}\eta) = \sup\left\{\int_\Omega (\eta \circ \overline{f}^{-1} \cdot \operatorname{div} \overrightarrow{\phi})\, dm \ : \ \overrightarrow{\phi} \in C_c^1(\Omega, \mathbb{R}^d),\ \|\overrightarrow{\phi}\|_\infty \le 1\right\}.$$

Hence, given any $\epsilon > 0$, there exists $\overrightarrow{\phi} \in C_c^1(\Omega, \mathbb{R}^d)$ with $\|\overrightarrow{\phi}\|_\infty \le 1$ such that

$$\begin{aligned}\operatorname{var}(\mathcal{L}_{\overline{f}}\eta) - \epsilon &\le \int_\Omega (\eta \circ \overline{f}^{-1} \cdot \operatorname{div} \overrightarrow{\phi})\, dm \\ &= \sum_{i=0}^{r} \int_{\overline{f}_i(\operatorname{int}\omega_i)} (\eta \circ \overline{f}_i^{-1} \cdot \operatorname{div} \overrightarrow{\phi})\, dm.\end{aligned}$$

Let $x = \overline{f}_i^{-1}(y)$ on each ω_i; we then have

$$\begin{aligned}&\int_{\overline{f}_i(\operatorname{int}\omega_i)} \eta(\overline{f}_i^{-1}(y)) \cdot \operatorname{div} \overrightarrow{\phi}(y)\, dm(y) \\ &= \int_{\operatorname{int}\omega_i} \eta(x) \cdot \operatorname{div}(\overrightarrow{\phi})(\overline{f}_i(x))\, dm(\overline{f}_i(x)) \\ &= \int_{\operatorname{int}\omega_i} \eta \cdot (\operatorname{div} \overrightarrow{\phi}) \circ \overline{f}_i\, dm = \int_{\operatorname{int}\omega_i} \eta \cdot \operatorname{div} \overrightarrow{\phi}^{\overline{f}_i}\, dm,\end{aligned}$$

where $\overrightarrow{\phi}^{\overline{f}_i} := (D\overline{f}_i)^{-1} \cdot \overrightarrow{\phi} \circ \overline{f}_i$. The second equality is due to $|\det D\overline{f}_i(x)| \equiv 1$, while the third is due to equation (6) in Lemma 6. Hence,

$$\operatorname{var}(\mathcal{L}_{\overline{f}}\eta) - \epsilon \le \sum_{i=1}^{r} \int_{\operatorname{int}\omega_i} \eta(x) \cdot \operatorname{div} \overrightarrow{\phi}^{\overline{f}_i}\, dm.$$

Moreover, each $\overline{f}_i$ can be written in the form $\overline{f}_i(x) = A_i \cdot x + c_i$, where the orthogonal matrix A_i preserves the Euclidean metric. Then

$$\begin{aligned}\sup_{x\in\operatorname{int}\omega_i} |\overrightarrow{\phi}^{\overline{f}_i}(x)| &= \sup_{x\in\operatorname{int}\omega_i} |A_i^{-1}\overrightarrow{\phi}(\overline{f}_i(x))| \\ &= \sup_{x\in\operatorname{int}\omega_i} |\overrightarrow{\phi}(\overline{f}_i(x))| = \sup_{y\in\overline{f}_i(\operatorname{int}\omega_i)} |\overrightarrow{\phi}(y)| \le 1.\end{aligned}$$

Since $\eta|_{N_\delta} = 0$ for some $\delta > 0$, for each ω_i, we let $\omega_i^{(\delta)} := N_\delta^c \cap \omega_i$, which is a compact subset of ω_i. Moreover, $\omega_i^{(\delta)} \subset \omega_i^{(\delta/2)} \subset \operatorname{int}\omega_i$. Therefore, there exists a bump function $B_i(x) \in C_c^\infty(\Omega)$ such that $0 \le B_i(x) \le 1$, $B_i(x) = 1$ on $\omega_i^{(\delta)}$ and $B_i(x) = 0$ on $(\operatorname{int}\omega^{(\delta/2)})^c$. We extend the function $\overrightarrow{\phi}^{\overline{f}_i}$ to be zero outside ω_i and define

$$\overrightarrow{\psi}(x) := \sum_{i=0}^{r} B_i(x) \cdot \overrightarrow{\phi}^{\overline{f}_i}(x), \quad x \in \Omega.$$

It is apparent to see that $\overrightarrow{\psi} \in C_c^1(\Omega, \mathbb{R}^d)$ and $\sup_{x\in\Omega} |\overrightarrow{\psi}| \le \sup_{x\in\Omega} |\overrightarrow{\phi}^{\overline{f}_i}(x)| \le 1$ for each i. Moreover, since $\overrightarrow{\psi}|_{\omega_\delta} = \overrightarrow{\phi}^{\overline{f}_i}|_{\omega_\delta}$ and $\eta|_{N_\delta} = 0$, we have

$$\operatorname{var}(\mathcal{L}_{\overline{f}}\eta) - \epsilon \le \sum_{i=1}^{r} \int_{\operatorname{int}\omega_i} \eta(x) \cdot \operatorname{div} \overrightarrow{\phi}^{\overline{f}_i}\, dm = \int_\Omega \eta \cdot \operatorname{div} \overrightarrow{\psi}\, dm \le \operatorname{var}(\eta).$$

Since ϵ is arbitrary, it follows that $\operatorname{var}(\mathcal{L}_{\overline{f}}\eta) \le \operatorname{var}(\eta) < \infty$. □

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Proof of Theorem 2. Given any $\eta \in BV^*(\Omega)$, from Lemma 4, we can write $\eta = \eta^{(1)} + \eta^{(2)}$, where $\eta^{(1)}|_{N_\delta} = 0$ for some $\delta > 0$ and $\eta^{(2)} \in W^{1,2}$. Moreover, by Lemma 5, there exists a subsequence $\{\eta^{(2)}_{n_k}\}$ of $\{\eta^{(2)}_n\}$ which converges to a function $\overline{\eta}^{(2)} \in W^{1,2}$ in $L^1(m)$.

Consider the function $\eta^{(1)}$. Given any $n \geq 1$, by analogous arguments to those used in Lemma 7, we have $\mathrm{var}(\mathcal{L}^n_{\overline{f}}\eta^{(1)}) \leq \mathrm{var}(\eta^{(1)})$. It follows that $\|\mathcal{L}^n_{\overline{f}}\eta^{(1)}\|_{BV} \leq \|\eta^{(1)}\|_{BV}$. Hence, $\|\eta^{(1)}_n\|_{BV} \leq \|\eta^{(1)}\|_{BV} < \infty$. By Helly's theorem [**9**], there exists a subsequence, for convenience say $\{\eta^{(1)}_{n_k}\}$, converging to a function $\overline{\eta}^{(1)} \in BV$ in $L^1(m)$.

By the triangle inequality, for each $\overline{\eta}^{(i)}$, $i = 1, 2$,

$$\|\mathcal{L}_{\overline{f}}\overline{\eta}^{(i)} - \overline{\eta}^{(i)}\|_1 \leq \|\mathcal{L}_{\overline{f}}\overline{\eta}^{(i)} - \mathcal{L}_{\overline{f}}\eta^{(i)}_{n_k}\|_1 + \|\mathcal{L}_{\overline{f}}\eta^{(i)}_{n_k} - \eta^{(i)}_{n_k}\|_1 + \|\eta^{(i)}_{n_k} - \overline{\eta}^{(i)}\|_1. \quad (7)$$

It is clear that the first and third terms tend to 0 as $k \to \infty$. Moreover, we note that

$$\begin{aligned}\|\mathcal{L}_{\overline{f}}\eta^{(i)}_{n_k} - \eta^{(i)}_{n_k}\|_1 &= \left\| \frac{1}{n_k}\sum_{j=1}^{n_k} \eta^{(i)} \circ \overline{f}^{-j} - \frac{1}{n_k}\sum_{j=0}^{n_k-1} \eta^{(i)} \circ \overline{f}^{-j} \right\|_1 \\ &= \frac{1}{n_k}\|\eta^{(i)} \circ \overline{f}^{-n_k} - \eta^{(i)}\|_1 \leq \frac{2}{n_k}\|\eta^{(i)}\|_1;\end{aligned}$$

hence, the second term tends to 0 as $k \to \infty$. Therefore, $\mathcal{L}_{\overline{f}}\overline{\eta}^{(i)} = \overline{\eta}^{(i)}$, implying that $\overline{\eta} := \overline{\eta}^{(1)} + \overline{\eta}^{(2)} \in BV(\Omega)$ is an invariant density of $\overline{f}$ from Lemma 2. Moreover, since $\eta|_X \geq 0$ and $\|\overline{\eta}|_X\|_1 = \|\eta|_X\|_1 > 0$, by normalization $d\mu = \overline{\eta}|_X\, dm \in \mathcal{M}_{IB}(f)$. □

Remark 4. For a given invertible PWI, if the set N_δ satisfies $f^{-1}(N_\delta) \subseteq N_\delta$, then the above accumulation point $\overline{\eta}$ can be shown to be in $BV^*(\Omega)$ for the following reason. Since the sequence $\{\eta^{(1)}_{n_k}\}$ converges to $\overline{\eta}^{(1)}$ in L^1, there will exist a subsequence which pointwise converges to $\overline{\eta}^{(1)}$. Moreover, since $\eta^{(1)}|_{N_\delta} = 0$ and $\overline{f}^{-1}(N_\delta) \subseteq N_\delta$, it follows that $\eta^{(1)}_n|_{N_\delta} = 0$ for any $n \geq 0$. Therefore, $\overline{\eta}^{(1)}|_{N_\delta} = 0$. By Lemma 4, this implies $\overline{\eta} = \overline{\eta}^{(1)} + \overline{\eta}^{(2)} \in BV^*(\Omega)$.

4.3. *Proof of Proposition 1.*

LEMMA 8. *Let $(X, \mathfrak{B}, m)$ be a probability space and $f : X \to X$ be an invertible PAP; then $d\mu := \varphi\, dm \in \mathcal{M}_I(f)$ if and only if $\varphi = \mathbb{E}(\varphi|\mathcal{I})$, where $\mathbb{E}(\varphi|\mathcal{I})$ is the conditional expectation on the σ-field $\mathcal{I} := \{B \in \mathfrak{B} | f^{-1}(B) = B \mod m\}$.*

Proof. Suppose $d\mu = \varphi\, dm \in \mathcal{M}_I(f)$; then, by Lemma 2, we have $\mathcal{L}_f\varphi = \varphi$, which implies $\varphi \circ f = \varphi$. Combining with the Birkhoff ergodic theorem, we obtain

$$\varphi = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ f^i = \mathbb{E}(\varphi|\mathcal{I}).$$

For the converse, given any fixed $L \in \mathbb{N}$, we define $\varphi_L(x) := \min\{\varphi(x), L\}$. By the Birkhoff ergodic theorem, for m-almost every $x \in X$,

$$\begin{aligned}\mathcal{L}_f(\mathbb{E}(\varphi_L|\mathcal{I}))(x) = \mathbb{E}(\varphi_L|\mathcal{I}) \circ f^{-1}(x) &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=-1}^{n-2} \varphi_L \circ f^i(x) \\ &= \lim_{n\to\infty}\left[\frac{1}{n}\left(\sum_{i=0}^{n-1} \varphi_L \circ f^i\right)(x) + \frac{1}{n}(\varphi_L \circ f^{-1}(x) - \varphi_L \circ f^{n-1}(x))\right].\end{aligned}$$

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Since

$$\lim_{n\to\infty}\frac{1}{n}(\varphi_L\circ f^{-1}(x)-\varphi_L\circ f^{n-1}(x))\le\lim_{n\to\infty}\frac{2L}{n}=0,$$

we have $\mathcal{L}_f\mathbb{E}(\varphi_L|\mathcal{I})=\mathbb{E}(\varphi_L|\mathcal{I})$. Then, by the monotone convergence theorem,

$$\mathcal{L}_f\mathbb{E}(\varphi|\mathcal{I})=\mathbb{E}(\varphi|\mathcal{I}).$$

Therefore, $\varphi=\mathbb{E}(\varphi|\mathcal{I})$ implies $d\mu:=\varphi\,dm\in\mathcal{M}_I(f)$ from Lemma 2. □

We say a σ-field $\mathcal{I}$ is *finitely generated* if there exists a partition $\mathcal{A}:=\{A_i\}_{i=0}^{r-1}\subseteq\mathcal{I}$ and, for each $B\in\mathcal{I}$, there exist finitely many $A_{i_1},\dots,A_{i_l}\in\mathcal{A}$ such that $B=\bigcup_{k=1}^{l}A_{i_k}$ mod m.

Proof of Proposition 1. We first show statement (i). For a topologically transitive IET f, since there are only finitely many ergodic measures, say $\{\nu_i\}_{i=0}^{r-1}\in\mathcal{M}_I(f)$ [**17**], the σ-field $\mathcal{I}$ is finitely generated by a partition, say $\mathcal{A}:=\{A_0,\dots,A_{r-1}\}$ with $0<m(A_i)<1$ and $\nu_i=m|_{A_i}\in\mathcal{M}_I(f)$. Therefore, for any $d\mu=\varphi\,dm\in\mathcal{M}_I$, by Lemma 8 and [**3**],

$$\varphi=\mathbb{E}(\varphi|\mathcal{I})=\sum_{i=0}^{r-1}\frac{1}{m(A_i)}\int_{A_i}\varphi\,dm\cdot\chi_{A_i}\quad\text{for all }\varphi\in L^1(m).\tag{8}$$

This implies that φ is a simple function.

Now we show statement (ii). Since each $m|_{A_i}\in\mathcal{M}_I(f)$ and $m(A_i)>0$, it is clear that for m-almost every $x\in A_i$, the orbit $O_f(x)\subset A_i$. Moreover, since f is minimal, each A_i is dense in X. Hence, int $A_i=\emptyset$. Based on (8), if $\mu:=\varphi\,dm\in\mathcal{M}_I(f)$ and $\mu\neq m$, then there exists $i\neq j$ such that $\int_{A_i}\varphi\,dm/m(A_i)\neq\int_{A_j}\varphi\,dm/m(A_j)$. Therefore, φ is discontinuous everywhere. Moreover, for any open set $U\subset[0,1)$, we have $m(A_i\cap U)>0$. Hence, $\mu(U)>0$. It follows that $\operatorname{supp}(\mu)=[0,1)$. □

4.4. *Proofs of Lemma 1 and Proposition 2.*

Proof of Lemma 1. We first show almost closedness of X^+. Since f_i is Lipschitz continuous, it can be continuously extended from int ω_i onto $\overline{\text{int }\omega_i}$. If we denote its continuous extension by $\widehat{f_i}:\overline{\text{int }\omega_i}\to\widehat{f_i}(\overline{\text{int }\omega_i})$, then each $\widehat{f_i}$ is also Lipschitz continuous. Together with the non-singularity of f, we know that†

$$\begin{aligned}\overline{X^+}:=\text{closure}\Big(\bigcap_{j=0}^{\infty}f^j\Big(\bigcup_{i=0}^{r-1}\omega_i\Big)\Big)&\subseteq\bigcap_{j=0}^{\infty}\text{closure}\Big(\bigcup_{i=0}^{r-1}\widehat{f_i}^{\,j}(\text{int }\omega_i)\Big)\ \text{mod }m\\&\subseteq\bigcap_{j=0}^{\infty}\bigcup_{i=0}^{r-1}\widehat{f_i}^{\,j}(\overline{\text{int }\omega_i})\ \text{mod }m\\&=\bigcap_{j=0}^{\infty}\bigcup_{i=0}^{r-1}f_i(\omega_i)\ \text{mod }m\\&=X^+\ \text{mod }m.\end{aligned}$$

This implies $m(X^+)=m(\overline{X^+})$.

† By $A\subseteq B$ mod m, we mean that for m-almost every $x\in A$, we have $x\in B$.

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Denote $\omega_i^+ := \omega_i \cap X^+$; then $X^+ = \bigcup_{i=0}^{r-1} \omega_i^+$. Consequently,

$$\sum_{i=0}^{r-1} m(f(\omega_i^+)) = \sum_{i=0}^{r-1} m(\omega_i^+) = m\left(\bigcup_{i=0}^{r-1} \omega_i^+\right) = m(X^+) = m(f(X^+))$$
$$= m\left(\bigcup_{i=0}^{r-1} f(\omega_i^+)\right).$$

It follows that $m(f(\omega_i^+) \cap f(\omega_j^+)) = 0$, implying $f|_{X^+}$ is m-almost everywhere invertible. □

Proof of Proposition 2. We note that f^+ is piecewise Lipschitz continuous; then, for any Borel subset $A \subset \overline{X^+}$ with $m(A) = 0$, we have $m(f^+(A)) = 0$. Moreover, since $f|_{\omega_i}$ is bi-Lipschitz, it follows that $m((f^+)^{-1}(A)) = 0$. Hence, f^+ is non-singular. The m-almost everywhere invertibility of f^+ comes directly from Lemma 1.

When $m(\overline{X^+}) > 0$, $f|_{\overline{X^+}}$ can be viewed as a first return map of f on $\overline{X^+}$. Therefore,

$$\{\mu(\cdot) := \nu(\cdot \cap \overline{X^+}), \forall \nu \in \mathcal{M}_I(f^+)\} \subseteq \mathcal{M}_I(f).$$

Moreover, for any $\mu \in \mathcal{M}_I(f)$, since $\mu(X) = 1$ and $f^{-1} \circ f(X) = X$, $\mu(f(X)) = 1$. This implies that $\mu(X^+) = 1$ and completes the proof of statement (i).

For statement (ii), we argue by contradiction. Suppose that there exists $\mu \in \mathcal{M}_I(f)$; since $m(\overline{X^+}) = 0$, it follows that $\mu(\overline{X^+}) = 0$. This is a contradiction with $\mu(X^+) = 1$. □

Acknowledgements. The authors are grateful to Professr Peter Ashwin, Professr Henk Bruin, Dr Mark Holland and Dr Corinna Ulcigrai as well as an anonymous referee for their very helpful comments. The authors also thank the Symposium on Ergodic Theory and Dynamical Systems organized in the University of Warwick during 2010–2011.

A. *Appendix*

A.1. *Bounded variation.* We introduce the usual notion of bounded variation in one dimension followed by definitions of multidimensional bounded variation.

Let $\eta \in L^1(\mathbb{R})$ and $[a, b]$ be an interval outside of which $\eta(x) = 0$. The *total variation* of η is defined to be

$$V(\eta) := \sup \sum_{i=0}^{i-1} |\eta(x_{i+1}) - \eta(x_i)|,$$

where 'sup' is taken over all possible finite partitions of the interval $[a, b]$ by points $x_0 = a < x_1 < \cdots < x_r = b$. The *essential total variation* of η is defined as $\bar{V}(\eta) := \inf_u V(\eta + u)$, where the 'inf' is taken over all functions u that equal zero almost everywhere on $[a, b]$ [**29**].

Next, we proceed to describe a definition of multidimensional bounded variation. Let Ω be an open subset of $\mathbb{R}^d$ and $\eta \in L^1(\Omega)$ be a function with compact support. We regard $\eta(x)$ as a function of the variable x_i for the other variables fixed and denote by $\bar{V}_i(x_i')$ the essential total variation of the function η with respect to x_i for a fixed point $x_i' = \{x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d\}$.

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Definition 3. [**29**] Suppose Ω is an open subset of $\mathbb{R}^d$. A function $\eta \in L^1(\Omega)$ with compact support is said to be of *bounded variation* if the integrals

$$\int \bar{V}_i(x_i')\, dx_i' < \infty \quad (i = 1, 2, \ldots, n).$$

We give a further definition of bounded variation below.

Definition 4. [**29**, p. 158] Suppose Ω is an open subset of $\mathbb{R}^d$ and a function $\eta \in L^1(\Omega)$; then η is said to be of *bounded variation* if there exists a constant K such that

$$\left| \int_\Omega \frac{\partial \phi}{\partial x_i} \eta \, dx \right| \le K \sup_{x \in \Omega} |\phi(x)| \quad (i = 1, 2, \ldots, d) \tag{9}$$

for all $\phi \in C_c^1(\Omega, \mathbb{R})$.

Definitions 3 and 4 coincide for functions $\eta \in L^1(\Omega)$ with compact support [**29**]. Moreover, we show the equivalence between Definitions 4 and 2 in §2.2 via the following proposition.

PROPOSITION 4. *Suppose Ω is an open subset of $\mathbb{R}^n$ and $\eta \in L^1(\Omega)$. Then* $\mathrm{var}(\eta) < \infty$ *(recall that* $\mathrm{var}(\cdot)$ *is defined in Definition 2) if and only if the inequality (9) holds for all* $\phi \in C_c^1(\Omega, \mathbb{R})$.

Proof. '$\Rightarrow$'. Suppose that $\mathrm{var}(\eta) < \infty$; then, by Definition 2,

$$\left| \int_\Omega \eta(x) \operatorname{div} \overrightarrow{\phi}(x)\, dx \right| \le \mathrm{var}(\eta) \| \overrightarrow{\phi} \|_\infty \quad \text{for all } \overrightarrow{\phi} \in C_c^1(\Omega, \mathbb{R}^d). \tag{10}$$

For any $\phi \in C_c^1(\Omega, \mathbb{R})$, let $\overrightarrow{\phi}_i = (\overbrace{0, \ldots,}^{i-1} \phi, \ldots, 0)$; then the inequality (10) holds for $\overrightarrow{\phi}_i$. This implies that inequality (9) holds for all $C_c^1(\Omega, \mathbb{R}^d)$.

'$\Leftarrow$'. By the inequality (9) and Definition 2, it is clear that $\mathrm{var}(\eta) \le d \cdot K < \infty$, where d is the dimension constant. □

A.2. *Sobolev space* $W^{1,2}$.

Definition 5. [**15**] Suppose Ω is an open set in $\mathbb{R}^d$. The Sobolev space $W^{1,2}(\Omega)$ is defined to be the set of all functions $u \in L^2(\Omega)$ such that for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_i \ge 0$ and $|\alpha| := \sum_{j=1}^d \alpha_j \le 1$, the weak partial derivative $D_\alpha u$ belongs to $L^2(\Omega)$, i.e.

$$W^{1,2}(\Omega) = \{u \in L^2(\Omega) : D_\alpha u \in L^2(\Omega), \forall\, |\alpha| \le 1\}.$$

We define a norm on $W^{1,2}(\Omega)$ by

$$\|u\|_{W^{1,2}} := \left(\sum_{|\alpha| \le 1} \int_\Omega |D_\alpha u|^2 \, dm \right)^{1/2}.$$

The Sobolev space $W^{1,2}(\Omega)$ with norm $\| \cdot \|_{W^{1,2}}$ is a Hilbert subspace of $BV(\Omega)$. We state the following Banach–Saks theorem, which is applied to the Sobolev space $W^{1,2}$ in Lemma 5.

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Banach–Saks theorem. [**15**] Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence from a Hilbert space H with $\|x_n\|_H \le K$ (independent of n); then there exist a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ and an $x \in H$ such that

$$\frac{1}{k}\sum_{j=1}^{k} x_{n_j} \to x \quad \text{as } k \to \infty$$

in $\|\cdot\|_H$.

A.3. *Piecewise rotations.*

Definition 6. [**14**] Let X be a compact subset of $\mathbb{C}$. A map $T : X \to X$ is called a *piecewise rotation* with a partition $\mathcal{P} := \{\omega_0, \ldots, \omega_{r-1}\}$ if

$$T|_{\omega_j} x = \rho_j x + z_j, \quad x \in \omega_j$$

for some complex numbers z_j and ρ_j such that $|\rho_j| = 1$ for all $j = 0, 1, \ldots, r-1$. The atoms are assumed to be mutually disjoint convex polygons.

It is clear that piecewise rotations are PWIs in $\mathbb{R}^2$ with a topological partition $\mathcal{P}$ and are homeomorphisms when restricting on each atom.

A.4. *Interval translation maps and interval exchange transformations.*

Definition 7. Let $I = [0, 1)$ be an interval and $0 = \beta_0 < \beta_1 < \cdots < \beta_r = 1$ be a finite partition of I. An interval map $T : I \to I$ is said to be an *interval translation map* [**4**] if

$$T(x) = x + \gamma_i, \ \beta_{i-1} \le x < \beta_i,$$

where each γ_i is a fixed real number. Particularly, if T maps I onto itself, then T is called an *interval exchange transformation.*

References

[1] P. Ashwin, X. C. Fu and J. R. Terry. Riddling and invariance for discontinuous parabolic maps of torus. *Nonlinearity* **13** (2002), 818–835.

[2] V. Baladi and S. Gouëzel. Good Banach spaces for piecewise hyperbolic maps via interpolation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009), 1453–1481.

[3] A. Boyarsky and P. Góra. *Laws of Chaos: Invariant Measures and Dynamical Systems in One Dimension*. Birkhäuser, Boston, MA, 1997.

[4] H. Bruin and S. Troubetzkoy. The Gauss map on a class of interval translation mappings. *Israel J. Math.* **137** (2003), 125–148.

[5] J. Buzzi. No or infinitely many a.c.i.p. for piecewise expanding C^r maps in higher dimensions. *Comm. Math. Phys.* **222** (2001), 495–501.

[6] J. Coffey. Some remarks concerning an example of a minimal non-unique ergodic interval exchange transformation. *Math. Z.* **199** (1988), 577–580.

[7] J. Chaika. Hausdorff dimension for ergodic measures of interval exchange transformations. *J. Mod. Dyn.* **2** (2008), 457–464.

[8] B. K. Driver. *Analysis Tools with Applications*. Springer, Berlin, 2003.

[9] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*. CRC Press, Boca Raton, FL, 1992.

[10] B. Fayad and A. Katok. Constructions in elliptic dynamics. *Ergod. Th. & Dynam. Sys.* **24** (2004), 1477–1520.

https://doi.org/10.1017/S0143385711001167 Published online by Cambridge University Press

[11] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*, 2nd edn. Wiley, Hoboken, NJ, 2003.
[12] X. C. Fu, F. Y. Chen and X. H. Zhao. Dynamical properties of 2-torus parabolic maps. *Nonlinear Dynam.* **50** (2007), 539–549.
[13] A. Goetz. Piecewise isometries—an emerging area of dynamical systems. *Fractals in Graz 2001: Analysis, Dynamics, Geometry, Stochastics*. Eds. P. Grabner and W. Woess. Birkhäuser, Basel, 2002, pp. 135–144.
[14] A. Goetz. Stability of piecewise rotations and affine maps. *Nonlinearity* **14** (2001), 205–219.
[15] J. Jost. *Postmodern Analysis*. Springer, Berlin, 2005.
[16] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, London, 1995.
[17] M. Keane. Interval exchange transformations. *Math. Z.* **141** (1975), 25–31.
[18] M. Keane. Non-ergodic interval exchange transformations. *Israel J. Math.* **26** (1977), 188–196.
[19] M. Keane and G. Rauzy. Stricte ergodicité des échanges d'intervalles. *Math. Z.* **174** (1980), 203–212.
[20] G. Keller. Coupled map lattices via transfer operators on functions of bounded variation. *Stochastic and Spatial Structures of Dynamical Systems (Amsterdam, 1995) (Koninklijke Nederlandse Akademie van Wetenschappen. Verhandelingen, Afd. Natuurkunde. Eerste Reeks, 45)*. Eds. S. J. van Strien and S. M. Verduyn Lunel. North-Holland, Amsterdam, 1996, pp. 71–80.
[21] H. B. Keynes and D. Newton. A minimal, non-uniquely ergodic interval exchange transformation. *Math. Z.* **148** (1976), 101–105.
[22] R. S. MacKay. *Renormalisation in Area-preserving Maps*. World Scientific, Singapore, 1993.
[23] H. Masur. Interval exchange transformations and measured foliations. *Ann. of Math.* **115** (1982), 169–200.
[24] A. Peris. Transitivity, dense orbit and discontinuous functions. *Bull. Belg. Math. Soc. Simon Stevin* **6** (1999), 391–394.
[25] B. Saussol. Absolutely continuous invariant measures for multidimensional expanding maps. *Israel J. Math.* **116** (2000), 223–248.
[26] M. Tsujii. Piecewise expanding maps on the plane with singular ergodic properties. *Ergod. Th. & Dynam. Sys.* **20** (2000), 1851–1857.
[27] W. Veech. Gauss measures for transformations on the space of interval exchange maps. *Ann. of Math. (2)* **115** (1982), 201–242.
[28] M. Viana. *Stochastic Dynamics of Deterministic Systems (Brazilian Mathematics Colloquium, 21)*. IMPA, Rio de Janeiro, 1997.
[29] A. I. Vol'pert and S. I. Hudjaev. *Analysis in Classes of Discontinuous Functions and Equations of Mathematical Physics*. Martinus Nijhoff, Dordrecht, 1985.
[30] P. Walters. *An Introduction to Ergodic Theory*. Springer, New York, 1982.

https://doi.org/10.1017/S0143385711001167 Published online by Cambridge University Press