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RELEVANCE LOGICS AND RELATION ALGEBRAS

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Abstract. *Relevance logics* are known to be sound and complete for relational semantics with a ternary accessibility relation. This paper investigates the problem of adequacy with respect to special kinds of *dynamic semantics* (i.e., proper relation algebras and relevant families of relations). We prove several *soundness* results here. We also prove the *completeness* of a certain positive fragment of R as well as of the first-degree fragment of relevance logics. These results show that some core ideas are shared between relevance logics and relation algebras. Some details of certain incompleteness results, however, pinpoint where relevance logics and relation algebras diverge. To carry out these semantic investigations, we define a *new tableaux* formalization and new *sequent calculi* (with the single cut rule admissible) for various relevance logics.

§1. Preliminary remarks. Relation algebras have been of interest in themselves for a long time. In logic, the problem of dealing with relations—as opposed to properties—goes back at least to the works of Augustus De Morgan. Relations were central to Charles S. Peirce's thinking about logical problems. The properties of binary relations were first formalized as an algebra by Tarski (1941). Soon afterward, Jónsson and Tarski (1948) showed that relation algebras form a variety. Tarski's aim was to fully characterize binary relations and the operations on them by relation algebras, hence it was natural to raise the question whether all relation algebras can be represented as algebras of binary relations, which are called proper relation algebras. Lyndon (1950) showed that not all relation algebras can be represented in this way. Tarski (1955) established that the class of representable relation algebras is a variety, but Monk (1964) proved that it is not a finitely axiomatizable variety.

Collections of *binary relations* can be viewed as a sort of dynamic interpretation for a logic, that is thought to describe the impact a sentence has on a situation via specifying a set of possible resulting situations. Special types of *dynamic semantics* are those in which the binary relations constitute a relation algebra or a relevant family of operations.

First, we briefly recall the definition of relation algebras and that of the logic of relevant implication (R) and of R-mingle (RM). Then we establish completeness and incompleteness results for various fragments and extensions of these logics with respect to classes of proper relation algebras and relevant families of relations. In conclusion, we briefly sketch

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the more immediate motivations and some preceding work that led to the writing of this paper.

§2. Relation algebras. Relation algebras are algebras rich in structure. Therefore, they may be defined using various sets of primitive operations, and accordingly, by different (but equivalent) sets of equations. Nowadays, a more or less standard set of operations includes some *Booleans* (e.g., $-$ and $+$) as well as *relational composition* (;), *converse* ($^{\cup}$) and *identity* (**1'**). The following set of equations comes from Tarski and Givant (1987); essentially, the same equations may be found in the recent comprehensive study on relation algebras by Maddux (2006).[1]

DEFINITION 2.1. *Let* $\mathfrak{A} = \langle A; +, -, ;, ^{\cup}, \mathbf{1'} \rangle$ *be an algebra of similarity type* $\langle 2, 1, 2, 1, 0 \rangle$. $\mathfrak{A}$ *is a* relation algebra *when the reduct* $\langle A; +, - \rangle$ *is a Boolean algebra (a* BA*) and* $\mathfrak{A}$ *satisfies the equations (r1)–(r7).*

(r1) $(a\,;b)\,;c = a\,;(b\,;c)$
(r2) $(a\,;c) + (b\,;c) = (a+b)\,;c$
(r3) $a = a\,;\mathbf{1'}$
(r4) $a = a^{\cup\cup}$
(r5) $a^{\cup} + b^{\cup} = (a+b)^{\cup}$
(r6) $a^{\cup}\,;b^{\cup} = (b\,;a)^{\cup}$
(r7) $a^{\cup}\,;-(a\,;b) \leq -b$

The class of relation algebras is denoted by RA.

Of course, there are infinitely many equations that are now derivable. We restrict ourselves to mentioning those that may help to position relation algebras among other well-known classes of algebras. We will denote the top and the bottom of the Boolean algebra by 1 and 0, that is, 1 is defined as $a + -a$ and 0 is -1. The "diversity element" of an RA is denoted by **0'**, and **0'** is defined as $-\mathbf{1'}$. Converse has *fixed points* on the four elements 0, **1'**, **0'** and 1, which is concisely expressed by the following equations.

$$0^{\cup} = 0 \qquad \mathbf{1'}^{\cup} = \mathbf{1'} \qquad \mathbf{0'}^{\cup} = \mathbf{0'} \qquad 1^{\cup} = 1.$$

1' is right identity for ; by (r3), and **1'** can be proven to be left identity as well (i.e., $\mathbf{1'}\,;a = a$), hence, **1'** is *the identity* for relational composition. (r2) expresses the additivity of ; in its first argument place, and ; may be proven to distribute over $+$ in its second argument place too, that is, $(a\,;b) + (a\,;c) = a\,;(b+c)$.[2]

An element a of a relation algebra is an *atom* if $0 \leq b \leq a$ implies $0 = b$ or $b = a$; and the algebra is *atomic* if each element is a (generalized) join of a set of atoms (below that element).[3] A relation algebra may or may not be atomic. Of course, an atomic relation algebra is atomic as a Boolean algebra, hence a description of the behavior of negation on

[1] The equations R_4–R_{10} on p. 289 in Maddux (2006) are (r1)–(r7) here, in the same order. We permit the reader to supply her (or his) favorite axioms for Boolean algebras (e.g., R_1–R_3, *ibid.*).

[2] These observations put relation algebras into the class of *Boolean gaggles* with operations that have distribution types $;: +, + \longrightarrow +$, $^{\cup}: + \longrightarrow +$, $^{\cup}: \cdot \longrightarrow \cdot$ (where $\cdot$ is meet) and respect the bounds. Compare Bimbó and Dunn (2008, chapter 1).

[3] $\leq$ is the usual lattice order of the underlying BA, that is, $a \leq b$ iff $a + b = b$. The other concepts we use here are also standard. Compare Davey and Priestley (2002).

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the atoms (in terms of atoms and join) completely determines the BA. However, atomic relation algebras possess the nonobvious and remarkable property that the behavior of ; and $^{\cup}$ can be completely characterized by their behavior on the atoms. The converse of an atom is an atom, and the relational composition of a pair of atoms is specifiable by a set of incidence triples of the form $a \leq b\,;c$ (where a, b and c are atoms). This suggests a ternary relation and a possible connection to the relational semantics of relevance logics (which is described in section 3).

We shall recall certain subvarieties of RAs that we will utilize later on. We also define an important subclass of RAs, namely, the class of integral RAs.

DEFINITION 2.2. *Let* $\mathfrak{A}$ *be a relation algebra.*

(1) $\mathfrak{A}$ *is* integral *iff* $a\,;b = 0$ *implies either* $a = 0$ *or* $b = 0$.

(2) $\mathfrak{A}$ *is* commuting *iff* $a\,;b = b\,;a$.

(3) $\mathfrak{A}$ *is* dense *iff* $a \leq a\,;a$.

(4) $\mathfrak{A}$ *is* transitive *iff* $a\,;a \leq a$.

(5) $\mathfrak{A}$ *is* symmetric *iff* $a = a^{\cup}$.

Equivalent characterizations of integral relation algebras include the following (cf. Jónsson & Tarski, 1952, Thm. 4.17).

(6) $\mathfrak{A}$ *is* integral *iff* $\mathbf{1'}$ *is an atom.*

(7) $\mathfrak{A}$ *is* integral *iff* $a \neq 0 \Rightarrow a\,;1 = 1$.

An equivalent characterization of transitive relation algebras is obtained using the notion of Boolean RAs*, which are defined in Maddux (2006, p. 291).*

(8) $\mathfrak{A}$ *is* Boolean *iff* $\mathbf{1'} = 1$.

Boolean relation algebras are particularly simple. An obvious consequence of (8) is that all elements a are subidentities, that is, $a \leq \mathbf{1'}$ (cf. Givant, 1994, chapter 1), therefore, ; coincides with $\cdot$. Converse is identity, that is, a Boolean RA is a BA with some of its operations repeated. The following lemma (that seems not to have been noted in print) justifies the equivalence of (4) and (8).

LEMMA 2.3. *An* RA *is* transitive *if and only if it is* Boolean.

Proof. The proof is straightforward. If $\mathbf{1'} = 1$, then from $a\,;a \leq 1\,;a = \mathbf{1'}\,;a$ we immediately obtain $a\,;a \leq a$. For the other direction, we start with $\mathbf{0'}\,;\mathbf{0'} \leq \mathbf{0'}$, that is, by Boolean laws, $(\mathbf{0'}\,;\mathbf{0'}) \cdot -\mathbf{0'} = 0$. We have noted that converse has fixed points on the constants we singled out, hence by the so-called "cycle law," $(\mathbf{0'}\,;\mathbf{1'}) \cdot \mathbf{0'} = 0$. $\mathbf{1'}$ is identity for ;, and so by the idempotence of $\cdot$ and by taking into account the relationship of these constants and Boolean negation, $\mathbf{1'} = 1$. □

Some of the labels in Definition 2.2 for special classes of relational algebras are widely used (cf. Givant, 1994, chapter 1), and some of them may be straightforwardly motivated when the elements of a relation algebra are binary relations.

ILLUSTRATION. Let us assume that a and b (in Definition 2.2) are binary relations. Figure 1 (below) illustrates the properties in question with the dashed lines indicating existential quantification. (The two equations in (2) and (5) are self-dual, hence may be treated as inequations.)
(2) If $\langle x, y\rangle \in a\,;b$ then $\langle x, z\rangle \in a$ and $\langle z, y\rangle \in b$ for some z. For the same pair $\langle x, y\rangle$ to be an element of $b\,;a$, $\langle x, v\rangle \in b$ and $\langle v, y\rangle \in a$ should be the case for some v. Formally,

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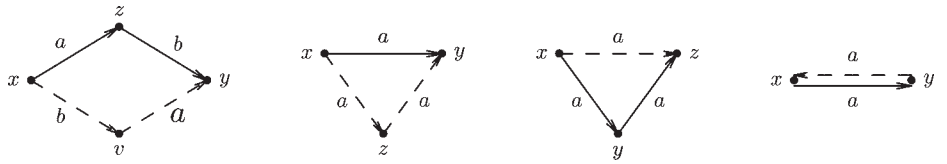


Fig. 1. Commutativity, density, transitivity and symmetry.

we get the formula $\forall x, y, z((\langle z, x\rangle \in \varrho_1 \wedge \langle z, y\rangle \in \varrho_2) \Rightarrow \exists v(\langle x, v\rangle \in \varrho_2 \wedge \langle y, v\rangle \in \varrho_1))$, which is known in modal logics as the "diamond property." This is a "global" property in the sense that it characterizes the *interaction* of relations rather that some property they have by themselves. (We changed the notation slightly, because in order to match the diamond property, we have to take ϱ_2 to be b, but ϱ_1 to be $a^{\cup}$.)
(5) Clearly, a relation a is symmetric when $\langle x, y\rangle \in a$ iff $\langle y, x\rangle \in a$. Moreover, if all relations are symmetric, then (2) holds, because if $\langle x, y\rangle \in a ; b$, so is $\langle y, x\rangle$. But from $\langle y, z\rangle \in a$ and $\langle z, x\rangle \in b$ by symmetry, $\langle x, y\rangle \in b ; a$ follows. Although symmetry is a sufficient "local" property to ensure the truth of (2), symmetry does not correspond to (2).
(3) Density is expressed by the formula $\forall x, y(\langle x, y\rangle \in a \Rightarrow \exists z(\langle x, z\rangle \in a \wedge \langle z, y\rangle \in a))$, and it is easily seen to be necessary and sufficient for (3).
(4) Transitivity is expressed by the converse of the previous formula, that is, by $\forall x, y, z((\langle x, y\rangle \in a \wedge \langle y, z\rangle \in a) \Rightarrow \langle x, z\rangle \in a)$. Again, this is the necessary and sufficient condition to ensure that (4) holds.

The elements of (abstract) relation algebras need not be binary relations. Hence we give the following definition.

DEFINITION 2.4. *A* proper relation algebra *is a relation algebra where the elements are binary relations with their natural operations as detailed below.*

A proper relation algebra need not contain all the binary relations over some set U. If it does, then it is a *full* PRA. Given a PRA there is a unique U that may be viewed as the carrier set of the binary relations. Let us denote by A' the set $A - \{\mathbf{1'}\}$, that is the carrier set of the PRA with **1'** omitted. The suitable U can be taken to be $\bigcup_{a\in A'} \pi_1 a \cup \bigcup_{a\in A'} \pi_2 a$, because if $\langle x, y\rangle \in a$ (where $a \in A'$) then $\langle x, x\rangle$ and $\langle y, y\rangle$ have to be elements of the identity **1'**.[4] This justifies the definition of PRAs given above, and allows us to sum up the definitions of the operations and constants of a PRA as follows.

DEFINITION 2.5. *If* $\mathfrak{A} = \langle A; +, \cdot, -, ;, {}^{\cup}, \mathbf{1'}\rangle$ *is a* PRA, *then the operations and constants are defined by (p1)–(p9).*

(p1) $a + b = a \cup b$, *that is,* $a + b = \{\langle x, y\rangle : \langle x, y\rangle \in a \vee \langle x, y\rangle \in b\}$,
(p2) $a \cdot b = a \cap b$, *that is,* $a \cdot b = \{\langle x, y\rangle : \langle x, y\rangle \in a \wedge \langle x, y\rangle \in b\}$,
(p3) $-a = \bigcup A - a$, *that is,* $-a = \{\langle x, y\rangle : \exists b \in A.\, \langle x, y\rangle \in b \wedge \langle x, y\rangle \notin a\}$,
(p4) $1 = \bigcup A$, *that is,* $1 = \{\langle x, y\rangle : \exists a \in A.\, \langle x, y\rangle \in a\}$,
(p5) $0 = \emptyset$, *that is,* $0 = \{\langle x, y\rangle : \exists x, y.\, \langle x, y\rangle \in 1 \wedge \langle x, y\rangle \notin 1\}$,
(p6) $a ; b = a \mid b$, *that is,* $a ; b = \{\langle x, y\rangle : \exists z.\, \langle x, z\rangle \in a \wedge \langle z, y\rangle \in b\}$,
(p7) $\mathbf{1'} = \mathrm{Id}$, *that is,* $\mathbf{1'} = \{\langle x, x\rangle : \exists a \in A\, \exists y.\, \langle x, y\rangle \in a \vee \langle y, x\rangle \in a\}$,

[4] π_1 and π_2 are the first and second projections, respectively. Concerning **1'** in PRAs, see also Jónsson and Tarski (1952, p. 142).

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(p8) $\mathbf{0'} = \mathbf{-1'}$, *that is,* $\mathbf{0'} = \{\langle x, y\rangle : \exists a \in A.\langle x, y\rangle \in a \wedge x \neq y\}$,
(p9) $a^{\cup} = a^{-1}$, *that is,* $a^{\cup} = \{\langle x, y\rangle : \langle y, x\rangle \in a\}$.

§3. Relevance logics *R* and *R*-mingle. *Relevance logics* are mainly concerned with the modeling of implication and entailment (both distinct from material "implication").[5] In relevance logics, just as in some other nonclassical logics, it is reasonable to have logical connectives that are not interdefinable (for instance, with negation and disjunction), though they may resemble one or another connective from classical logic to a certain extent. This also means that a logic, for example, the logic of relevant implication may be taken either to include or not to include some connectives. Accordingly, the expressivity of the logic may change, unlike in some variations on classical logic, such as including $\wedge$ and $\supset$ in addition to $\vee$ and $\neg$, but omitting $\boldsymbol{T}$ and $\boldsymbol{F}$.

The set of axioms and rules below defines $R^{\boldsymbol{tT}}$, that includes the connectives $\wedge, \vee, \rightarrow$ and $\sim$ as well as an "Ackermann constant" $\boldsymbol{t}$, and a "Church constant" $\boldsymbol{T}$. The two other constants $\boldsymbol{f}$ and $\boldsymbol{F}$ may be viewed either as defined or primitive, and so may fusion $\circ$ be treated. (D1)–(D3) are the three definitions that supplement R and RM if those three logical constants are omitted together with axiom (A15) and rules (R3) and (R5).

A nice feature of this formalization is that it can be dissected to obtain axiomatizations of some of the fragments of $R^{\boldsymbol{tT}}$. The three fragments that we mainly think of here are $R^{\boldsymbol{t}}$, $R^{\boldsymbol{T}}$ and R. The first may be obtained by excluding $\boldsymbol{T}$ (and $\boldsymbol{F}$) and axiom (A14) (and (A15), of course). The second results when $\boldsymbol{t}$ (and $\boldsymbol{f}$) is omitted together with the rules (R4) (and (R5), of course). R is the set of axioms and rules that does not contain either of the four constants. (Notice that each of (R3), (R4) and (R5) comprises two rules.)

(A1) $A \rightarrow A$
(A2) $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
(A3) $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$
(A4) $(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$
(A5) $(A \wedge B) \rightarrow A$
(A6) $(A \wedge B) \rightarrow B$
(A7) $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$
(A8) $A \rightarrow (B \vee A)$
(A9) $A \rightarrow (A \vee B)$
(A10) $((A \rightarrow B) \wedge (C \rightarrow B)) \rightarrow ((A \vee C) \rightarrow B)$
(A11) $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)$
(A12) $(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$
(A13) $\sim\sim A \rightarrow A$
(A14) $A \rightarrow \boldsymbol{T}$
(A15) $\boldsymbol{F} \rightarrow A$
(R1) $A \rightarrow B, A \Rightarrow B$

[5] Comprehensive volumes on relevance logics include Anderson and Belnap (1975), Anderson *et al.* (1992), Routley *et al.* (1982) and Brady (2003). More concise expositions are Dunn (1986), Mares and Meyer (2001), Mares (2002) and Bimbó (2007b). Some other nonclassical logics are described in Restall (2000).

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(R2) $A, B \Rightarrow A \wedge B$
(R3) $(A \circ B) \rightarrow C \Leftrightarrow A \rightarrow (B \rightarrow C)$
(R4) $A \Leftrightarrow \boldsymbol{t} \rightarrow A$
(R5) $A \Leftrightarrow {\sim}A \rightarrow \boldsymbol{f}$
(D1) ${\sim}\boldsymbol{t} \leftrightarrow \boldsymbol{f}$
(D2) ${\sim}\boldsymbol{T} \leftrightarrow \boldsymbol{F}$
(D3) $(A \circ B) \leftrightarrow {\sim}(A \rightarrow {\sim}B)$

Relevance logics characteristically do not include the *positive paradox*, that is, $A \rightarrow (B \rightarrow A)$ as a theorem, because this formula can introduce irrelevance between the antecedent and the consequent via detachment in just one step, as $C \rightarrow (A \rightarrow (B \rightarrow A))$, a theorem of classical logic, shows. RM includes all instances of the positive paradox in which A and B are the same formula. This logic is sometimes called "half-relevant" because of the surprising consequence that $C \rightarrow D$ is a theorem whenever both ${\sim}C$ and D are theorems—even when they do not share a propositional variable. In the system R an implication can be proven only when the antecedent and the consequent share some variable. RM may be axiomatized by adding (A16), the *mingle* axiom to R; similarly, for $RM^{\boldsymbol{t}}$, $RM^{\boldsymbol{T}}$ and $RM^{\boldsymbol{tT}}$. To refer to all these R and RM logics (as well as to some other fragments that we introduce in section 4) we will use the term 'R^* logics'.

(A16) $A \rightarrow (A \rightarrow A)$

A useful observation is that in $RM^{\boldsymbol{t}}$ or in RM (a fortiori in $R^{\boldsymbol{t}}$ or in R) $\boldsymbol{F}$ and $\boldsymbol{T}$ are not definable, that is, there is no formula B such that $B \rightarrow A$ is provable or $A \rightarrow B$ is provable, for all A. This is in sharp contrast with classical logic, where the constants $\boldsymbol{T}$ and $\boldsymbol{F}$ are definable. Indeed, $\boldsymbol{T}$ can be interpreted as truth and $\boldsymbol{T}$ is equivalent to all the theorems, whereas $\boldsymbol{F}$ can be interpreted as falsity and $\boldsymbol{F}$ is equivalent to all the contradictions. In relevance logics, $\boldsymbol{T}$ and $\boldsymbol{F}$ are more appropriately termed as "triviality" and "absurdity." An easy proof of these claims is by the models of $R^{\boldsymbol{t}}$ and $RM^{\boldsymbol{t}}$ pictured in Figure 2 below.[6]

The Ackermann constants are not definable in $R^{\boldsymbol{T}}$ or R (not even in $RM^{\boldsymbol{T}}$ or RM), although for concrete theorems (each having a finite proof) it is possible to find a formula that behaves like $\boldsymbol{t}$. (See Anderson and Belnap (1975, §28) for further details and for the proof of this claim.)

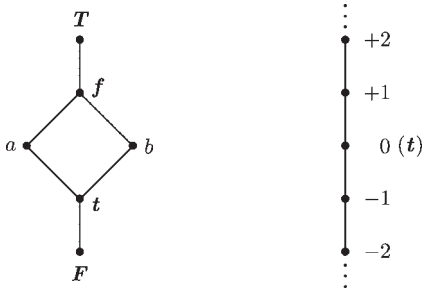


Fig. 2. CRYSTAL and $S_{\mathbb{Z}}$.

[6] Urquhart in Brady (2003, §11.6) uses the matrix CRYSTAL in a proof of the failure of interpolation in the positive fragment of R. Meyer proved that $S_{\mathbb{Z}}$ is characteristic for RM. See Anderson and Belnap, (1975, §29.3).

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The R^* logics may be given algebraic semantics, indeed the infinite Sugihara matrix in Figure 2 is characteristic for RM^t. However, the R^* logics have *relational semantics*, and we shall rely on them (and their variants) in what follows.[7] The first relational semantics utilizing a ternary accessibility relation for some of the R^* logics was introduced by Routley & Meyer (1973). The following semantics are our way of looking at those semantics with certain modifications and improvements incorporated.

DEFINITION 3.1. *A* normalized structure for R^{tT} *is* $\mathfrak{F} = \langle U; W, I, \sqsubseteq, R, {}^*\rangle$, *where the conditions (f0)–(f9) are satisfied.*

(f0) $\emptyset \neq W, \quad W \subseteq I \subseteq U, \quad \sqsubseteq \subseteq U^2, \quad R \subseteq U^3, \quad {}^*\colon U \longrightarrow U, \quad R \downdownarrows \uparrow,$
(f1) $\alpha \sqsubseteq \beta \;\Leftrightarrow\; \exists \imath \in I. R\imath\alpha\beta,$
(f2) $\exists\varepsilon.\, R\alpha\beta\varepsilon \wedge R\varepsilon\gamma\delta\,. \;\Rightarrow\; \exists\varepsilon.\, R\alpha\varepsilon\delta \wedge R\beta\gamma\varepsilon,$
(f3) $R\alpha\beta\gamma \;\Rightarrow\; \exists\delta.\, R\alpha\beta\delta \wedge R\delta\beta\gamma,$
(f4) $\exists\varepsilon.\, R\alpha\beta\varepsilon \wedge R\varepsilon\gamma\delta\,. \;\Rightarrow\; \exists\varepsilon.\, R\alpha\gamma\varepsilon \wedge R\varepsilon\beta\delta,$
(f5) $\exists \imath \in I.\, R\imath\alpha\alpha,$
(f6) $\exists \imath, \imath' \in I.\, R\imath\alpha\beta \wedge R\imath'\beta\gamma\;. \Rightarrow \exists \imath \in I.\, R\imath\alpha\gamma,$
(f7) $\alpha = \alpha^{**},$
(f8) $R\alpha\beta\gamma \;\Rightarrow\; R\alpha\gamma{}^*\beta^*,$
(f9) $w \in W \;\Rightarrow\; w^* = w.$

These structures are fairly standard (see Anderson *et al.*, 1992, §48) in that they include a set U of situations (which may be inconsistent or incomplete) and a set I of logical situations (where the theorems of the logic are true), but they are unusual for the inclusion of a subset of situations W, that are like possible worlds (cf. especially (f9)). Possible worlds are "normal" in the sense that they contain all the theorems, they are consistent and complete (i.e., $A \in w$ iff $\sim A \notin w$), and they also include a "witness" for each disjunct contained. An adequate semantics does not need to include possible worlds, hence we will refer to $\mathfrak{F}$ as above but without W as a *structure for* R^{tT}.[8]

DEFINITION 3.2. *A* (normalized) model for R^{tT} *is obtained by adding a valuation function* v *to a (normalized) structure.* v*'s type is* $\mathbb{P} \longrightarrow \wp(U)^{\uparrow}$, *where* $\mathbb{P}$ *is the set of propositional variables and* $\wp(U)^{\uparrow}$ *is the set of upward closed subsets of* U *(i.e., the set of cones on* U*).* v *is extended to all formulas according to (v1)–(v9).*

(v1) $v(A \wedge B) = v(A) \cap v(B),$
(v2) $v(A \vee B) = v(A) \cup v(B),$
(v3) $v(A \to B) = \{\, \alpha\colon \forall\beta, \gamma\, (R\alpha\beta\gamma \;\wedge \beta \in v(A)\,. \Rightarrow\; \gamma \in v(B))\,\},$
(v4) $v(\sim A) = \{\, \alpha\colon \alpha^* \notin v(A)\,\},$
(v5) $v(\boldsymbol{t}) = \{\, \alpha\colon \exists \imath, \imath' \in I.\, R\imath\imath'\alpha\,\},$
(v6) $v(\boldsymbol{T}) = U,$

[7] These semantics parallel the well-known possible worlds semantics of modal logics introduced by Kripke, as well as a step in the construction of the perfect extensions of Boolean algebras with operators introduced by Jónsson and Tarski. A comprehensive treatment of this approach is Bimbó and Dunn (2008). (Standard definitions in the relational semantics of relevance logics may be found in Dunn (1986, §4) and Bimbó (2007b, §3). Occasionally, we use them without further reference.)

[8] To be pedantic, (f9) is omitted, and the first two conditions in (f0) are replaced by $\emptyset \neq I \subseteq U$.

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(v7) $v(A \circ B) = \{\, \gamma : \exists \alpha, \beta.\, R\alpha\beta\gamma \wedge \alpha \in v(A) \wedge \beta \in v(B) \,\}$,
(v8) $v(\boldsymbol{f}) = \{\, \alpha : \neg\exists \imath \in I.\, \imath \sqsubseteq \alpha^* \,\}$,
(v9) $v(\boldsymbol{F}) = \emptyset$.

The previously mentioned fragments of $R^{\boldsymbol{t}\boldsymbol{T}}$ may be accommodated by simply omitting the clauses for (v5) and (v8), or (v6) and (v9). The semantics and the normalized semantics are sound and complete for $R^{\boldsymbol{t}\boldsymbol{T}}$ (and for $R^{\boldsymbol{t}}$, $R^{\boldsymbol{T}}$ and R).

THEOREM 3.3. (1) *If A is a theorem of $R^{\boldsymbol{t}\boldsymbol{T}}$, then for every valuation on a normalized structure for the logic, $I \subseteq v(A)$.*
(2) *If A is not a theorem of $R^{\boldsymbol{t}\boldsymbol{T}}$, then there is a normalized structure for the logic with a valuation v and a possible world w such that $w \notin v(A)$.*

From (1), soundness follows for the class of structures without possible worlds, and it is also implied that $W \subseteq v(A)$. From (2), we may infer that $I \not\subseteq v(A)$, and then 'normalized' may be omitted. Indeed, the normalization's contribution is exactly the admissibility of the so-called "disjunctive syllogism," which depends on every nontheorem's being refutable in a possible world. We do not include details of the proof of the theorem here, because a proof can be pieced together (with minor additions) from existing proofs, especially, from those in Dunn (1986, §2.4, §4.7–4.8) showing the admissibility of γ and the completeness of R.

The first relational semantics for RM was given by Dunn (1976b). It used a binary accessibility relation and it highlighted a special feature—linearity—of this logic. However, here we extend the relational semantics for $R^{\boldsymbol{t}\boldsymbol{T}}$ above that uses a ternary relation.

DEFINITION 3.4. *A* structure for $RM^{\boldsymbol{t}\boldsymbol{T}}$ *is a structure* $\mathfrak{F} = \langle U; I, \sqsubseteq, R, {}^*\rangle$ *for* $R^{\boldsymbol{t}\boldsymbol{T}}$ *(i.e., satisfies (f0$^{(\prime)}$)–(f8)) in which (f10) also holds.*

(f10) $R\alpha\beta\gamma \;\Rightarrow\; .\alpha \sqsubseteq \gamma \vee \beta \sqsubseteq \gamma\,.$

A model for $RM^{\boldsymbol{t}\boldsymbol{T}}$ *is defined as for* $R^{\boldsymbol{t}\boldsymbol{T}}$ *in Definition 3.2 but with* $RM^{\boldsymbol{t}\boldsymbol{T}}$ *frames instead of* $R^{\boldsymbol{t}\boldsymbol{T}}$ *frames.*

The semantics so obtained is adequate as stated by the next theorem. (Once again, we omit the details of the proof, however, see Dunn (1986, §4.10).)

THEOREM 3.5. *A is a theorem of $RM^{\boldsymbol{t}\boldsymbol{T}}$ iff $I \subseteq v(A)$ under all valuations on every structure for the logic.*

Obvious modifications of Theorem 3.5 provide adequacy for some of the other R^* logics, such as $RM^{\boldsymbol{t}}$, $RM^{\boldsymbol{T}}$ and RM.

§4. Completeness of some R^* logics. The relational semantics in the preceding section interpret formulas of a logic into a set of situations (which is not a binary relation in any obvious way). If we want to interpret the R^* logics into a PRA, then we have to specify the interpretation of the connectives using the operations of the PRAs. Some of the choices are rather straightforward, for instance, mapping conjunction into intersection. Furthermore, certain similarities between relation algebras and relevance logics suggest a plausible interpretation for $\sim$ too. (See Dunn (1966), Anderson and Belnap (1975, §28), and especially, Dunn (1982).)

DEFINITION 4.1. *The interpretation v of a formula of the R^* logics into a* PRA *is a function defined by the (appropriate) clauses from among (c0)–(c9).*

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(c0) $v(p) = a$ *where* $p \in \mathbb{P}$ *and* $a \in A$,
(c1) $v(A \wedge B) = v(A) \cap v(B)$,
(c2) $v(A \vee B) = v(A) \cup v(B)$,
(c3) $v(A \rightarrow B) = -(v(A)^{\cup}; -v(B))$,
(c4) $v(\sim A) = -v(A)^{\cup}$,[9]
(c5) $v(\boldsymbol{t}) = \mathbf{1'}$,
(c6) $v(\boldsymbol{T}) = 1$,
(c7) $v(A \circ B) = v(A) ; v(B)$,
(c8) $v(\boldsymbol{f}) = \mathbf{0'}$,
(c9) $v(\boldsymbol{F}) = 0$.

There are no primitive operations in relation algebras (when they are defined as in 2.1) that would interpret implication and negation directly. Of course, $\rightarrow$ and $\sim$ are definable in a PRA by (p10) and (p11).

(p10) $a \rightarrow b = -(a^{\cup}; -b)$, that is, $a \rightarrow b = \{\langle x, z\rangle \in 1 : \forall y.\, \langle y, x\rangle \in a \Rightarrow \langle y, z\rangle \in b\}$,
(p11) $\sim a = -a^{\cup}$, that is, $\sim a = \{\langle x, y\rangle \in 1 : \langle y, x\rangle \notin a\}$.

These definitions are not wholly unexpected if one notes that $\rightarrow$ in the R^* logics is the residual of fusion, and (p10) defines the residual of ; in RAs, and concretely in PRAs. De Morgan negation has been characterized as a "displaced" Boolean negation, where the shift is due to $^{\cup}$. More formally, a De Morgan negation is an orthonegation except that $a \wedge \sim a \leq b$ need not hold.

The other clauses in Definition 4.1 are straightforward. Intersection and union match the properties of conjunction and disjunction, the Church constants correspond to extremal elements, and $\boldsymbol{t}$ is a monoid identity exactly as **1'** is. However, fusion is commutative in the R^* logics, whereas ; in RAs is not, in general. Contraction and mingle make $\circ$ "square increasing" and "square decreasing," respectively—something that is not true of ;. Obviously, these features of fusion motivated us to carve out some of the subclasses of relation algebras in Definition 2.2.

Maddux (2007) proved the next theorem for R and RM with respect to what we will here call "relevant families of relations," that are like PRAs (and so are closed under $\sim$) but without the requirement of closure under $^{\cup}$ or $-$. More precisely, a *relevant family of relations* is a set of relations on some set U with the operations and constants being as on the right-hand side of the equations in Definition 4.1. The next theorem for R and RM then follows from the fact that a relevant family of relations can always be defined on a PRA as indicated above.[10] The addition of the constants to the R^* logics is conservative, and so we state the theorem with constants (that in turn, implies soundness for the logics with either of the constants omitted). Before we state the theorem though, we wish to reiterate that theoremhood in the R^* logics is indicated by $\boldsymbol{t}$ rather that $\boldsymbol{T}$, and so semantic validity with respect to PRAs is the same as $\mathbf{1'} \subseteq v(A)$ (for all interpretations v).

THEOREM 4.2.
(1) $R^{\boldsymbol{tT}}$ *is* sound *for the class of dense and commuting* PRA*s.*
(2) $RM^{\boldsymbol{tT}}$ *is* sound *for the class of dense, commuting and transitive* PRA*s.*

[9] Negation and converse commute in RAs, therefore, the ambiguity of $-a^{\cup}$ (that results from omitting parentheses in the absence of notational conventions) is harmless.
[10] This was anticipated by Dunn in an "informal publication" in 1992 (cf. section 7).

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Notice that (2), in effect, states that RM^{tT} is sound for Boolean relation algebras, that is, for Boolean algebras (where meet and join interpret fusion as well as conjunction and disjunction). (This observation immediately suggests that RM^{tT} cannot be complete for the class of dense, commuting and transitive relation algebras, because the intensional connectives of RM^{tT} are not notational variants of the extensional ones, for example, $A \circ B \,.\to A$ is not a theorem of RM^{tT}. We return to this problem in section 5.)

Maddux (2007) posed the question whether the converse of soundness, that is, completeness obtains with respect to certain classes of relevant families of relations. In this section we show that completeness obtains for some suitable fragments of R. The idea is to omit connectives that contain a "negative component," by which we mean negation and implication.

In order to be able to delineate certain fragments of the R^* logics, we start with formalizing the *binary consequence* relation of those fragments of the R^* logics that do not contain $\to$ or $\sim$. The axiomatization that we gave in section 3 includes implication in an essential way. The positive fragment of R has a sequent calculus formulation $L\mathbf{R}^+$ (see Dunn, 1986), that we (slightly) modify to obtain a formalization of the positive fragment without implication. A difference from $L\mathbf{R}^+$ is that no "deduction theorem" connects the binary consequence relation to theoremhood, because for some pairs of formulas that are in the consequence relation there is no equivalent formula that is a theorem.

The set of formulas (denoted by wff) is generated from a denumerable set of propositional variables and $\boldsymbol{t}$ using the binary connectives $\wedge$, $\vee$ and $\circ$. There are two structural connectives , and ; that allow us to combine structures into structures.

DEFINITION 4.3. *The set of structures (*str*) includes* extensional *(*estr*) and* intensional *(*istr*) structures.*[11]

(i) $\mathit{wff} \subseteq \mathrm{istr}^+, \quad \mathit{wff} \subseteq \mathrm{estr}, \quad \varepsilon \in \mathrm{istr}^0,$

(ii) $\alpha, \beta \in \mathrm{istr}^0 \;\Rightarrow\; (\alpha; \beta) \in \mathrm{istr}^0,$

(iii) $\alpha_1, \ldots, \alpha_n \in \mathrm{str}^+ \;\Rightarrow\; (\alpha_1, \ldots, \alpha_n) \in \mathrm{estr},$

(iv) $\alpha_1, \ldots, \alpha_n \in \mathrm{str} \wedge \exists_{1 \le i}^{i \le n}\, i.\, \alpha_i \in \mathrm{str}^+ \,. \;\Rightarrow\; (\alpha_1; \ldots; \alpha_n) \in \mathrm{istr}^+,$

(v) $\mathrm{istr}^+ \cup \mathrm{estr} = \mathrm{str}^+, \quad \mathrm{istr}^0 \cup \mathrm{istr}^+ = \mathrm{istr}, \quad \mathrm{istr} \cup \mathrm{estr} = \mathrm{str}.$

In order to avoid postulating too many structural rules, we stipulate the following notational conventions beyond omitting outmost parentheses.[12]

(vi) $\varepsilon \rightsquigarrow \;, \quad \varepsilon; \alpha \rightsquigarrow \alpha, \quad \alpha; \varepsilon \rightsquigarrow \alpha,$

(vii) *if* $\alpha, \beta \in \mathrm{estr}$ *then* $(\alpha), (\beta) \rightsquigarrow (\alpha, \beta)$, *if* $\alpha, \beta \in \mathrm{istr}$ *then* $(\alpha); (\beta) \rightsquigarrow (\alpha; \beta)$.

A *sequent* (i.e., a consecution) is a structure followed by $\vdash$, and then a formula (i.e., $\alpha \vdash A$). Informally, ε is the empty structure, and it may be thought of as a placeholder for $\boldsymbol{t}$, what explains why ε is an intensional structure. Due to the notational conventions according to which ε disappears, the empty structure cannot appear as one of the components of an extensional structure. An occurrence of a structure α in a structure β is denoted by $\alpha[\beta]$, and within a rule, the same notation is used for the result of the replacement of an occurrence of γ in α by β.

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[11] The following multi-inductive definition yields the same set of structures that are described—in much more understandable terms—in Dunn (1996, §3.10).

[12] The effect of the notational conventions is that both structural connectives behave like associative ones, and the empty structure is typically invisible, but occasionally, looks like a space.

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The axioms and rules of $R^{t\circ}_{\wedge\vee}$ are as follows.[13]

$$A \vdash A \qquad \varepsilon \vdash \boldsymbol{t}$$

$$\frac{\alpha \vdash A}{\alpha; \boldsymbol{t} \vdash A}\ \boldsymbol{t}\vdash \qquad \frac{\alpha[A; B] \vdash C}{\alpha[A \circ B] \vdash C}\ \circ\vdash \qquad \frac{\alpha \vdash A \quad \beta \vdash B}{\alpha; \beta \vdash A \circ B}\ \vdash\circ$$

$$\frac{\alpha[A] \vdash B}{\alpha[A \wedge C] \vdash B}\ \wedge\vdash \qquad \frac{\alpha[A] \vdash B}{\alpha[C \wedge A] \vdash B}\ \wedge\vdash \qquad \frac{\alpha \vdash A \quad \alpha \vdash B}{\alpha \vdash A \wedge B}\ \vdash\wedge$$

$$\frac{\alpha[A] \vdash B \quad \alpha[C] \vdash B}{\alpha[A \vee C] \vdash B}\ \vee\vdash \qquad \frac{\alpha \vdash B}{\alpha \vdash C \vee B}\ \vdash\vee \qquad \frac{\alpha \vdash B}{\alpha \vdash B \vee C}\ \vdash\vee$$

$$\frac{\alpha[\beta; \gamma] \vdash A}{\alpha[\gamma; \beta] \vdash A}\ \mathrm{C}\vdash \qquad \frac{\alpha[\beta; \beta] \vdash A}{\alpha[\beta] \vdash A}\ \mathrm{W}\vdash$$

$$\frac{\alpha[\beta] \vdash A}{\alpha[\beta, \gamma] \vdash A}\ K\vdash \qquad \frac{\alpha[\beta, \gamma] \vdash A}{\alpha[\gamma, \beta] \vdash A}\ C\vdash \qquad \frac{\alpha[\beta, \gamma, \gamma] \vdash A}{\alpha[\beta, \gamma] \vdash A}\ W\vdash$$

The presence of the empty structure in the calculus (together with the absence of thinning for intensional structures) means that we have to consider carefully if the cut rule has to take a more refined (than the usual) form. For the sake of transparency, we make explicit the two concrete forms that the cut rule can take in $L\mathbf{R}^+$, and only for this exemplification we stipulate that ν is a nonempty structure.

$$\frac{\vdash A \quad \alpha[A] \vdash B}{\alpha[\boldsymbol{t}] \vdash B}\ \mathrm{cut}_0 \qquad \frac{\nu \vdash A \quad \alpha[A] \vdash B}{\alpha[\nu] \vdash B}\ \mathrm{cut}_+$$

Cut_+ is the standard *single cut*; cut_0 is stipulated in $L\mathbf{R}^+$ to preclude the introduction of irrelevance between the antecedent and the succedent via "cutting into" an extensional structure with an empty antecedent. We can omit cut_0 here, because if $\alpha[A]$ is of the form $\alpha'[(A, B)]$ then cut_+ (with ε in place of ν) does not yield a lower sequent. (On the other hand, the lower sequent of cut_0 can always be recuperated via an extra $\boldsymbol{t} \vdash$ step.) In sum, the cut rule for $R^{t\circ}_{\wedge\vee}$ is cut_+ without the requirement that ν is nonempty.

The cut_+ rule is admissible in this calculus. The proof of this claim may be obtained by observing that the proof of the elimination theorem for $L\mathbf{R}^+$ (due to Dunn) nowhere requires applications of the $\rightarrow$ rules except in (some of the) cases, where a formula of this form is principal in the step immediately preceding the cut.

As a result, a provable sequent that has more than one formula in its antecedent is *equivalent* to a sequent consisting of exactly two formulas. First, we observe that the structural connective, may be turned into conjunction, whereas the other structural connective; becomes fusion. This proves that a consecution of the required form is implied; the other half of the equivalence follows by the cut theorem.

A *binary consequence relation* is defined by pairs of formulas in provable sequents. If the left-hand side of a provable consecution is empty, then the formula on the right is a *theorem*. (Equivalently, theorems may be defined as formulas implied by $\boldsymbol{t}$ alone.) However, there is no way, in general, to turn a provable consecution with nonempty left-hand side into (an equivalent) one with empty left-hand side.

[13] The notion of provable sequents is defined as usual, therefore, we omit the details.

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The addition of the rule *expansion* to $R^{t\circ}_{\wedge\vee}$ yields $RM^{t\circ}_{\wedge\vee}$.

$$\frac{\alpha[\beta] \vdash A}{\alpha[\beta;\beta] \vdash A}\ \text{exp}$$

The expansion rule could cause a difficulty with respect to the admissibility of the cut.[14] However, it is often forgotten that the cut theorem may be proven by various methods, and the admissibility or the lack of the admissibility of the cut is the property of the whole calculus. Furthermore, the admissibility of the cut rule is not a property that can be tightly bundled with (the presence or absence of) a connective or a structural rule independently of the other components of the calculus.

Therefore, it is remarkable that the cut_+ rule is *admissible* in $RM^{t\circ}_{\wedge\vee}$. It should be remembered that the cut theorem concerns *proofs* not merely derivations. The crucial insight for the proof of the admissibility of cut_+—that we extract from Bimbó (2005)—is that the proof of a cut theorem may (but need not) rely on "moving the cut upward until an axiom is detected," so to speak. Indeed, in certain cases an application of the cut rule can be replaced by a sequence of rule applications that "mimic" the rules in the proof of the left premise of the cut. The proof for $RM^{t\circ}_{\wedge\vee}$ can be constructed by a triple induction on the *degree* of the cut formula, on the *rank* and on the *contraction measure* of the application of the cut_+ together with an induction on the *height* of the left premise of the cut_+. (We do not have space to include the details of the proof here, but we recommend the inspection of the proof described in Bimbó (2007a).) That is, the following is true.

THEOREM 4.4 (ADMISSIBILITY OF CUT). *The* single cut rule is admissible *in the consecution calculi* $R^{t\circ}_{\wedge\vee}$ *and in* $RM^{t\circ}_{\wedge\vee}$.

We note that if we would not have the cut theorem (or if we desired to have yet another formalization of the same logics), then we could proceed to formalize these fragments of the R^* logics along the lines of the formalization of the Kleene logic in Bimbó and Dunn (2005). (Once again, we do not go into the details of this enterprise here.) However, we note that the cut theorem ensures that $R^{t\circ}_{\wedge\vee}$ is indeed a *fragment* of $R^{t\circ}_{+}$, and hence, also of $R^{t\circ}$.

Having suitable formalizations of these fragments of the R^* logics, we proceed to the proofs of completeness. Our proofs essentially rely on the standard set-theoretical semantics for these logics, variants of which have been mentioned in the previous sections (see also Routley & Meyer, 1973; Dunn, 1986).

THEOREM 4.5.
(1) $A \vdash B$ *is a provable sequent of* $R^{t\circ}_{\wedge\vee}$ *(or* $RM^{t\circ}_{\wedge\vee}$*) iff* $v(A) \subseteq v(B)$ *in all models* $\mathfrak{M} = \langle \mathfrak{F}, v\rangle$ *for the logic, where the models are defined as in* 3.2 *(in* 3.4*), but with omitting the superfluous clauses.*
(2) $\vdash A$ *is a provable sequent of* $R^{tT}_{\wedge\vee}$ *(or* $RM^{tT}_{\wedge\vee}$*) iff* $I \subseteq v(A)$ *in all models for the logic.*

The proof of the theorem is obtained by selectively omitting parts of the proofs of Theorems 3.3 and 3.5.

[14] Similar difficulties may be caused by special versions of the contraction rule where the lower consecution retains more than a single occurrence of the formulas (or of the structures) that are principal in the application of the special contraction rule. See Bimbó (2005) for certain related issues as well as for a new kind of proof of the cut theorem.

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In the relational semantics, fusion is modeled as the existential image of a ternary relation R, and each formula is interpreted as an upward closed set of situations. This reiterates the *discrepancy in types* between the interpretations of $R^{\boldsymbol{t}\circ}_{\wedge\vee}$ (and $RM^{\boldsymbol{t}\circ}_{\wedge\vee}$) into the relational semantics and into PRAs (or relevant families of relations). The first assigns sets of some sort of elements to formulas, whereas the second yields sets of ordered pairs of some kind of elements. We remedy this hapless situation by turning a set of situations into a set of pairs of situations.

The idea, informally, is that the ternary relation in the semantics of $R^{\boldsymbol{t}\circ}_{\wedge\vee}$ (and $RM^{\boldsymbol{t}\circ}_{\wedge\vee}$) is connected to fusion. Hence, we use this relation to connect situations representing A with those pairs the elements of which represent B and $A \circ B$, respectively. Since the extensional connectives are interpreted by set operations, it is not surprising that this trick allows us to prove completeness for a fragment that contains those operations. (We note that this move should not be confused with the modeling of the logic of first-degree entailments (**fde**) by pairs of sets of situations as in Dunn (1966, 1982). In those models of **fde**, certain pairs from the Cartesian product of the two sets of situations are linked by the function * that is canonically defined from negation rather than fusion.) Accordingly, we define v' as a derivative of v, that is the valuation in the semantics of the R^* logics (cf. Definitions 3.2 and 3.4). v interprets a formula into a set of situations, whereas v' maps a wff into a set of pairs of situations. (We continue to use $\alpha, \beta, \gamma, \ldots$ to refer to situations, however, we would like to warn the reader that context should be taken into consideration to determine if a structure or a situation is meant.)

$$v'A = \{\, \langle \beta, \gamma \rangle : \exists \alpha \in vA.\, R\alpha\beta\gamma \,\},$$

and we fix $v'(\boldsymbol{t})$ too as

$$v'(\boldsymbol{t}) = \{\, \langle \alpha, \beta \rangle : \alpha = \beta \,\},$$

which is the full identity relation on the set of situations.

THEOREM 4.6. *If A is not a theorem of $R^{\boldsymbol{t}\circ}_{\wedge\vee}$ then there is a commuting and dense* relevant family of relations *with a valuation v such that* $\mathrm{Id} \not\subseteq v(A)$.

Proof. 1. Given the condition $R^2(\alpha\beta)\gamma\,\delta \Leftrightarrow R^2\beta(\alpha\gamma)\delta$ in the semantics of $R^{\boldsymbol{t}\circ}_{\wedge\vee}$ (the two directions of which correspond to the combinator B' and the dual combinator $\mathsf{b(cb)}$), it follows that $v'(A \circ B) = v'A\,;v'B$, that is, fusion is interpreted as relational composition. It is easy to see that $v'(A \wedge B) = v'(A) \cap v'(B)$ and $v'(A \vee B) = v'(A) \cup v'(B)$. Due to the properties of propositions in the canonical model, $v'(\boldsymbol{t})\,;v'(A) = v'(A)$ holds too.
2. To prove that relational composition is commutative and square increasing, we express these properties in terms of the accessibility relation R and prove that $\mathfrak{F}_{\mathfrak{c}}(R^{\boldsymbol{t}\circ}_{\wedge\vee})$ satisfies those formulas. (R^2 indicates the relational composition of R with itself in its first or second argument place as indicated by grouping.)

(1) $R^2\alpha(\beta\gamma)\delta \Rightarrow R^2\beta(\alpha\gamma)\delta$ (commuting),
(2) $R\alpha\beta\gamma \Rightarrow R^2\alpha(\alpha\beta)\gamma$ (density),

The following sequences of implications are justifiable by known properties of the canonical accessibility relation on the frame that consists of proper nonempty prime filters.
re (1) If $R^2\alpha(\beta\gamma)\delta$ then $R^2(\alpha\beta)\gamma\,\delta$ and further $R^2(\beta\alpha)\gamma\,\delta$. The other associativity property yields $R^2\beta(\alpha\gamma)\beta$.
re (2) If $R\alpha\beta\gamma$ then $R^2(\alpha\alpha)\beta\gamma$ is immediate because $R\alpha\alpha\alpha$.
3. The proof that v' is injective depends on the known fact that v is injective. If $A \nvdash B$ then $\exists\delta \in vA$ such that $\delta \notin vB$. If $[A \circ C] \leq [B \circ C]$ for all $[C]$ then $A \vdash B$. The complement

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of an ideal (or co-theory) maximal with respect to not containing $[A \circ C]$ (but containing $[B \circ C]$) is prime, let us say γ. By squeezing on $[[A]]$ and $[[C]]$ (the cones generated by $[A]$ and $[C]$), one gets α and β that are prime filters, and it can be shown that $[B] \notin \alpha$. □

THEOREM 4.7. *If $A \vdash B$ is a consecution not provable in $R^{\boldsymbol{t}\circ}_{\wedge\vee}$ then there is a commuting and dense (and transitive)* relevant family of relations *with a valuation v such that $v(A) \not\subseteq v(B)$.*

The proof of this theorem is identical to the proof of the previous theorem. (One might wish to scrutinize step 3.)

Using insights from Bimbó (2009), we add $\boldsymbol{F}$ to $R^{\boldsymbol{t}\circ}_{\wedge\vee}$ and $RM^{\boldsymbol{t}\circ}_{\wedge\vee}$ by the following axiom. (The addition of $\boldsymbol{F}$ to R or RM is not interesting for our purposes here, because in the presence of negation $\boldsymbol{T}$ becomes definable leading to incompleteness (cf. Theorem 5.8). However, we show that the addition of $\boldsymbol{F}$ to the "purely positive fragment" of R is harmless not only in the sequent calculus but also from the point of view of completeness.)

$$\boldsymbol{F}; \alpha \vdash A$$

First of all we can prove the following theorem that comprises two claims and implies that the addition of $\boldsymbol{F}$ is conservative. (The details of the proofs of the cut theorems are omitted.)

THEOREM 4.8 (ADMISSIBILITY OF CUT). *The* single cut rule is admissible *in the sequent calculi $R^{\boldsymbol{t}\circ\boldsymbol{F}}_{\wedge\vee}$ and $RM^{\boldsymbol{t}\circ\boldsymbol{F}}_{\wedge\vee}$.*

The analogues of Theorems 4.6 and 4.7 are (1) and (2) in the next theorem.

THEOREM 4.9.
(1) *If A is a sequent not provable in $R^{\boldsymbol{t}\circ\boldsymbol{F}}_{\wedge\vee}$ then there is a commuting and dense* relevant family of relations *with a valuation v such that* $\mathrm{Id} \not\subseteq v(A)$.
(2) *If $A \vdash B$ is a sequent not provable in $R^{\boldsymbol{t}\circ\boldsymbol{F}}_{\wedge\vee}$ then there is a commuting and dense* relevant family of relations *with a valuation v such that $v(A) \not\subseteq v(B)$.*

The proofs of (1) and (2) are similar to the proofs of Theorems 4.6 and 4.7, when we note that there is nothing special about $\boldsymbol{F}$ with respect to v'.

The role of the $\boldsymbol{t}$ in $R^{\boldsymbol{t}\circ}_{\wedge\vee}$ and $RM^{\boldsymbol{t}\circ}_{\wedge\vee}$ is diminished when compared to that in $L\mathbf{R}^{+}$. Roughly speaking, $R^{\boldsymbol{t}\circ}_{\wedge\vee}$ and $RM^{\boldsymbol{t}\circ}_{\wedge\vee}$ do not have rules that would move a formula to the other side of the turnstile (either as a formula or as a subformula of a formula). This means that we may consider $R^{\circ}_{\wedge\vee}$ and $RM^{\circ}_{\wedge\vee}$, that is, the $\boldsymbol{t}$-less fragments of $R^{\boldsymbol{t}\circ}_{\wedge\vee}$ and $RM^{\boldsymbol{t}\circ}_{\wedge\vee}$. The *admissibility of the cut rule* follows from the previous cut theorem (more precisely, a careful inspection of the preceding proof yields a proof).[15] Obviously, the set of theorems of this logic is empty, but its consequence relation is not. It also remains true that every provable sequent can be equivalently viewed as a pair of formulas.

The ternary accessibility relation in the presence of an order relation on situations can be recast as $\alpha \circ \beta \leq \gamma$ provided $R\alpha\beta\gamma$ (where $\circ$ is lifted to situations in the obvious way). This suggests that to obtain a commuting dense set of relations, we could take a more direct route by picking the first two arguments of R instead of the second and third. For example, $\langle \alpha, \beta \rangle \in \varrho \Leftrightarrow \langle \beta, \alpha \rangle \in \varrho$ would follow immediately from $R\alpha\beta\gamma \Leftrightarrow R\beta\alpha\gamma$. We leave to the reader to verify that density obtains as well.

[15] Some definitions in the sequent calculus could be simplified, for example, the empty structure could be pruned altogether from the set of structures with all the complications that it necessitated.

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THEOREM 4.10. *If* $A \nvdash B$ *in* $R^{\circ}_{\wedge\vee}$ *then* $v(A) \not\subseteq v(B)$ *in a commuting dense* relevant family of relations.

Proof. As we hinted above, we will want to define a valuation v' that is derived from v via R. This time we assume that v is the canonical valuation in the usual semantics for relevance logics with a slight amendment. Bimbó and Dunn (2008, §2.2) have shown that it is possible to give relational semantics for positive relevance logics with the whole carrier set of the Lindenbaum algebra of the logic included in the set of prime filters. With this addition we define v' as follows.

$$v'A = \{\, \langle \alpha, \beta \rangle : \exists \gamma \,.\, R\alpha\beta\gamma \wedge \gamma \in vA \,\}$$

Now we prove that v' preserves the operations.
1. $v'(A \vee B) = v'(A) \cup v'(B)$ is nearly obvious. $\langle \alpha, \beta \rangle \in v'(A \vee B)$ iff $\exists \gamma\, (R\alpha\beta\gamma \wedge \gamma \in v(A \vee B))$ iff $\exists \gamma\, (R\alpha\beta\gamma \wedge (\gamma \in vA \vee \gamma \in vB))$ iff $\exists \gamma\, (R\alpha\beta\gamma \wedge \gamma \in vA) \vee \exists \gamma\, (R\alpha\beta\gamma \wedge \gamma \in vB)$ iff $\langle \alpha, \beta \rangle \in v'A \cup v'B$.
2. In the case of $\wedge$ we cannot appeal to 'iff's. For one of the inclusions, let us assume that $\langle \alpha, \beta \rangle \in v'(A \wedge B)$, that is, $\gamma \in v(A \wedge B)$ with $R\alpha\beta\gamma$ for some γ. v "maps" $\wedge$ into $\cap$, and $\exists$ distributes (in one direction) over $\wedge$. That is, $\exists \gamma\, (R\alpha\beta\gamma \wedge \gamma \in vA)$ and $\exists \gamma\, (R\alpha\beta\gamma \wedge \gamma \in vB)$, hence $\langle \alpha, \beta \rangle$ is an element of both $v'A$ and $v'B$, in short, $\langle \alpha, \beta \rangle \in v'A \cap v'B$.

For the converse we start with the last formula as assumption. Of course, we cannot retrace the steps, because two existentially quantified conjuncts cannot have their variable unified, as a rule. However, from $R\alpha\beta\gamma_1$ and $R\alpha\beta\gamma_2$ we may infer that $\gamma_1, \gamma_2 \sqsubseteq \gamma$ for some γ because—if nothing else—the carrier set is always available. Then by the tonicity of $R__\uparrow$, $R\alpha\beta\gamma$. On the other hand, the closure property of propositions ensures that $\gamma \in vA$ and $\gamma \in vB$—provided so are γ_1 and γ_2. In sum, $\exists \gamma\, (R\alpha\beta\gamma \wedge \gamma \in v(A \wedge B))$, that is, $\langle \alpha, \beta \rangle \in v'(A \wedge B)$.
3. The case of fusion falls into two parts too. Let us assume that $\langle \alpha, \beta \rangle \in v'(A \circ B)$. By the definition of v' and v we get $R\alpha\beta\gamma$, $R\delta\eta\gamma$, $\eta \in vA$ and $\delta \in vB$, for some γ, δ and η. From the first two, it follows that $R\alpha\vartheta\zeta$ and $R\vartheta\beta\varepsilon$ for some $\vartheta, \zeta, \varepsilon$ such that $\delta \sqsubseteq \varepsilon$ and $\eta \sqsubseteq \zeta$. Formulas are interpreted by v as cones, which means that $\zeta \in vA$ and $\varepsilon \in vB$. Then, $\langle \alpha, \vartheta \rangle \in v'A$, whereas $\langle \vartheta, \beta \rangle \in v'B$, that is, $\langle \alpha, \beta \rangle \in v'A \,;\, v'B$.

Having assumed the last membership and having retraced a few steps, we have $\zeta \in vA$ and $\varepsilon \in vB$, where $R\alpha\vartheta\zeta \wedge R\vartheta\beta\varepsilon$. From this we can infer that $\exists \gamma\, (R\alpha\beta\gamma \wedge R\delta\eta\gamma)$. Then both $\gamma \in v(A \circ B)$ and $\langle \alpha, \beta \rangle \in v'(A \circ B)$. □

§5. The problem of completeness of other R^* logics. As we foreshadowed earlier, the completeness results do not carry over to certain other fragments, especially, to those that include negation. In this section we prove and scrutinize a series of incompleteness results for the R^* logics with respect to classes of PRAs. Although not all the theorems are completely independent, we provide a whole range of conceptually differing proofs. The proofs show the various points at which the R^* logics are incomplete with respect to PRAs, thereby enhancing our understanding of the difference between the R^* logics and PRAs. Depending on the concrete proof, occasionally, we are able to use the same proof to demonstrate incompleteness with respect to relevant families of relations, as well.

THEOREM 5.1.
(1) $\mathbf{R^t}$ *is* incomplete *for the class of dense and commuting* PRAs.
(2) $\mathbf{RM^t}$ *is* incomplete *for the class of dense, commuting and transitive* PRAs.
(3) *Both* (1) *and* (2) *hold with respect to* relevant families of relations.

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Proof. Obviously, if a is a symmetric relation (i.e., $a^{\cup} = a$), then $\sim a = -a$. $\mathbf{1'}$ is always a symmetric relation, therefore, $\mathbf{1'} \cap \sim\mathbf{1'} = \emptyset$ in all PRAs. The interpretation of the connectives and constants is "fixed" by Definition 4.1; accordingly, this equation implies that $\boldsymbol{F}$ is definable (i.e., $\boldsymbol{t} \wedge \sim\boldsymbol{t} \,.\, \rightarrow A$ is a theorem).

1. To prove (2), we note that $\boldsymbol{F}$ is not definable in $RM^{\boldsymbol{t}}$ (let alone in $R^{\boldsymbol{t}}$). We appeal to the infinite Sugihara matrix $S_{\mathbb{Z}}$ (see Figure 2) that is characteristic for $RM^{\boldsymbol{t}}$, but has no least element.

A different proof results from the observation that a transitive relation algebra is Boolean, however, $\boldsymbol{T}$ is not definable in $RM^{\boldsymbol{t}}$. (Furthermore, the two constants are distinct in $RM^{\boldsymbol{tT}}$.)

2. $\nvdash_{R^{\boldsymbol{t}}} \boldsymbol{t} \wedge \sim\boldsymbol{t} \,.\, \rightarrow A$ can be proven independently from $RM^{\boldsymbol{t}}$ and its relationship to $R^{\boldsymbol{t}}$ by noting that $\boldsymbol{t} \wedge \sim\boldsymbol{t} = \boldsymbol{t}$ in the lattice CRYSTAL (see Figure 2 above). Thus if A is evaluated into the bottom element of the lattice then $v(A) \neq v(\boldsymbol{t} \wedge \sim\boldsymbol{t})$ but $v(A) \leq v(\boldsymbol{t} \wedge \sim\boldsymbol{t})$. This suffices for the truth of (1).

Alternatively, we may scrutinize the issue in the normalized relational semantics that is complete for the R^* logics. The existence of a nonempty set of possible worlds W (with $W \subseteq I$ true) implies that $\forall \imath \in I \,.\, \imath^* \notin I$ is false. That is, $v(\boldsymbol{t} \wedge \sim\boldsymbol{t}) \neq \emptyset$, though $v(\boldsymbol{F}) = \emptyset$ in the normalized relational semantics for $R^{\boldsymbol{tT}}$.

3. Relevant families of relations are closed under $\sim$ and $\cap$, and therefore, unchangeably $\mathbf{1'} \cap \sim\mathbf{1'} = \emptyset$. Thus the reasoning in 1. and 2. may be repeated for them. □

To further clarify what is happening with the monoid identity we recall a small finite PRA from Maddux (2007). If viewed as an abstract algebra (rather that a concrete algebra comprising binary relations), then 1_3 (more precisely, the relevant reduct of 1_3) is isomorphic to M_0 that is well-known as a model for R—see Anderson and Belnap (1975, §18). 1_3 is pictured in Figure 3 (below).

ILLUSTRATION. 1_3 is a dense and commuting PRA that has three atoms, which are L (strictly less than), G (strictly greater than) and $\mathbf{1'}$ (the identity relation) on $\mathbb{Q}$.

$\sim L = L \cup \mathbf{1'}$ and $\sim G = G \cup \mathbf{1'}$, however, $\sim\mathbf{1'} = L \cup G$ hence $\mathbf{1'} \cap \sim\mathbf{1'} = \mathbf{1'} \cap (L \cup G) = \emptyset$. 1_3 is a PRA in which negation is not Boolean complement except for $\mathbf{1'}$. (Incidentally, $\sim L$ and $\sim G$ nicely show the "shift" that is due to the De Morgan negation, because these two co-atoms are switched compared to $-L$ and $-G$.) Of course, the example does not prove what we proved above, but demonstrates that De Morgan negation need not be Boolean complement on all elements in order to be orthonegation on $\mathbf{1'}$.

An obvious question at this point is if we can formalize the fragment of R that contains only $\wedge$, $\sim$ and $\boldsymbol{t}$. (Of course, $\vee$ is definable in this fragment.) The answer is 'yes', and the

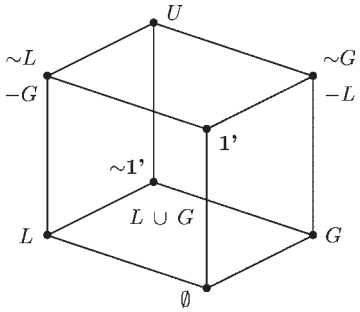


Fig. 3. The relation algebra 1_3.

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formalization is not too difficult to come by, given the tableaux for first-degree entailment introduced in Bimbó (2007b, §2). (The first tableaux for **fde** was given in Dunn (1976a)—see also Dunn (1980).) Here we modify the tableaux for first-degree entailment, that were introduced later, by adding $\boldsymbol{t}$, that is, we define a *tableaux for first- and zero-degree entailment* that we denote as **zfde**. As far as we know, **zfde** has not been isolated as a fragment of R and other relevance logics. It seems to us that the formalization here—in the form of tableaux—is especially neat, and perhaps more straightforward that adding $\boldsymbol{t}$ to $L\mathbf{E}_{\mathbf{fde}1}$ (see Anderson & Belnap, 1975, §17) would be.

In the presence of $\boldsymbol{t}$, every theorem of R may be viewed equivalently as an entailment, including zero-degree wffs, that become first-degree wffs when prefixed by $\boldsymbol{t} \rightarrow$. The tableaux extended with $\boldsymbol{t}$ have to prove all the zero-degree theorems in their first-degree form.

Now we define zero-degree formulas to comprise the set of propositional variables and $\boldsymbol{t}$, and to be closed under the operations $\wedge$, $\vee$ and $\sim$. If A and B are zero-degree formulas, then $A \rightarrow B$ is a first-degree formula (henceforth, a formula). The signs are T_0, T_1, F_0 and F_1, and *signed formulas* are defined as usual (see Smullyan, 1968). Tableaux are defined as for **fde** (but with the larger set of wffs), in particular, the following 12 connective rules are stipulated (with $i \in \{0, 1\}$).

$$\frac{T_i(A \wedge B)}{T_i A,\ T_i B}\ T_i\wedge \qquad \frac{T_i(A \vee B)}{T_i A \mid T_i B}\ T_i\vee \qquad \frac{T_i{\sim}A}{F_i A}\ T_i\sim$$

$$\frac{F_i(A \wedge B)}{F_i A \mid F_i B}\ F_i\wedge \qquad \frac{F_i(A \vee B)}{F_i A,\ F_i B}\ F_i\vee \qquad \frac{F_i{\sim}A}{T_i A}\ F_i\sim$$

A *tableau* for $A \rightarrow B$ has T_0A, F_1B as its root, that is, entailments between zero-degree formulas are treated as before. A zero-degree formula A will be called a theorem when $\boldsymbol{t} \rightarrow A$ has a *tableau proof*. Although the rules—formulated with metavariables, thereby, concealing occurrences of $\boldsymbol{t}$—remain the same, *conjugates* have to be defined anew in order to give $\boldsymbol{t}$ the status of truth rather that that of a propositional variable. As before, T_iA, F_jA are conjugates whenever $i \neq j$; and T_jA, F_jA are conjugates whenever $T_i\boldsymbol{t}$ occurs on the same branch with $i \neq j$.[16]

Beyond the *entailment relation* of **fde** (with $\boldsymbol{t}$ added into the signature), **zfde** has a (nonempty) set of *theorems*. Since $\boldsymbol{t} \rightarrow .\ A \vee {\sim}A$ is easily provable, **zfde** inherits $\aleph_0$ many copies of all theorems of classical logic—in the form of zero-degree theorems—from the R^* logics under the translation that maps $\vee$ into $\vee$ and $-$ into $\sim$.

For the next theorem we suppose that the interpretation of the operations ($\wedge$, $\vee$ and $\sim$) is by the applicable clauses from Definition 4.1. Entailment is interpreted as inclusion, that is, $A \rightarrow B$ if $v(A) \subseteq v(B)$.

THEOREM 5.2. ***zfde** is* incomplete *for the class of commuting dense* PRA*s, indeed, **zfde** is* incomplete *for the class of* PRA*s.*

Proof. The proof of this theorem is the same as that of the previous one—we sharpened Theorem 5.1 to maximally exploit the yield of the preceding proof.

We note that the claim and the proof may be adapted to relevant families of relations just as in the previous theorem. □

[16] Alternatively, T_jA, F_jA, $T_i\boldsymbol{t}$ could be thought of as a "conjugate triple" when $i \neq j$.

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Now we turn to considering $R^{\boldsymbol{T}}_{+}$. Our idea is that we can find an (in)equation that holds in all representable RAs (i.e., in all RRAs), and such that the formula that is interpreted into the (in)equation is not provable in $R^{\boldsymbol{T}}_{+}$. Using *serial commuting labeled graphs* (SCLGs) it is possible to generate (in)equations that are true in all RRAs with the terms containing $\cdot$, ; and $^{\cup}$ only. To start with, we take the following inequation from Maddux (1996).[17]

$$a \cdot (((b\,;c)\cdot d)\,;(e\cdot(f\,;g))) \leq b\,;((((b^{\cup}\,;a)\cdot(c\,;e))\,;g^{\cup})\cdot(c\,;f)\cdot(b^{\cup}\,;((a\,;g^{\cup})\cdot(d\,;f))))\,;g$$

Of course, we run into a difficulty if we want to come up with a formula that would be interpreted into this inequation—unless we have constants. $^{\cup}$ is not in the language of the R^{*} logics, and there is no obvious way to introduce $-$ in the appropriate places to force $^{\cup}$ to dissolve into $\sim$. However, we noted in section 2 that $^{\cup}$ has fixed points on 0, **1'**, **0'** and 1. Thus, substituting 1 for b and g, the term on the right simplifies into a term without $^{\cup}$. Furthermore, if we substitute $1\,;c$ for d and $f\,;1$ for e, then by idempotence we obtain

$$a \cdot (1\,;c\,;f\,;1) \leq 1\,;((((1\,;a)\cdot(c\,;f\,;1))\,;1)\cdot(c\,;f)\cdot(1\,;((a\,;1)\cdot(1\,;c\,;f))))\,;1.$$

The inequation now can be viewed as an interpretation of the formula

$$A \wedge (\boldsymbol{T}\circ C\circ F\circ \boldsymbol{T})\,.\,\rightarrow\,.$$
$$\boldsymbol{T}\circ((((\boldsymbol{T}\circ A)\wedge(C\circ F\circ \boldsymbol{T}))\circ \boldsymbol{T})\wedge(C\circ F)\wedge(\boldsymbol{T}\circ((A\circ \boldsymbol{T})\wedge(\boldsymbol{T}\circ C\circ F))))\circ \boldsymbol{T}.$$

Fusion is commutative in the R^{*} logics, which allows some further simplifications that yield the next formula.

$$A \wedge (\boldsymbol{T}\circ \boldsymbol{T}\circ C\circ F)\,.\,\rightarrow\,.\boldsymbol{T}\circ \boldsymbol{T}\circ(\boldsymbol{T}\circ((\boldsymbol{T}\circ A)\wedge(\boldsymbol{T}\circ C\circ F))\wedge(C\circ F))$$

Lastly, $\boldsymbol{T}\circ\boldsymbol{T}\circ B$ may be replaced (equivalently) by $\boldsymbol{T}\circ B$, because $\boldsymbol{T}\circ\boldsymbol{T}$ is provably equivalent to $\boldsymbol{T}$, which gives (18).

(18) $$A \wedge (\boldsymbol{T}\circ C\circ F)\,.\,\rightarrow\,.\boldsymbol{T}\circ(\boldsymbol{T}\circ((\boldsymbol{T}\circ A)\wedge(\boldsymbol{T}\circ C\circ F))\wedge(C\circ F))$$

$R^{\boldsymbol{T}}_{+}$ is known to be undecidable, but this does not prevent us from performing a "backward proof search" in the sequent calculus for $R^{\boldsymbol{T}}_{+}$. The unprovability of the formula (18) hinges upon the unprovability of $\boldsymbol{T}\circ P\,.\,\rightarrow\,P$ (for a propositional variable P or for an arbitrary formula P). This suggests that we consider the serial commuting labeled graph depicted on the left-hand side in Figure 4 (below).
The corresponding inequation is

$$a\cdot(b\,;c)\leq b\,;((b^{\cup}\,;a)\cdot c).$$

After eliminating the occurrence of $^{\cup}$ as before, we obtain the next inequation (re) and the corresponding formula (re′).

(re) $a\cdot(1\,;b)\leq 1\,;((1\,;a)\cdot b)$

(re′) $A\wedge(\boldsymbol{T}\circ B)\,.\,\rightarrow\,.\boldsymbol{T}\circ((\boldsymbol{T}\circ A)\wedge B)$

THEOREM 5.3. $R^{\boldsymbol{T}}_{+}$ *is* incomplete *with respect to the class of dense and commuting* PRAs.

[17] This is one of the three (nonequivalent) inequations, namely, (M) (modulo a commutation in a conjunction), that emerge from Lyndon's representability condition for $n = 5$—see Maddux (1996, (18), 2006, p. 30).

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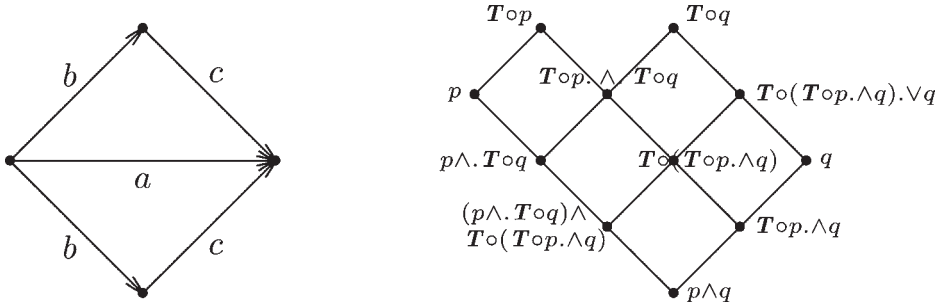


Fig. 4. An SCLG and a (partial) Hasse diagram.

Proof. 1. The inequation (re) is true in all RRAs, however, (re′) is not a theorem of $R^{\boldsymbol{T}}_{+}$. We leave to the reader the recreation of our verification (that is slightly tedious) that the sequent $A \wedge (\boldsymbol{T} \circ B) \vdash \boldsymbol{T} \circ ((\boldsymbol{T} \circ A) \wedge B)$ (or equivalently, the sequent $\vdash A \wedge (\boldsymbol{T} \circ B) \,.\, \rightarrow .\, \boldsymbol{T} \circ ((\boldsymbol{T} \circ A) \wedge B)$) is not provable in $L\mathbf{R}^{+}$ with $\boldsymbol{T}$ added. A formulation of $L\mathbf{R}^{+}$ may be found in Dunn (1986, §3.10), and $\boldsymbol{T}$ may be added as for $\mathbf{E}_{+}$ (see Bimbó, 2009).
2. As a somewhat different proof we show that if A and B are chosen to be the propositional variables p and q, respectively, then (the equivalence classes of) $p \wedge (\boldsymbol{T} \circ q)$ and $\boldsymbol{T} \circ (\boldsymbol{T} \circ p \,.\wedge q)$ are incomparable in the Lindenbaum algebra of $R^{\boldsymbol{T}}_{+}$. This implies that (re′) is not a valid formula on the canonical frame of $R^{\boldsymbol{T}}_{+}$. Obviously, p and q (that we stipulate to be distinct) are incomparable. By tonicity properties of $\wedge$ and $\circ$ the series of inequations $p \wedge q \leq p \wedge .\boldsymbol{T} \circ q \leq p \leq \boldsymbol{T} \circ p$ and $p \wedge q \leq \boldsymbol{T} \circ p. \wedge q \leq q \leq \boldsymbol{T} \circ q$ are obvious. (This establishes the relationships between nearly all the outer points in the lattice that is partially diagrammed on the right-hand side in Figure 4 above.)

$\boldsymbol{T} \circ (\boldsymbol{T} \circ p. \wedge q) \vdash \boldsymbol{T} \circ p$ because $\boldsymbol{T}; ((\boldsymbol{T}; p), q) \vdash \boldsymbol{T} \circ p$ may be obtained from the axioms $\boldsymbol{T}; \boldsymbol{T} \vdash \boldsymbol{T}$ and $p \vdash p$. Also, $\boldsymbol{T} \circ (\boldsymbol{T} \circ p \,.\wedge q) \vdash \boldsymbol{T} \circ q$, because $\boldsymbol{T} \circ p, q \vdash q$ is provable. Then the following three inequations are also true. $\boldsymbol{T} \circ p. \wedge q \leq \boldsymbol{T} \circ (\boldsymbol{T} \circ p. \wedge q) \leq \boldsymbol{T} \circ p. \wedge .\boldsymbol{T} \circ q \leq \boldsymbol{T} \circ p, \boldsymbol{T} \circ q$.

Further, none of the following is provable: $\boldsymbol{T} \circ p \vdash \boldsymbol{T} \circ (\boldsymbol{T} \circ p. \wedge q)$, $\boldsymbol{T} \circ q \vdash \boldsymbol{T} \circ (\boldsymbol{T} \circ p. \wedge q)$, $\boldsymbol{T} \circ p. \wedge .\boldsymbol{T} \circ q \vdash \boldsymbol{T} \circ (\boldsymbol{T} \circ p. \wedge q)$, $\boldsymbol{T} \circ (\boldsymbol{T} \circ p. \wedge q) \vdash q$, $q \vdash \boldsymbol{T} \circ (\boldsymbol{T} \circ p. \wedge q)$ and $\boldsymbol{T} \circ (\boldsymbol{T} \circ p. \wedge q) \vdash p \wedge .\boldsymbol{T} \circ q$. (This allows us to complete the labeling of the right-hand diagram in figure 4, that also implies the unprovability of (18).) Therefore, the incompleteness for $R^{\boldsymbol{T}}_{+}$ follows. □

To provide additional insights into the workings of this proof, we note that if we use one of the two constants **1'** or 0 in place of 1, then we do not obtain a proof along these lines. Moreover, a substitution of 0 for only one of b or g already destroys the counterexample.

LEMMA 5.4. *The substitution of* $\boldsymbol{F}$ *or* $\boldsymbol{t}$ *instead of* $\boldsymbol{T}$ *yields formulas that are provable in* $R^{\boldsymbol{F}}_{+}$ *(or equivalently, in* $R^{\boldsymbol{T}}_{+}$*) and in* $R^{\boldsymbol{t}}_{+}$*, respectively.*

Proof. 1. The only elements that occur in the scope of $^{\cup}$ are b and g. (Of course, they also occur "without converse," so to speak; otherwise, $^{\cup}$ could be simply dropped from the terms.) If we substitute 0 for b and g, then both sides of the inequality are easily seen to equal to 0, because ; is a normal operation. Furthermore, either of these two substitutions turns the left-hand side of the equation into 0. This means that independently of what the right-hand side of the inequation is, the corresponding sequent will be provable in $R^{\boldsymbol{F}}_{+}$, since 0 is the bottom element of the lattice in the logic's Lindenbaum algebra.

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2. We might try to substitute **1'** for b and g. Then we get the inequality

$$a \cdot (\mathbf{1'} ; c \cdot d) ; (e \cdot f ; \mathbf{1'}) \leq \mathbf{1'} ; ((\mathbf{1'}^{\cup} ; a \cdot c ; e) ; \mathbf{1'}^{\cup} \cdot c ; f \cdot \mathbf{1'}^{\cup}(a ; \mathbf{1'}^{\cup} \cdot d ; f)) ; \mathbf{1'}.$$

1' is left–right identity for ;, therefore, the term further simplifies to

$$a \cdot ((c \cdot d) ; (e \cdot f)) \leq (a \cdot c ; e) \cdot c ; f \cdot (a \cdot d ; f).$$

Then the following wff should be provable in the logic.

$$A \wedge ((B \wedge C) \circ (D \wedge E)) \,.\, \rightarrow \,.\, A \wedge (B \circ D) \wedge (B \circ E) \wedge (C \circ E).$$

Viewed as a sequent, the above entailment is easily provable in $L\mathbf{R}^{+}$, indeed, already in the sequent calculus $R^{t\circ}_{\wedge\vee}$, that we introduced in section 4.

The following semantic argument also shows that every situation that verifies the formula on the left of the $\rightarrow$ verifies the wff on its right. Let us assume $\alpha \vDash A$, $\alpha \vDash (B \wedge C) \circ (D \wedge E)$ and $\alpha \nvDash A \wedge (B \circ D) \wedge (B \circ E) \wedge (C \circ E)$. The definition of satisfaction of a conjunctive formula implies that $\alpha \nvDash B \circ D$ or $\alpha \nvDash B \circ E$ or $\alpha \nvDash C \circ E$—by the first and the last assumptions. But from the second one, we have $\exists \beta, \gamma \,.\, R\beta\gamma\alpha \wedge \beta \vDash B \wedge \beta \vDash C \wedge \gamma \vDash D \wedge \gamma \vDash E$, by the definitions of $\circ$ and $\wedge$. The last conjunction can be broken into smaller conjunctions by conjunction elimination (after the quantifiers have been instantiated, perhaps, with the same variables). Namely, we may obtain (i) $R\beta\gamma\alpha \wedge \beta \vDash B \wedge \gamma \vDash D$, (ii) $R\beta\gamma\alpha \wedge \beta \vDash B \wedge \gamma \vDash E$ and (iii) $R\beta\gamma\alpha \wedge \beta \vDash C \wedge \gamma \vDash E$. Each case leads to a contradiction that completes the proof. □

LEMMA 5.5. $\vDash_{R^{T},\, RM^{T}} A \wedge (\boldsymbol{T} \circ B) \,.\, \rightarrow \,.\, \boldsymbol{T} \circ (\boldsymbol{T} \circ A \,.\, \wedge B)$.

Proof. 1. To prove the claim for $R^{\boldsymbol{T}}$, let $\gamma \in v(A)$, $\gamma \in v(\boldsymbol{T} \circ B)$ without $\gamma \in v(\boldsymbol{T} \circ ((\boldsymbol{T} \circ A) \wedge B))$. Then $R\alpha\beta\gamma$ and $\beta \in v(B)$ for some α, β. From the third assumption, we get that $\beta \notin v((\boldsymbol{T} \circ A) \wedge B)$, hence $\beta \notin v(\boldsymbol{T} \circ A)$. However, from $R\alpha\beta\gamma$ we may obtain that $R\delta\gamma\beta$ (for some δ) as follows. If $R\alpha\beta\gamma$, then $R(\alpha\beta)\gamma^{*}\gamma^{*}$, hence $R(\alpha\gamma^{*})\beta\gamma^{*}$. By the condition that corresponds to antilogism, $R(\alpha\gamma^{*})\gamma\beta^{*}$. Then also $R((\alpha\gamma^{*})\beta)\gamma\beta$, that is, $R\delta\gamma\beta$ for some δ. Since $\gamma \in v(A)$, $R\delta\gamma\beta$ and $\beta \notin v(\boldsymbol{T} \circ A)$ lead to a contradiction.
2. Let $\alpha \in v(A)$, $\alpha \in v(\boldsymbol{T} \circ B)$ but $\alpha \notin v(\boldsymbol{T} \circ (\boldsymbol{T} \circ A \,.\, \wedge B))$. Then $R\beta\gamma\alpha$ for some β and γ such that also $\gamma \in v(B)$. By (f10), either $\beta \sqsubseteq \alpha$ or $\gamma \sqsubseteq \alpha$. If the second, then by $R\alpha\alpha\alpha$ we obtain $\alpha \notin v(\boldsymbol{T} \circ A \,.\, \wedge B)$ from the last assumption. But $\gamma \in v(B)$ together with $\gamma \sqsubseteq \alpha$ implies that $\alpha \in v(B)$, therefore, $\alpha \notin v(\boldsymbol{T} \circ A)$. From the latter we easily get a contradiction.

If $\gamma \sqsubseteq \alpha$ does not hold, then $\beta \sqsubseteq \alpha$ should hold, and even more importantly, so should $\alpha \sqsubseteq \gamma$. (The connectedness of $\sqsubseteq$ corresponds to the mingle axiom.) Then now using $R\beta\gamma\alpha$ and $\beta \in v(\boldsymbol{T})$, we have that $\gamma \notin v((\boldsymbol{T} \circ A) \wedge B)$, but then $\gamma \notin v(\boldsymbol{T} \circ A)$ and further $\gamma \notin v(A)$. Therefore, by $\alpha \sqsubseteq \gamma$, $\alpha \notin v(A)$—a contradiction. □

The formula in question was derived from an inequation that in a PRA involves *reflection*. Thus it is interesting to note that the condition $R\alpha\beta\gamma \Rightarrow R\alpha^{*}\gamma\beta$ that corresponds in the relational semantics of RAs to reflection is false on the canonical structure of $R^{\boldsymbol{T}}$ as well as $RM^{\boldsymbol{T}}$. This condition (too) is called reflection in Maddux (2007), and a condition that in R is equivalent to reflection, $R\alpha\beta\gamma \Rightarrow R\alpha^{*}\beta^{*}\gamma^{*}$, is called *tagging* in Dunn (2001). (Reflection may be pictured for binary relations as the SCLG in Figure 4 above.)

LEMMA 5.6 (REFLECTION ON CANONICAL STRUCTURES).

(1) $\mathfrak{F}_{\mathfrak{c}}(R^{\boldsymbol{T}}) \nvDash R\alpha\beta\gamma \Rightarrow R\alpha^{*}\gamma\beta$.
(2) $\mathfrak{F}_{\mathfrak{c}}(RM^{\boldsymbol{T}}) \nvDash R\alpha\beta\gamma \Rightarrow R\alpha^{*}\gamma\beta$.

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(3) *The same holds for* $\mathfrak{F}_c(R)$ *and for* $\mathfrak{F}_c(RM)$.

Proof. 1. We choose two propositional variables, let us say, p and q. $[p)$, obviously, does not contain $\sim q$ (because $p \to \sim q$ is not a theorem of R due to the lack of variable sharing). Maximizing $[p)$ to a theory that excludes $\sim q$, we obtain a prime theory α such that $p \in \alpha$ but $\sim q \notin \alpha$. (By the usual definition of *, $q \in \alpha^*$ though.) Next we choose two other propositional variables r and s. $q \circ s \notin [r)$, because $\nvdash r \to . q \circ s$. Therefore, there is a prime theory β that extends $[r)$ and excludes $q \circ s$. The theory $[\alpha \circ \beta, s)$ is contained in a prime theory γ (and we can choose any such γ). By the definition of γ, $R\alpha\beta\gamma$ is obvious, however, $R\alpha^*\gamma\beta$ is false.
2. For the sake of transparency, in the proof for $RM^{\boldsymbol{T}}$ we use the labels from $S_{\mathbb{Z}}$ directly. (The addition of $\boldsymbol{T}$ and $\boldsymbol{F}$ means bounding $S_{\mathbb{Z}}$ with bottom and top.) Let α be $[+2)$, hence $\alpha^* = [-1)$. We choose γ to be $[-3)$, whereas we take $[+4)$ for β. $+2 \circ +4 = +4$, therefore, $[+2) \circ [+4) \subseteq [-3)$. On the other hand, $[-1) \circ [-3) = [-3)$, that is, obviously, not contained in $[+4)$.
3. Having scrutinized 1. and 2., we see that nothing depends on the presence of $\boldsymbol{T}$, therefore, steps 1. and 2. establish the truth of (3) too. □

THEOREM 5.7.

(1) $R^{\boldsymbol{tT}}$ *is* incomplete *for the class of dense and commuting* PRA*s.*
(2) $RM^{\boldsymbol{tT}}$ *is* incomplete *for the class of dense, commuting and transitive* PRA*s.*

Proof. The proof of this theorem is easy, since from the proof of Theorem 5.1 follows that $\mathbf{1'} \cap \sim\mathbf{1'}$ is the bottom of PRAs, however, $\boldsymbol{t} \wedge \sim\boldsymbol{t} . \to \boldsymbol{F}$ remains unprovable both in $R^{\boldsymbol{tT}}$ and in $RM^{\boldsymbol{tT}}$.[18] □

Notice that forcing $\sim$ to be an orthonegation in RM collapses RM into classical logic, which means that the above construction cannot be adapted for RM—at least not without compensating for the fact that all RM algebras are linear.

A proof of the completeness could proceed by a concrete construction of a PRA from the canonical model of $RM^{(\boldsymbol{T})}$. Figure 5 (below) shows some components of the canonical embedding algebra of the Sugihara matrix $S_{\mathbb{Z}^*}$ that is like $S_{\mathbb{Z}}$ in Figure 2, except that 0 is dropped (cf. also Dunn, 1970). Figure 5 shows some of the atoms and co-atoms; the dotted line is the embedded Sugihara matrix and the dashed curves indicate the converses of some of the atoms.

Given the canonical definition of * together with the definitions of $\sim$ and $-$, the converse operation on atoms is defined as

$$a^{\cup} = \begin{cases} a & \text{if } a = \{[1)\}, \\ \{[1 - m)\} & \text{if } a = \{[m)\} \text{ and } m \neq 1. \end{cases}$$

Unfortunately, this Boolean algebra is not a relation algebra, let alone a transitive or a representable one. To show that (r6) is not true, we give a concrete counterexample. $(\{[1)\}; \{[3)\})^{\cup}$ is $3^{\cup}$, that is, $\{[i): -2 \leq i\}$. On the other hand, $\{[1)\}^{\cup} = \{[1)\}$ and $\{[3)\}^{\cup} = \{[-2)\}$. Then $\{[1)\}^{\cup}; \{[3)\}^{\cup} = -2$, that is, $\{[i): i \leq -2\}$.

[18] A formula that looks somewhat like the dual of the above formula was mentioned as unprovable both in R and RM by Kowalski (2007, p. 155).

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Although we cannot use the canonical embedding algebra of RM to prove the incompleteness of RM, nevertheless, the following theorem obtains.

THEOREM 5.8. $RM^{\boldsymbol{T}}$, RM, $RM^{\boldsymbol{t}\circ}_{\wedge\vee}$, $RM^{\boldsymbol{t}\circ\boldsymbol{F}}_{\wedge\vee}$ *are* incomplete *for the class of dense, commuting and transitive* PRAs*, as well as for the class of dense, commuting and transitive relevant families of relations.*

Proof. In all transitive (i.e., Boolean) relation algebras $1 = \mathbf{1'}$ (cf. Definition 2.2 and Lemma 2.3). However, $\boldsymbol{t}$ and $\boldsymbol{f}$ are not definable (nor contained in) $RM^{\boldsymbol{T}}$. Further, in a Boolean RA ; and $\cdot$ coincide, however, $\circ$ and $\wedge$ are not provably equivalent in RM (or its fragments), which establishes the claim for the $\boldsymbol{t}$- and $\boldsymbol{T}$-free R-mingle. A slight modification of Lemma 2.3 shows that all relevant families of relations are Boolean relation algebras, therefore, the same argument may be repeated for them. □

§6. Extensions of *R* and their dynamic semantics. A way to react to the results of the preceding section is to say that the logic in question has to be extended by adding further axioms, perhaps, starting with those formulas that were used in the proofs. Extensions may seem desirable also from the point of view of making the logic more expressive. It seems natural, for instance, to consider the addition of $-$ or of $^{\cup}$ to the language of R so that all operations of RAs become definable—at least in principle. Of course, the enlargement of the vocabulary of R may cause incompleteness itself, exactly because the greater expressive power entices us (to try) to define the missing operations of RAs. However, sometimes the forthright definitions do not yield operations that have the needed properties.

First, we consider two preexisting extensions of R, CR and KR, that both contain an orthonegation. In CR, $\sim$ and $-$ are distinct, that is, $\sim A$ and $-A$ are not provably equivalent. On the other hand, KR identifies $\sim$ and $-$ (thus, KR can be viewed as an extension of the logic R without an extension of its vocabulary).

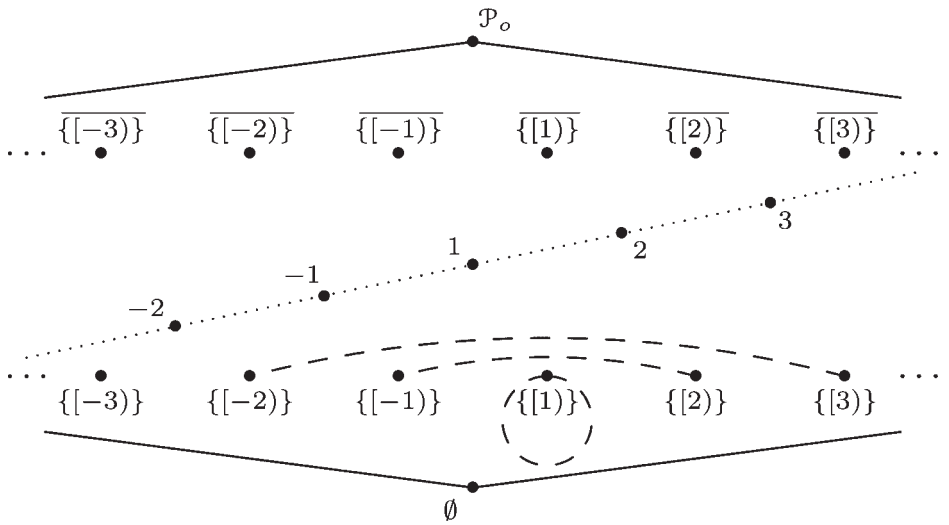


Fig. 5. The canonical embedding algebra of $S_{\mathbb{Z}^*}$.

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Classical relevance logic, *CR* has as its logical constants $\wedge$, $\vee$, $\rightarrow$, $\sim$ and $-$.[19] The system *CR* comprises the axioms (A1)–(A13) (from section 3) and (A17) (below) as well as the rules (R1)–(R2) and (R6) (together with Definition (D3), if desired). $\boldsymbol{T}$ and $\boldsymbol{F}$ are definable in *CR* as $A \vee -A$ and $A \wedge -A$, respectively, and with these formulas in place for the constants, axioms (A14) and (A15) become theorems.

(A17) $--A \rightarrow A$

(R6) $(A \wedge B) \rightarrow -C \Rightarrow (A \wedge C) \rightarrow -B$

Of course, if $-$ and $\sim$ are both in the language, then we may attempt to define converse. From $\sim a = -a^{\cup}$ in (p11), exploiting the ambiguity of the expression and the involutive character of $-$, we may define $a^{\cup}$ as $-\sim a$. Before we turn to investigating the properties of the so-defined $^{\cup}$, we provide a semantics for *CR* that demonstrates the reasonableness of this extension.

The notation $-$ for the added unary operation is intended to suggest that this operation behaves like Boolean complement. In the terminology of Bimbó and Dunn (2008), the Lindenbaum algebra of *CR* is a *Boolean gaggle*; and the semantics we define now is built utilizing this insight.

DEFINITION 6.1. *A* structure *for CR is defined to be* $\mathfrak{F} = \langle U; I, R, {}^{*} \rangle$ *where (f0)–(f6) hold.*

(f0) $\emptyset \neq I \subseteq U, \quad R \subseteq U^3, \quad {}^{*}\colon U \longrightarrow U,$

(f1) $\alpha = \beta \;\Leftrightarrow\; \exists \imath \in I.\, R\imath\alpha\beta,$

(f2) $R\alpha\alpha\alpha,$

(f3) $R\alpha\beta\gamma \;\Rightarrow\; R\beta\alpha\gamma,$

(f4) $R^2(\alpha\beta)\gamma\delta \;\Rightarrow\; R^2\alpha(\beta\gamma)\delta,$

(f5) $\alpha^{**} = \alpha,$

(f6) $R\alpha\beta\gamma \;\Rightarrow\; R\alpha\gamma^{*}\beta^{*}.$

A model *for CR is defined to be* $\mathfrak{M} = \langle \mathfrak{F}, v \rangle$ *where* $\mathfrak{F}$ *is a structure for CR,* $v\colon \mathbb{P} \longrightarrow \wp(U)$ *that is extended to compound formulas according to (v1)–(v6). If* $\boldsymbol{t}$ *is included, so is (v7).*

(v1) $v(A \wedge B) = v(A) \cap v(B),$

(v2) $v(A \vee B) = v(A) \cup v(B),$

(v3) $v(-A) = U - v(A),$

(v4) $v(A \circ B) = \{\, \alpha\colon \exists\beta, \gamma\,.\, R\beta\gamma\alpha \wedge \beta \in v(A) \wedge \gamma \in v(B) \,\},$

(v5) $v(A \rightarrow B) = \{\, \alpha\colon \forall\beta, \gamma\,(R\alpha\beta\gamma \wedge \beta \in v(A)\,.\, \Rightarrow \gamma \in v(B)) \,\},$

(v6) $v(\sim A) = \{\, \alpha^{*}\colon \alpha \notin v(A) \,\},$

(v7) $v(\boldsymbol{t}) = \{\, \alpha\colon \alpha \in I \,\}.$

We combine the soundness and the completeness theorems into one.

THEOREM 6.2. *If the formula A is a theorem of CR, that is,* $\vdash_{CR} A$ *then A is valid in all models on all structures for CR. If the formula A is not a theorem of CR then there is a model of CR in which* $I \nsubseteq v(A)$.

[19] Versions of *CR* were introduced by Meyer and Routley (1973, 1974). Our axiom set is, probably, the same as theirs for CR^{*} (though we use different primitives), and the semantics presented here is very similar to theirs. (However, we do not follow their notation or terminology.)

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Proof. To prove that (A17) is valid, let us assume that $\iota \notin v(--A \to A)$, that is, $\alpha \in v(--A)$ though $\alpha \notin v(A)$, for some α. However, $\alpha \in v(--A)$ iff $\alpha \notin v(-A)$ iff $\alpha \in v(A)$. From the contradiction, we conclude that $\iota \in I$ implies $\iota \in v(--A \to A)$.

Now let $\alpha \in v(A \wedge C)$ and $\alpha \in v(B)$, while we assume that $I \subseteq v(A \wedge B \to -C)$. Then $\alpha \in v(A)$ and $\alpha \in v(C)$, that is, $\alpha \notin v(-C)$. Obviously, $\alpha \in v(A \wedge B)$, hence for some ι, $\iota \notin v(A \wedge B \to -C)$ contrary to the starting assumption.

The rest of the proof may be compiled from proofs in Dunn (1986, §4.11), and some further details in Meyer and Routley (1973, 1974). □

Converse in RAs has two distribution types and also reversely distributes over $\circ$ into $\circ$. In other words, for the Lindenbaum algebra of CR to be an RA, the equations (r4)–(r7) have to hold with $^{\cup}$ defined from $-$ and $\sim$. The absence of $\boldsymbol{t}$ would also prevent the Lindenbaum algebra of CR from being a relation algebra, however, $\boldsymbol{t}$ is easy to add, hence we ignore the issue of $\boldsymbol{t}$ for a moment.

LEMMA 6.3. *In CR,* (1)–(4) *hold when* $A^{\cup}$ *is defined as* $-\sim A$.

(1) $\vDash A \to A^{\cup\cup}$ *and* $\vDash A^{\cup\cup} \to A$, *that is,* (r4) *is true (in the algebra of CR).*
(2) $\vDash A^{\cup} \vee B^{\cup}. \to (A \vee B)^{\cup}$ *and* $\vDash (A \vee B)^{\cup} \to . A^{\cup} \vee B^{\cup}$, *that is,* (r5) *holds.*
(3) $\vDash -\sim A \to \sim -A$ *and* $\vDash \sim -A \to -\sim A$, *that is,* $-$ *and* $\sim$ commute.
(4) *If* (r6) *holds, then* (3) *implies* (r7).

Proof. 1. To prove the first implication, let us assume that $\alpha \in v(A)$ but $\alpha \notin v(-\sim -\sim A)$. Then $\alpha \in v(\sim -\sim A)$, $\alpha^* \notin v(-\sim A)$, $\alpha^* \in v(\sim A)$ and lastly, $\alpha^{**} \notin v(A)$. Since $\alpha = \alpha^{**}$, the latter (together with the starting assumption) constitutes a contradiction. The proof of the other implication is alike, and the details are omitted here.
2. To prove that $-\sim A \vee -\sim B$ and $-\sim(A \vee B)$ coimply each other, we assume that $\alpha \in v(-\sim A \vee -\sim B)$. Then $\alpha^* \in v(A)$ or $\alpha^* \in v(B)$, and so $\alpha^* \in v(A \vee B)$. By the definition of $\sim$, $\alpha \notin v(\sim(A \vee B))$, that is, $\alpha \in v(-\sim(A \vee B))$ as desired. The other direction is similar.
3. If $\alpha \in v(-\sim A)$ then $\alpha \notin v(\sim A)$, and further, $\alpha^* \in v(A)$. This is the same as $\alpha^* \notin v(-A)$, and so $\alpha \in v(\sim -A)$. The proof of the other implication is alike.
4. In CR, $\to$ and $\circ$ are interdefinable via $\sim$ (by Definition (D3) from R^*), and $\to$ is the residual of $\circ$ in the Lindenbaum algebra of CR. That is, if we denote the equivalence class of A by $[A]$, then $[(A \to B) \circ A] \leq [B]$ which is the same as $[\sim(A \circ B) \circ A] \leq [\sim B]$. $-$ is an involution, and so $[--(\sim(A \circ B) \circ A)] \leq [--\sim B]$. Assuming that (r6) is true (with $-\sim$ in place of $^{\cup}$) together with (3) and double-negation elimination, we get that $[-\sim(-\sim A \circ -(A \circ B))] \leq [-\sim -B]$. Then it is immediate that $[A^{\cup} \circ -(A \circ B)] \leq [-B]$, because both $-$ and $\sim$ are order-inverting operations. □

We stress that (3) does not imply that the two negations are the same, rather it is yet another way of viewing the De Morgan negation as a "dislodged Boolean negation." Of course, the distinctness of the negations also follows when we observe that M_0 is not only an $R^{\boldsymbol{t}}$ algebra but also a $CR^{\boldsymbol{t}}$ algebra.

LEMMA 6.4. $A \wedge (\boldsymbol{T} \circ B) . \to . \boldsymbol{T} \circ (\boldsymbol{T} \circ A . \wedge B)$ *is* provable *in* $CR^{(t)}$.

Proof. We show that the above formula is valid on structures for CR. This is easily seen to hinge upon whether $R\alpha\beta\gamma$ implies $R\delta\gamma\beta$ for some δ—when we restrict our considerations to first-order conditions. (Compare step 1. in the proof of Lemma 5.6. We leave the reconstruction of the calculations leading up to this fact to the reader.)

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Let us assume $R\alpha\beta\gamma$. For any β, $R\imath\beta\beta$ for some $\imath \in I$. Hence, by permutation (f3) and antilogism (f6), $R\beta^*\beta\imath^*$. In a structure for CR (as defined in 6.1), $\imath^* = \imath$, because $R\imath\imath\imath$ due to contraction (f2), and then by (f6) and (f1) the desired equality follows. That is, we have $R\beta^*\beta\imath$. From the starting assumption, we obtain $R\gamma^*\alpha\beta^*$ via a few similar steps. Using the usual notation for composition in the first two argument places of R, the last two formulas amount to $R(\gamma^*\alpha)\beta\imath$. The frame condition corresponding to right associativity (i.e., to B) is (f4), that gives us $R\gamma^*(\alpha\beta)\imath$. Then $R\gamma^*\varepsilon\imath$, and by (f6), (f3), (f5) and $\imath = \imath^*$, we obtain that $\gamma = \varepsilon$, that is, $R\imath\gamma^*\gamma^*$. Composing with $R\beta^*\beta\imath$, we get $R(\beta^*\beta)\gamma^*\gamma^*$, which implies that $R\beta^*\delta\gamma^*$. Therefore, $R\delta\gamma\beta$ is immediate after applications of (f3) and (f6). □

The incompleteness of CR (and of CR^t) runs deeper though, because there is a new formula (that includes $-$, of course) that is not provable.

LEMMA 6.5. *The Lindenbaum algebra of* CR^t *is* not a relation algebra.

Proof. Our goal is to show that (r6) with the defined $^\cup$ fails in CR^t. We rely on the relational semantics for CR^t given above.

For $A \circ B . \leftrightarrow . \sim -(-\sim A \circ -\sim B)$, that is the same as (r6) in (the Lindenbaum algebra of) CR^t, the following equivalence has to be true.

$$\exists\alpha, \beta.\, R\alpha\beta\gamma \wedge \alpha \in A \wedge \beta \in B \;\Leftrightarrow\; \exists\delta, \varepsilon.\, R\delta\varepsilon\gamma^* \wedge \delta \notin \sim A \wedge \varepsilon \notin \sim B$$

The natural corresponding condition is $R\alpha\beta\delta$ iff $R\alpha^*\beta^*\delta^*$, given (v6). However, there are CR models in which this condition fails. In fact, it may be verified by finite combinatorial reasoning that this condition is not derivable from the set of conditions postulated for a structure for CR. □

The truth of the claim of the next theorem is now immediate.

THEOREM 6.6. *CR and CR^t are* incomplete *with respect to the class of dense and commuting* PRAs.

Now we turn to the other promised extension of R, KR. If the two negations of CR are identified, or if (A18) is added to R, then we obtain KR.

(A18) $A \wedge \sim A . \rightarrow B$

KR can hardly be counted as a relevance logic according to the view that provable implications comprise an antecedent and a consequent that share some information. Nevertheless, it is remarkable that the "surprise move" of adding (A18), does not collapse R into a "baroque version" of classical logic (with a superfluous stock of connectives).[20] The semantic condition that corresponds to axiom (A18) on an R structure (that we defined in 3.1) is (f11).

(f11) $\alpha^* = \alpha$.

KR is known to be sound and complete for the resulting semantics, and the proofs of adequacy may be assembled from proofs in the literature (with slight modifications). A consequence of the addition of (f11) is that all situations satisfy (f9), thus in a sense, KR is *normalized* R. For our purposes, we need the following lemma in which we assume once again that $\boldsymbol{t}$ has been added.

[20] KR plays a central role in the proofs of important theorems *about* relevance logics, and so information about KR may be found, for example, in Anderson *et al.* (1992, §65) and Urquhart (1993)—despite the nonrelevant character of KR.

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LEMMA 6.7. *The Lindenbaum algebra of* KR *is a* relation algebra*, however, it possesses additional properties not shared by all relation algebras.*[21]

Proof. Condition (f11) implies (together with the other R conditions) the frame condition that we discovered (in the proof of Lemma 6.5) as sufficient for the analogue of (r6) to be valid in a model of CR. Axiom (A17) with $\sim$ in place of $-$ is a theorem of KR, indeed, that formula is a theorem of R^* because (A13) is an axiom. Finally, the De Morgan negation variant of (R6) is easily derivable in KR. □

Given that (the algebra of) KR is a relation algebra (as well as an extension of R), it is useful to note that the algebra of KR possesses additional properties not only with respect to the class of relation algebras, but even for the class of dense and commuting relation algebras.

The canonical model of KR is a relation algebra independently of whether $\boldsymbol{t}$ is or is not in the language of KR. (I is determined by the set of theorems, because all the elements of I contain all theorems. The addition of $\boldsymbol{t}$ simply means that there is a least theorem.) But our reservations concerning the inclusion of $\boldsymbol{t}$ (see Theorem 5.1) now have disappeared, since in KR, $\boldsymbol{F}$ is definable by $\sim$ and $\wedge$. Thus, perhaps, somebody could still expect that KR can be proven complete with respect to a subclass of PRAs. However, we used KR frames (indirectly) in the proof of the incompleteness of R, and in the proof of the next incompleteness theorem, we appeal to the algebra $\mathscr{R}_1$ that emerges from a KR frame.

The following quasi-inequation (J), that is an instance of Lyndon's general condition, holds in all RRAs (cf. Maddux, 2006, p. 30).

$$a \leq b\,;c \cdot d\,;e \wedge b^{\cup}\,;d \cdot c\,;e^{\cup} \leq f\,;g \;\Rightarrow\; a \leq (b\,;f \cdot d\,;g^{\cup})\,;(f^{\cup}\,;c \cdot g\,;e^{\cup})$$

$^{\cup}$ is not a primitive connective of KR, however, the definition of $^{\cup}$ as $-\sim$ that we utilized to add converse to CR now yields a connective that interacts with $\circ$ as $^{\cup}$ does with ; in RAs. (The truth of this claim is immediate from Lemma 6.7 above.) Now we can give a simple proof of the following theorem.

THEOREM 6.8. KR *is* incomplete *with respect to the class of symmetric, commuting, dense* PRAs.

Proof. We appeal to the relation algebra $\mathscr{R}_1$ from Urquhart (1993). This algebra is constructed from a frame that is a frame for KR, which means that it contains a subalgebra that is a KR algebra, in other words, a homomorphic image of the Lindenbaum algebra of KR. The particular frame from which $\mathscr{R}_1$ emerges is defined (with an additional situation) from a projective geometry in which Desargues's theorem is not true. Therefore, by Jónsson's results (cf. Jónsson, 1959), there is an instance of (J) that does not hold in $\mathscr{R}_1$. □

§7. Concluding remarks. As the reader has surely realized by now, constructing a dynamic semantics with certain closure properties is not a trivial enterprise because of the nonrepresentability result of Lyndon (1950). The investigations into the connections of relevance logics and relation algebras have a long history. Dunn (1966) algebraized R as what he called a "De Morgan monoid," which is a certain type of residuated monoid grafted onto a distributive lattice with a De Morgan negation. Dunn tried (with no success) to link R to relation algebras in a similar vein as two-valued logic is coupled with Boolean

[21] In other words, the Lindenbaum algebra of KR is not a free relation algebra of cardinality $\aleph_0$ (with $\aleph_0$-many generators).

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algebras. However, Dunn (1982) made progress by showing that De Morgan lattices can be represented as a collection of relations, where meet is $\cap$, join is $\cup$, and De Morgan complement is the (Boolean) complement of the converse (like (c4) on p. 110).

In 1991, Dunn responded to a posting by Thomas Streicher on the TYPES listserv that posed the question if it was well-known that relation algebras were a model of noncommutative exponential-free linear logic. As a reply, Dunn sketched his attempts to discover the precise relationship between relation algebras and relevance logic. He also outlined how the same approach would apply to variants of linear logic.[22]

Dunn (2001) gave representations for various algebras—generalizing the approach of Jónsson and Tarski (1951), who modeled an additive operator on a BA as the existential image of a relation. In particular, all relation algebras can be modeled in this way, where the elements of the algebra are mapped into a set of relations rather that into a relation. This representation is exactly at the "type level" and in the style of the Kripke–Routley–Meyer–Dunn semantics for substructural logics, including relevance logics. We should mention here also van Benthem (1995, 1996) for his recognition of the connection between relation algebras and dynamic semantics.

The immediate motivation for this paper comes from Maddux's talk at the *International Conference on Order, Algebra and Logics* that was held at the Vanderbilt University in June, 2007.[23] Maddux has worked on representations of relation algebras for a long time, and in his talk raised completeness questions for R and RM with respect to collections of relations that he called "families of relations." (That notion is "officially" termed here "relevant families of relations.") In this paper, we investigated analogous (i.e., (in)completeness) questions for the R^* logics, their fragments and their extensions—with respect to PRAs and relevant families of relations. By November 2007, when Maddux visited Bloomington, Indiana, some of the main results of this paper had been obtained, including the incompleteness of $R^{\boldsymbol{T}}_{+}$ as well as the completeness results in section 4. Then Maddux visited England in January 2008, and gave a talk on some of the same topics.[24]

The semantical investigations of relevance logics and the algebraic problems of the representation of relation algebras meet and entwine in the soundness and (in)completeness results for relevance logics with respect to PRAs and relevant families of relations as dynamic semantics.

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[22] http://www.seas.upenn.edu/~sweirich/types/archive/1992/msg00028.html is the URL for the archived form of the message of Dunn posted on February 5, 1992.

[23] Another paper inspired by the same talk is Kowalski (2007), which deals with weakly associative RAs as an algebraic semantics for B_+.

[24] We would like to thank Sz. Mikulás for his comments on a draft of our paper.

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