Metric relations in crossed trapezia

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The trapezium (trapezoid in American English) is one of the seven most well-known quadrilaterals that is studied in school geometry. However, it is always assumed that it is convex. In this note we shall derive metric formulas for the most important quantities in *crossed* trapezia and compare them to the similar formulas for convex trapezia. The corresponding investigation regarding crossed cyclic quadrilaterals was conducted in [1]. Any crossed quadrilateral has a pair of opposite sides intersecting and both of its diagonals are outside of itself.

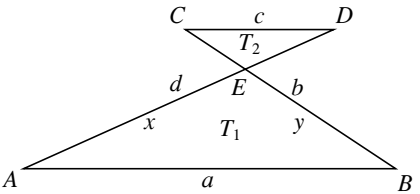


FIGURE 1: A crossed trapezium

We start with the area of a crossed trapezium. Regarding the area of crossed quadrilaterals, then there is something important to note directly. The area of a crossed quadrilateral $ABCD$ is according to [2, pp. 52-53] and [3] defined to be equal to the *difference* between the two triangles it is composed of. This is the same as saying that its area is equal to the difference of the areas of triangles ABC and CDA. To avoid negative area, we will consider the absolute value of this difference. We denote the sides of a crossed trapezium $ABCD$ by $a = AB$, $b = BC$, $c = CD$, $d = DA$ and its area by K. Then we have

$$K = \left|\tfrac{1}{2}ah - \tfrac{1}{2}ch\right| = \tfrac{1}{2}h\left|a - c\right| \tag{1}$$

where h is the height, that is, the distance between the parallel sides a and c. This formula is similar to the one for convex trapezia, $K = \frac{1}{2}h(a + c)$. They are the same except for the sign on one of the bases.

It is possible to express the area in terms of the lengths of the four sides alone. To do so, we first need the lengths of the parts of the intersecting sides b and d. Let them intersect at E and denote $AE = x$ and $BE = y$, so $DE = d - x$ and $CE = b - y$ (see Figure 1). By similarity we have

$$\frac{b - y}{y} = \frac{d - x}{x} = \frac{c}{a}.$$

Solving for x and y yields

$$x = \frac{ad}{a + c}, \quad y = \frac{ab}{a + c}, \quad d - x = \frac{cd}{a + c}, \quad b - y = \frac{cb}{a + c}.$$

We shall use Heron's formula to get expressions for the areas T_1 and T_2 of

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triangles ABE and CDE respectively. Let us first put

$$k_1 = \frac{a}{a+c}, \qquad k_2 = \frac{c}{a+c}$$

so that the sides of ABE and CDE are $k_1 d$, $k_1 b$, $k_1(a+c)$ and $k_2 d$, $k_2 b$, $k_2(a+c)$ respectively. Now applying a version of Heron's formula, we get

$$16T_1^2 = k_1^4(d+b+a+c)(-d+b+a+c)(d-b+a+c)(d+b-a-c)$$

and

$$16T_2^2 = k_2^4(d+b+a+c)(-d+b+a+c)(d-b+a+c)(d+b-a-c).$$

Defining

$$Q_1 = (-a+b-c+d)(a+b+c+d)(a-b+c+d)(a+b+c-d),$$

the areas of triangles ABE and CDE are

$$T_1 = \frac{a^2\sqrt{Q_1}}{4(a+c)^2}, \qquad T_2 = \frac{c^2\sqrt{Q_1}}{4(a+c)^2}.$$

Hence the crossed trapezium has the area

$$K = |T_1 - T_2| = \frac{|a^2 - c^2|\sqrt{Q_1}}{4(a+c)^2} = \frac{|a-c|\sqrt{Q_1}}{4(a+c)}. \tag{2}$$

In comparison, a convex trapezium with consecutive sides a, b, c, d where $a \,//\, c$ has an area given by the formula

$$K = \frac{(a+c)\sqrt{Q_2}}{4|a-c|}$$

where

$$Q_2 = (-a+b+c+d)(a+b-c+d)(a-b-c+d)(a+b-c-d).$$

We see that the only difference between the two formulas for the area of a crossed and a convex trapezium is the sign with the base c. Note that it is unfortunately common to denote the consecutive sides in a convex trapezium as a, c, b, d (see for example [4]), in which case the notations in the formulas for K and Q_2 need to have c and b interchanged.

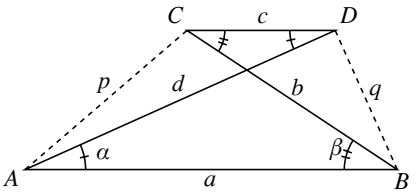


FIGURE 2: The two exterior diagonals

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Combining formula (2) with (1), we directly get an expression for the height of a crossed trapezium in terms of the four sides. This is

$$h = \frac{\sqrt{(-a+b-c+d)(a+b+c+d)(a-b+c+d)(a+b+c-d)}}{2(a+c)}.$$

A non-factorised version of this for a convex trapezium was derived in [5, pp. 34-35].

In order to get formulas for the diagonal lengths $p = AC$ and $q = BD$ in a crossed trapezium, we use the notation of Figure 2 and apply the cosine rule, which yields

$$q^2 = a^2 + d^2 - 2ad\cos\alpha = b^2 + c^2 - 2bc\cos\beta$$

and

$$p^2 = a^2 + b^2 - 2ab\cos\beta = c^2 + d^2 - 2cd\cos\alpha.$$

From these we get

$$cq^2 = a^2c + cd^2 - 2acd\cos\alpha, \qquad ap^2 = ac^2 + ad^2 - 2acd\cos\alpha$$

so

$$cq^2 - ap^2 = ac(a-c) + d^2(c-a). \tag{3}$$

Similarly

$$aq^2 = ab^2 + ac^2 - 2abc\cos\beta, \qquad cp^2 = a^2c + b^2c - 2abc\cos\beta$$

and so

$$aq^2 - cp^2 = ac(c-a) + ab^2 - b^2c. \tag{4}$$

Multiplying (3) and (4) by appropriate factors gives

$$c^2q^2 - acp^2 = ac^2(a-c) + cd^2(c-a),$$

$$a^2q^2 - acp^2 = a^2c(c-a) + ab^2(a-c)$$

and it follows that

$$c^2q^2 - a^2q^2 = \left(ac^2 - ab^2\right)(a-c) + \left(cd^2 - a^2c\right)(c-a).$$

Solving for q yields

$$q = \sqrt{\frac{ab^2 + cd^2 - ac(a+c)}{a+c}}. \tag{5}$$

By a similar approach we get a formula for the other diagonal p, or we can simply use symmetry and need only interchange b and d, to get

$$p = \sqrt{\frac{ad^2 + cb^2 - ac(a+c)}{a+c}}. \tag{6}$$

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In a convex trapezium, according to [5, p. 31] it holds that

$$q = \sqrt{\frac{ab^2 - cd^2 + ac(a - c)}{a - c}} \tag{7}$$

and

$$p = \sqrt{\frac{ad^2 - cb^2 + ac(a - c)}{a - c}}. \tag{8}$$

Again we see that the only difference between the formulas for crossed and convex trapezia is the sign with c.

There is another way of deriving the diagonal lengths in a crossed trapezium that start from the formulas for the convex trapezium. From (8) and (7) we get

$$p^2(a - c)c = ac^2(a - c) + acd^2 - c^2b^2,$$
$$q^2(a - c)a = a^2c(a - c) + a^2b^2 - acd^2$$

and thus

$$(p^2c + q^2a)(a - c) = (a - c)(a^2c + ac^2) + b^2(a^2 - c^2).$$

It follows that

$$b^2 = \frac{p^2c + q^2a - ac(a + c)}{a + c}, \tag{9}$$

which is valid in convex trapezia, and gives one of the non-parallel sides expressed in terms of the two bases and the two diagonal lengths. Now comparing a convex and a crossed trapezium (see Figure 3), we see that one of these is converted into the other if we interchange p and d, and also q and b. Making these changes of letters in (9), we get (5) again.

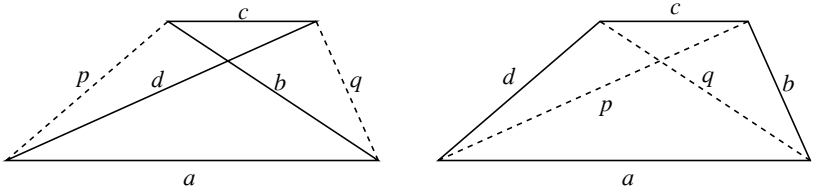


FIGURE 3: Comparison of crossed and convex trapezia

Next we consider the distance v between the midpoints of the two diagonals and the distance n between the midpoints of the two non-parallel sides (the midsegment, also called one of the bimedians). It was derived in [5, pp. 30-31] that in a convex trapezium these formulas are

$$v = \frac{|a - c|}{2}, \qquad n = \frac{a + c}{2}.$$

Comparing a convex trapezium and a crossed trapezium (see Figure 4), we see that v and n interchange their roles. Thus for a crossed trapezium, the corresponding formulas are

$$n = \frac{|a - c|}{2}, \qquad v = \frac{a + c}{2}.$$

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Another way of interpreting these is that those for the convex trapezium transform into those for the crossed trapezium by changing the sign with c (and including an absolute value if necessary), which we have observed several times by now.

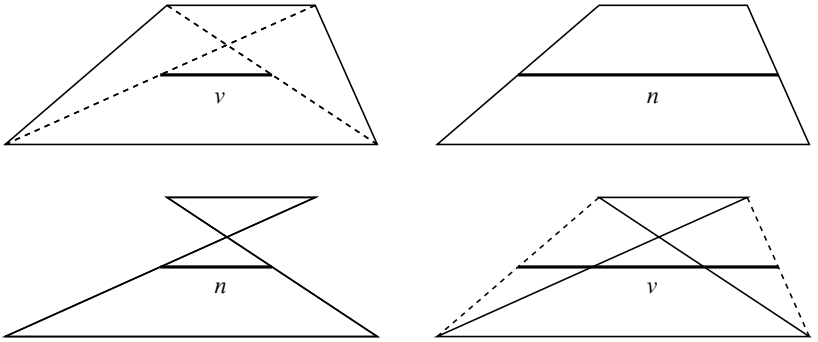


FIGURE 4: The distances v and n in convex and crossed trapezia

We conclude this exploration of the crossed trapezium by answering in two different ways why the base c has reversed sign in the metric formulas in comparison with the convex trapezium. From the formulas (6) and (5), we get for a crossed trapezium

$$p^2 + q^2 = b^2 + d^2 - 2ac. \tag{10}$$

The corresponding relation in a convex trapezium is

$$p^2 + q^2 = b^2 + d^2 + 2ac \tag{11}$$

according to [5, p. 30]. We have previously noted that the two cases of crossed and convex trapezia interchange when we interchange p and d, and also q and b. This explains the reversed sign on c, since the term $2ac$ only changes side when comparing (10) and (11). This also means that any one of the two bases a or c can be the one that changes sign. Another way of understanding the negative c is by studying Figure 5. In a convex trapezium the sides are all in a counter clockwise order, whereas in a crossed trapezium, three of the sides are in a counter-clockwise order but c is in a clockwise order when travelling around the figure along its sides. This is the reason why c is negative in the metric relations for crossed trapezia in comparison to convex trapezia.

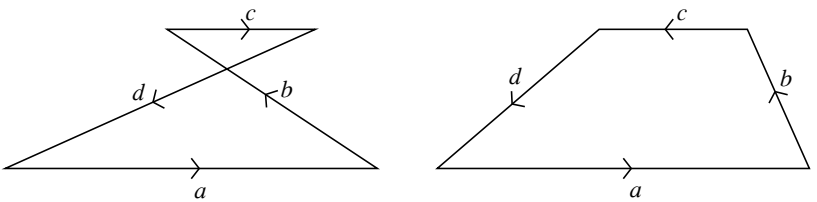


FIGURE 5: The reversed direction of c in a crossed trapezium

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