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JOINT DISTRIBUTION OF A LÉVY PROCESS AND ITS RUNNING SUPREMUM

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Abstract

Let X be a jump-diffusion process and X^* its running supremum. In this paper we first show that for any $t > 0$, the law of the pair (X_t^*, X_t) has a density with respect to the Lebesgue measure. This allows us to show that for any $t > 0$, the law of the pair formed by the random variable X_t and the running supremum X_t^* of X at time t can be characterized as a weak solution of a partial differential equation concerning the distribution of the pair (X_t^*, X_t). Then we obtain an expression of the marginal density of X_t^* for all $t > 0$.

Keywords: Lévy process; partial differential equation; running supremum process; first hitting time

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1. Introduction

Consider a Lévy process $(X_t,\ t \geq 0)$, starting from zero, which is right continuous and left limited. If, moreover, X is the sum of a drifted Brownian motion and a compound Poisson process, it is called a mixed diffusive-jump process. As in any Lévy process, X has stationary and independent increments and is characterized by its Laplace transform. The mixed diffusive-jump processes and the notion of the first-passage time are very useful and widely studied.

Introducing X_t^*, the running supremum at time t and the first-passage time τ_b of X at level b, the probability $\mathbb{P}(X_t \geq a, X_t^* \geq b) = \mathbb{P}(X_t \geq a, \tau_b \leq t)$ for some fixed real numbers (a, b), $a \leq b$ and $b > 0$, is of great importance. For example, in pricing barrier options where the logarithm of the underlying asset price is modeled by a jump-diffusion process. In this context, Kou and Wang [6] stated the explicit expression of the Laplace transform of the joint distribution of the double exponential mixed diffusive-jump process and its running supremum.

Jeanblanc *et al.* [5] considered the first-passage time by a diffusion at a deterministic function h that depends on time and the authors defined a function of τ_h and X which satisfied the Fokker–Planck equation.

In [1], it was noted (see Theorem 2.2.9 and Exercise 2.2.10 of that paper) that the $\frac{1}{2}$-stable subordinator is the first-passage time of a standard Brownian motion and the inverse Gaussian subordinator is the first-passage time of a standard Brownian motion with a drift.

Veillette and Taqqu [11] computed numerically the expectation of $E(t)$, where $E(t) = \inf\{s\colon D_s > t\}$ and D is a nondecreasing Lévy process.

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Carr and Cousot [3] used a partial differential equation approach to show that the calibration of an implied volatility surface and the pricing of contingent claims can be as simple in the mixed diffusive-jump framework as it is in a diffusion framework. Another approach is the one of Kuznetsov *et al.* [7]. There, with a spectrally negative Lévy process $(X_t,\ t \geq 0)$ and Borel set A, the authors used their Theorem 2.7 to provide a Laplace transform of the indicator $\mathbf{1}_{\{X_t \in A,\ X_t^* \leq a\}}$. This involved the so-called scale functions.

In this paper we consider a mixed diffusion-jump process $(X_t,\ t \geq 0)$. Here it is characterized as the density of the law of the pair U_t formed by the random variable X_t and the running supremum X_t^* of X at time t. This density is a weak solution of a partial differential equation; see Corollary 2.1. An explicit expression for the density function of this pair is also stated. Then we provide the marginal density of X_t^*.

The paper is organized as follows. In Section 2 we provide the main result. In Section 3 we give the density function of the law of the pair formed by the random variable X_t and its running supremum X_t^*. Section 4 is devoted to the proof of the main result of Section 2. Finally, we conclude with Section 5 and provide auxiliary results in Appendix A.

2. Partial differential equation for the joint law

We first introduce some preliminary concepts for the diffusion part. For a standard Brownian motion W and a real number m, let

$$\tilde{X}_t = mt + W_t, \qquad \tilde{X}_t^* = \sup_{s \leq t} \tilde{X}_s. \tag{2.1}$$

Jeanblanc *et al.* [5, p. 147] showed that the pair $(\tilde{X}_t^*, \tilde{X}_t)$ has a density with respect to the Lebesgue measure on $\mathbb{R}^2$ denoted by $\tilde{p}(\cdot, \cdot;\ t)$, where

$$\tilde{p}(b, a;\ t) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left[-\frac{(2b-a)^2}{2t} + ma - m^2\frac{t}{2}\right]\mathbf{1}_{\{\max(0,a)<b\}}. \tag{2.2}$$

Throughout, Φ_{G} means the standard normal Gaussian distribution. We will often use the following:

$$1 - \Phi_{\mathrm{G}}(x) = \Phi_{\mathrm{G}}(-x) \leq \frac{1}{x\sqrt{2\pi}} \exp -\frac{x^2}{2} \quad \text{for all } x > 0.$$

In order to have a Lévy process with a nonzero jump part, we introduce

$$X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i, \qquad X_t^* = \sup_{s \leq t} X_t,$$

where N is a Poisson process with constant positive intensity λ, $(Y_i,\ i \in \mathbb{N}^*)$ is a sequence of independent and identically distributed random variables with the distribution function F_Y, and the sequence of jump times of N is denoted by $(T_i,\ i \geq 0,\ T_0 = 0)$. Let $(U_t;\ t \geq 0)$ be the $\mathbb{R}^2$-value process defined by

$$U_t = (X_t^*, X_t), \qquad t \geq 0. \tag{2.3}$$

Our aim is to prove the following theorem.

Theorem 2.1. (i) *For all $t > 0$, the law of the pair (X_t^*, X_t) has a density with respect to the Lebesgue measure, denoted by $p(b, a;\ t)$.*

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(ii) *For all $t > 0$ and $a \in \mathbb{R}$, the map $h \mapsto p(a+h, a;\ t)$ has a limit when h goes to 0, denoted by $p(a+, a;\ t)$.*

(iii) *Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be a C_b^3-bounded function with a support in $\{(b, a),\ b > 0,\ b \geq a\}$ such that there exists $\delta > 1$ satisfying $\int_{\mathbb{R}} |\partial_1 \varphi(a, a)|^\delta \,\mathrm{d}a < \infty$. For any $t > 0$,*

$$\begin{aligned}\mathbb{E}(\varphi(U_t)) = \varphi(0,0) &+ \int_0^t \mathbb{E}\left(m\partial_2\varphi(U_s) + \frac{1}{2}\partial_{22}^2\varphi(U_s)\right)\mathrm{d}s \\ &+ \int_0^t \frac{1}{2}\mathbb{E}\left(\partial_1\varphi(X_s, X_s)\frac{p(X_s+, X_s;\ s)}{g(X_s;\ s)}\right)\mathrm{d}s \\ &+ \lambda\int_0^t \mathbb{E}\left(\int_{\mathbb{R}}[\varphi(U_s(y)) - \varphi(U_s)]F_Y(\mathrm{d}y)\right)\mathrm{d}s, \end{aligned} \tag{2.4}$$

where $g(\cdot;\ s)$ is the density of the random variable X_s, $p(x+, \cdot;\ \cdot) = \lim_{y\downarrow x} p(y, \cdot;\ \cdot)$, and

$$U_s = (X_s^*, X_s), \qquad U_s(y) = (\max(X_s^*, X_s + y), X_s + y), \quad s \geq 0. \tag{2.5}$$

Using the definition of a weak solution (see [2]), we have the following corollary.

Corollary 2.1. *The density of the law of the pair (X_t^*, X_t) is a weak solution of the following partial derivative equation on $\{(x^*, x) \in \mathbb{R}^2,\ x^* > \max(x, 0)\}$:*

$$\begin{aligned}\partial_t p(x^*, x;\ t) = &-m\partial_2 p(x^*, x;\ t) + \frac{1}{2}\partial_{22}^2 p(x^*, x; t) \\ &+ \int_{\mathbb{R}}[p(x^*, x - y;\ t)\mathbf{1}_{\{x^* > \max(0, x, x-y)\}} - p(x^*, x;\ t)]\,\mathrm{d}F_Y(y)\end{aligned}$$

for all $x^* > \max(x, 0)$ with the boundary condition

$$0 = mp(x, x;\ t) - \frac{1}{2}\partial_1 p(x, x;\ t) - \partial_2 p(x, x;\ t) + \int_{\mathbb{R}} p(x^*, x - y;\ t)\mathbf{1}_{\{x^* < x\}}\,\mathrm{d}F_Y(\mathrm{d}y).$$

The detailed proof of Theorem 2.1 can be found in Proposition 3.1 and Section 4.

3. Existence of the density of the law of (X_t^*, X_t) and its properties

We note that

$$X_t^* = \max\left\{\left(\sup_{u\in[T_i, \inf(T_{i+1}, t))} X_u,\ i = 0, \ldots, N_t\right), X_t\right\}, \qquad T_0 = 0$$

and use the joint density of $(\tilde{X}_t^*, \tilde{X}_t)$ given by (2.2) to show that the law of the pair (X_t^*, X_t) law has a density which is right continuous on the diagonal; see Proposition 3.1 below which is actually the proof of Theorem 2.1(i) and 2.1(ii).

Proposition 3.1. (i) *For all $t > 0$, the law of the random vector (X_t^*, X_t) admits a density with respect to the Lebesgue measure*

$$\begin{aligned}p(b, a;\ t) = \mathbb{E}\Bigg(\sum_{k=0}^{N_t} \tilde{p}(&b - X_{T_k}, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{T_{k+1}\leq t\}} \\ &- (X_t - X_{T_{k+1}\wedge t}), t \wedge T_{k+1} - T_k)\mathbf{1}_{\Delta_{k,t}}(b, a)\Bigg),\end{aligned}$$

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where $\tilde{p}$ is given by (2.2) and

$$\Delta_{k,t} = \left\{(b,a) \mid b > \max\left(X^*_{T_k}, a + \left[X_{t\wedge T_{k+1}} - \sup_{u\in[T_{k+1},t]} X_u\right]\mathbf{1}_{\{T_{k+1}<t\}}\right)\right\}. \tag{3.1}$$

(ii) *Moreover, for all $a \in \mathbb{R}$ and $t > 0$, the map $h \mapsto p(a+h, a;\ t)$ has a limit when h goes to 0, denoted by $p(a+, a;\ t)$, and*

$$p(a+, a;\ t) = \mathbb{E}\Bigg(\sum_{k=0}^{N_t} \tilde{p}((a - X_{T_k})+, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{T_{k+1}\le t\}} - (X_t - X_{T_{k+1}\wedge t}), t \wedge T_{k+1} - T_k)\mathbf{1}_{\{a\in D_{k,t}\}}\Bigg),$$

where $D_{k,t} := \{a : a \ge \max(X^*_{T_k}, a + [X_{t\wedge T_{k+1}} - \sup_{u\in[T_{k+1},t]} X_u]\mathbf{1}_{\{T_{k+1}<t\}})\}$.

The proof of (i) relies on the following lemma.

Lemma 3.1. *Almost surely for all t,*

$$X^*_t = \max\Big(X_{T_k} + \sup_{u\in[T_k, T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}),\ k = 0, \dots, N_t\Big). \tag{3.2}$$

Moreover, almost surely for all t, there exists a unique k, denoted as N^*_t, such that

$$X^*_t = X_{T_k} + \sup_{u\in[T_k, T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}).$$

Proof. Let t be fixed. We treat the proof in three parts.

(i) Note that

$$X^*_t = \max\Big\{\max\Big(X_{T_k} + \sup_{u\in[T_k, T_{k+1}\wedge t)} (X_u - X_{T_k}),\ k = 0, \dots, N_t\Big), X_t\Big\}. \tag{3.3}$$

For $k \in \mathbb{N}$ and for all $u \in [T_k, T_{k+1})$, $X_u - X_{T_k} = \tilde{X}_u - \tilde{X}_{T_k}$, where $\tilde{X}$ is the continuous process defined in (2.1). Thus, for $k \le N_t$,

$$\sup_{u\in[T_k, T_{k+1}\wedge t)} (X_u - X_{T_k}) = \sup_{u\in[T_k, T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}), \tag{3.4}$$

$$\max\Big(X_{T_{N_t}} + \sup_{u\in[T_{N_t}, T_{N_t+1}\wedge t)} (X_u - X_{T_{N_t}}), X_t\Big) = X_{T_{N_t}} + \sup_{u\in[T_{N_t}, T_{N_t+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_{N_t}}). \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3) yields (3.2).

(ii) Given two integers $i < j$, we have

$$\begin{aligned} &X_{T_j} + \sup_{u\in[T_j, T_{j+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}) \\ &\quad = X_{T_i} + (\tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}) + Y_{i+1} + (X_{T_j} - X_{T_{i+1}}) + \sup_{u\in[T_j, T_{j+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}) \end{aligned}$$

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and

$$\begin{aligned} X_{T_j} + \sup_{u\in[T_j,T_{j+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}) - X_{T_i} - \sup_{u\in[T_i,T_{i+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_i}) \\ = \Big\{ Y_{i+1} + (X_{T_j} - X_{T_{i+1}}) + \sup_{u\in[T_j,T_{j+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}) \Big\} \\ + \Big\{ (\tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}) - \sup_{u\in[T_i,T_{i+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_i}) \Big\}. \end{aligned}$$

The following two random vectors are independent:

$$\begin{aligned} &\Big(\sup_{u\in[T_i,T_{i+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_i});\ \tilde{X}_{T_{i+1}} - \tilde{X}_{T_i} \Big), \\ &Y_{i+1} + (X_{T_j} - X_{T_{i+1}}) + \sup_{u\in[T_j,T_{j+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}) \end{aligned}$$

and the law of the vector $(\sup_{u\in[T_i,T_{i+1}\wedge t]}(\tilde{X}_u - \tilde{X}_{T_i});\ \tilde{X}_{T_{i+1}} - \tilde{X}_{T_i})$ admits a density with respect to the Lebesgue measure; hence, the law of the random variable

$$\sup_{u\in[T_i,T_{i+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_i}) + \tilde{X}_{T_{i+1}} - \tilde{X}_{T_i}$$

has a density with respect to the Lebesgue measure and is independent of

$$Y_{i+1} + (X_{T_j} - X_{T_{i+1}}) + \sup_{u\in[T_j,T_{j+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}).$$

Therefore, $X_{T_j} + \sup_{u\in[T_j,T_{j+1}\wedge t]}(\tilde{X}_u - \tilde{X}_{T_j}) - X_{T_i} - \sup_{u\in[T_i,T_{i+1}\wedge t]}(\tilde{X}_u - \tilde{X}_{T_i})$ is the sum of two independent random variables, each having a density. So, for all t, almost surely, whenever $i \neq j$,

$$X_{T_j} + \sup_{u\in[T_j,T_{j+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_j}) \neq X_{T_i} + \sup_{u\in[T_i,T_{i+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_i}).$$

(iii) In the above statement, we can exchange the 'for all $t > 0$' and 'almost surely', since the processes $(N_t,\ t \geq 0)$ and $((\max(X_{T_k} + \sup_{u\in[T_k,T_{k+1}\wedge t]}(\tilde{X}_u - \tilde{X}_{T_k})),\ k \leq N_t),\ t \geq 0)$ are right continuous. □

Proof of Proposition 3.1. (i) According to Lemma 3.1, let N_t^* denote the index k below which the maximum is reached by

$$X_t^* = \max\Big(X_{T_k} + \sup_{u\in[T_k,T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}),\ k = 0, \dots, N_t \Big).$$

The fact $N_t^* = k$ is equivalent to: the supremum is reached on the interval $[T_k, T_{k+1} \wedge t]$, actually meaning that $X_t^* = \sup_{[T_k,T_{k+1}\wedge t]} X_u$ and we remark that $\sup_{[T_k,T_{k+1}\wedge t]} X_u \geq X_{T_k}^* \vee \sup_{[T_{k+1}\wedge t,t]} X_u$.

On the interval $[T_k, T_{k+1} \wedge t]$, $X_u = X_{T_k} + \tilde{X}_u - \tilde{X}_{T_k}$. Thus, the following inequalities are equivalent to $N_t^* = k$ on the set $\{N_t \geq k\}$:

$$X_{T_k} + \sup_{[T_k,T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \geq X_{T_k}^*,$$

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and

$$\begin{aligned} X_{T_k} + \sup_{[T_k, T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) &\geq \sup_{[T_{k+1}\wedge t, t]} X_u \\ &= \Big[X_{T_{k+1}\wedge t} + \sup_{[T_{k+1}, t]} (X_u - X_{T_{k+1}\wedge t})\Big]\mathbf{1}_{\{T_{k+1}<t\}} + X_t \mathbf{1}_{\{T_{k+1}\geq t\}}. \end{aligned} \tag{3.6}$$

Using $X_{T_{k+1}} = X_{T_k} + \tilde{X}_{T_{k+1}} - \tilde{X}_{T_k} + Y_{k+1}$, (3.6) is equivalent to

$$\begin{aligned} &\sup_{[T_k, T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \\ &\quad \geq \Big[\tilde{X}_{T_{k+1}} - \tilde{X}_{T_k} + Y_{k+1} + \sup_{[T_{k+1}, t]} (X_u - X_{T_{k+1}})\Big]\mathbf{1}_{\{T_{k+1}<t\}} + (\tilde{X}_t - \tilde{X}_{T_k})\mathbf{1}_{\{T_{k+1}\geq t\}}. \end{aligned}$$

As a conclusion, we have

$$\begin{aligned} \{N_t^* = k\} = \Big\{ &\sup_{[T_k, T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \geq X_{T_k}^* - X_{T_k} \Big\} \\ &\cap \Big\{ \sup_{[T_k, T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}) \geq \tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k} \\ &\quad + \Big[Y_{k+1} + \sup_{[T_{k+1}, t]} (X_u - X_{T_{k+1}})\Big]\mathbf{1}_{\{T_{k+1}\leq t\}} \Big\}. \end{aligned}$$

Thus,

$$\{N_t^* = k\} = \Big\{ \Big(\sup_{u\in[T_k, T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}), \tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k} \Big) \in \bar{\Delta}_{k,t} \Big\} \cap \{N_t \leq k\},$$

where

$$\bar{\Delta}_{k,t} = \Big\{ (b, a) \mid b > \max\Big(X_{T_k}^* - X_{T_k}, a + \Big[Y_{k+1} + \sup_{u\in[T_{k+1}, t]} (X_u - X_{T_{k+1}})\Big]\mathbf{1}_{\{T_{k+1}\leq t\}} \Big) \Big\}.$$

Moreover, on the set $\{k \leq N_t\}$, and so on its subset $\{N_t^* = k\}$, we have

$$X_t = X_{T_k} + (\tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k}) + Y_{k+1}\mathbf{1}_{\{t\geq T_{k+1}\}} + (X_t - X_{t\wedge T_{k+1}}).$$

Let Φ be a bounded Borel function, hence

$$\begin{aligned} \mathbb{E}(\Phi(X_t^*, X_t)) &= \mathbb{E}\Big(\sum_{k=0}^{N_t} \Phi(X_t^*, X_t)\mathbf{1}_{\{N_t^*=k\}} \Big) \\ &= \mathbb{E}\Big(\sum_{k=0}^{N_t} \mathbf{1}_{\{N_t^*=k\}} \Phi\Big(X_{T_k} + \sup_{u\in[T_k, T_{k+1}\wedge t]} (\tilde{X}_u - \tilde{X}_{T_k}), X_{T_k} + (\tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k}) \\ &\qquad + Y_{k+1}\mathbf{1}_{\{t\geq T_{k+1}\}} + (X_t - X_{t\wedge T_{k+1}}) \Big) \Big). \end{aligned}$$

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The following four random vectors are independent:

$$(X_{T_k}, X^*_{T_k}), \qquad Y_{k+1}, \qquad \Big(X_t - X_{t\wedge T_{k+1}}, \sup_{u\in[T_{k+1}\wedge t, t[}(X_u - X_{T_{k+1}\wedge t})\Big),$$
$$\Big(\sup_{u\in[T_k, T_{k+1}\wedge t]} \tilde{X}_u - \tilde{X}_{T_k}, \tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k}\Big),$$

and conditionally on $\sigma(\mathcal{F}_{T_k}, Y_{k+1}, (X_u - X_{T_{k+1}},\ u \geq T_{k+1}\wedge t),\ T_k, T_{k+1})$, the law of the random vector

$$\Big(\sup_{u\in[T_k, T_{k+1}\wedge t]} \tilde{X}_u - \tilde{X}_{T_k}, \tilde{X}_{t\wedge T_{k+1}} - \tilde{X}_{T_k}\Big)$$

has a density with respect to the Lebesgue measure $\tilde{p}(b, a, T_{k+1}\wedge t - T_k)$, where $\tilde{p}$ is defined by (2.2). We obtain

$$\mathbb{E}(\Phi(X^*_t, X_t)) = \int \mathbb{E}\Bigg(\sum_{k=0}^{N_t} \Phi(X_{T_k} + b, X_{T_k} + a + Y_{k+1}\mathbf{1}_{\{t\geq T_{k+1}\}} + (X_t - X_{T_{k+1}\wedge t}))\tilde{p}(b, a, T_{k+1}\wedge t - T_k)\mathbf{1}_{\bar{\Delta}_{k,t}}(b, a)\Bigg)\,\mathrm{d}a\,\mathrm{d}b.$$

The change of variable formulas $v = b + X_{T_k}$ and $u = X_{T_k} + a + Y_{k+1}\mathbf{1}_{\{t\geq T_{k+1}\}} + (X_t - X_{T_{k+1}\wedge t})$ conclude the proof.

(ii) Let $a \in \mathbb{R}$ and $t > 0$. The map

$$h \mapsto \sum_{k=0}^{N_t} \tilde{p}(a + h - X_{T_k}, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{T_{k+1}\leq t\}} - (X_t - X_{T_{k+1}\wedge t}), t\wedge T_{k+1} - T_k)\mathbf{1}_{\Delta_{k,t}}(b, a)$$

has a right limit when h goes to 0 since both functions

$$h \mapsto \tilde{p}(a + h, a;\ t) \quad \text{and} \quad h \to \mathbf{1}_{\Delta_k,t}(a + h, a)$$

admit a limit when h decreases to 0. According to Proposition A.1, the family

$$\Bigg(\sum_{k=0}^{N_t} \tilde{p}(a + h - X_{T_k}, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{T_{k+1}\leq t\}} - (X_t - X_{T_{k+1}\wedge t}), t\wedge T_{k+1} - T_k)\mathbf{1}_{\Delta_{k,t}}(b, a)\Bigg)_{h\in[0,1]}$$

is uniformly integrable.

Exchanging the order of the limit and expectation, it follows that $h \mapsto p(a + h, a;\ t)$ has a limit when h decreases to 0 and

$$p(a+, a; t) = \mathbb{E}\Bigg(\sum_{k=0}^{N_t} \tilde{p}((a - X_{T_k})_+, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{T_{k+1}\leq t\}} - (X_t - X_{T_{k+1}\wedge t}), t\wedge T_{k+1} - T_k)\mathbf{1}_{\{a\in D_{k,t}\}}\Bigg),$$

where $D_{k,t} \coloneqq \{a\colon a \geq \max(X^*_{T_k}, a + [X_{t\wedge T_{k+1}} - \sup_{u\in[T_{k+1},t]} X_u]\mathbf{1}_{\{T_{k+1}<t\}})\}$. □

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As a corollary we obtain the law of X_t^*.

Corollary 3.1. *For any $t > 0$, the law of the random variable X_t^* has a density $p^*(\cdot, t)$ given by*

$$p^*(b,t) = 2\mathbb{E}\bigg(\sum_{k=0}^{N_t} \mathrm{e}^{2(b-X_{T_k})m} H_m[(C_{t,k})^+ \mathbf{1}_{\{T_{k+1}<t\}} + (b - X_{T_k}) + m(t \wedge T_{k+1} - T_k), t \wedge T_{k+1} - T_k]\mathbf{1}_{\{b>X^*_{T_k}\}}\bigg),$$

where

$$H_m\colon (x,t) \to \frac{1}{\sqrt{2\pi t}} \exp\bigg[-\frac{x^2}{2t}\bigg] - m\Phi_{\mathrm{G}}\bigg(-\frac{x}{\sqrt{t}}\bigg),$$
$$C_{t,k} = \Big(Y_{k+1} + \sup_{u\in[T_{k+1},t]} (X_u - X_{T_{k+1}})\Big)\mathbf{1}_{\{T_{k+1}\le t\}}.$$

Proof. Let $\tilde{q}$ be a function such that $\tilde{p}(b,a,\cdot) = \tilde{q}(b,a,\cdot)\mathbf{1}_{\{b>a\vee 0\}}$, where

$$\tilde{p}(b,a;\,t) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\bigg[-\frac{(2b-a)^2}{2t} + ma - m^2\frac{t}{2}\bigg]\mathbf{1}_{\{\max(0,a)<b\}}.$$

Note that

$$(2b-a)^2 - 2mta + m^2t^2 = [a-(2b+mt)]^2 - 4bmt;$$

thus, we obtain

$$\tilde{q}(b,a;\,t) = \frac{2\mathrm{e}^{2bm}}{\sqrt{2\pi t}}\bigg(\frac{2b+mt-a}{t}\exp\bigg[-\frac{[a-(2b+mt)]^2}{2t}\bigg] - m\exp\bigg[-\frac{[a-(2b+mt)]^2}{2t}\bigg]\bigg).$$

Hence, for any $A \in \mathbb{R}$,

$$0 \le \int_{-\infty}^{A} \tilde{p}(b,a;\,t)\,\mathrm{d}a = 2\mathrm{e}^{2bm}\mathbf{1}_{\{b>0\}} H_m(x,t), \qquad x = b \wedge A - 2b - mt. \tag{3.7}$$

Let k be fixed and denote $P_k^*(b,t)$ by

$$P_k^*(b,t) := \int_{\mathbb{R}} \tilde{p}(b - X_{T_k}, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{T_{k+1}\le t\}} - (X_t - X_{T_{k+1}\wedge t}), t \wedge T_{k+1} - T_k)\mathbf{1}_{\Delta_{k,t}}(b,a)\,\mathrm{d}a.$$

Then the density of the law of X_t^* is given by

$$p^*(b,t) = \mathbb{E}\bigg(\sum_{k=0}^{N_t} P_k^*(b,t)\bigg).$$

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With the change of variables $u = a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{T_{k+1}\le t\}} - (X_t - X_{t\wedge T_{k+1}})$, it follows that

$$P_k^*(b,t) \coloneqq \int_{\mathbb{R}} \tilde{p}(b - X_{T_k}, u, t \wedge T_{k+1} - T_k)\mathbf{1}_{\Delta_{k,t}}(b, u + X_{T_k} + Y_{k+1}\mathbf{1}_{\{T_{k+1}\le t\}} + (X_t - X_{t\wedge T_{k+1}}))\,\mathrm{d}u.$$

According to the definition of $\Delta_{k,t}$ in (3.1),

$$\mathbf{1}_{\Delta_{k,t}}(b, u + X_{T_k} + Y_{k+1}\mathbf{1}_{\{T_{k+1}\le t\}} + (X_t - X_{t\wedge T_{k+1}})) = \mathbf{1}_{\{b>X^*_{T_k}\}}\mathbf{1}_{\{b>u+X_{T_k}+[Y_{k+1}+\sup_{u\in[T_{k+1},t]}(X_u - X_t)]\mathbf{1}_{\{T_{k+1}\le t\}}\}}.$$

On the event $\{T_{k+1} \le t\}$ (which is equal to $\{k < N_t\}$), we have

$$P_k^*(b,t) = \int_{\mathbb{R}} \tilde{q}(b - X_{T_k}, u, T_{k+1} - T_k)\mathbf{1}_{\{b>X^*_{T_k}\}}\mathbf{1}_{(-\infty,\min(b-X_{T_k}, b-X_{T_k}-C_{t,k})}(u)\,\mathrm{d}u$$

since $C_{t,k} = (Y_{k+1} + \sup_{u\in[T_{k+1},t]}(X_u - X_t))\mathbf{1}_{\{T_{k+1}\le t\}}$. And on the event $\{T_{k+1} > t\}$,

$$P_k^*(b,t) = \int_{\mathbb{R}} \tilde{q}(b - X_{T_k}, u, t - T_k)\mathbf{1}_{\{b>X^*_{T_k}\}}\mathbf{1}_{(-\infty,b-X_{T_k})}(u)\,\mathrm{d}u.$$

We have the following two cases.

Case 1. Applying (3.7) to $A = b - X_{T_k} - C_{t,k}$ with $T_{k+1} - T_k$ and $b - X_{T_k}$ instead of t and b, and $(b - X_{T_k}) \wedge A = b - X_{T_k} - (C_{t,k})^+$ (since $0 \wedge (-x) = -x^+$) on the event $\{T_{k+1} \le t\}$, we have

$$P_k^*(b,t) = 2\mathrm{e}^{2m(b-X_{T_k})}H_m[(C_{t,k})^+ + (b - X_{T_k}) + m(T_{k+1} - T_k), T_{k+1} - T_k].$$

Case 2. Applying (3.7) to $A = b - X_{T_k}$, and taking $t - T_k$ and $b - X_{T_k}$ instead of t and b, on the event $\{T_{k+1} > t\}$, we have

$$P_k^*(b,t) = 2\mathrm{e}^{2(b-X_{T_k})m}H_m[-(b - X_{T_k}) - m(t - T_k), t - T_k].$$

We summarize both cases as

$$P_k^*(b,t) = 2\mathrm{e}^{2(b-X_{T_k})m}H_m[(C_{t,k})^+\mathbf{1}_{\{T_{k+1}\le t\}} + (b - X_{T_k}) + m(t \wedge T_{k+1} - T_k), t \wedge T_{k+1} - T_k]$$

and the proof of Corollary 3.1 is complete. □

4. Proof of Theorem 2.1(iii)

To prove the part (iii) of Theorem 2.1, we proceed as follows. We will compute

$$\lim_{h\to 0} h^{-1}A(t,h) = a(t),$$

where

$$A(t,h) \coloneqq \mathbb{E}(\varphi(U_{t+h}) - \varphi(U_t)).$$

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Then we will use the result of Titchmarsh [10, Section 11.82, p. 368]. If f is a function such that f' is finite everywhere and integrable, then for all $a \le b$, $f(b) - f(a) = \int_a^b f'(s)\,\mathrm{d}s$. From the study of $a(t) := \lim_{h\to 0} h^{-1}A(t,h)$, it is possible to prove that for all $t > t_0 > 0$,

$$\mathbb{E}(\varphi(U_t)) = \int_{t_0}^{t} a(s)\,\mathrm{d}s + \mathbb{E}(\varphi(X^*_{t_0}, X_{t_0})) \quad \text{for all } t > t_0.$$

The last step is to prove the convenient properties of the function a. For all $T > 0$, there exist $v \in (0,1)$, $\xi \ge 0$, and a constant C such that

$$|a(t)| \le C\left[\frac{1}{t^v} + 1 + t^{\xi}\right] \quad \text{for all } t \in (0,T]. \tag{4.1}$$

Now let t_0 go to 0. Using the fact that φ is a continuous and bounded function, X and X^* are right continuous, (4.1), the Lebesgue dominated theorem, and for all $t > 0$, we have

$$\mathbb{E}(\varphi(U_t)) = \int_0^t a(s)\,\mathrm{d}s + \varphi(0,0).$$

Proposition 4.1. *The function $a : t \mapsto \mathbb{E}(\varphi(U_t))$ admits a derivative on $(0,+\infty)$, denoted as a and satisfying (4.1).*

Proof. The idea is to split $A(t,h)$ into three parts according to the values of $N_{t+h} - N_t$; thus,

$$A(t,h) = \sum_{i=0}^{2} A_i(t,h),$$

where

$$A_i(t,h) := \mathbb{E}([\varphi(U_{t+h}) - \varphi(U_t)]\mathbf{1}_{\{N_{t+h}-N_t=i\}}), \qquad i = 0,1,$$
$$A_2(t,h) := \mathbb{E}([\varphi(U_{t+h}) - \varphi(U_t))]\mathbf{1}_{\{N_{t+h}-N_t\ge 2\}}).$$

□

Lemma 4.1. *Under the hypothesis of Theorem 2.1,*

$$\lim_{h\to 0} h^{-1}A_2(t,h) = 0.$$

Proof. By our hypothesis, φ is bounded and we obtain

$$|A_2(t,h)| \le 2\|\varphi\|_\infty \mathbb{P}(N_{t+h} - N_t \ge 2) \le 2\|\varphi\|_\infty(1 - \mathrm{e}^{-\lambda h} - \lambda h \mathrm{e}^{-\lambda h}).$$

Thus, $\lim_{h\to 0} h^{-1}A_2(t,h) = 0$. □

Using this lemma, in addition to the next three propositions, we can prove Theorem 2.1(iii): Lemma 4.1 treats the $A_2(t,h)$ term while Proposition 4.2 treats the $A_1(t,h)$ term. Propositions 4.3 and 4.4 treat the $A_0(t,h)$ term.

Proposition 4.2. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$, a C_b^3-bounded function. Then*

$$\lim_{h\to 0} h^{-1}A_1(t,h) = \lambda\mathbb{E}\int_{\mathbb{R}} [\varphi(U_t(y)) - \varphi(U_t)]F_Y(\mathrm{d}y),$$

where U_t is defined by (2.3), $U_t(y)$ by (2.5), and F_Y is the distribution function of the random variable Y_1.

In this proposition, the term on the right-hand side comes from the last term of (2.4).

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Proof of Proposition 4.2. Introducing the $\varphi(U_t(Y_{N_{t+h}}))$ term, let $A_1(t,h) \coloneqq A_{1,1}(t,h) + A_{1,2}(t,h)$, where

$$A_{1,1}(t,h) = \mathbb{E}(\{\varphi(U_t(Y_{N_{t+h}})) - \varphi(U_t)\}\mathbf{1}_{\{N_{t+h}=N_t+1\}}),$$
$$A_{1,2}(t,h) = \mathbb{E}(\{\varphi(U_{t+h}) - \varphi(U_t(Y_{N_{t+h}}))\}\mathbf{1}_{\{N_{t+h}=N_t+1\}}).$$

Since φ is of class $\mathcal{C}^1$ with bounded derivative, Lemma A.1 implies that on the event $\{N_{t+h} - N_t = 1\}$,

$$|\varphi(U_{t+h}) - \varphi(U_t(Y_{N_{t+h}}))| \le \|\nabla\varphi\|_\infty\Big(2\sup_{0\le u\le h}|\tilde{X}_{t+u} - \tilde{X}_t| + h\Big).$$

Thus,

$$h^{-1}A_{1,2}(t,h) \le \lambda \mathrm{e}^{-\lambda h}\|\nabla\varphi\|_\infty\Big(2\sup_{0\le u\le h}|\tilde{X}_{t+u} - \tilde{X}_t| + h\Big) \to 0 \quad \text{when } h \to 0.$$

Indeed, the process $\tilde{X}$ satisfies $0 \le \tilde{X}_h \le |m|h + W_h^*$ and the Burkholder–Davis–Gundy inequality for all $i \ge 1$ implies that there exists a constant $C_i > 0$ such that $\mathbb{E}((W_h^*)^i) \le C_i h^{i/2}$; hence, for $h \le 1$,

$$\mathbb{E}((\tilde{X}_h^*)^i) \le C_i h^{i/2}; \tag{4.2}$$

thus, $\mathbb{E}(\tilde{X}_h^*) \le C\sqrt{h}$.

We now deal with $A_{1,1}(t,h)$ in order to show that

$$\lim_{h\to 0} h^{-1}A_{1,1}(t,h) = \lambda\mathbb{E}\int_{\mathbb{R}}[\varphi(U_t(y)) - \varphi(U_t)]F_Y(\mathrm{d}y).$$

On the event $\{N_t = n, N_{t+h} = n+1\}$, the equality $U_t(Y_{N_{t+h}}) = U_t(Y_{n+1})$ holds. The independence properties arising from the structure of the process X, the use of the laws of Y_i and T_n, the decomposition of T_{n+1}, $T_{n+1} = T_n + S_{n+1}$, and the conditioning to $\mathcal{F}_{T_n}$ yield

$$\begin{aligned} A_{1,1}(t,h) &= \sum_{n=0}^{+\infty}\mathbb{E}(\mathbf{1}_{\{T_n\le t<T_{n+1}\le t+h<T_{n+2}\}}(\varphi(U_t(Y_{n+1})) - \varphi(U_t))) \\ &= \sum_{n=0}^{+\infty}\mathbb{E}\bigg(\mathbf{1}_{\{T_n\le t\}}\int_{t-T_n}^{t+h-T_n}\mathrm{d}s\int_{\mathbb{R}}(\varphi(U_t(y)) - \varphi(U_t))F_Y(\mathrm{d}y)\lambda\mathrm{e}^{-\lambda(t+h-T_n)}\bigg). \end{aligned}$$

By assumption, the function φ is bounded and when h goes to 0, Lebesgue's dominated convergence theorem yields

$$\lim_{h\to 0} h^{-1}A_{1,1}(t,h) = \sum_{n=0}^{+\infty}\mathbb{E}\bigg(\lambda\mathrm{e}^{-\lambda(t-T_n)}\,\mathbf{1}_{\{T_n\le t\}}\int_{\mathbb{R}}(\varphi(U_t(y)) - \varphi(U_t))F_Y(\mathrm{d}y)\bigg).$$

Since $\mathrm{e}^{-\lambda(t-T_n)}\mathbf{1}_{\{T_n\le t\}} = \mathbb{E}(\mathbf{1}_{\{T_n\le t<T_{n+1}\}}/\mathcal{F}_t)$, it follows that

$$\begin{aligned} \lim_{h\to 0} h^{-1}A_{1,1}(t,h) &= \sum_{n=0}^{+\infty}\mathbb{E}\bigg(\lambda\,\mathbf{1}_{\{T_n\le t<T_{n+1}\}}\int_{\mathbb{R}}(\varphi(U_t(y)) - \varphi(U_t))F_Y(\mathrm{d}y)\bigg) \\ &= \mathbb{E}\bigg(\lambda\int_{\mathbb{R}}(\varphi(U_t(y)) - \varphi(U_t))F_Y(\mathrm{d}y)\bigg). \end{aligned}$$

□

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Now we turn our attention to $h^{-1}A_0(t,h)$ when h goes to 0. On the event $\{N_{t+h}-N_t=0\}$, $T_{N_t}=T_{N_{t+h}}$; hence, $X^*_{T_{N_{t+h}}}=X^*_{T_{N_t}}$, $X_{T_{N_{t+h}}}=X_{T_{N_t}}$, and

$$X_{t+h}=X_t+mh+W_{t+h}-W_t, \qquad X^*_{t+h}=\max(X^*_t, X_t+mh+W_{t+h}-W_t).$$

Using the Markov property at time t and the fact that the processes N and $\tilde{X}$ are independent, we have

$$A_0(t,h)=\mathrm{e}^{-\lambda h}\mathbb{E}(\mathbb{E}(\varphi(\max(x^*, x+\tilde{X}^*_h), x+\tilde{X}_h)-\varphi(x^*,x))_{|x^*=X^*_t,\, x=X_t}).$$

Introduce

$$a_0(h,x^*,x) \coloneqq \mathbb{E}(\varphi(\max(x^*, x+\tilde{X}^*_h), x+\tilde{X}_h)-\varphi(x^*,x)).$$

To study the $a_0(h,x^*,x)$ term, we make a Taylor expansion at a neighborhood of (x^*,x); thus,

$$\begin{aligned} a_0(h,x^*,x) \coloneqq{}& \partial_2\varphi(x^*,x)mh+\tfrac{1}{2}\partial^2_{22}\varphi(x^*,x)[m^2h^2+h] \\ &+\partial^2_{1,2}\varphi(x^*,x)\mathbb{E}([\max(x^*,x+\tilde{X}^*_h)-x^*]\tilde{X}_h) \\ &+\tfrac{1}{2}\partial^2_{1,1}\varphi(x^*,x)\mathbb{E}([\max(x^*,x+\tilde{X}^*_h)-x^*]^2)+R_0(h,x^*,x) \\ &+\partial_1\varphi(x^*,x)\mathbb{E}([\max(x^*,x+\tilde{X}^*_h)-x^*]), \end{aligned}$$

where (with ∇^i denoting the derivative of order i)

$$|R_0(h,x^*,x)|\le 4\|\nabla^3\varphi\|_\infty[\mathbb{E}(|\max(x^*,x+\tilde{X}^*_h)-x^*|^3)+\mathbb{E}(|\tilde{X}_h|^3)].$$

This allows us to write

$$A_0(t,h)=\sum_{i=1}^{3}A_{0,i}(t,h), \qquad A_{0,i}(t,h)\coloneqq\mathbb{E}(a_{0,i}(h,x^*;\,x)_{|x^*=X^*_t,\,x=X_t}), \tag{4.3}$$

where

$$\begin{aligned} a_{0,1}(h,x^*,x) \coloneqq{}& \partial_2\varphi(x^*,x)mh+\tfrac{1}{2}\partial^2_{22}\varphi(x^*,x)[m^2h^2+h], \\ a_{0,2}(h,x^*,x) \coloneqq{}& \partial^2_{1,2}\varphi(x^*,x)\mathbb{E}([\max(x^*,x+\tilde{X}^*_h)-x^*]\tilde{X}_h) \\ &+\tfrac{1}{2}\partial^2_{1,1}\varphi(x^*,x)\mathbb{E}([\max(x^*,x+\tilde{X}^*_h)-x^*]^2)+R_0(h,x^*,x), \\ a_{0,3}(h,x^*,x) \coloneqq{}& \partial_1\varphi(x^*,x)\mathbb{E}([\max(x^*,x+\tilde{X}^*_h)-x^*]). \end{aligned}$$

Proposition 4.3. *Let $\varphi\colon\mathbb{R}^2\to\mathbb{R}$ be a C^3_b-bounded function. Then, for any $t>0$,*

$$\lim_{h\to 0}h^{-1}(A_{0,1}+A_{0,2})(t,h)=\mathbb{E}\big(\partial_2\varphi(U_t)m+\tfrac{1}{2}\partial^2_{22}\varphi(U_t)\big).$$

Proof. Since $\tilde{X}$ and $\tilde{X}^*$ are continuous processes and φ is a three-times differentiable function with bounded derivatives, it follows that

$$\lim_{h\to 0}h^{-1}A_{0,1}(t,h)=\mathbb{E}\big(\partial_2\varphi(U_t)m+\tfrac{1}{2}\partial^2_{22}\varphi(U_t)\big).$$

The second term satisfies, under hypothesis of Theorem 2.1,

$$\lim_{h\to 0}h^{-1}A_{0,2}(t,h)=0.$$

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Indeed, we first note that $\max(x^*, x + \tilde{X}_h^*) - x^* = (\tilde{X}_h^* - (x^* - x))^+ \le \tilde{X}_h^* \mathbf{1}_{\{\tilde{X}_h^* > x^* - x\}}$. Using the Cauchy–Schwarz inequality and (4.2), there exists a constant D_i such that

$$\mathbb{E}([\max(x^*, x + \tilde{X}_h^*) - x^*]^i) \le D_i h^{i/2} \sqrt{\mathbb{P}(\tilde{X}_h^* > x^* - x)}.$$

The function φ is three-times differentiable with bounded differential. From the expression of $a_{0,2}$, we deduce that there exists a constant $C > 0$ such that

$$A_{0,2}(t,h) \le \left[\sum_{i=2}^{3} \|\nabla^i \varphi\|\right] C \sum_{i=2}^{3} h^{i/2} \mathbb{E}\Big(\sqrt{\mathbb{P}(\tilde{X}_h^* > x^* - x)}_{|x^* = X_t^*,\, x = X_t}\Big).$$

As the law of the pair (X_t^*, X_t) has a density with respect to the Lebesgue measure on $\mathbb{R}^2$ (see Proposition 3.1) almost surely $X_t^* > X_t$, and it follows from the Lebesgue dominated convergence theorem that

$$\lim_{h \to 0} h^{-1} A_{0,2}(t,h) = 0. \qquad \square$$

Now we deal with the $h^{-1} A_{0,3}(t,h)$ term.

Proposition 4.4. *Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be a C_b^3-bounded function such that there exists $\delta > 1$ satisfying $\int_{\mathbb{R}} |\partial_1 \varphi(a,a)|^\delta \, \mathrm{d}a < \infty$. Then, for any $t > 0$,*

$$\lim_{h \to 0} \frac{1}{h} \mathbb{E}(\partial_1 \varphi(U_t) \mathbb{E}(\max(x^*, x + \tilde{X}_h^*) - x^*)_{x^* = X_t^*,\, x = X_t}) = \frac{1}{2} \mathbb{E}\left(\partial_1 \varphi(X_t, X_t) \frac{p(X_t+, X_t, t)}{g(X_t, t)}\right).$$

The proof of Proposition 4.4 relies on the following two lemmas.

Lemma 4.2. *Recall that U_t denotes (X_t^*, X_t). For all $t > 0$, when h goes to 0,*

$$A_{0,3}(t,h) = \mathbb{E}(\partial_1 \varphi(U_t)(\sqrt{h}|G| - (X_t^* - X_t))^+) + o(h),$$

where G is a Gaussian random variable with mean 0 and variance 1 that is independent of (X_t^*, X_t).

Proof. Recall that $\tilde{X}_u = mu + W_u$. Then $\sup_{0 \le u \le h} \tilde{X}_u \le |m|h + \sup_{0 \le u \le h} W_u$. The roles of $\tilde{X}$ and W are symmetric and so

$$\sup_{u \le h} W_u \le |m|h + \sup_{0 \le u \le h} \tilde{X}_u, \qquad \Big| \sup_{0 \le u \le h} W_u - \sup_{0 \le u \le h} \tilde{X}_u \Big| \le |m|h.$$

We can write the last term in (4.3) as

$$A_{0,3}(t,h) = \mathbb{E}\Big(\partial_1 \varphi(U_t) \mathbb{E}\Big(\Big(\sup_{u \le h} \tilde{X}_u - \delta\Big)^+\Big)_{|\delta = X_t^* - X_t}\Big).$$

Using

$$|(g - \delta)^+ - (g' - \delta)^+| \le |g - g'| [\mathbf{1}_{\{g \ge \delta\}} + \mathbf{1}_{\{g' \ge \delta\}}]$$

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for $g = \sup_{u\le h} \tilde{X}_u$, $g' = \sup_{u\le h} W_u$, and the fact that $\partial_1\varphi$ is bounded, we deduce that

$$\Big|A_{0,3}(t,h) - \mathbb{E}\Big(\partial_1\varphi(U_t)\mathbb{E}\Big(\Big(\sup_u W_u - \delta\Big)^+\Big)_{|\delta=X_t^*-X_t}\Big)\Big|$$
$$\le |m|h\|\partial_1\varphi\|_\infty \mathbb{E}\Big(\Big[\mathbb{P}\Big(\sup_{0\le u\le h} W_u \ge \delta\Big) + \mathbb{P}\Big(\sup_{0\le u\le h} \tilde{X}_u \ge \delta\Big)\Big]_{\delta=X_t^*-X_t}\Big).$$

Since the distribution function of the law of the pair (X_t^*, X_t) admits a density (see Proposition 3.1), almost surely

$$\lim_{h\to 0}\Big[\mathbb{P}\Big(\sup_{0\le u\le h} W_u \ge \delta\Big) + \mathbb{P}\Big(\sup_{0\le u\le h} \tilde{X}_u \ge \delta\Big)\Big]_{\delta=X_t^*-X_t} = 0$$

and the Lebesgue dominated theorem yields

$$\mathbb{E}\Big(\Big[\mathbb{P}\Big(\sup_{0\le u\le h} W_u \ge \delta\Big) + \mathbb{P}\Big(\sup_{0\le u\le h} \tilde{X}_u \ge \delta\Big)\Big]_{\delta=X_t^*-X_t}\Big) = o_h(1).$$

The random variable $\sup_{0\le u\le h} W_u$ has the same law as $|W_h|$ (see [9, Proposition 3.7]) which is the law of $\sqrt{h}|G|$, where G is a Gaussian random variable $\mathcal{N}(0,1)$ and

$$A_{0,3}(t,h) = \mathbb{E}(\partial_1\varphi(U_t)\mathbb{E}((\sqrt{h}|G| - \delta)^+)_{|\delta=X_t^*-X_t}) + o(h).$$

Using Fubini's theorem

$$A_{0,3}(t,h) = \mathbb{E}(\partial_1\varphi(U_t)(\sqrt{h}|G| - (X_t^* - X_t))^+) + o(h),$$

where G is a Gaussian random variable with mean 0 and variance 1 that is independent of (X_t^*, X_t). □

Lemma 4.3. *For all $t > 0$, when h goes to 0,*

$$A_{0,3}(t,h) = \mathbb{E}(\partial_1\varphi(X_t, X_t)(\sqrt{h}|G| - (X_t^* - X_t))^+) + o(h),$$

where G is a Gaussian random variable with mean 0 *and variance* 1 *that is independent of* (X_t^*, X_t).

Proof. The function φ is of class C_b^3. Then

$$|\mathbb{E}([\partial_1\varphi(X_t, X_t) - \partial_1\varphi(U_t)](\sqrt{h}|G| - (X_t^* - X_t))^+)|$$
$$\le \|\partial_{1,2}^2\varphi\|_\infty \mathbb{E}(|X_t^* - X_t|(\sqrt{h}|G| - (X_t^* - X_t))^+)$$
$$\le \|\partial_{1,2}^2\varphi\|_\infty h\mathbb{E}(|G|^2\mathbf{1}_{\{|X_t^*-X_t|\le\sqrt{h}|G|\}}).$$

Since the law of the pair (X_t^*, X_t) has a density (see Proposition 3.1), almost surely

$$\lim_{h\to 0} |G|^2\mathbf{1}_{\{|X_t^*-X_t|\le\sqrt{h}|G|\}} = 0.$$

From the Lebesgue dominated theorem and Lemma 4.2, it follows that

$$A_{0,3}(t,h) = \mathbb{E}(\partial_1\varphi(X_t, X_t)(\sqrt{h}|G| - (X_t^* - X_t))^+) + o(h).$$ □

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Proof of Proposition 4.4. Using the density of the law of the pair (X_t^*, X_t) (see Proposition 3.1) and Lemma 4.3,

$$A_{0,3}(t,h) = \int_{\mathbb{R}^2\times[0,+\infty)} \partial_1\varphi(a,a)(\sqrt{h}g - (b-a))^+ p(b,a;\,t)\frac{2}{\sqrt{2\pi}}\mathrm{e}^{-g^2/2}\,\mathrm{d}b\,\mathrm{d}a\,\mathrm{d}g + o(h).$$

We perform the change of variable $b = a + u\sqrt{h}$. Thus,

$$A_{0,3}(t,h) = h\int_{\mathbb{R}\times[0,+\infty)^2} \partial_1\varphi(a,a)(g-u)^+ p(a+\sqrt{h}u,a;\,t) \times \frac{2}{\sqrt{2\pi}}\mathrm{e}^{-g^2/2}\,\mathrm{d}a\,\mathrm{d}u\,\mathrm{d}g + o(h). \tag{4.4}$$

From Proposition 3.1(ii), $\lim_{h\to 0} p(a+\sqrt{h}u,a;\,t) = p(a+,a;\,t)$.

To exchange the order of the limit and the integral, we prove that the family of applications

$$l_h\colon (a,u,g) \mapsto \varphi(a,a)(g-u)^+ p(a+\sqrt{h}u,a;\,t)\frac{2}{\sqrt{2\pi}}\mathrm{e}^{-g^2/2}, \qquad h\in(0,1]$$

is uniformly integrable or that there exists $\delta > 1$ such that $(l_h,\ h\in(0,1])$ is bounded in $\in L^\delta$.

According to Proposition A.1 and (A.6), for any $\delta > 1$ there exist $(\gamma,\beta,\nu) \in [0,1)^3$ and $\xi \geq 0$ such that

$$p^\delta(a+u\sqrt{h},a,t) \leq C(\delta,T,m)\left[\frac{1}{t^\nu}\frac{1}{|a+2u\sqrt{h}+mt|^\gamma} + \frac{1}{t^\beta} + 1 + t^\xi\right]\mathrm{e}^{6(m^+)^2\delta t}.$$

We bound $|a+2u\sqrt{h}+mt|^{-\gamma}$ by 1 on $\{u, |a+2u\sqrt{h}+mt| \geq 1\}$; thus,

$$\frac{1}{|a+2u\sqrt{h}+mt|^\gamma} \leq 1 + \frac{1}{|a+2u\sqrt{h}+mt|^\gamma}\mathbf{1}_{\{|a+2u\sqrt{h}+mt|\leq 1\}}.$$

Then we bound $(g-u)^+$ by g for $u\in[0,g]$ and integrate with respect to u between 0 and g. Thus,

$$\begin{aligned}\int_0^g p^\delta(a+u\sqrt{h},a,t)(g-u)^\delta\,\mathrm{d}u &\leq g^\delta C(\delta,T,m)\int_0^g\left[\frac{1}{t^\nu} + \frac{1}{t^\nu|a+2u\sqrt{h}+mt|^\gamma}\mathbf{1}_{\{|a+2u\sqrt{h}+mt|\leq 1\}}\right.\\ &\qquad\left. + \frac{1}{t^\beta} + 1 + t^\xi\right]\mathrm{e}^{6(m^+)^2\delta t}\,\mathrm{d}u.\end{aligned}$$

Setting $x = u + (a+mt)/2\sqrt{h}$, we have

$$\int_0^g \frac{1}{|a+2u\sqrt{h}+mt|^\gamma}\mathbf{1}_{\{|a+2u\sqrt{h}+mt|\leq 1\}}\,\mathrm{d}u \leq \int_{\mathbb{R}} |x|^{-\gamma}\,\mathrm{d}x,$$

$$\begin{aligned}&\int_0^g p^\delta(a+u\sqrt{h},a,t)(g-u)^\delta\,\mathrm{d}u\\ &\quad\leq C(\delta,T,m)g^\delta\left[g\left(t^{-\nu}+1+\frac{1}{t^\beta}+t^\xi\right) + \frac{1}{t^\nu}\int_{\mathbb{R}}|x|^{-\gamma}\,\mathrm{d}x\right]\mathrm{e}^{6(m^+)^2\delta t},\end{aligned}$$

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and using the fact that on $\mathbb{R}^+$, with both g^δ and $g^{\delta+1} \le 1 + g^{\delta+1}$, this leads to

$$\int_0^g p^\delta(a + u\sqrt{h}, a, t)(g-u)^\delta \,\mathrm{d}u \le C(\delta, T, m)(1 + g^{\delta+1})\left[t^{-\nu} + 1 + \frac{1}{t^\beta} + t^\xi + \frac{1}{t^\nu}\frac{1}{1-\gamma}\right]\mathrm{e}^{6(m^+)^2\delta t}.$$

Since $a \mapsto \partial_1\varphi(a, a)$ has a finite δ moment, according to the assumption in Proposition 4.4,

$$\begin{aligned}
&\sup_{h\in(0,1]} \int_{\mathbb{R}\times[0,+\infty)^2} \partial_1\varphi(a,a)^\delta((g-u)^+)^\delta p(a+\sqrt{h}u, a;\, t)^\delta \left(\frac{2}{\sqrt{2\pi}}\mathrm{e}^{-g^2/2}\right)^\delta \mathrm{d}a\,\mathrm{d}u\,\mathrm{d}g \\
&\quad\le \|\partial_1\varphi(\cdot,\cdot)^\delta\|_1 C(\delta, T, m)\left[t^{-\nu} + 1 + \frac{1}{t^\beta} + t^\xi + \frac{1}{t^\nu}\frac{1}{1-\gamma}\right] \\
&\qquad\times \mathrm{e}^{6(m^+)^2\delta t}\int_0^\infty (1 + g^{\delta+1})\left(\frac{2}{\sqrt{2\pi}}\mathrm{e}^{-g^2/2}\right)^\delta \mathrm{d}g \\
&\quad< +\infty. \qquad (4.5)
\end{aligned}$$

Then we can exchange the order of the limit and integration in (4.4) to obtain

$$\lim_{h\to 0} h^{-1}A_{0,3}(t,h) = \int_{\mathbb{R}\times[0,+\infty)^2} \partial_1\varphi(a,a)(g-u)^+ p(a+, a; t)\frac{2}{\sqrt{2\pi}}\mathrm{e}^{-g^2/2}\,\mathrm{d}a\,\mathrm{d}u\,\mathrm{d}g.$$

Integrating initially in u and noting that

$$\int_0^\infty g^2 \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-g^2/2}\,\mathrm{d}g = \frac{1}{2},$$

we deduce

$$\lim_{h\to 0} h^{-1}A_{0,3}(t,h) = \frac{1}{2}\int_{\mathbb{R}} \partial_1\varphi(a,a)p(a+, a;\, t)\,\mathrm{d}a.$$

Recalling that $g(\cdot, t)$ denotes the density of X_t, we obtain

$$\lim_{h\to 0} h^{-1}A_{0,3}(t,h) = \frac{1}{2}\mathbb{E}\left(\partial_1\varphi(X_t, X_t)\frac{p(X_t+, X_t;\, t)}{g(X_t;\, t)}\right). \qquad \square$$

Proof of Proposition 4.1. Using Propositions 4.2–4.4, we arrive at

$$\begin{aligned}
&\lim_{h\to 0}\frac{\mathbb{E}(\varphi(U_{t+h}) - \varphi(U_t))}{h} = a(t) \quad \text{for all } t \in (0, T], \\
&a(t) := \lambda\mathbb{E}\int_{\mathbb{R}}[\varphi(U_t(y)) - \varphi(U_t)]F_Y(\mathrm{d}y) + \mathbb{E}\left(\partial_2\varphi(U_t)m + \frac{1}{2}\partial_{22}^2\varphi(U_t)\right) \\
&\qquad + \frac{1}{2}\mathbb{E}\left(\partial_1\varphi(X_t, X_t)\frac{p(X_t+, X_t;\, t)}{g(X_t, t)}\right).
\end{aligned}$$

The first two terms are bounded, so we have to check only (4.1) on the third term. Using (4.5) and the bound $t^{-\nu} \le 1 + t^{-v}$, $t^{-\beta} \le 1 + t^{-v}$ for all $v \in [\nu \vee \beta, 1)$, for all $T > 0$ there exist $\xi \ge 0$ and a constant C such that

$$\left|\frac{1}{2}\mathbb{E}\left(\partial_1\varphi(X_t, X_t)\frac{p(X_t+, X_t, t)}{g(X_t, t)}\right)\right| \le C\left[\frac{1}{t^v} + 1 + t^\xi\right] \quad \text{for all } t \in (0, T].$$

Thus, the function a satisfies (4.1) and the proof of Proposition 4.1 is complete. $\square$

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5. Conclusion

The aim of this paper was to have a complete study of the law of one Lévy process X and its running supremum. Recall that X^* is not Markovian, but the pair $U \coloneqq (X^*, X)$ is. In Section 2 we stated the main result (Theorem 2.1): the density of the law of U_t, its right continuity on the diagonal, and its characterization as a weak solution of a partial differential equation.

To complete the study of the survival probability initiated by Coutin and Dorobantu [4], we stated the marginal density of the law of X_t^* (Corollary 3.1). One interpretation of this could lead to the proof of regularity of the survival probability on $\mathbb{R}^+ \times \mathbb{R}$, meaning

$$(b, t) \to \mathbb{P}(\tau_b > t) = \mathbb{P}(X_t^* \le b) = \int_0^b p^*(x, t)\,\mathrm{d}x.$$

Another interpretation could lead to the study of filtering, for instance, to generalize the work of Ngom [8] to the case of incomplete observation.

Appendix A.

Lemma A.1. *On the event* $\{N_{t+h} = N_t + 1\}$*, we have*

$$|U_t(Y_{N_{t+h}}) - U_{t+h}| \le 2 \sup_{0\le u\le h} |\tilde{X}_{t+u} - \tilde{X}_t|,$$

where $U_s(y)$ *is defined in (2.5) and* $\tilde{X}$ *is defined in (2.1).*

Proof. We treat the proof by dealing with each event in turn.

(i) On the event $\{N_{t+h} - N_t = 1\} = \bigcup_n \{N_t = n, N_{t+h} - N_t = 1\}$, we compute

$$U_t(Y_{N_{t+h}}) = (\max(X_t^*, X_t + Y_{N_{t+h}});\ X_t + Y_{N_{t+h}}) \quad \text{and} \quad U_{t+h} = (X_{t+h}^*, X_{t+h}).$$

(ii) On the event $\{N_t = n, N_{t+h} = n + 1\}$, we have

$$U_t(Y_{N_{t+h}}) = (\max(X_t^*, X_t + Y_{n+1});\ X_t + Y_{n+1}) \quad \text{and} \quad U_{t+h} = (X_{t+h}^*, X_{t+h}).$$

(iii) We bound $|U_t(Y_{N_{t+h}}) - U_{t+h}|$ component by component.

Second component. On the event $\{N_t = n, N_{t+h} = n + 1\}$, we have

$$X_{t+h} = X_t + Y_{n+1} + (X_{t+h} - X_t - Y_{n+1}).$$

Since there is only one jump at time T_{n+1} for the process X between t and $t + h$, we have $X_{t+h} - X_t - Y_{n+1} = \tilde{X}_{t+h} - \tilde{X}_t$ and

$$|X_{t+h} - X_t - Y_{n+1}|\mathbf{1}_{\{N_t=n,\, N_{t+h}=n+1\}} \le \sup_{0\le u\le h} |\tilde{X}_{t+u} - \tilde{X}_t|\mathbf{1}_{\{N_t=n,\, N_{t+h}=n+1\}}. \tag{A.1}$$

First component. We have

$$X_{t+h}^* - \max(X_t^*, X_t + Y_{n+1})$$

with

$$\begin{aligned} X_{t+h}^* &= \max\Big(X_t^*, X_t + \sup_{t\le u<T_{n+1}} (\tilde{X}_u - \tilde{X}_t), X_t + Y_{n+1} + (\tilde{X}_{T_{n+1}} - \tilde{X}_t) \\ &\qquad + \sup_{T_{n+1}\le u\le t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}})\Big) \\ &= \max\Big(X_t^*, X_t + \sup_{t\le u<T_{n+1}} (\tilde{X}_u - \tilde{X}_t), X_t + Y_{n+1} + \sup_{T_{n+1}\le u\le t+h} (\tilde{X}_u - \tilde{X}_t)\Big). \end{aligned}$$

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(iv) On the event $\{X_t^* \geq X_t + Y_{n+1}\}$,

$$X_{t+h}^* - \max(X_t^*, X_t + Y_{n+1}) = 0 \vee (X_t + \sup_{t \leq u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - X_t^*) \vee \Big(X_t + Y_{n+1} + \sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t) - X_t^*\Big).$$

Since $X_t \leq X_t^*$, we have

$$X_t + \sup_{t \leq u \leq T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - X_t^* \leq \sup_{t \leq u \leq T_{n+1}} (\tilde{X}_u - \tilde{X}_t) \leq \sup_{t \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t)$$

and, on the event $\{X_t^* \geq X_t + Y_{n+1}\}$,

$$X_t + Y_{n+1} + \sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t) - X_t^* \leq \sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t) \leq \sup_{t \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_t).$$

Globally, on this event,

$$0 \leq X_{t+h}^* - \max(X_t^*, X_t + Y_{n+1}) \leq \sup_{0 \leq u \leq h} |\tilde{X}_{t+u} - \tilde{X}_t|. \tag{A.2}$$

(v) On the event $\{X_t^* < X_t + Y_{n+1}\}$, the first component is equal to

$$(X_t^* - X_t - Y_{n+1}) \vee \Big(\sup_{t \leq u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - Y_{n+1}\Big) \vee \Big(\sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}})\Big). \tag{A.3}$$

On this event, with the first element in (A.3) being $(X_t^* - X_t - Y_{n+1}) \leq 0$ and the third one being nonnegative, we see that the first component is $(\sup_{t \leq u < T_{n+1}} (\tilde{X}_u - \tilde{X}_t) - Y_{n+1}) \vee (\sup_{T_{n+1} \leq u \leq t+h} (\tilde{X}_u - \tilde{X}_{T_{n+1}}))$.

Globally, as a conclusion,

$$|X_{t+h}^* - \max(X_t^*, X_t + Y_{n+1})| \mathbf{1}_{\{N_t = n,\, N_{t+h} = n+1\}} \leq \sup_{t \leq u \leq t+h} |\tilde{X}_u - \tilde{X}_t|. \tag{A.4}$$

The result follows from (A.1), (A.2), and (A.4). □

A.1. Integrability properties

In what follows, P is the random field defined by

$$P(b, a; t) := \sum_{k=0}^{N_t} \tilde{p}(b - X_{T_k}, a - X_{T_k} - Y_{k+1} \mathbf{1}_{\{t > T_{k+1}\}} - (X_t - X_{\min(t, T_{k+1})});\ \min(t, T_{k+1}) - T_k), \tag{A.5}$$

which satisfies (from Proposition 3.1)

$$p(b, a, t) \leq \mathbb{E}(P(b, a;\ t)). \tag{A.6}$$

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Proposition A.1. *Let $\delta \in [1, \frac{3}{2})$. Then there exist $(v, \gamma, \beta) \in [0, 1)^3$ and $\xi \geq 0$ such that, for all $T > 0$, there exists a constant $C(\delta, T, m)$ satisfying, for all $t \in (0, T]$ and $b > \max(a, 0)$,*

$$\mathbb{E}(P(b, a;\, t)^{\delta}) \leq C(\delta, T, m)\left[\frac{1}{t^{v}}\frac{1}{(2b - a + mt)^{\gamma}} + \frac{1}{t^{\beta}} + 1 + t^{\xi}\right]\mathrm{e}^{6(m^{-})^2\delta t}, \tag{A.7}$$

where $x^{+} = \max(x, 0)$ and $x^{-} = \max(-x, 0)$ for any $x \in \mathbb{R}$.

The proof is based on the following three lemmas. We have the following estimations on $\tilde{p}$.

Lemma A.2. *Recall that, for all $t > 0$, the law of the pair $(\tilde{X}_t^{*}, \tilde{X}_t)$ has a density with respect to the Lebesgue measure given by*

$$\tilde{p}(b, a;\, t) = \frac{2(2b - a)}{\sqrt{2\pi t^3}}\exp\left(-\frac{(2b - a)^2}{2t} + ma - \frac{m^2 t}{2}\right)\mathbf{1}_{\{b > \max(0,a)\}}.$$

For all $\alpha \in (0, 1)$, there exists a constant $C(\alpha, m)$ such that, for all $t > 0$,

$$\begin{aligned}\tilde{p}(b, a;\, t) \leq C(\alpha, m)&\left[\frac{1}{t^{1-\alpha/2}(2b - a + mt)^{\alpha}} + (m^{-})\frac{1}{\sqrt{t}}\right] \\ &\times \exp\left(-\frac{(2b - a + mt)^2}{8t} + 6(m^{+})^2 t\right)\mathbf{1}_{\{b > \max(0,a)\}}.\end{aligned} \tag{A.8}$$

Proof. We factorize

$$(2b - a)^2 - 2mat + m^2t^2 = [a - (2b + mt)]^2 - 4bmt$$

and write

$$\begin{aligned}\tilde{p}(b, a;\, t) = \frac{2}{\sqrt{2\pi}}&\left[\frac{(2b - a + mt)}{t\sqrt{t}} - \frac{m}{\sqrt{t}}\right] \\ &\times \exp\left(-\frac{(2b - a + mt)^2}{2t} + 2bm\right)\mathbf{1}_{\{b > \max(0,a)\}}.\end{aligned} \tag{A.9}$$

Let $C_{\beta} := \sup_{x \in [0, +\infty[} x^{\beta}\exp(-x^2/4) < +\infty$.

- For $m \leq 0$ and since $b > 0$, we obtain

$$\begin{aligned}\tilde{p}(b, a;\, t) \leq \frac{2}{\sqrt{2\pi}}&\left[\frac{C_{1+\alpha}}{t^{1-\alpha/2}(2b - a + mt)^{\alpha}} + \frac{m^{-}}{\sqrt{t}}\right] \\ &\times \exp\left(-\frac{(2b - a + mt)^2}{4t}\right)\mathbf{1}_{\{b > \max(0,a)\}},\end{aligned}$$

  and also (A.8) for $m \leq 0$.

- For $m > 0$, using (A.9),

$$\begin{aligned}\tilde{p}(b, a;\, t) \leq \frac{2}{\sqrt{2\pi}}&\left[\frac{C_{1+\alpha}}{t^{1-\alpha/2}(2b - a + mt)^{\alpha}}\right] \\ &\times \exp\left(-\frac{(2b - a + mt)^2}{4t} + 2bm\right)\mathbf{1}_{\{b > \max(0,a)\}}.\end{aligned}$$

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Since

$$\frac{(2b-a+mt)^2}{8t} - 2bm = \frac{1}{8t}((2b-a)^2 + 2(2b-a)mt + m^2t^2 - 16bmt),$$

using $2b - a \geq b$ and $m > 0$, we have

$$\begin{aligned}\frac{(2b-a+mt)^2}{8t} - 2bm &\geq \frac{1}{8t}(b^2 + 2bmt + m^2t^2 - 16bmt)\\ &= \frac{1}{8t}((b-7mt)^2 - 48m^2t^2)\\ &\geq -6m^2t^2.\end{aligned}$$

Thus,

$$\begin{aligned}\tilde{p}(b,a;\,t) \leq{}& \frac{2}{\sqrt{2\pi}}\left[\frac{C_{1+\alpha}}{t^{1-\alpha/2}(2b-a+mt)^{\alpha}}\right]\\ &\times \exp\left(-\frac{(2b-a+mt)^2}{8t} + 6m^2t\right)\mathbf{1}_{\{b>\max(0,a)\}},\end{aligned}$$

and also (A.8) for $m > 0$.

This completes the proof. □

Lemma A.3. *Let $0 < \alpha < 1$. There exists a constant $C(\alpha)$ such that, for all $T > 0$ and $\sigma > 0$,*

$$\sup_{c\in\mathbb{R}} \mathbb{E}\left(|c+\sigma G|^{-\alpha}\exp\left(-\frac{(c+\sigma G)^2}{2T}\right)\right) \leq C(\alpha)\frac{T^{(1-\alpha)/2}}{\sigma}, \tag{A.10}$$

where G is a standard Gaussian variable.

Proof. First we prove (A.10) for $\sigma = 1$. Let

$$I(c,T) := \mathbb{E}\left(|c+G|^{-\alpha}\exp\left(-\frac{(c+G)^2}{2T}\right)\right) = I(c,T,+) + I(c,T,-), \tag{A.11}$$

where

$$I(c,T,\pm) := \mathbb{E}\left([(c+G)^{\pm}]^{-\alpha}\exp\left(-\frac{(c+G)^2}{2T}\right)\right).$$

Using the density of G, we have

$$I(c,T,+) = \int_{-c}^{\infty}(c+g)^{-\alpha}\exp\left(-\frac{(c+g)^2}{2T}\right)\frac{\exp(-g^2/2)}{\sqrt{2\pi}}\,\mathrm{d}g$$

and integration by part yields

$$I(c,T,+) = \int_{-c}^{\infty}\frac{1}{1-\alpha}\left[\frac{(c+g)^{2-\alpha}}{T} + g(c+g)^{1-\alpha}\right]\exp\left(-\frac{(c+g)^2}{2T}\right)\frac{\exp(-g^2/2)}{\sqrt{2\pi}}\,\mathrm{d}g.$$

Note that $x \mapsto x^{\beta}\exp(-x^2/4)$ is bounded on $\mathbb{R}^+$ by C_{β}. Then

$$\begin{aligned}I(c,T,+) \leq \int_{-c}^{\infty}\frac{1}{1-\alpha}\Bigg[&C_{2-\alpha}T^{-\alpha/2}\frac{\exp(-(c+g)^2/4T)}{\sqrt{2\pi}}\\ &+ C_1C_{1-\alpha}T^{(1-\alpha)/2}\frac{\exp(-g^2/4)}{\sqrt{2\pi}}\Bigg]\mathrm{d}g.\end{aligned}$$

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Integrating with respect to g yields

$$I(c, T, +) \leq 2[C_{2-\alpha} T^{(1-\alpha)/2} + C_1 C_{1-\alpha} T^{(1-\alpha)/2}]. \tag{A.12}$$

Along the same lines, we obtain

$$I(c, T, -) \leq 2[C_{2-\alpha} T^{(1-\alpha)/2} + C_1 C_{1-\alpha} T^{(1-\alpha)/2}]. \tag{A.13}$$

Substituting (A.12) and (A.13) into (A.11) yields (A.10) for $\sigma = 1$.

Replacing c by c/σ and T by T/σ^2 yields (A.10). □

Lemma A.4. *For all $\alpha < 1$, $\beta < 1$, $\delta > 0$, and $T > 0$, there exists a constant $C(\alpha, \beta, \delta, T)$ such that, for all $t \in (0, T]$,*

$$\mathbb{E}\left(\mathbf{1}_{\{N_t>0\}}(N_t+1)^{\delta}\left[T_{N_t}^{-\beta}(t-T_{N_t})^{-\alpha} + \sum_{k=0}^{N_t-1}(t-T_{k+1})^{-\alpha}(T_{k+1}-T_k)^{-\beta}\right]\right) \leq C(\alpha, \beta, \delta, T)t^{1-\alpha-\beta}. \tag{A.14}$$

Proof. Let $S(1, \alpha, \beta, \delta, t)$ be defined as

$$S(1, \alpha, \beta, \delta, t) := \mathbb{E}(\mathbf{1}_{\{N_t>0\}}(N_t+1)^{\delta}[T_{N_t}^{-\beta}(t-T_{N_t})^{-\alpha}]).$$

Note that $(T_{k+1} - T_k)_{k\geq 0}$ are independent variables, identically distributed, and T_1 follows an exponential distribution with parameter λ.

Then $S(1, \alpha, \beta, \delta, t) = \sum_{n=1}^{\infty} S(1, \alpha, \beta, \delta, t; n)$, where, for $n \geq 1$,

$$\begin{aligned} S(1, \alpha, \beta, \delta, t;\, n) = (n+1)^{\delta} \int_{[0,\infty[^{n+1}} & (u_1 + \cdots + u_n)^{-\beta}(t - u_1 - \cdots - u_n)^{-\alpha}\lambda^{n+1} \\ & \times \exp(-\lambda(u_1 + \cdots + u_{n+1})) \\ & \times \mathbf{1}_{\{u_1+\cdots+u_n\leq t\leq u_1+\cdots+u_{n+1}\}}\, \mathrm{d}u_1 \cdots \mathrm{d}u_{n+1}. \end{aligned}$$

We integrate with respect to u_{n+1} between $t - u_1 - \cdots - u_n$ and ∞; thus,

$$\begin{aligned} S(1, \alpha, \beta, \delta, t;\, n) = \lambda^n (n+1)^{\delta} \mathrm{e}^{-\lambda t} \int_{[0,\infty[^{n}} & (u_1 + \cdots + u_n)^{-\beta}(t - u_1 - \cdots - u_n)^{-\alpha} \\ & \times \mathbf{1}_{\{u_1+\cdots+u_n\leq t\}}\, \mathrm{d}u_1 \cdots \mathrm{d}u_n. \end{aligned}$$

We perform the change of variable $t v_i = u_1 + \cdots + u_i$, $i = 1, \ldots, n$ to obtain

$$S(1, \alpha, \beta, \delta, t;\, n) = \lambda^n t^{n-\alpha-\beta} \mathrm{e}^{-\lambda t} \int_{[0,1]^n} (v_n)^{-\beta}(1 - v_n)^{-\alpha} \mathbf{1}_{\{v_1<\cdots<v_n\leq 1\}}\, \mathrm{d}v_1 \cdots \mathrm{d}v_n.$$

We integrate with respect to $v_1, \ldots, v_{n-1}$ yielding

$$\begin{aligned} S(1, \alpha, \beta, \delta, t;\, n) &= \frac{(n+1)^{\delta}\lambda^n t^{n-\alpha-\beta}}{(n-1)!} \mathrm{e}^{-\lambda t} \int_0^1 (v_n)^{-\beta+n-1}(1 - v_n)^{-\alpha}\, \mathrm{d}v_n \\ &= \frac{(n+1)^{\delta}\lambda^n t^{n-\alpha-\beta}}{(n-1)!} \mathrm{e}^{-\lambda t} B(n - \beta, 1 - \alpha). \end{aligned} \tag{A.15}$$

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The radius of convergence of the entire series

$$\left(\sum_{n\geq 1}\frac{(n+1)^{\delta}\lambda^{n}t^{n-1}}{(n-1)!}B(n-\beta,1-\alpha)\right)$$

is infinite, so the series is continuous on $\mathbb{R}$ and then bounded on $[0,T]$. There exists a constant $C_1(\alpha,\beta,\delta,T)$ such that for all $t\in(0,T]$ and adding (A.15) yields

$$S(1,\alpha,\beta,\delta,t)\leq C_1(\alpha,\beta,\delta,T)t^{1-\alpha-\beta}. \tag{A.16}$$

Let $S(2,\alpha,\beta,\delta,t)$ be defined by

$$S(2,\alpha,\beta,\delta,t):=\mathbb{E}\left(\mathbf{1}_{\{N_t>0\}}(N_t+1)^{\delta}\sum_{k=0}^{N_t-1}[(t-T_{k+1})^{-\beta}(T_{k+1}-T_k)^{-\alpha}]\right).$$

Note that $(T_{k+1}-T_k)_{k\geq 0}$ are independent variables, identically distributed, and T_1 follows an exponential distribution with parameter λ.

Then

$$S(2,\alpha,\beta,\delta,t)=\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}S(2,\alpha,\beta,\delta,t;\,n,k),$$

where, for $n\geq 1$ and $0\leq k\leq n-1$,

$$\begin{aligned}S(2,\alpha,\beta,\delta,t;\,n,k)=(n+1)^{\delta}\int_{[0,\infty[^{n+1}}&(t-u_1-\cdots-u_{k+1})^{-\beta}(u_{k+1})^{-\alpha}\lambda^{n+1}\\&\times\exp(-\lambda(u_1+\cdots+u_{n+1}))\\&\times\mathbf{1}_{\{u_1+\cdots+u_n\leq t\leq u_1+\cdots+u_{n+1}\}}\,\mathrm{d}u_1\cdots\mathrm{d}u_{n+1}.\end{aligned}$$

We integrate with respect to u_{n+1} between $t-u_1-\cdots-u_n$ and ∞ to obtain

$$\begin{aligned}S(2,\alpha,\beta,\delta,t;\,n,k)=(n+1)^{\delta}\mathrm{e}^{-\lambda t}\lambda^{n}\int_{[0,\infty[^{n}}&(t-u_1-\cdots-u_{k+1})^{-\beta}(u_{k+1})^{-\alpha}\\&\times\mathbf{1}_{\{u_1+\cdots+u_n\leq t\}}\,\mathrm{d}u_1\cdots\mathrm{d}u_n.\end{aligned}$$

We perform the change of variable $tv_i=u_1+\cdots+u_i$, $i=1,\ldots,n$ yielding

$$\begin{aligned}S(2,\alpha,\beta,\delta,t;\,n,k)=(n+1)^{\delta}t^{n-\alpha-\beta}\mathrm{e}^{-\lambda t}\lambda^{n}\int_{[0,1]^{n}}&(1-v_{k+1})^{-\beta}(v_{k+1}-v_k)^{-\alpha}\\&\times\mathbf{1}_{\{v_1<\cdots<v_n\leq 1\}}\,\mathrm{d}v_1\cdots\mathrm{d}v_n.\end{aligned}$$

We integrate with respect to $v_1,\ldots,v_{k-1}$ and $v_{k+2},\ldots,v_n$ with the convention $v_0=0$; thus,

$$\begin{aligned}&S(2,\alpha,\beta,\delta,t;\,n,k)\\&\quad=\frac{(n+1)^{\delta}t^{n-\alpha-\beta}\mathrm{e}^{-\lambda t}\lambda^{n}}{(k-1)^{+}!\,((n-k-1))!}\\&\qquad\times\int_{[0,1]^2}(1-v_{k+1})^{-\beta+(n-k-1)}(v_{k+1}-v_k)^{-\alpha}v_k^{(k-1)^{+}}\,\mathrm{d}v_k\,\mathrm{d}v_{k+1}\\&\quad\leq\frac{(n+1)^{\delta}t^{n-\alpha-\beta}\mathrm{e}^{-\lambda t}\lambda^{n}}{(n-2)!}\frac{(n-2)!}{((k-1)^{+})!\,(n-k-1)!}\max(1,B(1-\beta,1-\alpha))\end{aligned} \tag{A.17}$$

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since

$$0 \le \int_{[0,1]^2} (1-v_{k+1})^{-\beta+(n-2-(k-1))_+} (v_{k+1}-v_k)^{-\alpha} v_k^{(k-1)^+} \,\mathrm{d}v_k \,\mathrm{d}v_{k+1}$$
$$\le \max\left(1, \int_{[0,1]^2} (1-v_{k+1})^{-\beta} (v_{k+1}-v_k)^{-\alpha} \,\mathrm{d}v_k \,\mathrm{d}v_{k+1}\right).$$

Then adding (A.17) from $k=0$ to $n-1$, using the binomial Newton formula for $k=1$ to $n-1$, and looking to the $k=0$ case separately, we have

$$\sum_{k=0}^{n-1} S(2,\alpha,\beta,\delta,t;\, n,k)$$
$$\le 2\max\left(1, \frac{\Gamma(-\alpha+1)\Gamma(-\beta+1)}{\Gamma(-\alpha-\beta+1)}\right) \frac{(n+1)^\delta 2^n t^{n-\alpha-\beta} \mathrm{e}^{-\lambda t}\lambda^n}{(n-2)!}. \tag{A.18}$$

The radius of convergence of the entire series

$$\left(\sum_{n\ge 1} \frac{(n+1)^\delta \lambda^n 2^n t^{n-1}}{(n-2)!}\right)$$

is infinite, so the series is continuous on $\mathbb{R}$ and then bounded on $[0,T]$: there exists a constant $C_2(\alpha,\beta,\delta,T)$ such that for all $t\in(0,T]$ and adding (A.18) yields

$$S(2,\alpha,\beta,\delta,t) \le C_2(\alpha,\beta,\delta,T) t^{1-\alpha-\beta}. \tag{A.19}$$

Equation (A.14) is a consequence of (A.16) and (A.19). □

Proof of Proposition A.1. Recall that

$$P(b,a;\,t) \coloneqq \sum_{k=0}^{N_t} \tilde{p}(b-X_{T_k}, a-X_{T_k}-Y_{k+1}\mathbf{1}_{\{t>T_{k+1}\}}$$
$$-(X_t - X_{\min(t,T_{k+1})});\, \min(t,T_{k+1})-T_k).$$

- Note that, from Lemma A.2, there exists a constant $C(\alpha,m)$ such that

$$P(b,a;\,t)\mathbf{1}_{\{N_t=0\}}$$
$$\le C(\alpha,m)\left[\frac{1}{t^{1-\alpha/2}(2b-a+mt)^\alpha} + (m)^- \frac{1}{\sqrt{t}}\right] \exp(6(m^+)^2 t)\mathbf{1}_{\{b>\max(0,a)\}}.$$

Set $\xi=0$ and $\beta=\frac{1}{2}\delta$. Since $\delta\in[1,\frac{3}{2})$ then $2(1-1/\delta)<1/\delta$. If $\alpha=1-1/2\delta$ then $\gamma\coloneqq\alpha\delta=\delta-\frac{1}{2}<1$ and $v\coloneqq\delta(1-\alpha/2)<1$. Then, using $(x+y)^\delta\le 2^{\delta-1}[x^\delta+y^\delta]$, there exists a constant $C_0(\delta)$ such that, for all $t>0$ and $b>\max(a,0)$,

$$\mathbb{E}(P(b,a;\,t)^\delta \mathbf{1}_{\{N_t=0\}})$$
$$\le C_0(\delta)\left[\frac{1}{t^v}\frac{1}{(2b-a+mt)^\gamma} + \frac{((m)^-)^\delta}{t^\beta}\right] \exp(6(m^+)^2 t\delta). \tag{A.20}$$

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- Using (A.8) and the development (A.5), we have

$$P(b,a;\,t) \coloneqq \sum_{k=0}^{N_t} \tilde{p}(b - X_{T_k}, a - X_{T_k} - Y_{k+1}\mathbf{1}_{\{t>T_{k+1}\}} - (X_t - X_{\min(t,T_{k+1})});\, \min(t,T_{k+1}) - T_k),$$

and

$$P(b,a;\,t)\mathbf{1}_{\{N_t>0\}} \le C(\alpha,m)\mathbf{1}_{\{N_t>0\}} \sum_{k=0}^{N_t} P_k(b,a,t),$$

where

$$P_k(b,a;\,t) \le \left[\frac{1}{(t\wedge T_{k+1} - T_k)^{1-\alpha/2} Z_k(t)} + \frac{m^-}{\sqrt{t\wedge T_{k+1} - T_k}}\right] \times \exp\left(-\frac{Z_k(t)^2}{8t\wedge T_{k+1} - T_k}\right) \exp(6(m^+)^2(\min(t,T_{k+1}) - T_k))$$

and

$$Z_k(t) \coloneqq 2b - a - X_{T_k} + Y_{k+1}\mathbf{1}_{\{t>T_{k+1}\}} + X_t - X_{t\wedge T_{k+1}} + m(T_{k+1}\wedge t - T_k).$$

We define a family $(C_k(t), \sigma_k(t), S_k(t), G_k(t))_{k\in\mathbb{N}\cup\{0\}}$ as follows:

- $(C_k(t), \sigma_k(t), S_k(t), G_k(t)) = (0,0,0,0)$ for $k > N_t$;
- for $k < N_t$,

$$C_k(t) \coloneqq 2b - a - X_{T_k} + \sum_{i=k+1}^{N_t} Y_i + m(t - T_{k+1}) + m(T_{k+1} - T_k),$$

$$\sigma_k(t) \coloneqq \sqrt{t - T_{k+1}}, \qquad S_k(t) \coloneqq T_{k+1} - T_k, \qquad G_k(t) \coloneqq \frac{B_t - B_{T_{k+1}}}{\sqrt{t - T_{k+1}}};$$

- for $k = N_t$,

$$C_k(t) \coloneqq 2b - a - \sum_{i=0}^{N_t} Y_i - mT_{N_t} + m(t - T_{N_t}),$$

$$\sigma_k(t) \coloneqq \sqrt{T_{N_t}}, \qquad S_k(t) \coloneqq t - T_{N_t}, \qquad G_k(t) \coloneqq \frac{B_{T_{N_t}}}{\sqrt{T_{N_t}}}.$$

Then, from (A.8) and the definition of $(C_k(t), \sigma_k(t), S_k(t), G_k(t))$, we have

$$P(b,a;\,t)\mathbf{1}_{\{N_t\neq 0\}} \le \mathbf{1}_{\{N_t\neq 0\}} \sum_{k=0}^{N_t} \left[\frac{1}{S_k^{1-\alpha/2}(C_k + \sigma_k G_k)^\alpha} + m^- \frac{1}{\sqrt{S_k}}\right] \mathrm{e}^{6(m^+)^2 t}\mathbf{1}_{\{b>a^+\}}.$$

Note that $S_k(t) \le t$ and using the Jensen inequality leads to

$$P(b,a;\,t)^\delta \mathbf{1}_{\{N_t\neq 0\}} \le \mathbf{1}_{\{N_t\neq 0\}}(N_t+1)^\delta \sum_{k=0}^{N_t} \left[\frac{1}{S_k^{\delta(1-\alpha/2)}(C_k + \sigma_k G_k)^{\alpha\delta}} + (m^-)^\delta \frac{1}{\sqrt{S_k}^\delta}\right] \mathrm{e}^{6\delta(m^+)^2 t}\mathbf{1}_{\{b>a^+\}}.$$

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Conditional on $\{N_t \neq 0\}$, the random variables $G_k(t)$ are independent of $C_k(t)$, $\sigma_k(t)$, and $S_k(t)$ and their common law is the standard Gaussian distribution. Using Lemma A.4 for $\sigma = \sigma_k(t)$ and $T = S_k(t)$, there exists a constant $C(\alpha, \delta)$ such that

$$\begin{aligned}
&\mathbb{E}(P(b,a;\,t)^\delta \mathbf{1}_{\{N_t\neq 0\}}) \\
&\quad \le C(\alpha,\delta)\mathbb{E}\Bigg(\mathbf{1}_{\{N_t\neq 0\}}(N_t+1)^\delta \sum_{k=0}^{N_t-1}\Bigg[\frac{1}{(T_{k+1}-T_k)^{\delta-1/2}\sqrt{t-T_{k+1}}} \\
&\qquad\qquad + (m^-)^\delta \frac{1}{\sqrt{t-T_{k+1}}^{\,\delta}}\Bigg]\Bigg) \\
&\qquad \times \mathrm{e}^{6\delta(m^+)^2 t}\mathbf{1}_{\{b>\max(0,a)\}} \\
&\qquad + C_(\alpha,\delta)\mathbb{E}\Bigg(\mathbf{1}_{\{N_t\neq 0\}}(N_t+1)^\delta\Bigg[\frac{1}{(t-T_{N_t})^{\delta-1/2}\sqrt{T_{N_t}}} + (m^-)^\delta\frac{1}{\sqrt{T_{N_t}}^{\,\delta}}\Bigg]\Bigg) \\
&\qquad\quad \times \mathrm{e}^{6\delta(m^+)^2 t}\mathbf{1}_{\{b>\max(0,a)\}}.
\end{aligned}$$

According to Lemma A.4, there exists a constant $C(\alpha, \delta, T)$ such that, for all $t \in (0, T]$,

$$\mathbb{E}(P(b,a;\,t)^\delta \mathbf{1}_{\{N_t\neq 0\}}) \le C(\alpha,\delta,T)[t^{1-\delta} + t^{(1-\delta)/2}]\mathrm{e}^{6\delta(m^+)^2 t}\mathbf{1}_{\{b>\max(0,a)\}}. \tag{A.21}$$

Adding (A.20) and (A.21) yields (A.7). □

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