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On the global linearization theorem of Charles Pugh

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The global C^0 linearization theorem on Banach spaces was first proposed by Pugh [26], but it requires that the nonlinear term is globally bounded. In the present paper, we discuss global linearization of semilinear autonomous ordinary differential equations on Banach spaces assuming that the linear part is hyperbolic (including contraction as a particular case) and that the nonlinear term is only Lipschitz with a sufficiently small Lipschitz constant. To overcome the difficulties arising in this problem, in this paper, we rely on a splitting lemma to decouple the hyperbolic system into a contractive system along the stable manifold and an expansive system along the unstable manifold. We then construct a transformation to linearize a contractive/expansive system, which is defined by the crossing time with respect to the unit sphere. To demonstrate the strength of our result, we apply our results to a nonlinear Duffing oscillator without external excitation.

Keywords: global linearization; splitting lemma; invariant manifolds; exponential dichotomy; spectrum

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1. Introduction

The earliest contributions devoted to linearization can be traced back to Poincaré's work on complex analytic conjugacy in 1890. Given an analytic diffeomorphism F near a fixed point, Poincaré [25] proved that F can be analytically conjugated to its linear part $A := DF(O)$ if A is contractive (i.e., the real parts of all eigenvalues

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$\lambda_1, \ldots, \lambda_p$ of A are less than 1) and satisfies nonresonance conditions of any orders (i.e., $\lambda_i \neq \lambda^m$ for all $i \in \{1, ..., p\}$, where $\lambda = (\lambda_1, ..., \lambda_p)$, and $m = (m_1, ..., m_p)$ is a nonnegative integer vector satisfying $|m| = \sum_{i=1}^{p} m_i \geq 2$).

Later, Siegel [29], Brjuno [3], and Yoccoz [36] contributed to the case of the eigenvalues on the unit circle. Sternberg [30, 31] initially investigated C^k linearization for C^n $(1 \leq k \leq n \leq \infty)$ diffeomorphisms on $\mathbb{R}^n$. Additional conditions for smooth linearization were obtained by Belitskii [2], Elbialy [9], Lu et al. [18], Rodrigues and Solà-Morales [28], Castañeda and Robledo [4], Dragičević et al. [8], and Zhang et al. [37–39] (see also references therein). Observe that all these linearization results involving smoothness of conjugacies require some form of nonresonant or spectral gap conditions.

Without these conditions, the well-known linearization result is the Hartman–Grobman theorem (see [11, 12]), which answers whether a continuous system or discrete system can be locally linearized by a C^0 conjugacy. For the infinite-dimensional system, Lu [17] presented a classical Hartman–Grobman theorem for the scalar reaction–diffusion equations. By using the methods in Moser's proof in [20] of the structural stability of Anosov diffeomorphisms, Pugh [26] gave a global version of Hartman–Grobman theorem on arbitrary Banach spaces. Moreover, the global Hartman–Grobman theorem has been extended by Palmer [23], Tan [32], Jiang [15], Hein and Prüss [13], Reinfelds and Šteinberga [27], Backes et al. [1], Xia et al. [35], and others.

We remark that the global C^0 linearization works mentioned above all require that the nonlinear term is globally bounded. A natural question is what happens when the nonlinear term is not necessarily globally bounded. Usually, considering unbounded perturbations can destroy the structure and stability of the system. Hence, it is natural to ask whether unbounded perturbations destroy the topological conjugacy? The answer is negative for systems with uniformly contractive linear part [16]. Recently, Huerta [14] established a topological conjugacy between a nonlinear system and its linear part by using the crossing time defined by the Lyapunov functions associated with a nonuniformly contractive system, without requiring that the nonlinear part is bounded. The corresponding discrete version is given by Castañeda and Robledo [5]. Nevertheless, these results are limited to the case when the linear part is contractive. Thus, it is natural to ask whether it is possible to construct conjugacies between a hyperbolic linear system and a nonlinear system obtained by perturbing the linear system by (not necessarily bounded) nonlinearity? Recently, Wu and Xia [34] attempted to solve this problem for a coupled hyperbolic system. They employed a spectral approach to make a moderate progress by assuming *partially unbounded* nonlinear perturbations, but failed to solve this problem completely. Moreover, in this situation, the difficulty arises from the fact that the results on the semiflow given by [14] cannot be extended to a flow.

In the present paper, we investigate a global linearization of autonomous semilinear ordinary differential equations whose linear part is hyperbolic and the nonlinear term is not necessarily bounded. We first decouple a nonlinear hyperbolic system into a contractive system on a stable manifold and an expansive system on an unstable manifold, which is based on a splitting lemma given by [6, Theorem 4.12]. Then, inspired by [24, Section 4], we can construct an appropriate transformation to linearize a contractive/expansive system, which is defined by the crossing time

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with respect to the unit sphere. On the other hand, it is difficult to discuss the regularity of conjugacies in [5, 14] due to the fact that the construction of conjugacies in [5, 14] is implicit. However, in our result, we show the Hölder continuity of the conjugacy at the fixed point (see (2.4)).

The rest of this paper is organized as follows. In §2 we state our main result. In §3, we give a decoupling lemma and then prove our main result. In §4, the linearization result for a contractive system on the stable manifold is established. Finally in §5, in order to demonstrate the strength of our result, we present an application to a nonlinear Duffing oscillator without external excitation.

2. Main result

In this section, we state our main result. Consider a nonlinear differential equation on a Banach space $X = (X, \|\cdot\|)$ over $\mathbb{C}$,

$$x'(t) = Ax(t) + f(x(t)), \quad t \in \mathbb{R}, \tag{2.1}$$

where A is a bounded linear operator on X and $f : X \to X$ is a continuous mapping with $f(O) = O$, where $O \in X$ denotes the origin. Provided that f is globally Lipschitz, as in the case when X is finite-dimensional, we have the existence and uniqueness of a global solution of Eq. (2.1) with the prescribed initial condition (see [7, Section 1] for details). Suppose that O is a hyperbolic fixed point for the linear part of Eq. (2.1), i.e., that the spectrum $\sigma(A)$ of A does not intersect the imaginary axis. Then the main result of this paper reads as follows.

THEOREM 2.1. *Suppose that $f : X \to X$ is such that $f(O) = O$ and with the property that there exists $L_f > 0$ such that*

$$\|f(x) - f(\tilde{x})\| \le L_f \|x - \tilde{x}\|, \quad \forall x, \tilde{x} \in X. \tag{2.2}$$

Then, provided that $\mathrm{L_f}$ *is sufficiently small, there exists a homeomorphism $H : X \to X$ which conjugates Eq. (2.1) topologically to its linear part*

$$x'(t) = Ax(t), \quad t \in \mathbb{R}. \tag{2.3}$$

More precisely,

$$H \circ S(t) = e^{At} \circ H, \quad t \in \mathbb{R},$$

where $S(t) : X \to X$ is the nonlinear solution operator of (2.1). Moreover, H *and its inverse* H^{-1} *satisfy*

$$H(x) = \mathcal{O}(\|x\|^{\varrho}) \quad \text{and} \quad H^{-1}(x) = \mathcal{O}(\|x\|^{\varrho}), \quad \text{as } x \to O, \tag{2.4}$$

for some $\varrho \in (0, 1)$.

REMARK 2.2. We point out that the global Hartman–Grobman theorem on Banach spaces was established independently by Pugh [26] and Palis [22], under an additional assumption that f is bounded, i.e., $\sup_{x \in X} \|f(x)\| < +\infty$. Huerta [14] and Castañeda and Robledo [5] established a C^0-conjugacy between Eq. (2.1) and its

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linear part, respectively, when f is not necessarily bounded. However, these results are limited to the case when the linear part is exponentially stable, sacrificing the hyperbolicity. Recently, Wu and Xia [34] addressed this problem for a coupled hyperbolic system by considering partially unbounded nonlinear perturbations, but failed to provide a complete solution due to the difficulties mentioned in the introduction. In this respect, theorem 2.1 firstly formulates a global linearization result on the flow of hyperbolic system where f is not necessarily bounded. On the other hand, we also note that it is difficult to discuss the regularity of conjugacies in [5, 14] due to the fact that the conjugacies in these works are constructed in an implicit manner. On the other hand, our linearization result gives the Hölder continuity of the conjugacy in a fixed point O (see (2.4)).

Remark 2.3. The proof of theorem 2.1 is partially inspired by the recent work of Dragičević–Zhang–Zhang [8]. Dragičević et al. resolved the open problem proposed by van Strein in [33]; they demonstrated that the linearization of a $C^{1,1}$ hyperbolic system can be locally differentiable and Hölder continuous. Whereas we provide a global linearization result for $C^{0,1}$ differential equations, complementing Pugh's result [26].

Example 2.4. Notice that theorem 2.1 gives that the following systems on $\mathbb{R}^2$ are topologically conjugated

$$\begin{cases} x' = -ax + f_1(x,y), \\ y' = by + f_2(x,y), \end{cases} \quad \text{and} \quad \begin{cases} x' = -ax, \\ y' = by, \end{cases}$$

provided that the Lipschitz constant L_f of $f = (f_1, f_2)$ is sufficiently small.

3. Proof of our main result

In this section, we give the proof of our main result.

Proof of theorem 2.1. Since A is a hyperbolic linear operator, the spectrum of A is given by $\sigma(A) := \sigma_s \cup \sigma_u$ where

$$\sigma_s := \{\varrho \in \sigma(A) : \operatorname{Re}\varrho \leq -\lambda\} \quad \textit{and} \quad \sigma_u := \{\varrho \in \sigma(A) : \lambda \leq \operatorname{Re}\varrho\}, \tag{3.1}$$

and $\lambda > 0$ is a constant. It follows from the Spectral Decomposition Theorem (see [10, p. 9]) that the space X has a direct sum decomposition $X := X_s \oplus X_u$ with A-invariant closed subspaces X_s and X_u, where $A_s := A|_{X_s} : X_s \to X_s$ and $A_u := A|_{X_u} : X_u \to X_u$ are bounded linear operators such that $\sigma(A_s) = \sigma_s$ and $\sigma(A_u) = \sigma_u$, respectively. Then by (3.1), we have that

$$\begin{cases} \|e^{A_s(t-s)}\| \leq Ke^{-\lambda(t-s)}, & \forall t \geq s, \\ \|e^{A_u(t-s)}\| \leq Ke^{\lambda(t-s)}, & \forall t \leq s, \end{cases} \tag{3.2}$$

where $K \geq 1$ is a constant. We note without any loss of generality we may assume that (3.2) holds with $K = 1$ by passing to a norm which is equivalent to $\|\cdot\|$. Indeed,

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let $\pi_s\colon X \to X_s$ and $\pi_u\colon X \to X_u$ be the projections associated with the splitting $X = X_s \oplus X_u$. For $x \in X$, we set

$$\|x\|_* = \max\left\{\sup_{t\geq 0}\left(e^{\lambda t}\|e^{A_s t}\pi_s x\|\right),\ \sup_{t\leq 0}\left(e^{-\lambda t}\|e^{A_u t}\pi_u x\|\right)\right\},$$

where $\lambda > 0$ is such that (3.1) holds. Note that (3.2) implies that $\|x\|_* \leq K\max\{\|\pi_s\|, \|\pi_u\|\}\|x\|$ for $x \in X$. Moreover,

$$\|x\|_* \geq \max\{\|\pi_s x\|, \|\pi_u x\|\} \geq \frac{1}{2}(\|\pi_s x\| + \|\pi_u x\|) \geq \frac{1}{2}\|x\|.$$

In addition, for $x_s \in X_s$ and $t \geq 0$ we have

$$\|e^{A_s t}x_s\|_* = \sup_{r\geq 0}\left(e^{\lambda r}\|e^{A_s r}e^{A_s t}x_s\|\right) = e^{-\lambda t}\sup_{r\geq 0}\left(e^{\lambda(t+r)}\|e^{A_s(t+r)}x_s\|\right) \leq e^{-\lambda t}\|x_s\|_*,$$

yielding the first estimate in (3.2) with $K = 1$. Similarly, one can establish the second estimate in (3.2). Observe that

$$\|x\|_* = \max\{\|x_s\|_*, \|x_u\|_*\}, \quad \text{where } x_s = \pi_s x \text{ and } x_u = \pi_u x. \tag{3.3}$$

To simplify the notation, we will assume that we passed to this new norm and $\|\cdot\|_*$ will be denoted by $\|\cdot\|$.

Given an initial value $x \in X$, the solution of (2.1) with $x(O) = x$ is given by $t \mapsto S(t)x$ where $S(t) : X \to X$ satisfies

$$S(t)x = e^{At}x + \int_0^t e^{A(t-\tau)}f(S(\tau)x)\,d\tau, \quad \text{for } t \in \mathbb{R} \text{ and } x \in X. \tag{3.4}$$

We note that the integral appearing in (3.4) is the Bochner integral, which represents an extension of the classical Lebesgue integral to functions that take values in a Banach space (see [19]). Then we have the following result.

LEMMA 3.1. *Suppose that $f : X \to X$ is such that $f(O) = O$ and that (2.2) holds with $2L_f < \lambda$. Then, there exist Lipschitz maps $h_s\colon X_s \to X_u$ and $h_u\colon X_u \to X_s$ satisfying $h_s(O) = O$, $h_u(O) = O$, $\max\{L_{h_s}, L_{h_u}\} \leq \frac{L_f}{\lambda - L_f}$, and with the property that Eq. (2.1) is conjugated to the system*

$$\begin{cases} x_s' = A_s x_s + \pi_s f(x_s + h_s(x_s)), \\ x_u' = A_u x_u + \pi_u f(h_u(x_u) + x_u). \end{cases} \tag{3.5}$$

More precisely, there is a homeomorphism $\Phi\colon X \to X$ with $\Phi(O) = O$ satisfying

$$\Phi \circ S(t) = (S^s(t) \circ \pi_s + S^u(t) \circ \pi_u) \circ \Phi \quad \forall t \in \mathbb{R}, \tag{3.6}$$

where

$$S^s(t)x_s = e^{A_s t}x_s + \int_0^t e^{A_s(t-\tau)}\pi_s f(S^s(\tau)x_s + h_s(S^s(\tau)x_s))\,d\tau,$$

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and

$$S^u(t)x_u = e^{A_u t}x_u + \int_0^t e^{A_u(t-\tau)}\pi_u f(S^u(\tau)x_u + h_u(S^u(\tau)x_u))\,d\tau.$$

Moreover, there exists $\tilde{\varrho} \in (0,1)$ *and for every bounded set* $B \subset X$ *a constant* $\mathrm{K} > 0$ *such that*

$$\|\Phi(x_1) - \Phi(x_2)\| \le K\|x_1 - x_2\|^{\tilde{\varrho}} \quad \textit{and} \quad \|\Phi^{-1}(x_1) - \Phi^{-1}(x_2)\| \le K\|x_1 - x_2\|^{\tilde{\varrho}},$$

for $x_1, x_2 \in B$ *with* $\|x_1 - x_2\| \le 1$.

Remark 3.2. We notice that lemma 3.1 follows from [6, Theorem 4.12] and [6, Proposition 4.16] by considering $Z = X_s \times X_u$ (endowed with the norm of the maximum as in [6]), which can be identified with $X = X_s \oplus X_u$ by the canonical isomorphism $(x_s, x_u) \to x_s + x_u$. Moreover, $C\colon Z \to Z$ and $H\colon Z \to Z$ in [6, Eq. (2.8)] are given by

$$C(x_s, x_u) = (A_s x_s, A_u x_u) \quad \text{and} \quad H(x_s, x_u) = (\pi_s f(x_s + x_u), \pi_u f(x_s + x_u)).$$

Then, note that (3.2) gives [6, Eq. (2.4)] with $\alpha = -\lambda$ and $\beta = \lambda$. In addition, observe that

$$\begin{aligned}\|H(x_s, x_u) - H(x_s', x_u')\| &= \max\{\|\pi_s(f(x_s + x_u) - f(x_s' + x_u'))\|,\\ &\qquad \|\pi_u(f(x_s + x_u) - f(x_s' + x_u'))\|\}\\ &= \|f(x_s + x_u) - f(x_s' + x_u')\|\\ &\le L_f\|x_s + x_u - x_s' - x_u'\|\\ &\le 2L_f\|(x_s, x_u) - (x_s', x_u')\|,\end{aligned}$$

where we have taken into account (3.3). Hence, H is Lipschitz and the Lipschitz constant L_H satisfies $2L_H < 2\lambda = \beta - \alpha$. We conclude that [6, Eq. (2.7)] is verified. Next, observe that Eq. (2.1) can be written as

$$(x_s'(t), x_u'(t)) = C(x_s(t), x_u(t)) + H(x_s(t), x_u(t)), \quad t \in \mathbb{R}. \tag{3.7}$$

Namely, if $x\colon \mathbb{R} \mapsto X$ is a solution of Eq. (2.1), then $t \mapsto (x_s(t), x_u(t)) := (\pi_s(x(t)), \pi_u(x(t)))$ is a solution of Eq. (3.7). Conversely, if $t \mapsto (x_s(t), x_u(t))$ is a solution of Eq. (3.7), then $t \mapsto x(t) := x_s(t) + x_u(t)$ is a solution of Eq. (2.1). In other words, the relationship between the solution operator $S(t)$ of Eq. (2.1) and of the solution operator $\bar{S}(t)$ of Eq. (3.7) is given by

$$\bar{S}(t)(x_s, x_u) = (\pi_s S(t)(x_s + x_u), \pi_u S(t)(x_s + x_u))$$

and

$$S(t)(x) = p_1\bar{S}(t)(\pi_s x, \pi_u x) + p_2\bar{S}(t)(\pi_s x, \pi_u x),$$

where $p_1\colon Z = X_s \times X_u \to X_s$ and $p_2\colon Z = X_s \times X_u \to X_u$ are projections on the first and the second coordinate, respectively. The existence of $h_s\colon X_s \to X_u$ and

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$h_u\colon X_u \to X_s$ with Lipschitz constants $\frac{L_f}{\lambda - L_f}$ and $h_s(O) = h_u(O) = O$, together with a homeomorphism $\bar{\Phi}\colon Z \to Z$ satisfying

$$\bar{\Phi} \circ \bar{S}(t) = (S^s(t), S^u(t)) \circ \bar{\Phi} \quad \text{for } t \in \mathbb{R},$$

follows from [6, Theorem 4.12]. We now define $\Phi\colon X \to X$ by

$$\Phi(x) = p_1\bar{\Phi}(\pi_s x, \pi_u x) + p_2\bar{\Phi}(\pi_s x, \pi_u x).$$

It is straightforward to verify that Φ is a homeomorphism satisfying (3.6). The desired Hölder continuity of $\bar{\Phi}$ (and consequently of Φ) follows from [6, Proposition 4.16]. Finally, we note that without any loss of generality we may suppose that the Hölder constant $\tilde{\varrho}$ belongs to $(0, 1)$ as for $0 < r_1 \leq r_2$ and $\|z\| \leq 1$ we have $\|z\|^{r_2} \leq \|z\|^{r_1}$.

By applying lemma 3.1, one only needs to linearize the contractive system in the stable manifold (i.e., the first equation of (3.5)), as the linearization of the second equation in (3.5) can be discussed similarly. In what follows, we consider the subsystem

$$S^s(t)x_s = e^{A_s t}x_s + \int_0^t e^{A_s(t-\tau)}\pi_s f(S^s(\tau)x_s + h_s(S^s(\tau)x_s))\, d\tau, \quad \text{for} \quad x_s \in X_s. \tag{3.8}$$

Note that Eq. (3.8) describes the flow of Eq. (2.1) on the stable invariant manifold $\mathcal{M}_s$ and $h_s : X_s \to X_u$ is Lipschitz continuous such that $h_s(O) = O$ and

$$\|h_s(x_s) - h_s(\tilde{x}_s)\| \leq L_{h_s}\|x_s - \tilde{x}_s\|, \quad \forall x_s, \tilde{x}_s \in X_s. \tag{3.9}$$

Then we have the following result.

Proposition 3.3. *Suppose that $f : X \to X$ is continuous such that $f(O) = O$ and (2.2) hold. Then there exists a homeomorphism $\Psi_s : X_s \to X_s$ such that*

$$\Psi_s(S^s(t)x_s) = e^{A_s t}\Psi_s(x_s). \tag{3.10}$$

Moreover, Ψ_s and its inverse Ψ_s^{-1} satisfy

$$\Psi_s(x_s) = \mathcal{O}(\|x_s\|^\alpha) \quad \text{and} \quad \Psi_s^{-1}(x_s) = \mathcal{O}(\|x_s\|^\alpha) \quad \text{as} \quad x_s \to O, \tag{3.11}$$

for some $\alpha \in (0, 1)$.

We postpone the proof of proposition 3.3 to §4. Similarly to proposition 3.3, the following proposition can be obtained.

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PROPOSITION 3.4. *Suppose that $f : X \to X$ is continuous such that $f(O) = O$ and (2.2) hold. Then there exists a homeomorphism $\Psi_u : X_u \to X_u$ such that*

$$\Psi_u(S^u(t)x_u) = e^{A_u t}\Psi_u(x_u).$$

Moreover, Ψ_u and its inverse Ψ_u^{-1} satisfy

$$\Psi_u(x_u) = \mathcal{O}(\|x_u\|^\alpha) \quad \text{and} \quad \Psi_u^{-1}(x_u) = \mathcal{O}(\|x_u\|^\alpha) \quad \text{as} \quad x_u \to O.$$

We now proceed to prove theorem 2.1. From propositions 3.3 and 3.4, one knows that the homeomorphism $\Psi : X \to X$ defined by

$$\Psi(x) := \Psi_s(\pi_s x) + \Psi_u(\pi_u x)$$

satisfies

$$\begin{aligned}\Psi \circ (S^s(t) \circ \pi_s + S^u(t) \circ \pi_u) &= \Psi_s \circ S^s(t) \circ \pi_s + \Psi_u \circ S^u(t) \circ \pi_u \\ &= e^{A_s t} \circ \Psi_s \circ \pi_s + e^{A_u t} \circ \Psi_u \circ \pi_u \\ &= e^{At} \circ \Psi_s \circ \pi_s + e^{At} \circ \Psi_u \circ \pi_u \\ &= e^{At} \circ \Psi,\end{aligned}$$

for $t \in \mathbb{R}$ as $e^{At}|_{X_*} = e^{A_* t}$ for $* = s, u$. Moreover,

$$\Psi(x) = \mathcal{O}(\|x\|^\alpha) \quad \text{and} \quad \Psi^{-1}(x) = \mathcal{O}(\|x\|^\alpha) \quad \text{as} \quad x \to O.$$

This result together with lemma 3.1 enable us to define a homeomorphism $H : X \to X$ by

$$H = \Psi \circ \Phi, \tag{3.12}$$

where Φ is given by lemma 3.1. Therefore, the homeomorphism H satisfies (2.4) with $\varrho := \alpha\tilde{\varrho} \in (0, 1)$. This completes the proof of theorem 2.1. □

4. Linearization on the stable manifold

In this section, we give a linearization result for a contractive system on the stable manifold (i.e., proposition 3.3). In order to simplify the notation, in what follows, we will drop the subscript s when denoting an element from X_s, and we write

$$Y(t, y) := S^s(t)y \text{ for all } t \in \mathbb{R} \text{ and } y \in X_s.$$

We note that $\{Y(t, \cdot) : t \in \mathbb{R}\}$ is a group of (nonlinear) operators, i.e., $Y(0, \cdot) = id_{X_s}$ and

$$Y(t + s, y) = Y(t, Y(s, y)) \quad \text{for } t, s \in \mathbb{R} \text{ and } y \in X_s. \tag{4.1}$$

Now we are ready to prove the proposition.

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Proof of proposition 3.3. Firstly, we give the estimate for $Y(t,y)$. Given $y \in X_s$, it follows from (2.2), (3.2), (3.8), and (3.9) that

$$\begin{aligned}\|Y(t,y)\| &\le e^{-\lambda t}\|y\| + \int_0^t e^{-\lambda(t-\tau)} L_f(\|Y(\tau,y)\| + \|h_\tau(Y(\tau,y))\|)\, d\tau \\ &\le e^{-\lambda t}\|y\| + \int_0^t e^{-\lambda(t-\tau)} L_f(1+L_{h_s})\|Y(\tau,y)\|\, d\tau \end{aligned} \tag{4.2}$$

for $t \ge 0$. By Gronwall's inequality, we get

$$\|Y(t,y)\| \le e^{-(\lambda - L_f(1+L_{h_s}))t}\|y\|, \quad t \ge 0,\ y \in X_s.$$

Set

$$\mu := \lambda - L_f(1+L_{h_s}).$$

Provided that L_f is sufficiently small, we have that $\mu > 0$. Then,

$$\|Y(t,y)\| = \|Y(t-t_0, Y(t_0,y))\| \le e^{-\mu(t-t_0)}\|Y(t_0,y)\|, \quad t \ge t_0,\ y \in X_s, \tag{4.3}$$

which means that for any given $y \ne O$, $t \mapsto \|Y(t,y)\|$ is a strictly decreasing function. Furthermore, we obtain from (4.3) that

$$\lim_{t\to+\infty} \|Y(t,y)\| = 0 \text{ and } \lim_{t\to-\infty} \|Y(t,y)\| = +\infty.$$

Indeed, the first equality above follows readily from (4.3), while for the second, it is sufficient to note that (4.3) gives (by replacing the roles of t and t_0) that

$$e^{\mu(t_0-t)}\|Y(t_0,y)\| \le \|Y(t,y)\| \quad t \le t_0,$$

yielding the desired claim. Hence, for each $y \ne O$, there is a unique time $\kappa(y) \in \mathbb{R}$ such that

$$\|Y(\kappa(y),y)\| = 1. \tag{4.4}$$

Clearly, when $y \to O$,

$$\kappa(y) \to -\infty. \tag{4.5}$$

Indeed, otherwise there would exist a sequence $(y_n)_n$ in $X_s \setminus \{O\}$ converging to O such that $\kappa(y_n) \nrightarrow -\infty$. Consequently, there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ with the property that $\kappa(y_{n_k}) \to L \in (-\infty, \infty]$. If L is finite, the continuity of $Y(\cdot,\cdot)$ gives that $Y(\kappa(y_{n_k}), y_{n_k}) \to Y(L,O) = O$, which yields a contradiction since $\|Y(\kappa(y_{n_k}), y_{n_k})\| = 1$ for each $k \in \mathbb{N}$. Similarly, if $L = \infty$ we have

$$1 = \|Y(\kappa(y_{n_k}), y_{n_k})\| \le e^{-\mu\kappa(y_{n_k})}\|y_{n_k}\| \quad \text{for all } k \in \mathbb{N} \text{ large enough.}$$

By passing to the limit when $k \to \infty$, we again obtain a contradiction. Further, we claim that $y \mapsto \kappa(y)$ is continuous on $X_s \setminus \{O\}$. To this end, we fix $y \ne O$ and $\epsilon > 0$ arbitrary. We may find $\varrho > 0$ such that $\|Y(\kappa(y) - \epsilon, y)\| > 1 + \varrho$ and

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$\|Y(\kappa(y)+\epsilon,y)\|<1-\varrho$. Since $Y(t,\cdot)$ is continuous, there exists $\delta>0$ such that for each $z\in X_s$ with $\|y-z\|<\delta$,

$$\|Y(\kappa(y)-\epsilon,z)\|>1+\frac{\varrho}{2}\quad\text{and}\quad\|Y(\kappa(y)+\epsilon,z)\|<1-\frac{\varrho}{2}.$$

Consequently, $|\kappa(y)-\kappa(z)|<\epsilon$ for $z\in X_s\setminus\{O\}$ with $\|y-z\|<\delta$, proving the desired continuity.

Similarly, for any given $x\neq O$, $t\mapsto\|e^{A_st}x\|$ is a strictly decreasing function,

$$\lim_{t\to+\infty}\|e^{A_st}x\|=0\,\text{and}\,\lim_{t\to-\infty}\|e^{A_st}x\|=+\infty.$$

Therefore, for each $x\neq O$, there is a unique time $\sigma(x)\in\mathbb{R}$ such that

$$\|e^{A_s\sigma(x)}x\|=1. \tag{4.6}$$

As for κ, we have that $x\mapsto\sigma(x)$ is continuous on $X_s\setminus\{O\}$ and

$$\sigma(x)\to-\infty,\quad\text{when } x\to O. \tag{4.7}$$

Next, we claim that

$$\kappa(Y(t,y))=\kappa(y)-t\,\text{and}\,\sigma(e^{A_st}x)=\sigma(x)-t. \tag{4.8}$$

In fact, by using the definition of $\kappa(y)$ and (4.1), we have

$$1=\|Y(\kappa(Y(t,y)),Y(t,y))\|=\|Y(\kappa(Y(t,y))+t,y)\|.$$

Due to the uniqueness of $\kappa(y)$, we have

$$\kappa(Y(t,y))+t=\kappa(y).$$

Similarly, by using (4.6) and the uniqueness of $\sigma(x)$, we obtain that $\sigma(e^{A_st}x)=\sigma(x)-t$.

Now, we can define two mappings $\Psi_s\colon X_s\to X_s$ and $\tilde{\Psi}_s\colon X_s\to X_s$ by

$$\Psi_s(y)=\begin{cases}e^{-A_s\kappa(y)}Y(\kappa(y),y), & y\neq O,\\ O, & y=O,\end{cases} \tag{4.9}$$

and

$$\tilde{\Psi}_s(x)=\begin{cases}Y(-\sigma(x),e^{A_s\sigma(x)}x), & x\neq O,\\ O, & x=O.\end{cases} \tag{4.10}$$

Recall that

$$Y(t,y)=e^{A_st}y+\int_0^t e^{A_s(t-\tau)}\pi_sf(Y(\tau,y)+h_s(Y(\tau,y)))\,d\tau.$$

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Then, when $y \neq O$, we can write $\Psi_s(y)$ as

$$\begin{aligned}\Psi_s(y) &= e^{-A_s\kappa(y)}\left(e^{A_s\kappa(y)}y + \int_0^{\kappa(y)} e^{A_s(\kappa(y)-\tau)}\pi_s f(Y(\tau,y)+h_s(Y(\tau,y)))\,d\tau\right)\\ &= y + \int_0^{\kappa(y)} e^{-A_s\tau}\pi_s f(Y(\tau,y)+h_s(Y(\tau,y)))\,d\tau.\end{aligned}$$

Likewise, we have

$$\tilde{\Psi}_s(x) = x + \int_0^{-\sigma(x)} e^{-A_s(\sigma(x)+\tau)}\pi_s f(Y(\tau, e^{A_s\sigma(x)}x)+h_s(Y(\tau,e^{A_s\sigma(x)}x)))\,d\tau.$$

We claim that Ψ_s and $\tilde{\Psi}_s$ are both continuous. Indeed, due to the continuity of κ and σ on $X_s \setminus \{O\}$, we only need to prove their continuity in O. From (4.4) and (4.5), it follows that when $y \to O$, $\Psi_s(y) \to O$, which gives the continuity of Ψ_s in O. Indeed, for $y \in X_s$ with $y \neq O$, we have using (4.4) that

$$\|\Psi_s(y)\| = \|e^{-A_s\kappa(y)}Y(\kappa(y),y)\| \leq \|e^{-A_s\kappa(y)}\|,$$

while $\lim\limits_{y\to O}\|e^{-A_s\kappa(y)}\| = 0$ by (4.5) and the first inequality in (3.2). Therefore, $\lim\limits_{y\to O}\Psi_s(y) = O$, yielding the continuity of Ψ_s in O. Similarly, from (4.3), (4.6), and (4.7), we see that when $x \to O$, $\tilde{\Psi}_s(x) \to O$, which implies that $\tilde{\Psi}_s$ is continuous in O.

In the following, we claim that

$$\sigma(\Psi_s(y)) = \kappa(y),\ y \neq O \text{ and } \kappa(\tilde{\Psi}_s(x)) = \sigma(x),\ x \neq O. \tag{4.11}$$

In view of (4.4), we have

$$\|Y(\kappa(\tilde{\Psi}_s(x)), \tilde{\Psi}_s(x))\| = 1, \quad x \neq O.$$

By (4.1) and (4.10), we see that

$$\begin{aligned}1 &= \|Y(\kappa(\tilde{\Psi}_s(x)), Y(-\sigma(x), e^{A_s\sigma(x)}x))\|\\ &= \|Y(\kappa(\tilde{\Psi}_s(x)) - \sigma(x), e^{A_s\sigma(x)}x)\|, \quad x \neq O.\end{aligned}$$

Since $t \mapsto \|Y(t,y)\|$ is a strictly decreasing function for each $y \neq O$, we assert (see (4.6)) that

$$\kappa(\tilde{\Psi}_s(x)) - \sigma(x) = 0, \quad x \neq O.$$

Similarly, one can obtain that $\sigma(\Psi_s(y)) = \kappa(y)$ for each $y \neq O$.

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Then, we are going to prove that

$$\Psi_s(\tilde{\Psi}_s(x)) = x,\ x \in X_s \text{ and } \tilde{\Psi}_s(\Psi_s(y)) = y,\ y \in X_s.$$

In fact, for $y = O$, $\tilde{\Psi}_s(\Psi_s(y)) = \tilde{\Psi}_s(\Psi_s(O)) = O = y$. For $y \neq O$, by (4.1), (4.9), (4.10), and (4.11), we have

$$\begin{aligned}\tilde{\Psi}_s(\Psi_s(y)) &= Y(-\sigma(\Psi_s(y)), e^{A_s\sigma(\Psi_s(y))}\Psi_s(y))\\ &= Y(-\kappa(y), e^{A_s\kappa(y)}e^{-A_s\kappa(y)}Y(\kappa(y), y))\\ &= Y(-\kappa(y), Y(\kappa(y), y)) = y.\end{aligned}$$

For $x = O$, $\Psi_s(\tilde{\Psi}_s(x)) = \Psi_s(\tilde{\Psi}_s(O)) = O = x$. For $x \neq O$, we also obtain from (4.1) (4.9), (4.10), and (4.11) that

$$\begin{aligned}\Psi_s(\tilde{\Psi}_s(x)) &= e^{-A_s\kappa(\tilde{\Psi}_s(x))}Y(\kappa(\tilde{\Psi}_s(x)), \tilde{\Psi}_s(x))\\ &= e^{-A_s\sigma(x)}Y(\sigma(x), Y(-\sigma(x), e^{A_s\sigma(x)}x)) = x.\end{aligned}$$

In this step, we establish (3.10). If $y = O$, (3.10) is obvious. For $y \neq O$, by (4.1), (4.8), and (4.9), we see that

$$\begin{aligned}\Psi_s(Y(t, y)) &= e^{-A_s\kappa(Y(t,y))}Y(\kappa(Y(t, y)), Y(t, y))\\ &= e^{-A_s(\kappa(y)-t)}Y(\kappa(y) - t, Y(t, y))\\ &= e^{A_st}e^{-A_s\kappa(y)}Y(\kappa(y), y) = e^{A_st}\Psi_s(y).\end{aligned}$$

Finally, it remains to prove (3.11). First, we claim that

$$y \in X_s,\ 0 < \|y\| < 1 \implies \kappa(y) < 0. \tag{4.12}$$

In order to prove (4.12) we take $y \in X_s$ with $0 < \|y\| < 1$. Suppose that $\kappa(y) \geq 0$. Then, (4.4) together with (4.3) yields

$$1 = \|Y(\kappa(y), y)\| \leq e^{-\mu\kappa(y)}\|y\| \leq \|y\| < 1,$$

which yields a contradiction. Thus, (4.12) holds. Next, note that $\|e^{A_st}\| \leq e^{|t|\|A_s\|}$ for all $t \in \mathbb{R}$. In the sequel, we will assume that $\lambda \leq \|A_s\|$ as if this is not the case, we replace $\|A_s\|$ with any number which is greater than λ. Then, similar to (4.2), one can verify that

$$\|Y(t, y)\| \leq e^{(\|A_s\|+L_f(1+L_{h_s}))|t|}\|y\|, \quad \text{for all } t \in \mathbb{R}.$$

Then, we see from (4.4) that

$$1 = \|Y(\kappa(y), y)\| \leq e^{(\|A_s\|+L_f(1+L_{h_s}))|\kappa(y)|}\|y\|,$$

which yields that

$$e^{-|\kappa(y)|} \leq \|y\|^{\frac{1}{\|A_s\|+L_f(1+L_{h_s})}}. \tag{4.13}$$

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Therefore, (4.9) together with (3.2), (4.4), and (4.13) gives

$$\begin{aligned}\|\Psi_s(y)\| &= \|e^{-A_s\kappa(y)}Y(\kappa(y),y)\| \\ &\le e^{\lambda\kappa(y)}\|Y(\kappa(y),y)\| \\ &= e^{-\lambda|\kappa(y)|} \le \|y\|^{\frac{\lambda}{\|A_s\|+L_f(1+L_{h_s})}},\end{aligned}$$

which implies that $\Psi_s(y) = \mathcal{O}(\|y\|^{\alpha_1})$ as $\|y\| \to O$, where

$$\alpha_1 = \frac{\lambda}{\|A_s\| + L_f(1+L_{h_s})} \in (0,1).$$

Similar to (4.13), we have

$$e^{-|\sigma(x)|} \le \|x\|^{1/\|A_s\|}, \tag{4.14}$$

and $\sigma(x) < 0$ if $0 < \|x\| < 1$. Then, (4.10) together with (4.3), (4.6), and (4.14) gives

$$\|\tilde{\Psi}_s(x)\| = \|Y(-\sigma(x), e^{A_s\sigma(x)}x)\| \le e^{\mu\sigma(x)} \le \|x\|^{\mu/\|A_s\|},$$

which indicates that $\tilde{\Psi}_s(x) = \mathcal{O}(\|x\|^{\alpha_2})$ as $\|x\| \to O$, where

$$\alpha_2 = \frac{\lambda - L_f(1+L_{h_s})}{\|A_s\|} \in (0,1).$$

Therefore, one can take $\alpha = \min\{\alpha_1, \alpha_2\}$ such that (3.11) holds. Consequently, the proof of proposition 3.3 is completed. □

5. Application to semilinear second-order equations

In this section, in order to demonstrate the strength of our result, we present an application to a nonlinear Duffing oscillator without external excitation.

Example 1. Consider a nonlinear Duffing oscillator without external excitation

$$\ddot{u} + c\dot{u} + ku + \alpha g(u) = 0, \quad u \in \mathbb{R}, \tag{5.1}$$

where $c > 0$ is a damping coefficient, k, α are constant coefficients with physical significance, respectively, and $g(u)$ is given by

$$g(u) = \begin{cases} 3\epsilon u - 2\epsilon, & u > 1, \\ \epsilon u^3, & u \in [-1,1], \\ 3\epsilon u + 2\epsilon u, & u < -1. \end{cases}$$

Here $\epsilon > 0$ is a constant and

$$\|g(u) - g(\tilde{u})\| \le 3\epsilon\,\|u - \tilde{u}\|, \quad \forall u, \tilde{u} \in \mathbb{R}.$$

In Eq. (5.1), for convenience, we take $k = -1$. For $k = -1 < 0$, the oscillator can be regarded as a system of steel beam that is deflected towards the two magnets, see

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[21]. In order to linearize Eq. (5.1), as usual, setting $v = (u, \dot{u})^T$, one can transform Eq. (5.1) into the first-order problem

$$\dot{v} = Av + f(v), \tag{5.2}$$

where $f(v) = (0, -\alpha g(u))^T$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}.$$

We see that there exist two eigenvalues of A:

$$\lambda_1 = -(c/2) - (\sqrt{c^2+4}/2) < 0, \quad \lambda_2 = -(c/2) + (\sqrt{c^2+4}/2) > 0,$$

which implies that A is hyperbolic. Then by applying theorem 2.1, we have the following linearization result for the nonlinear second-order equation (5.1).

THEOREM 5.1. *Let* $\mathrm{c} > 0$ *and* $k = -1$. *Provided that* ϵ *is sufficiently small, the induced first-order problem (5.2) can be completely linearized by means of a homeomorphism* H. *Moreover,* $H^{\pm 1}(x) = \mathcal{O}(\|x\|^{\varrho})$, *as* $\mathrm{x} \to 0$.

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