Ergod. Th. & Dynam. Sys. (2002), **22**, 1061–1078
DOI: 10.1017/S0143385702000627
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Printed in the United Kingdom

Absolutely continuous invariant measures for most piecewise smooth expanding maps

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(*Received* 18 *April* 2001 *and accepted in revised form* 22 *February* 2002)

Abstract. We introduce transversality conditions for a piecewise smooth map in any dimension with a sufficiently regular discontinuity set. We prove that maps satisfying these conditions have certain nice properties and, among all maps with a common set of discontinuities, form an open and dense subset. This, in conjunction with other works of the author, implies the existence of stochastically stable invariant densities for an open and dense set of piecewise smooth expanding maps.

1. *Introduction*

Discontinuities arise in many dynamical systems. Examples include, but are not limited to, certain billiards systems, the first-return map induced on a subset $X \subset M$ by a smooth map $f : M \to M$ on a manifold, and maps of number-theoretic interest such as the beta-transformation $x \mapsto \beta x \pmod{\mathbb{Z}^d}$ on the torus $\mathbf{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, where β is a $d \times d$ matrix with real entries. Standard methods and facts used to study smooth systems (e.g. distortion estimates, existence of stable/unstable manifolds) are not readily applicable when discontinuities are present. The first obvious problem is that when $T : M \to M$ is not continuous, $x \approx y$ does not imply $T(x) \approx T(y)$. This by itself might not be so bad; however, we are interested in iterates T^n of such a map and, in general, the discontinuity set of T^n grows and can become very complicated as n increases (consider a β-transformation with β irrational and $\det(\beta) > 1$). The purpose of this paper is to establish conditions on T which (1) allow us to control the 'complexity' of the discontinuity set of T^n and (2) which hold for 'most' maps T. We will explain these statements after we describe an application to piecewise expanding maps which motivates this work.

Lasota and Yorke proved in [**12**] that every piecewise C^2 map $f : [0, 1] \to [0, 1]$ with $|f'|$ uniformly greater than 1 leaves invariant a probability measure μ which is absolutely continuous with respect to the Lebesgue measure m. Such a measure is called an *absolutely continuous invariant measure* (we consider only finite measures throughout). They proved this by constructing an *invariant density* for f; that is, a function $h \geq 0$ with

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$\int h\,dm = 1$, such that h is a fixed point of the associated Perron–Frobenius operator, P_f. They accomplished this by showing that P_f acting on functions of bounded variation is essentially a contraction.

The fundamental difficulty in extending this technique to higher dimensions is that the domains of smoothness no longer have trivial geometry. Góra and Boyarsky [**9**] extended the method of [**12**] to derive sufficient conditions for the existence of an invariant density for a piecewise smooth expanding map of a domain in $\mathbb{R}^d$; these were later relaxed and simplified in [**1**] and [**6**]. Sufficient conditions of a more technical nature, but of wider applicability, are given in [**13**]. As far as positive results for entire classes of maps are concerned, Keller proved in [**11**] that any piecewise complex-analytic expanding map of a planar domain has an invariant density; this was extended to the planar piecewise real-analytic case in both [**3**] and [**14**]. In [**2**], Buzzi shows that almost every expanding piecewise affine map on a domain in $\mathbb{R}^d$ has an invariant density. One might suspect that, as in one dimension, expansion alone is sufficient to guarantee the existence of an invariant density in higher dimensions—this is not the case. Tsujii constructed, for any $r \in [2, \infty)$, piecewise C^r expanding maps in the plane (with finitely many connected domains of continuity with piecewise smooth boundaries) with very singular properties [**15**]; in fact, as observed in [**4**], some of these maps have the unit mass at a fixed point as their only invariant measure. This, however, is not typical behavior. One conclusion of the current work is that among all expanding maps which are piecewise C^2 with respect to some fixed, sufficiently nice partition of the domain, those in an open and dense subset have at least one invariant density.

What does 'open and dense' mean? Consider the simple case where $\Omega \subset \mathbb{R}^d$ is an open rectangle subdivided by finitely many smooth surfaces into finitely many open and connected subsets $\pi_1, \ldots, \pi_s$. The collection $\Pi = \{\pi_1, \ldots, \pi_s\}$ is almost a partition of Ω. (Our results hold when Π is a piecewise C^r, or 'nice', partition of an open and bounded domain $\Omega \subset \mathbb{R}^d$, see §2 for details.) For each π_i, let $E^r(\pi, \Omega)$ be the space of maps $\pi \to \overline{\Omega}$ which extend to a neighborhood of $\overline{\pi}_i$ as a C^r embedding. Then

$$\mathcal{D}^r(\Pi) = E^r(\pi_1, \Omega) \times \cdots \times E^r(\pi_s, \Omega)$$

endowed with the product topology is the space of all maps which are *piecewise C^r with respect to* Π. Note that the discontinuity set of any map in $\mathcal{D}^r(\Pi)$ is a subset of $\Omega \setminus \cup\pi_i$.

The first result that we state concerns expanding maps. If $T : \Omega \to \Omega$ is differentiable on a subset Ω_T of full Lebesgue measure, the *expansion constant* of T is the largest number $\lambda = \lambda(T)$ such that $|DT(x) \cdot v| \geq \lambda |v|$ for all $x \in \Omega_T$ and $v \in T_x\Omega$. We say that T is *expanding* if $\lambda(T) > 1$ and let $\mathcal{D}^r_{\exp}(\Pi) \subset \mathcal{D}^r(\Pi)$ denote the open subset of expanding maps.

THEOREM 1.1. *Every map T in an open and dense subset of $\mathcal{D}^r_{\exp}(\Pi)$, $2 \leq r \leq \infty$, has an absolutely continuous invariant measure.*

Tsujii's example shows that for $r < \infty$ this theorem is, in some sense, topologically optimal (the question of whether or not every piecewise C^∞ expanding map admits an invariant density appears to be open). The proof of Theorem 1.1 falls into three parts, two of which are contained here. We briefly describe the third part, which can be found in [**6**].

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Let Π be a nice partition as above. Part of the definition of nice is that the boundary of each π_i is the union of finitely many smooth 'faces' of codimension 1. Postponing precision, if we let $F(\Pi)$ denote the maximum number of these faces which meet at a point, then one of the results of [**6**] (simplifying results in [**9**] and [**1**]†) is as follows.

PROPOSITION 1.2. *If $T \in \mathcal{D}^2(\Pi)$ and $\lambda(T) > F(\Pi) + 1$, then T has an absolutely continuous invariant measure.*

For a given Π, this condition may fail for every $T \in \mathcal{D}^r(\Pi)$ (consider the partition of the unit ball by the coordinate planes). However, since T has an invariant density if T^n does for some $n > 1$, we can ask whether $\lambda(T^n) \geq \lambda(T)^n$ is larger than $F(\Pi(T^{n-1})) + 1$ for some n, where $\Pi(T^{n-1})$ is the dynamical partition obtained from the intersections of $T^{-k}(\Pi), 0 \leq k \leq n-1$ (see §2 for details). Thus, for the expanding map T to have an invariant density, it is sufficient that the sequence $F_n(T) = F(\Pi(T^n))$ does not grow too rapidly as n increases. In general, $F_n(T)$ might grow exponentially, but this is not initially the case if T is *n-transverse*. Roughly speaking, T is n-transverse if for each $k = 1, \ldots, n$ the pullback partition $T^{-k}(\Pi)$ intersects $\Pi(T^{k-1})$ transversely.

PROPOSITION 1.3. *There is a constant $B = B(\Pi)$ depending only on Π such that $F_n(T) \leq B$ for any $T \in \mathcal{D}^2(\Pi)$ which is n-transverse.*

We prove this in §3. Propositions 1.2 and 1.3 yield the following corollary.

COROLLARY 1.4. *If $T \in \mathcal{D}^2(\Pi)$ is n-transverse and $\lambda(T^{n+1}) > B(\Pi) + 1$, then T^{n+1}, hence T, has an absolutely continuous invariant measure.*

Theorem 1.1 follows from this and the following theorem.

THEOREM 1.5. *For every $n \geq 1$, the set of n-transverse maps is open and dense in $\mathcal{D}^r(\Pi), 2 \leq r \leq \infty$. In particular, the set of maps which are n-transverse for all n is residual.*

Proof of Theorem 1.1. Let $\mathcal{E}_n$ denote the open set of maps T for which $\lambda(T)^{n+1} > B + 1$, with B as in Proposition 1.3, and let $\mathcal{T}_n$ denote the set of n-transverse maps. By Corollary 1.4, every map in

$$\mathcal{T} = \bigcup_{n=1}^{\infty} (\mathcal{E}_n \cap \mathcal{T}_n)$$

has an invariant density, and $\mathcal{T}$ is open and dense in $\mathcal{D}^r_{\exp}(\Pi) = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ by Theorem 1.5. □

The main obstacle in our proof of Theorem 1.5 is the potential existence of periodic points of discontinuity; some care is required in making perturbations in a neighborhood of such a point.

Up to now we have only considered the iteration of a single map. In fact, all of our definitions may be formulated for arbitrary compositions of maps. The results we have

† Actually, the results as stated in [**6**] do not strictly generalize those in [**9**] and [**1**], as a certain 'extendability' assumption on faces is made; however, it is clear from the proof there that this is unnecessary. See §2 for details.

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stated here also hold in the case where maps which are sufficiently close to a good map are composed. This allows us to prove a result about the stochastic stability of the density obtained in Theorem 1.1. To be precise, let ν be a Borel measure on $\mathcal{D} = \mathcal{D}^r(\Pi)$. Then ν induces a discrete-time Markov process on the domain Ω with transition probabilities

$$P_\nu^x(A) = \nu(\{g \in \mathcal{D} : g(x) \in A\}),$$

for $x \in \cup \pi_i$ and all Borel sets $A \subset \Omega$. A measure μ on Ω is a *stationary distribution* for this process if

$$\mu(A) = \int_\Omega P_\nu^x(A)\, d\mu(x)$$

for every Borel set A. If $\mu \prec m$ we say that $h = d\mu/dm$ is an *invariant density* for the ν-process (such a density is not necessarily unique). We say that a map T is *locally L^1-stochastically stable* if it has a neighborhood $\mathcal{U} \subset \mathcal{D}$ such that (1) the ν-process for any ν supported on $\mathcal{U}$ has an invariant density and (2) if $\{\nu_k\}_{k=1}^\infty$ is a sequence of such measures with $\nu_k \to \delta_T$ (weak convergence), then the $L^1(m)$-limit set of any corresponding sequence of invariant densities $\{h_k\}$ is non-empty and consists of invariant densities for T. We now state a stronger version of Theorem 1.1.

THEOREM 1.6. *Every map T in an open and dense subset of $\mathcal{D}^r_{\exp}(\Pi)$, $2 \le r \le \infty$, is locally L^1-stochastically stable.*

The open and dense set here is the same as in Theorem 1.1. Note that we do not assume that the supports of the measures ν_k necessarily get smaller, only that they are restricted to a neighborhood of T. In fact, this restriction can be relaxed a great deal to allow, for example, perturbations by stochastic flows, see [**6**] and [**7**] for details.

In the next section we give precise definitions of n-transversality, nice partitions, the cut index, and so forth. We develop some useful properties of n-transversality and prove Proposition 1.3 in §3. In §4 we show that n-transversality is an open condition and in §5 we show that it is dense, completing the proof of Theorem 1.1. In the last section we verify that the conditions for L^1-stochastic stability given in [**6**] hold on the open and dense set $\mathcal{T}$, proving Theorem 1.6.

2. *Definitions*

We begin with a detailed description of the allowed domains of smoothness. The goal is to define a class of domains which is restrictive enough to be closed under transverse intersections yet flexible enough to allow for cusps on the boundaries.

Definition 2.1. Let $A \subset \mathbb{R}^d$ be compact. A C^r *stratification of* A is a (genuine) partition $\mathcal{A}$ of A into finitely many connected C^r submanifolds of $\mathbb{R}^d$. The elements of $\mathcal{A}$ are called *strata* and the unique stratum which contains a given $x \in A$ is denoted $\mathcal{A}_x$. The stratification $\mathcal{A}$ is *Whitney regular* if the following holds. For every $x \in A$ and any stratum $Y \in \mathcal{A}$, if $(x_i) \subset \mathcal{A}_x$ and $(y_i) \subset Y$ are sequences such that (i) $x_i, y_i \to x$, (ii) the lines $\overrightarrow{x_i y_i} \subset \mathbb{R}^d$ converge to a line L and (iii) the subspaces $T_{y_i} Y \subset \mathbb{R}^d$ converge to a subspace E, then $L \subset T$.

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Whitney transversality implies, among other things, that if K is an affine subspace transverse to $\mathcal{A}_x$ at x, then K is transverse to $\overline{L}_y$. Our main reference about stratified sets is [**8**].

Definition 2.2. A k-dimensional C^r *stratified submanifold* of $\mathbb{R}^d$ is a compact topological submanifold $K \subset \mathbb{R}^d$ with a Whitney regular C^r stratification $\mathcal{K}$ which is generated, as an algebra, by the closures of its k-dimensional strata.

'As an algebra' means that each stratum is obtained from k-strata by taking finitely many intersections and complements.

Definition 2.3. A smooth $(d-1)$-dimensional submanifold $W \subset \mathbb{R}^d$ has a *one-sided collar* if there exists $\delta > 0$ and a unit-normal vector field $\mathbf{n}_x$ on W such that the map $(x, t) \mapsto x + t\mathbf{n}_x$ from $W \times [0, \delta]$ to $\mathbb{R}^d$ is an embedding.

Note there are two potential choices for $\mathbf{n}_x$; everything is fine so long as one of these works. For example, the cone $\{(r, \theta, z) \in \mathbb{R}^3 : 0 < r = z < 1\}$ has a one-sided (but not a two-sided) collar.

Definition 2.4. A *piecewise C^r domain* is an open, bounded, and connected subset $\pi \subset \mathbb{R}^d$ such that $\partial\pi$ is a stratified submanifold and the $(d-1)$-dimensional strata, called *faces of* π, have one-sided collars.

The stratification of $\partial\pi$ extends in an obvious way to a stratification of $\overline{\pi}$. Abusing notation, we refer to strata of $\overline{\pi}$ as strata of π and let π_x denote the unique stratum of $\overline{\pi}$ which contains x. (Thus, $\pi_x \subset \partial\pi$ if $x \in \partial\pi$ and $\pi_x = \pi$ if $x \in \pi$.)

Definition 2.5. A *piecewise C^r partition* of an open and bounded domain $\Omega \subset \mathbb{R}^d$ is a collection $\Pi = \{\pi_i\}$ of disjoint, piecewise C^r domains $\pi_i \subset \Omega$ such that $\overline{\Omega} = \cup\overline{\pi}_i$.

All the results stated in the introduction hold when Π is a piecewise C^r partition. From this point on we fix a domain $\Omega \subset \mathbb{R}^d$, a degree of differentiability $2 \le r \le \infty$, and a piecewise C^r partition $\Pi = \{\pi_1, \ldots, \pi_s\}$. All maps considered are elements of $\mathcal{D}^r(\Pi)$.

Definition 2.6. The *cut index* of a piecewise smooth partition Σ is

$$F(\Sigma) = \sup\{F(\Sigma; x) : x \in \overline{\Omega}\},$$

where, for each $\sigma \in \Sigma$ and each $x \in \overline{\Omega}$, $F(\sigma; x)$ is the number of faces W of σ with $x \in \overline{W}$, and $F(\Sigma; x)$ is the sum, over all $\sigma \in \Sigma$, of $F(\sigma; x)$.

Proposition 1.2 states that if $\lambda(T) > F(\Pi)+1$, then T has an invariant density. This was proved in [**6**] with slightly stronger assumptions on the faces of the partition elements. However, the proof for the more general situation we consider here is virtually identical. (In [**6**] it is assumed that each face has a two-sided collar and the result is that T has an invariant density if $\lambda > F$. If we allow faces with only one-sided collars, then the same argument works except we require $\lambda > F + 1$.) As described already, we will apply this result to T^n for some $n \ge 1$.

For $T \in \mathcal{D}^r(\Pi)$ let

$$\Pi(T^n) = \left(\bigvee_{i=0}^{n} T^{-i}\Pi\right)^*,$$

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where

$$\bigvee_{i=0}^{n} T^{-i}\Pi = \{\pi_{i_0} \cap T^{-1}\pi_{i_1} \cap \cdots \cap T^{-n}\pi_{i_n} : \pi_{i_j} \in \Pi,\ j = 0, \ldots, n\}$$

and, for a partition Σ, Σ^* denotes the partition whose elements are the connected components of elements of Σ. Note that $\Pi = \Pi(T^0)$ for any $T \in \mathcal{D}^r(\Pi)$. More generally, for a sequence $T_1, T_2, \ldots, T_n \in \mathcal{D}^r(\Pi)$, let

$$\Pi(T_n \circ \cdots \circ T_1) = \left(\Pi \vee \bigvee_{i=1}^{n} (T_i \circ \cdots \circ T_1)^{-1}\Pi\right)^*.$$

These partitions arise in our study of random maps and are compatible with the previous definition, i.e. $\Pi(T_n \circ \cdots \circ T_1) = \Pi(T^n)$, if $T_i = T$ for $1 \le i \le n$. Note that $T_n \circ \cdots \circ T_1$ is smooth on each element of $\Pi(T_n \circ \cdots \circ T_1)$ and maps each of these into an element of Π. As a consequence, $S \circ T_n \circ \cdots \circ T_1$ is smooth on elements of $\Pi(T_n \circ \cdots \circ T_1)$ for any $S \in \mathcal{D}^r(\Pi)$.

These dynamical partitions are not always piecewise smooth. In particular, one can always find a map T so that $\Pi(T)$ has infinitely many elements (as in Tsujii's examples). Transversality prevents this from happening.

Definition 2.7. Let π and σ be piecewise smooth domains. We say that π and σ are *transverse at* x if the strata π_x and σ_x are transverse at x. We say that π *is transverse to* σ if they are transverse at every point $x \in \overline{\pi} \cap \overline{\sigma}$. If Σ is a piecewise smooth partition, we say that π is *transverse to* Σ if it is transverse to σ for each $\sigma \in \Sigma$.

If π and σ are piecewise smooth domains, then $f(\pi)$ is transverse to σ for an open and dense set of smooth embeddings f defined in a neighborhood of π (since our definition of transversality amounts to simultaneous transversality of two finite collections of smooth submanifolds, see [**10**]).

Definition 2.8. A sequence of maps $T_1, T_2, \ldots, T_n \in \mathcal{D}^r(\Pi)$ is *transverse* if the sequence $T_2, \ldots, T_n$ is transverse, the partition $\Pi(T_n \circ \cdots \circ T_2)$ is piecewise C^r and $T_1(\overline{\pi})$ is transverse to $\Pi(T_n \circ \cdots \circ T_2)$ for all $\pi \in \Pi$. The empty sequence is transverse by convention. We say T is *n-transverse* if the sequence $T, T, \ldots, T$ (n times) is transverse.

For example, T is 1-transverse if it carries each element of Π to a domain which is transverse to each element of Π. It is not difficult to show that the requirement in the definition of sequence transversality that $\Pi(T_n \circ \cdots \circ T_2)$ be piecewise C^r is redundant; that is, the pullback of a nice partition by a transverse sequence is a nice partition (Corollary 3.3). Moreover, the obvious generalization of Proposition 1.3 holds with the same value of B. In addition to showing that n-transversality is an open property, we also show in §4 that any sequence of maps near an n-transverse map is a transverse sequence.

3. *Iteration of transverse maps*

We show that the pullback of a nice partition by a transverse sequence of maps is a nice partition and prove a generalization of Proposition 1.3.

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3.1. *Transverse intersections.*

PROPOSITION 3.1. *The transverse intersection of piecewise C^r domains is a finite union of piecewise C^r domains.*

Proof. Transversality implies that $M \cap N$ has only finitely many connected components and that the induced stratification of the boundary of each component is a Whitney regular C^r stratification (see [**8**]); the faces of this stratification clearly have one-sided collars and generate the stratification.

We now show that $\partial(M \cap N)$ is a topological submanifold. It is enough to show that for $x \in \partial M \cap \partial N$, there is a neighborhood U of x and a homeomorphism

$$\psi : (U, M, N, x) \to (\mathbb{R}^d, H_d, H_1, 0),$$

where $H_i = \{(x_1, \ldots, x_d) : x_i > 0\}$, $i = 1, \ldots, d$. In fact, the ψ that we construct has the property of being smooth when restricted to strata of $M \cap N$. Following ψ by the map $(x_1, y, x_d) \mapsto (x_1^2 - x_d^2, y, 2x_1x_d)$ gives the required chart at x. The point is that even though ∂M and ∂N are each linearizable in a neighborhood of x, we need a chart which linearizes them simultaneously. This is possible because regular stratifications are locally topologically trivial; that is, the stratification is locally a product.

LEMMA 3.2. *Let $A \subset \mathbb{R}^d$ be a closed subset with a regular stratification, let X be a stratum and let $\rho : \mathbb{R}^d \to X$ be a smooth map, such that ρ restricted to $\overline{Y}$ is a proper submersion for every stratum Y. Then for any $x \in X$ there is a neighborhood U, a regularly stratified set $F \subset \mathbb{R}^{d-k}, k = \dim(X)$ and a stratified diffeomorphism $h : U \to X \times \mathbb{R}^{d-k}$, which takes strata of A to strata of $X \times F$ and satisfies $\rho = pr_X \circ h$.*

This is a special case of Thom's First Isotopy Lemma [**8**, Theorem 2.5.2]. With $(\partial M, M_x)$ and $(\partial N, N_x)$ in place of (A, X), we obtain transverse foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of U such that leaves of $\mathcal{F}_1$ are diffeomorphic to M_x and foliate strata of M, and leaves of $\mathcal{F}_2$ are diffeomorphic to N_x and foliate strata of N.

Let $k = \dim(M_x)$ and let $D \subset N_x$ be a $(d-k)$-dimensional disk transverse to M_x at x. Since ∂M is a stratified submanifold, $D \cap \partial M$ is as well and we can use the product structure to lift a stratified diffeomorphism $(D, D \cap M, x) \to (\{0\}^k \times \mathbb{R}^{d-k}, \{0\}^k \times \mathbb{R}^{d-k-1} \times (0, \infty), 0)$ to a stratified diffeomorphism

$$\psi_1 : (U, M, x) \to (\mathbb{R}^d, H_d, 0).$$

Note that ψ_1 is not smooth on strata of N. All that is important for us, however, is that it is smooth on strata of $M \cap N$ and $\psi_1(N_x)$ is transverse to $\psi_1(M_x) = \mathbb{R}^k \times \{0\}^{d-k}$. We have reduced $M \cap N$ to the case $\overline{H}_d \cap N$ and we continue in these coordinates, writing N_0 in place of $\psi_1(N_x)$ and so on.

Let $\ell = \dim(N_0)$; without loss of generality, we let $N_0 = \{0\}^{d-\ell} \times \mathbb{R}^\ell$. We now unfold $N \cap \mathbb{R}^{d-1}$ in the same way that we unfolded M above, but with $\mathbb{R}^{d-1-\ell}$ in place of D; that is, we have a stratified chart $(\mathbb{R}^{d-1-\ell}, N \cap \mathbb{R}^{d-1-\ell}, 0) \to (\mathbb{R}^{d-1-\ell} \times \{0\}^{d-1}, (0, \infty) \times \mathbb{R}^{d-2-\ell} \times \{0\}^{d-1}, 0)$ which lifts to a chart $(\mathbb{R}^{d-1} \times \{0\}, N \cap \mathbb{R}^{d-1} \times \{0\}, 0) \to (\mathbb{R}^{d-1} \times \{0\}, (0, \infty) \times \mathbb{R}^{d-2} \times \{0\}, 0)$. Refining the foliation $\mathcal{F}_2$ by copies of $\{0\}^{d-1} \times \mathbb{R}$, we extend this chart to

$$\psi_2 : (\mathbb{R}^d, N, 0) \to (\mathbb{R}^d, H_1, 0)$$

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which is smooth on strata of N and satisfies $\psi_2(H_d) = H_d = \psi_1(M)$. The desired chart is $\psi = \psi_2 \circ \psi_1$. □

COROLLARY 3.3. *If $T_1, \ldots, T_n$ is a transverse sequence of maps, the dynamical partition $\Pi(T_n \circ \cdots \circ T_1)$ is piecewise smooth. In particular, if T is n-transverse, then $\Pi(T^n)$ is piecewise smooth.*

Proof. We proceed by induction, assuming that $\Pi(T_n \circ \cdots \circ T_2)$ (or Π when $n = 1$) is piecewise smooth; we know that Π is piecewise smooth. Each element of $\Pi(T_n \circ \cdots \circ T_1)$ is a connected component of a set of the form $\pi \cap T_1^{-1}(\pi')$ for some $\pi \in \Pi$ and $\pi' \in \Pi(T_n \circ \cdots \circ T_2)$ (or Π when $n = 1$). This set is diffeomorphic to its image, $T(\pi) \cap \pi'$, which is a finite union of piecewise smooth domains by Proposition 3.1. □

3.2. *Proof of Proposition 1.3.* We prove a generalization of Proposition 1.3 for transverse sequences. The proof is essentially the same as that for iterating a single map.

PROPOSITION 3.4. *Let Π be a piecewise smooth partition of a domain Ω. There exists a number $B = B(\Pi)$ such that $F(\Pi(T_n \circ \cdots \circ T_1)) \leq B$ for any transverse sequence of maps $T_1, \ldots, T_n \in \mathcal{D}^r(\Pi)$.*

The proof is based on a simple idea. As a partition is pulled back by transverse maps, the dimension of a stratum through a point must decrease with every intersection. Thus, only finitely many intersections at a point can occur, each of which can increase the cut index at the point by, at most, some fixed amount. This is not a proof since we must keep track of the intersections along different branches.

Let $T_1, \ldots, T_n \in \mathcal{D}^r(\Pi)$ be a transverse sequence and let $\Pi_i = \Pi(T_n \circ \cdots \circ T_{n-i+1})$ for $1 \leq i \leq n$, $\Pi_0 = \Pi$, and $F_i = F(\Pi_i)$. For $x \in \overline{\Omega}$ define

$$k_i(x) = \#\{\pi \in \Pi_i : x \in \overline{\pi}\} \quad \text{and} \quad m_i(x) = \max\{F(\pi; x) : \pi \in \Pi_i\},$$

and let

$$M_i = \sup\{m_i(x) : x \in \overline{\Omega}\} \quad \text{and} \quad K_i = \sup\{k_i(x) : x \in \overline{\Omega}\}.$$

It is clear that $F_i \leq M_i K_i$ whenever these are all defined, so it suffices to show that there are bounds on M_n and K_n which are independent of $T_1, \ldots, T_n$. We will show that

$$M_n \leq d M_0 \quad \text{and} \quad K_n \leq K_0^d.$$

The proposition follows with $B = d M_0 K_0^d$.

For a piecewise smooth domain D and $x \in \overline{D}$, the *index of x in D* is $\operatorname{ind}(x, D) = \dim(D_x)$.

LEMMA 3.5. *If M and N are piecewise C^r domains with transverse intersection at x, then $\operatorname{ind}(x, M \cap N) = \operatorname{ind}(x, M) + \operatorname{ind}(x, N) - d$, $F(M \cap N; x) = F(M; x) + F(N; x)$ and $B_\delta(x) \setminus (\partial M \cup \partial N)$ has exactly four connected components for $\delta > 0$ sufficiently small.*

Proof. Each claim is immediate from transversality and Proposition 3.1. □

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For a partition Σ and $x \in \overline{\Omega}$, let

$$\mathrm{Ind}(x, \Sigma) = \min\{\mathrm{ind}(x, \sigma) : x \in \overline{\sigma},\ \sigma \in \Sigma\}.$$

Fix $x \in \overline{\Omega}$, let $\pi_j \in \Pi,\ 1 \le j \le k_0(x)$, denote the elements of Π with x in their closures, and let $y_j = T_1|_{\pi_j}(x)$. We claim that

$$\mathrm{Ind}(x, \Pi_n) \le \mathrm{Ind}(y_j, \Pi_{n-1}) \tag{1}$$

for all $j = 1, \ldots, k_0(x)$, and that equality holds if and only if x is an interior point of a partition element, i.e. $k_0(x) = 1$ and $m_0(x) = 0$. By the definition of $\mathrm{Ind}(x, \Pi_n)$ and Lemma 3.5, for $1 \le j \le k_0(x)$,

$$\begin{aligned}\mathrm{Ind}(x, \Pi_n) &\le \mathrm{ind}(x, \pi_j) + \mathrm{Ind}(y_j, \Pi_{n-1}) - d \\ &\le \mathrm{Ind}(y_j, \Pi_{n-1}),\end{aligned}$$

with equality if and only if $\mathrm{ind}(x, \pi_j) = d$. This proves the claim (1).

It is now easy to show that

$$k_n(x) \le K_0^{d-\mathrm{Ind}(x,\Pi_n)} \quad \text{and} \quad m_n(x) \le M_0(d - \mathrm{Ind}(x, \Pi_n)),$$

from which the bounds on K_n and M_n are immediate. We proceed by induction and assume that these inequalities hold for all $z \in \overline{\Omega}$ with $n-1$ in place of n. By Lemma 3.5 and our induction hypothesis we have

$$k_n(x) = \sum_{j=1}^{k_0(x)} k_{n-1}(y_j) \le \sum_{j=1}^{k_0(x)} K_0^{d-\mathrm{Ind}(y_j,\Pi_{n-1})}.$$

The bound for $k_n(x)$ now follows from the claim (1).

For each $j = 1, \ldots, k_0(x)$ and $l = 1, \ldots, k_{n-1}(y_j)$, let $\pi_{j,l} \in \Pi_{n-1}$ denote the elements of Π_{n-1} with y_j in their closures. It follows from Lemma 3.5 that

$$\begin{aligned}m_n(x) &= \max_{j,l}\{F(\pi_j; x) + F(\pi_{j,l}; y_j)\} \\ &\le m_0(x) + \max_j\{m_{n-1}(y_j)\} \\ &\le m_0(x) + \max_j\{M_0(d - \mathrm{Ind}(y_j, \Pi_{n-1}))\},\end{aligned}$$

and the inequality follows from the claim (1). This completes the proof. □

4. *Openness of n-transversality*

We prove that the set of transverse sequences $(T_1, \ldots, T_n)$ is C^1-open in $(\mathcal{D}^r(\Pi))^n$. The proof is by induction on n once we know that the partition $\Pi(T_n \circ \cdots \circ T_1)$ varies continuously with respect to $(T_1, \ldots, T_n)$ when this sequence is transverse.

Let I_X denote the identity map on a set X.

Definition 4.1. A *stratified diffeomorphism* of piecewise C^r domains M and M' is a map $\varphi : M \to M'$ which is continuous and one-to-one on a neighborhood U of M, is C^r on strata of M and on $U \setminus M$, and takes strata of M to strata of M'. φ is called an *ϵ-similarity* if, in addition, $\|\varphi - I_U\|_{C^0(U)} < \epsilon$ and $\|\varphi - I_L\|_{C^1(L)} < \epsilon$ for any stratum $L \subset \overline{M}$, and likewise for φ^{-1}.

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We say that M and M' as above are *ϵ-similar*. Two piecewise smooth partitions Σ_1 and Σ_2 are *ϵ-similar* if there is a bijection of elements $\tau : \Sigma_1 \to \Sigma_2$ such that σ and $\tau(\sigma)$ are ϵ-similar for all $\sigma \in \Sigma_1$. Our goal is to prove the following proposition.

PROPOSITION 4.2. *Let $T_1, \ldots, T_n$ be a transverse sequence. Given $\epsilon > 0$, there exists $\delta > 0$ such that every sequence $T_1', \ldots, T_n'$ with $\|T_i' - T_i\|_{C^1} < \delta$ for all i is transverse and $\Pi(T_n' \circ \cdots \circ T_1')$ is ϵ-similar to $\Pi(T_n \circ \cdots \circ T_1)$. In particular, the set of n-transverse maps is C^1-open.*

Note that corresponding elements of $\Pi(T^n)$ and $\Pi(T_n \circ \cdots \circ T_1)$ have the same Π-trajectory. That is, if $\pi \in \Pi(T^n)$ corresponds to $\tilde{\pi} \in \Pi(T_n \circ \cdots \circ T_1)$, then $T^k(\pi)$ and $T_k \circ \cdots \circ T_1(\tilde{\pi})$ are contained in the same element of Π for $k = 1, \ldots, n$. The proof of the proposition is based on the following lemma.

LEMMA 4.3. *Let $M_0, N_0 \subset \mathbb{R}^d$ be transverse piecewise smooth domains. For any $\epsilon > 0$, there exists $\delta > 0$ such that if M and N are piecewise smooth domains which are δ-similar to M_0 and N_0, respectively, then M is transverse to N, and $M \cap N$ is ϵ-similar to $M_0 \cap N_0$. In particular, there is a natural correspondence between the connected components of $M_0 \cap N_0$ and $M \cap N$.*

Assuming this for now, we proceed with the proof.

Proof of Proposition 4.2. When $n = 1$, transversality is clear and similarity follows from Lemma 4.3. Proceeding by induction, we assume the proposition holds for sequences of $(n-1)$ maps, let $T_1, \ldots, T_n$ be transverse, and fix $\epsilon > 0$. We wish to find an appropriate δ.

$T_1, \ldots, T_n$ is transverse, therefore $T_2, \ldots, T_n$ is transverse, and our inductive hypothesis implies that for any $\epsilon_0 > 0$ there exists $\delta_0 > 0$ such that any sequence $T_2', \ldots, T_n'$ with $\|T_i' - T_i\|_{C^1} < \delta_0$, $2 \le i \le n$, is transverse and $\Pi(T_n' \circ \cdots \circ T_2')$ is ϵ_0-similar to $\Pi(T_n \circ \cdots \circ T_2)$. We will see in a moment how to choose ϵ_0, but suppose for now that we have done so and we have such a sequence. By transversality, $T_1(M)$ is transverse to N_0 for any $M \in \Pi$ and $N_0 \in \Pi(T_n \circ \cdots \circ T_2)$. Let N denote the element of $\Pi(T_n' \circ \cdots \circ T_2')$ corresponding to N_0 (so N is ϵ_0-similar to N_0). By Lemma 4.3, we know that if ϵ_0 is sufficiently small, then there is a number $\delta_{N_0} > 0$ so that $T_1'(M)$ is transverse to N and $M \cap T_1'^{-1}(N)$ is ϵ-similar to $M \cap T_1^{-1}(N_0)$ whenever $\|T_1' - T_1\|_{C^1} < \delta_{N_0}$.

Let $\delta_1 = \min\{\delta_{N_0} : N_0 \in \Pi(T_n \circ \cdots \circ T_2)\}$ and let $\delta = \min\{\delta_0, \delta_1\}$. It follows that every sequence of maps $T_1', \ldots, T_n'$ with $\|T_i' - T_i\|_{C^1} < \delta$ is transverse and that $\Pi(T_n' \circ \cdots \circ T_1')$ is ϵ-similar to $\Pi(T_n \circ \cdots \circ T_1)$. □

Proof of Lemma 4.3. M is transverse to N for δ small enough and therefore $M \cap N$, as well as $M_0 \cap N_0$, is a piecewise C^r domain by Proposition 3.1.

A δ-similarity $M_0 \to M$ restricts to a δ-similarity $M_0 \cap N_0 \to M \cap \varphi(N_0)$. Without loss of generality, then, we let $M = M_0$ and construct a ξ-similarity $\Phi : M \cap N \to M \cap N_0$, where $\xi \to 0$ as $\delta \to 0$.

The idea is the following. If $\varphi : N \to N_0$ is a δ-similarity, then we can define $\Phi : M \cap N \to M \cap N_0$ to be equal to φ on the interior of M, away from ∂M; on the interior of N, away from ∂N, let Φ be the identity; and near $\partial M \cap \partial N$, working in charts as in the proof of Proposition 3.1, Φ 'slides' N onto N_0 along strata of M.

https://doi.org/10.1017/S0143385702000627 Published online by Cambridge University Press

To be more precise, since $\partial M \cap \partial N$ is compact we may choose finitely many points $x_1, \ldots, x_m \in \partial M \cap \partial N$ and radii $r_1, \ldots, r_m$ such that the balls $B_{r_i}(x_i)$ cover $\partial M \cap \partial N$ and there are charts

$$\psi_i : (B_{2r_i}(x_i), M, N, x_i) \to (\mathbb{R}^d, H_1, H_d, 0)$$

as in the proof of Proposition 3.1. Let $R = \min\{r_i\}$.

Now we define domains N_i and maps Φ_i such that

$$N = N_m \xrightarrow{\Phi_m} N_{m-1} \xrightarrow{\Phi_{m-1}} N_{m-2} \to \cdots \to N_1 \xrightarrow{\Phi_1} N_0$$

and each Φ_i is supported on $B_{2r_i}(x_i)$. For $i = m$, let $\Phi_m = \psi_m^{-1} \circ \gamma_m \circ \psi_m$, where γ_i (in local coordinates) takes strata of $N \cap B_{r_m}(x_m)$ into strata of N_0 and preserves strata of M. For example, let $\mathcal{F}$ be a smooth foliation of $B_{2r_m}(x_m)$ by smooth disks $\mathcal{F}_y$ of dimension $d - \dim(N_{x_m})$ such that $\mathcal{F}$ restricts to a foliation of each stratum of M, and let $\gamma_m(x_m)$ be the unique point of intersection in $\mathcal{F}_{x_m} \cap (N_0)_x$. Similarly, by sequentially refining $\mathcal{F}$ to foliations by lower-dimensional disks and using that N and N_0 are similar (in particular, there is a well-defined correspondence between strata), we define γ_m on $N \cap B_{r_m}(x_m)$. As defined, γ_m is smooth on strata of $M \cap N$ with C^1 norm of the order of δ since $\mathcal{F}$ and its refinements are smooth, and N_x is transverse to M_x. Finally, extend γ_m to B_{2r_m}, using the local triviality of N, so that it is smooth on strata of $M \cap N$ and leaves M invariant.

For $i < m$, define γ_i in the same way as γ_m, but with N_i in place of N.

The composition $\Phi' = \Phi_1 \circ \cdots \circ \Phi_m$ takes $\partial M \cap N$ to $\partial M \cap N_0$. For $r > 0$ let $U_r = B_r(\partial M)$, and let R be a Lebesgue number for the cover of $\partial M \cap \partial N$ by the balls $B_{r_i}(x_i)$ which is less than $\min\{r_i\}$. By Urysohn's Lemma there exists a C^∞ function $\tau : \mathbb{R}^d \to [0, 1]$ which is equal to 1 on $\overline{U}_R$ and is supported on U_{2R}. Define

$$\widetilde{\Phi} = \tau \cdot \Phi' + (1 - \tau) \cdot \varphi,$$

where $\varphi : N \to N_0$ is a δ-similarity. Φ is obtained from Φ' by making small perturbations supported on $U_{2R} \setminus U_R$. □

We will conclude this section with a local version of Proposition 4.2 that is used in the proof that n-transverse maps are dense.

Definition 4.4. Let $\pi \in \Pi$. A *π-neighborhood* is a set of the form $O \cap \overline{\pi}$, where $O \subset \mathbb{R}^d$ is open. For such a set U, if $x \in U$ we say that U is a π-neighborhood of x. When the particular element π is not important, we speak of a *Π-neighborhood* (of x).

Definition 4.5. Let U be a π-neighborhood, $\pi \in \Pi$. A map T is *n-transverse on U* if T is $(n-1)$-transverse and $T(\pi)$ is transverse to $\Pi(T^{n-1})$ at each point in $\overline{T(U)}$.

Local transversality is used in the proof of density when we wish to make local perturbations to a given map without destroying desirable properties.

PROPOSITION 4.6. *Let T_0 be a piecewise C^r map which is n-transverse on a Π-neighborhood U. There exists a neighborhood $\mathcal{U}$ of T_0 such that every $T \in \mathcal{U}$ is n-transverse on U.*

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Proof. The case $n = 1$ is straightforward and we proceed by induction. Assume the result holds for a given value of n, let T_0 be $(n+1)$-transverse on a π-neighborhood U, and choose $\epsilon > 0$. T_0 is n-transverse and, by Proposition 4.2, not only are all maps in a neighborhood of T_0 n-transverse, but for any $\epsilon_0 > 0$ there exists $\delta > 0$ such that $\Pi(T^n)$ is ϵ_0-similar to $\Pi(T_0^n)$ for all $T \in B_\delta(T_0)$. Since T_0 is $(n+1)$-transverse on U, $T_0(\pi)$ is transverse to $\Pi(T_0^n)$ on $\overline{T_0(U)}$. That is, for any stratum W of $\partial\pi$, $T_0(\overline{W})$ is transverse to $\Pi(T_0^n)$ on $\overline{T_0(U)}$. It follows from standard theorems that if ϵ_0 and δ are sufficiently small, then $T(\pi)$ is transverse to $\Pi(T^n)$ on $\overline{T(U)}$ and we are done. □

5. *Density of n-transversality*

It is simple to prove, given the results of the last section, that the set of transverse sequences is dense in $(\mathcal{D}^r(\Pi))^n$. This, however, does not imply density on the diagonal, i.e. that the set of n-transverse maps is dense in $\mathcal{D}^r(\Pi)$. Our proof of this is by induction. Given a map which is $(n-1)$-transverse, we obtain one which is n-transverse by making arbitrarily small local perturbations. Care is required in choosing the supports of these perturbations, primarily in cases where a point in $\partial\Pi$ is periodic.

5.1. *Strategy.* We begin with some notation. Given a map T, a set $E \subset \overline{\Omega}$, and a positive integer j, if no particular branch of T^j is specified, then $T^j(E)$ denotes the union of images of E under all branches of T^j. That is, we consider T to be multi-valued on $\partial\Pi$. In the same context, for any positive integer n let

$$\mathcal{O}_n(E; T) := \bigcup_{j=1}^{n} T^j(E)$$

denote the (strictly) forward orbit of E up to and including time n. When there is no ambiguity, we write $\mathcal{O}_n(E)$. Note that E is not necessarily contained in $\mathcal{O}_n(E)$. When $E = \{x\}$ we write $\mathcal{O}_n(x; T)$ or $\mathcal{O}_n(x)$.

Definition 5.1. A point $x \in \overline{\Omega}$ is *n-aperiodic* for a map T if $x \notin \mathcal{O}_n(x)$. T is *n-aperiodic on a set* $E \subset \overline{\Omega}$ if every point in E is n-aperiodic. T is simply *n-aperiodic* if it is n-aperiodic on $\partial\Pi$.

If x is n-aperiodic for T, then we can choose a neighborhood U of x so small that the forward images $T(U), \ldots, T^n(U)$ are disjoint from U (though not necessarily from each other). That is, we may assume $\overline{U} \cap \overline{\mathcal{O}_n(U)} = \emptyset$. We will use this fact repeatedly.

We show in the next section that if T_0 is n-transverse *and* $(n+1)$-aperiodic, then all maps in an open and dense subset of a neighborhood of T_0 are $(n+1)$-transverse. We then show that any n-transverse map can be approximated by one which is also $(n+1)$-aperiodic. In this way an arbitrary n-transverse map is approximated by one which is $(n+1)$-transverse, and the density of n-transversality for arbitrary n follows by induction and the openness of n-transversality. We accomplish all this by making local perturbations to obtain local versions of these statements, then passing to the global versions we require.

A *local property* of a map is a property, such as n-transversality or n-aperiodicity, which can be verified to hold or not by looking at some iterate of the map restricted to arbitrarily

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small neighborhoods of each point in its domain. The statement and proof of the next lemma hold either with or without the statements in brackets.

LEMMA 5.2. *Let T_0 be a map, $K \subset \overline{\Omega}$ compact, and $\mathcal{P}$ a local property. If for every $x \in K$ there exists a neighborhood, U_x, of x and a number $\delta_x > 0$ such that all maps in [an open and dense subset of] $B_{\delta_x}(T_0)$ satisfy $\mathcal{P}$ on U_x, then $\mathcal{P}$ holds on all of K for maps in an open [and dense subset of a] neighborhood of T_0.*

Proof. K is compact, so we can cover it by finitely many of the neighborhoods $U_i = U_{x_i}, i = 1, \ldots, m$. Let $\delta_i = \delta_{x_i}$ and let $\mathcal{G}_i$ denote the set of maps in $B_{\delta_i}(T_0)$ which satisfy $\mathcal{P}$ on U_i. Then $\mathcal{G} = \bigcap_{i=1}^m \mathcal{G}_i$ is [an open and dense subset of] $B_\delta(T_0)$, where $\delta = \min_i\{\delta_i\}$, and every $T \in \mathcal{G}$ satisfies $\mathcal{P}$ on each U_i, hence on all of K. □

5.2. *Aperiodicity implies density.*

PROPOSITION 5.3. *Let T_0 be n-transverse and $(n+1)$-aperiodic. Then all maps in an open and dense subset of a neighborhood of T_0 are $(n+1)$-transverse.*

Proof. By Lemma 5.2 it suffices to find, for each $x \in \overline{\Omega}$, a neighborhood U_x and a number $\delta_x > 0$ such that every map in an open and dense subset of $B_{\delta_x}(T_0)$ is $(n+1)$-transverse on U_x. Let $x \in \overline{\Omega}$; there are two cases.

Case 1: $x \notin \partial\Pi$. This is trivial since the stratum through x is an open set and n-transversality is an open condition.

Case 2: $x \in \partial\Pi$. For each $\pi \in \Pi$ such that $x \in \partial\pi$, we find a π-neighborhood U_π and a number $\delta_\pi > 0$ so that T is $(n+1)$-transverse on U_π for all T in an open and dense subset of $B_{\delta_\pi}(T_0)$. We let U_x be a neighborhood of x contained in the union of the U_π's and let δ_x be the smallest of the δ_π's.

For $x \in \partial\pi$ and $\epsilon > 0$, let $U = U_\epsilon := B_\epsilon(x) \cap \overline{\pi}$. By $(n+1)$-aperiodicity we may choose ϵ so small that $\eta := d(\overline{U}, \overline{\mathcal{O}_{n+1}(U; T_0)}) > 0$. It follows from Proposition 4.2 that for some $\delta > 0$, all maps $T \in B_\delta(T_0)$ are n-transverse and satisfy $d(\overline{U}, \overline{\mathcal{O}_{n+1}(U; T)}) > \eta/2$. We claim that $U_\pi = U_{\epsilon/2}$ and $\delta_\pi = \delta$ have the desired properties.

For any $T \in B_\delta(T_0)$ we will find a map T_1 which is arbitrarily close to T and is $(n+1)$-transverse on U. By our choice of δ, there is a neighborhood $V \supset \overline{T_0(U)}$ such that $\overline{(V \cup \mathcal{O}_n(V; T))} \cap \overline{U} = \emptyset$ and $\overline{T(U)} \subset V$ for all $T \in B_\delta(T_0)$. Therefore, given $T \in B_\delta(T_0)$, if $T_1 \in B_\delta(T_0)$ is any map which differs from T only on U, then the restrictions of $\Pi(T^n)$ and $\Pi(T_1^n)$ to V are identical. By standard transversality theorems we can find such a T_1 that carries the restriction of $\partial\pi$ to $\overline{U_{\epsilon/2}}$ transversely to $\Pi(T_1^n)$. In other words, T_1 is $(n+1)$-transverse on $U_{\epsilon/2}$. This shows that the set of maps which are $(n+1)$-transverse on U is dense in $B_\delta(T_0)$. By Proposition 4.6, the set of maps which are $(n+1)$-transverse on U is open. □

5.3. *Aperiodicity is open.* For a set E, a map T, and $n > 0$, set

$$\Delta_T^{(n)}(E) := d(E, \mathcal{O}_n(E; T)),$$

and just $\Delta_T^{(n)}(x)$ when $E = \{x\}$.

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LEMMA 5.4. *If T is n-transverse, then $\Delta_T^{(n+1)}(x)$ is lower semicontinuous (as a function of x) on $\overline{\Omega}$. It follows that T is $(n+1)$-aperiodic if and only if there exists $\eta > 0$ such that $\Delta_T^{(n+1)}(x) \geq \eta$ for all $x \in \partial\Pi$.*

Proof. The restriction of $\Delta := \Delta_T^{(n+1)}$ to any element of $\Pi(T^n)$ is continuous since Δ is the minimum of the continuous functions $d(x, T^j(x))$, $j = 1, \ldots, n+1$, on each element of $\Pi(T^n)$. To see that Δ is lower semicontinuous on $\overline{\Omega}$, let $x \in \partial\Pi(T^n)$, and let $\pi_1, \ldots, \pi_k$ denote all the elements of $\Pi(T^n)$ with x in their closure. Then

$$\Delta(x) = \min_{1 \leq i \leq k} \lim_{\substack{z \to x \\ z \in \pi_i}} \Delta(z),$$

and the lower semicontinuity follows from the fact that Δ is continuous on each element of $\Pi(T^n)$.

In particular, Δ is lower semicontinuous on $\partial\Pi$. T is $(n+1)$-aperiodic if and only if $\Delta(x) > 0$ for all $x \in \partial\Pi$; by the lower semicontinuity of Δ and the compactness of $\partial\Pi$, this is true if and only if $\Delta \geq \eta$ for some $\eta > 0$. □

PROPOSITION 5.5. *The set of n-transverse maps which are $(n+1)$-aperiodic on a given compact subset $K \subset \partial\Pi$ is open. In particular, the set of maps which are n-transverse and $(n+1)$-aperiodic is open.*

Proof. Let T_0 be n-transverse and $(n+1)$-aperiodic on K. By the lemma, $\Delta_{T_0} := \Delta_{T_0}^{(n+1)} \geq \eta$ on K for some $\eta > 0$. We construct an open cover of K as follows. For each $x \in K$ let U_x be a neighborhood of x such that $\Delta_{T_0}(U_x) \geq \eta/2$. We may assume that for any particular branch of T_0^j, $j = 1, \ldots, n$, $T_0^j(\overline{U}_x)$ intersects $\partial\Pi$ if and only if $T_0^j(x) \in \partial\Pi$. By Proposition 4.2 there exists $\delta > 0$ such $\Delta_T^{(n+1)}(U_x) \geq \eta/4$ for all $T \in B_\delta(T_0)$, and the proposition follows from Lemma 5.2. □

5.4. *Aperiodicity is dense.* We begin with a lemma which shows how to destroy local $(n+1)$-periodicity with arbitrarily small perturbations. We apply the lemma in the case that U is a π-neighborhood of a point $x \in \partial\Pi$ which is n-aperiodic but $(n+1)$-periodic. One complication is that x may return to $\partial\Pi$ (though not to itself) at some time before the $(n+1)$th iterate. V is a suitably chosen neighborhood of the last place this occurs.

LEMMA 5.6. *Let T be n-transverse, $\pi \in \Pi(T^n)$, U a π-neighborhood, and V a Π-neighborhood satisfying:*

(1) *$\mathcal{O}_n(\overline{U})$ is disjoint from $\overline{U}$;*
(2) *there exists a unique k between 1 and n such that, for $1 \leq j \leq n$, $\overline{T^j(U)} \cap \overline{V} \neq \emptyset$ if and only if $j = k$;*
(3) *$\overline{T^k(U)} \subset V$; and*
(4) *$T^j(U) \cap \partial\Pi = \emptyset$ for $k < j \leq n$.*

Then for any $\epsilon > 0$ there exists a map T_1 within ϵ of T, such that U is contained in an element of $\Pi(T_1^n)$ and $T_1^{n+1}(y) \neq y$ for all $y \in \overline{U} \cap \partial\Pi$.

Proof. If T_1 is sufficiently close to T and agrees with T outside V, then, by conditions (2) and (4), $\Pi(T_1^n)$ restricted to $\overline{U}$ is identical to $\Pi(T^n)$ restricted to $\overline{U}$. Let W be the closure

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of a connected component of the restriction of a face of π to $\overline{U}$; we find a suitable map satisfying $T_1^{n+1}(y) \neq y$ for all $y \in W$. The result follows from this and the openness of local $(n+1)$-aperiodicity. Simply repeat this procedure for all such W to produce a finite sequence $T_1, T_2, \ldots$, the last of which is $(n+1)$-aperiodic on $\overline{U}$.

Let $W' = T^k(W) \subset \overline{V}$. This is a closed subset of a $(d-1)$-dimensional manifold contained in $\overline{V}$ and is bounded away from $\partial V \setminus \partial\Pi$. We want a map $\Psi : \overline{V} \to \overline{\Omega}$ such that $\Psi = T$ on $\partial V \setminus \partial\Pi$ and $T^{n-k} \circ \Psi \circ T^k(y) \neq y$ for all $y \in W$. Since U is a $\Pi(T^n)$-neighborhood, $T^k : U \to V$ has a local inverse, call it S_k, and we want $T^{n-k} \circ \Psi(z) \neq S_k(z)$ for all $z \in W'$. By condition (4), there is a neighborhood $\widetilde{U}$ of $\overline{T^{k+1}(U)}$ such that $T^{n-k} : \widetilde{U} \to \Omega$ has a local inverse, call it S_{n-k}. For any Ψ sufficiently close to $T|_{\overline{V}}$, $\overline{\Psi(T^k(U))} \subset \widetilde{U}$, and we want $\Psi(z) \neq S_{n-k} \circ S_k(z)$ for all $z \in W'$. Since W' is a closed subset of a $(d-1)$-dimensional manifold, the set of maps Ψ for which this holds is open and dense in the space of maps $\overline{V} \to \overline{\Omega}$; in particular, we can find one which is arbitrarily close to $T|_{\overline{V}}$ and equal to it on a neighborhood of $\partial V \setminus \partial\Pi$. Let T_1 be equal to Ψ on V and equal to T elsewhere. □

PROPOSITION 5.7. *The set of $(n+1)$-aperiodic maps is dense in the set of n-transverse maps.*

Proof. The case $n = 0$ is straightforward and we proceed by induction. Assume that n-aperiodic maps are dense in the set of $(n-1)$-transverse maps; by the openness of n-transversality these are dense in the set of n-transverse maps, so let T_0 be n-aperiodic and n-transverse. By Lemma 5.2, it suffices to find for each $x \in \partial\Pi$ a neighborhood U_x of x and a number $\delta_x > 0$ such that an open and dense subset of maps in the δ_x-ball around T_0 are $(n+1)$-aperiodic on U_x.

Given $x \in \partial\Pi$, let U be a neighborhood of x such that:

U1. $\overline{\mathcal{O}_n(U; T_0)} \cap \overline{U} = \emptyset$;

U2. for any $\pi \in \Pi(T_0^{j-1})$, $1 \leq j \leq n$, if $T_0^j|_\pi(x) \notin \partial\Pi$, then $\overline{T_0^j(U \cap \pi)} \cap \partial\Pi = \emptyset$; and

U3. for any $\pi \in \Pi(T_0^n)$, if $T_0^{n+1}|_\pi(x) \neq x$, then $\overline{T_0^{n+1}(U \cap \pi)} \cap \overline{U} = \emptyset$.

These conditions are stable in the following sense. If T is close enough to T_0, then by Proposition 4.2 each $\pi \in \Pi(T_0^{j-1})$, $j = 1, \ldots, n+1$, corresponds to a unique $\pi' \in \Pi(T^{j-1})$. If the closure of $T_0^j(U \cap \pi)$ is disjoint from any fixed compact set (e.g. $\overline{U}$ or $\partial\Pi$), then the same is true for the closure of $T^j(U \cap \pi')$ for any T close enough to T_0. So, for example, condition U3 could be written as follows. For any $\pi \in \Pi(T_0^n)$, if $T_0^{n+1}|_\pi(x) \neq x$, then $\overline{T^{n+1}(U \cap \pi)} \cap \overline{U} = \emptyset$ for all T sufficiently close to T_0. Without further remark, we assume all maps T are sufficiently close to T_0.

Suppose that x is $(n+1)$-aperiodic for T_0. By the definition of U and the above stability properties, all points in U are $(n+1)$-aperiodic for all maps T in some neighborhood of T_0. So we are done in this case, with $U_x = U$ and δ_x sufficiently small.

Assume then that $x \in \mathcal{O}_{n+1}(x; T_0)$. This means there is at least one orbit of the form $x = x_0, x_1, \ldots, x_n, x_{n+1} = x$, where $x_{i+1} \in T_0(\{x_i\})$. Of the points $x_0, x_1, \ldots, x_n$, let $z = x_k$ be the last one which is in $\partial\Pi$, i.e. $x_j \notin \partial\Pi$ for $k < j \leq n$. We call z a *last cut point*. If we consider all possible orbits of this form, we obtain the set of last cut

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points, $\mathrm{LCP}_n(x)$. This set is defined in terms of T_0 and *shall remain fixed throughout our discussion*. We now make a further condition on the neighborhood U:

U4. for any $\pi \in \Pi(T_0^{j-1})$, $1 \le j \le n$, and any $z \in \mathrm{LCP}_n(x)$, if $T_0^j|_\pi(x) \neq z$, then $z \notin \overline{T_0^j(U \cap \pi)}$.

That is, the orbit of U avoids $\mathrm{LCP}_n(x)$ whenever possible. We now construct Π-neighborhoods of our last cut points. These are where we will make perturbations to remove $(n+1)$-aperiodicity. Given a last cut point z, let $k = k(z)$ be the smallest number i such that $z \in \mathcal{O}_i(x)$ (that is, the first hitting time of z starting from x). If $k < n$, then there is at least one partition element $\pi \in \Pi$ with $z \in \partial\pi$ such that $T_0^j|_\pi(z) \notin \partial\Pi$ for $j = 1, \ldots, n-k$ and $T_0^{n-k+1}|_\pi(z) = x$. When $k = n$ there is simply at least one $\pi \in \Pi$ with $z \in \partial\pi$ and $T_0|_\pi(z) = x$. In either case let Σ_z denote the union of all such partition elements and let $\{V_z\}$, $z \in \mathrm{LCP}_n(x)$, be a collection of neighborhoods, $z \in V_z$, such that the following hold:

V1. $\overline{V}_z \cap \overline{V}_{z'} = \emptyset$ for $z \neq z'$;

V2. if $k(z) < n$, then $\overline{\mathcal{O}_{n-k(z)}(V_z \cap \Sigma_z)}$ is disjoint from $\partial\Pi$ and $\overline{V}_{z'}$ for all $z, z' \in \mathrm{LCP}_n(x)$; and

V3. for each $\pi \in \Pi(T_0^j)$, $1 \le j \le n$, if $T_0^j|_\pi(x) \neq z$, then $\overline{T_0^j(U \cap \pi)}$ is disjoint from $\overline{V}_z$.

This last condition is possible by condition U4. Note that neighborhoods defined in this way enjoy the same stability properties as U (remember, $\mathrm{LCP}_n(x)$ is fixed).

We claim that most of the work is now done. However, we may have to shrink U one last time:

U5. let U be so small that, for any $\pi \in \Pi(T_0^{j-1})$, $1 \le j \le n$, if $T_0^j(U \cap \pi)$ intersects $V_z \cap \Sigma_z$ for some z, then $\overline{T_0^j(U \cap \pi)} \subset V_z$.

Again, this is a stable property of U. Let T be a map close enough to T_0 so that the properties of U and $\{V_z\}$ still hold. For each $z \in \mathrm{LCP}_n(x)$, let

$$\Lambda_z = U \cap T^{-k(z)}(V_z \cap \Sigma_z).$$

These are the only places where $(n+1)$-periodicity after a perturbation could occur. However, now we can apply the previous lemma to sets of the form $\Lambda_z \cap \pi$, with $\pi \in \Pi(T^n)$. Since the perturbations made in that lemma do not alter the restriction of $\Pi(T^n)$ on U and aperiodicity is an open condition, we can make these perturbations sequentially until we have a map T' which is arbitrarily close to T, differs from T only on the sets $V_z \cap \Sigma_z$, and is $(n+1)$-aperiodic on $\overline{U}$. Since T was arbitrary in a neighborhood of T_0 and $(n+1)$-aperiodicity is a locally open condition, this completes the proof. □

PROPOSITION 5.8. *The set of n-transverse maps is dense in $\mathcal{D}^r(\Pi)$.*

Proof. The case $n = 1$ is straightforward and we proceed by induction. Assume that the set of n-transverse maps is dense and let $T_0 \in \mathcal{D}^r(\Pi)$ be arbitrary. By hypothesis we can find an n-transverse map T_1 arbitrarily close to T_0. By Propositions 4.2 and 5.7 we can approximate T_1 arbitrarily well by a map T_2 which is $(n+1)$-aperiodic and n-transverse. By Proposition 5.3, we can find an $(n+1)$-transverse map T_3 as close as we like to T_2 and we are done. □

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6. *Proof of Theorem 1.6*

Recall that a measure on the space of maps induces a Markov process on Ω in a natural way. We say that a function $h \in L^1(m)$ is an invariant density for this process if $h = d\mu/dm$ for some stationary distribution μ.

As with Proposition 1.2, the following result is proved in [**6**] (the constants $F_n(\mathcal{U}) \approx F(\Pi(T^n))$ and $\Lambda_n(\mathcal{U}) \approx \lambda(T^n)$ are defined below).

THEOREM 6.1. *Let Π be a piecewise C^r partition of Ω and $T \in \mathcal{D}^r(\Pi)$. If there is an integer $n \geq 0$ and a sufficiently small neighborhood $\mathcal{U}$ of T such that $\Lambda_{n+1}(\mathcal{U}) > F_n(\mathcal{U}) + 1$, then T is locally L^1-stochastically stable.*

To prove Theorem 1.6 we will show that the hypotheses of Theorem 6.1 are satisfied for the open and dense set $\mathcal{T}$ defined in the proof of Theorem 1.1. Before doing this we must explain the conditions.

For $T \in \mathcal{T}$, let $\mathcal{U} \subset \mathcal{T}$ be a neighborhood of T. Let $\mathcal{U}^{\circ n}$ denote the set of all compositions $\mathbf{T} = T_n \circ \cdots \circ T_1$ with $T_i \in \mathcal{U}$, $1 \leq i \leq n$, and let

$$\Lambda_n(\mathcal{U}) = \inf\{\lambda(\mathbf{T}) : \mathbf{T} \in \mathcal{U}^{\circ n}\}$$

be the smallest possible expansion obtained by composing n maps chosen from $\mathcal{U}$. $\Lambda_{n+1}(\mathcal{U})$ can be made as close to $\lambda(T^{n+1})$ as desired by choosing $\mathcal{U}$ sufficiently small.

We will define $F_n(\mathcal{U})$ for $\mathcal{U}$ a sufficiently small neighborhood of an n-transverse map T, since this is the case we are interested in. If Σ is a piecewise smooth partition of Ω, then, for all $\delta > 0$ sufficiently small, every face W of every element $\sigma \in \Sigma$ has a δ-collar W_δ. For $x \in \overline{\Omega}$, let $F_\delta(\sigma; x)$ denote the number of faces W of σ for which $x \in \overline{W}_\delta$, let $F_\delta(\Sigma; x)$ denote the sum of these over $\sigma \in \Sigma$, and let $F_\delta(\Sigma)$ be the supremum over $x \in \overline{\Omega}$. At last we define

$$F_n(\mathcal{U}) = \lim_{\delta \to 0} \sup\{F_\delta(\Pi(\mathbf{T})) : \mathbf{T} \in \mathcal{U}^{\circ n}\}.$$

This is well-defined for T n-transverse and $\mathcal{U}$ sufficiently small by Proposition 4.2, and the fact that the normal bundle of a manifold varies continuously with the tangent bundle.

Now that $\Lambda_n(\mathcal{U})$ and $F_n(\mathcal{U})$ are defined we can prove the theorem. Let T be in $\mathcal{T}$. We must find an integer n and a neighborhood $\mathcal{U}$ of T satisfying the hypotheses of Theorem 6.1. By the definition of $\mathcal{T}$, we can choose n so that T is n-transverse and $\lambda(T)^{n+1} > F(\Pi(T^n))$. Let $\mathcal{U}$ be a neighborhood of T which is so small that $F_n(\mathcal{U})$ is defined. Since $\Lambda_{n+1}(\mathcal{U})$ can be made arbitrarily close to $\lambda(T^{n+1})$ by choosing $\mathcal{U}$ sufficiently small, it suffices to show that $F_n(\mathcal{U})$ can be made arbitrarily close to $F(\Pi(T^n))$ by choosing $\mathcal{U}$ sufficiently small. For $\mathcal{U}$ sufficiently small these are actually equal as the next lemma, which completes the proof of the theorem, shows.

LEMMA 6.2. *Let T be n-transverse. For any sufficiently small neighborhood $\mathcal{U}$ of T,*

$$F_n(\mathcal{U}) = F(\Pi(T^n)).$$

Proof. For any $x \in \partial\Pi(T^n)$ there is a $\delta_x > 0$ such that for any face W of any element of $\Pi(T^n)$,

$$W \cap B_{\delta_x}(x) \neq \emptyset \quad \Longleftrightarrow \quad x \in \overline{W}.$$

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By the compactness of $\partial\Pi(T^n)$ we can choose $x_1, \ldots, x_m \in \partial\Pi(T^n)$ so that the balls $B_i = B_{\delta_i}(x_i)$, $\delta_i = \delta_{x_i}$, cover $\partial\Pi(T^n)$. In particular,

$$F(\Pi(T^n)) = \max\{F(\Pi(T^n); x_i) : 1 \le i \le m\}.$$

Choose ϵ so small that for every x in the ϵ-neighborhood of $\partial\Pi(T^n)$, the ball $B_{2\epsilon}(x)$ is contained in B_i for some i. By Proposition 4.2, we can choose a neighborhood $\mathcal{U}$ of T such that $\Pi(\mathbf{T})$ is ϵ-similar to $\Pi(T^n)$ for all $\mathbf{T} \in \mathcal{U}^{\circ n}$. If $\mathbf{T} \in \mathcal{U}^{\circ n}$ and W is a face of $\Pi(T^n)$, let $\tilde{W}$ denote the corresponding face of $\Pi(\mathbf{T})$. For x in the ϵ-neighborhood of $\partial\Pi(\mathbf{T})$ we have

$$\begin{aligned} F_\epsilon(\Pi(\mathbf{T}); x) &\le \#\{W : \tilde{W} \cap B_\epsilon(x) \ne \emptyset\} \\ &\le \#\{W : W \cap B_{2\epsilon}(x) \ne \emptyset\} \\ &\le \#\{W : W \cap B_i \ne \emptyset\} \quad \text{for some } i \\ &\le F(\Pi(T^n)), \end{aligned}$$

where the first inequality is obvious, the second is by ϵ-similarity, and the third is by our choice of ϵ. This completes the proof. □

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