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Phase transition, non-Gibbsianness and subexponential instability

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Abstract. In this paper, we consider piecewise smooth dynamical systems with generalized indifferent periodic orbits associated to a given potential function. We show that the presence of such orbits causes non-uniqueness of equilibrium states (phase transitions) and non-Gibbsianness of equilibrium measures. If, in addition, the system exhibits *local exponential instability*, we show that the distribution of the stopping times over hyperbolic regions has powerlike tails with respect to any s-conformal measure, where s is the smallest zero of Bowen's equation. Finally, we prove that the Hausdorff dimension of the limit set is equal to s provided the dynamics is piecewise conformal.

0. *Introduction*

Let $T : X \to X$ be a non-invertible map of a compact metric space X which is not necessarily continuous and piecewise C^1-invertible with respect to a countable generating Markov partition $Q = \{X_i\}_{i\in I}$ of X. Such a system $(T, X, Q = \{X_i\}_{i\in I})$ is called a *piecewise C^1-invertible Markov system* and provides a countable-state symbolic dynamics which is Markov. In this paper, we shall consider the following problem:

(P) How are statistical properties of the system (T, X, Q) influenced by *subexponential instability* of the dynamics?

Here instability is treated in terms of sensitive dependence on initial positions. We stress that the system under consideration is a general Markov system (not necessarily Bernoulli) and may be non-conformal. We direct our attention to:

(i) non-Gibbsian behaviour of equilibrium states in the sense of Bowen;
(ii) phase transitions (non-differentiability of the pressure function);
(iii) powerlike tails of the distribution of the stopping times over hyperbolic regions;
(iv) the Hausdorff dimension of level sets associated to pointwise dimension.

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Under the *finite range condition* subexponential instability of (T, X, Q) is observed when contraction rates of the local inverses are subexponential (see §1 for the definitions). Bowen's definition of a Gibbs measure ν for a potential function $\phi : X \to \mathbb{R}$ is given by the following boundedness condition: there exist $0 < K < \infty$ and $P \in \mathbb{R}$ such that

$$K^{-1} \leq \frac{\nu(X_{i_1 \dots i_n}(x))}{\exp\left(\sum_{i=0}^{n-1} \phi T^i(x) + nP\right)} \leq K \text{ (for all } x \in X, \text{ for all } n \geq 1),$$

where $X_{i_1 \dots i_n}(x) \in \bigvee_{h=0}^{n-1} T^{-h}(Q)$ denotes the cylinder containing x. Then, non-Gibbsianness can be described in the following two ways:

(NG1) for a fixed Borel probability measure ν, there is no potential function with respect to which ν is Gibbs;

(NG2) for a fixed potential ϕ, there is no Borel probability measure which is Gibbs for ϕ.

Even if the Gibbs property fails to hold for ν, it is still possible for a potential ϕ to choose a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n\to\infty}(1/n)\log K_n = 0$ instead of the uniform constant K. Such a Borel probability measure ν is called a *weak Gibbs measure* for ϕ (see the definition in §1). The non-Gibbsian weak Gibbs property can be observed for *Sinai–Ruelle–Bowen measures* invariant under certain non-hyperbolic maps which exhibit subexponential instability [**17, 19–22**]. To give an appropriate setting for solving the problem (P), we shall consider *weak bounded variation* (WBV) potentials and give a reasonable definition of the *topological pressure* $P_{\text{top}}(T, \phi)$. Then we define *indifferency* with respect to ϕ as follows: x is a *generalized indifferent point* with respect to ϕ if the ergodic averages $(1/n)\sum_{h=0}^{n-1} \phi T^h(x)$ converge to $P_{\text{top}}(T, \phi)$. Under the existence of a generalized indifferent point with respect to ϕ, we relate non-Gibbsianness of weak Gibbs measures for ϕ to subexponential instability of the system (T, X, Q) (see Theorem 1 and Lemma 3.1). Then we shall discuss our main problem: *when does subexponential instability of dynamics have direct influence upon statistical properties of equilibrium states?*

For this purpose we shall solve the following problems.

(P-1) Find a zero $s > 0$ of Bowen's equation

$$P_{\text{top}}(T, -s\log\|DT\|) = 0. \tag{BE}$$

(P-2) Among all zeros of (BE), characterize a zero $s > 0$ for which there exists a weak Gibbs measure ν for $-s\log\|DT\|$.

(P-3) Establish the existence of a T-invariant ergodic probability measure μ equivalent to ν and statistical properties for (T, μ).

The essential issue for solving these problems is to clarify the local nature of the system. More specifically, under a *locally uniformly expanding property* for (T, X, Q) and *local bounded distortion* for $-s\log\|DT\|$ we construct a uniformly expanding Schweiger's jump transformation [**11**] on a subset X^* of X which provides a countable generating Bernoulli partition Q^* (see §1 for the definitions). Then we prove (see Theorems 2 and 3) the following.

(A-1) There exists a unique zero s of Bowen's equation

$$P_{\text{top}}(T^*, -s\log\|DT^*\|) = 0 \tag{BE}*$$

with respect to (T^*, X^*, Q^*) and s is a zero of (BE) with respect to (T, X, Q).

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(A-2) For the unique zero s of (BE)*, there exist a weak Gibbs measure ν for $-s\log\|DT\|$ and a T-invariant ergodic probability measure μ which is equivalent to ν and satisfies $s = h_\mu(T)/\int_X \log\|DT\|\,d\mu > 0$.

(A-3) The unique zero of (BE)* coincides with the smallest zero of (BE) which can be characterized by a certain variational principle when a marginal set with respect to (T^*, X^*, Q^*) consists of periodic orbits satisfying

$$\sum_{h=0}^{q-1} \log\|DT(T^h x_0)\| = 0 \text{ for } T^q x_0 = x_0. \tag{1}$$

(A-4) Under piecewise conformality of (T, X, Q) the weak Gibbs measure ν is exact-dimensional and the pointwise dimension of ν is given by the smallest zero of (BE) which coincides with the Hausdorff dimension of X.

Combining these results, we can conclude for piecewise C^1-invertible Markov systems that as long as our reference measures are s-conformal measures for the smallest zero s of (BE) the distributions of the stopping times over hyperbolic regions have powerlike tails because of subexponential instability caused by periodic points with (1). Such statistics of the stopping times are connected with interesting statistical phenomena of the original systems (T, X, Q) (see Corollary 1). Although the system (T, X, Q) may not be (piecewise) conformal, we will obtain some additional information on the system if it is indeed (piecewise) conformal. We see that under piecewise conformality the smallest zero s of (BE) is equal to $\dim_H X$ (cf. [**2**]). We should remark that in the case of area-expanding ($|\det DT| \geq 1$) systems which are not necessarily (piecewise) conformal, the periodic points x_0 with (1) satisfy the property $|\det DT^q(x_0)| = 1$ which coincides with the definition of *indifferent* periodic point introduced by the author in previous papers [**14–22**]. Example A, a piecewise conformal countable-to-one Markov transformation, admits such periodic points which cause polynomial instability of the dynamics. By applying our results in §2, the subexponential instability has direct influence upon statistical properties of a weak Gibbs measure ν for $-2\log\|DT\|$, where 2 (= $\dim X$) is the smallest zero of Bowen's equation (BE) and ν is just the normalized Lebesgue measure of X (cf. [**17**]).

Example A. (Complex continued fraction [**1, 12, 14, 19–21**]) Let $X = \{z = x_1\alpha + x_2\overline{\alpha} \mid -1/2 \leq x_1, x_2 \leq 1/2\}$, where $\alpha = 1 + i$, and define $T : X \to X$ by $T(z) = 1/z - [1/z]_1$. Here $[z]_1$ denotes $[x_1 + 1/2]\alpha + [x_2 + 1/2]\overline{\alpha}$, where z is written in the form $z = x_1\alpha + x_2\overline{\alpha}$, $[x] = \max\{n \in \mathbb{Z} \mid n \leq x\}(x \in \mathbb{N})$ and $[x] = \max\{n \in \mathbb{Z} \mid n < x\}(x \in \mathbb{Z} - \mathbb{N})$. This transformation has generalized indifferent periodic points with respect to $-2\log\|DT\|$, that is, a periodic orbit $\{1, -1\}$ of period 2 and two fixed points $i, -i$ satisfy (1). For each $n\alpha + m\overline{\alpha} \in I := \{m\alpha + n\overline{\alpha} \mid (m, n) \in \mathbb{Z}^2 - (0, 0)\}$, we define $X_{n\alpha+m\overline{\alpha}} = \{z \in X \mid [1/z]_1 = n\alpha + m\overline{\alpha}\}$. Then we have a countable Markov partition $Q = \{X_i\}_{i\in I}$ of X.

There are many interesting limit sets arising from dynamical systems described via number-theoretical algorithms like Example A. More specifically, for each subset J of I, we can define a limit set $X_J := \bigcap_{k=0}^{\infty} T^{-k} \bigcup_{i\in J} \mathrm{cl}(\mathrm{int}\, X_i)$ which is compact if J is a finite set. Even if J is a countable set, we can still calculate $\dim_H X_J$ under

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piecewise conformality of T. The next example, which arises from Brun's algorithm, is not (piecewise) conformal. However, we still observe fractal-like limit sets (see [**17**]).

Example B. (Brun's map [**11, 17, 18**]) Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_2 \le x_1 \le 1\}$, and let

$$X_i = \{(x_1, x_2) \in X \mid x_i + x_1 \ge 1 \ge x_{i+1} + x_1\}$$

for $i = 0, 1, 2$, where we put $x_0 = 1$ and $x_3 = 0$. T is defined by

$$T(x_1, x_2) = (x_1/(1 - x_1), x_2/(1 - x_1)) \text{ on } X_0,$$
$$T(x_1, x_2) = (1/x_1 - 1, x_2/x_1) \text{ on } X_1,$$
$$T(x_1, x_2) = (x_2/x_1, 1/x_1 - 1) \text{ on } X_2.$$

Then, $Q = \{X_i\}_{i=0}^2$ is a Markov partition and T is a continuous piecewise C^2-map. $(0, 0)$ is a fixed point satisfying $\|DT(0, 0)\| = |\det DT(0, 0)| = 1$ and the dynamical instability is polynomial. Since $P_{\text{top}}(T, -\log|\det DT|) = 0$, $(0, 0)$ is a generalized indifferent fixed point with respect to $-\log|\det DT|$ and the normalized Lebesgue measure of X is a weak Gibbs measure for $-\log|\det DT|$. On the other hand, by our results we can construct weak Gibbs measures supported on the limit sets X_J which may reflect subexponential instability caused by $(0, 0)$.

Mauldin and Urbański, in [**4, 6, 7**], considered infinite iterated functional systems which are particular cases of our piecewise conformal systems such that $Q = \{X_i\}_{i \in I}$ is a Bernoulli partition. Indeed, Example A is not Bernoulli and Example B is non-conformal. In establishing the Hausdorff dimension of the limit set X_J, even in the case when $Q = \{X_i\}_{i \in I}$ is a Bernoulli partition Mauldin and Urbański require the bounded distortion property for $|T'|$. However, this property 'typically' fails under the appearance of parabolic periodic orbits so that certain techniques should be employed. Since our systems are Markov, some restrictions should apply (finite-range condition as well as WBV for potentials) so that the thermodynamic formalism can be effected. Although Sarig [**10**] established the thermodynamic formalism for countable Markov shifts which are particular cases of our countable-to-one Markov systems, the condition imposed on potentials typically fails under the appearance of generalized indifferent periodic orbits. In Theorem 4, we shall relate the existence of equilibrium states which are equivalent to weak Gibbs measures for ϕ admitting generalized indifferent periodic points to a lack of differentiability of the pressure function at ϕ. The generalized indifferent periodic points cause *phase transition* (non-uniqueness of equilibrium states) which gives a sufficient condition for non-differentiability of the pressure function. Moreover, we shall discuss how non-differentiability of pressure functions is related to the multifractal problem.

The paper is organized as follows. In §1 we present necessary background and introduce definitions for later sections. In §2 we state our results (Theorems 1–4), and the proofs of all results are given in §3.

1. *Preliminaries*

Let $(T, X, Q = \{X_i\}_{i \in I})$ be a *piecewise C^0-invertible system*, i.e., X is a compact metric space with metric d, $T : X \to X$ is a non-invertible map which is not necessarily

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continuous, and $Q = \{X_i\}_{i\in I}$ is a countable disjoint generating partition of X such that $\bigcup_{i\in I} \operatorname{int} X_i$ is dense in X and the following properties are satisfied.

(01) For each $i \in I$ with $\operatorname{int} X_i \neq \emptyset$, $T|_{\operatorname{int} X_i} : \operatorname{int} X_i \to T(\operatorname{int} X_i)$ is a homeomorphism and $(T|_{\operatorname{int} X_i})^{-1}$ extends to a homeomorphism v_i on $\operatorname{cl}(T(\operatorname{int} X_i))$.

(02) $T\big(\bigcup_{\operatorname{int} X_i=\emptyset} X_i\big) \subset \bigcup_{\operatorname{int} X_i=\emptyset} X_i$.

(03) $\sigma(n) := \sup\big\{\operatorname{diam} Y \mid Y \in \bigvee_{h=0}^{n-1} T^{-h}(Q)\big\} \to 0$ $(n \to \infty)$.

Let $\underline{i} = (i_1 \dots i_n) \in I^n$ satisfy $\operatorname{int}(X_{i_1} \cap T^{-1}X_{i_2} \cap \cdots \cap T^{-(n-1)}X_{i_n}) \neq \emptyset$. Then, we define $X_{\underline{i}} := X_{i_1} \cap T^{-1}X_{i_2} \cap \cdots \cap T^{-(n-1)}X_{i_n}$, which is called a *cylinder of rank n* and we denote $v_{\underline{i}} := v_{i_1} \circ v_{i_2} \circ \cdots \circ v_{i_n}$, which is a homeomorphism from $\operatorname{cl}(T^n(\operatorname{int} X_{\underline{i}}))$ onto $\operatorname{cl}(\operatorname{int} X_{\underline{i}})$. Let $X_{i_1\dots i_n}(x)$ be the unique cylinder containing x and let $\sigma(x)$ be the asymptotic growth rates of $\sigma_n(x) := \operatorname{diam} X_{i_1\dots i_n}(x)$, i.e., $\sigma(x) := \limsup_{n\to\infty}(1/n)\log\sigma_n(x)$. The condition (03) implies that $\sup_{x\in X}\sigma(x) \le \lim_{n\to\infty}(1/n)\log\sup_{x\in X}\sigma_n(x) \le 0$.

Definition. We say that (T, X, Q) satisfies *subexponential instability* (in distances) if there exist $x \in X$ and a positive sequence $\{s_n\}_{n\ge1}$ with $\lim_{n\to\infty}(1/n)\log s_n = 0$ satisfying $\sigma_n(x) \ge s_n$ so that $\lim_{n\to\infty}(1/n)\log\sigma(n) = 0$. If $\lim_{n\to\infty}(1/n)\log\sigma(n) < 0$ then we say that (T, X, Q) satisfies *exponential instability* (in distances).

We say that (T, X, Q) is a *piecewise C^1-invertible system* if for each $i \in I$ with $\operatorname{int} X_i \neq \emptyset$, both $T|_{\operatorname{int} X_i}$ and v_i on $\operatorname{cl}(T(\operatorname{int} X_i))$ are C^1-diffeomorphisms. In particular, if $T|_{\operatorname{int} X_i}$ and v_i on $\operatorname{cl}(T(\operatorname{int} X_i))$ are conformal maps, then we say that (T, X, Q) is a *piecewise conformal system*.

Remark A. Typical examples of systems with exponential instability are piecewise uniformly expanding systems (i.e., $\sup_{i\in I}\sup_{x\in TX_i}\|Dv_i(x)\| < 1$). On the other hand, subexponential instability may be observed if there exists certain non-hyperbolic periodic orbit. Indeed, maps with indifferent periodic points in the introduction (Examples A and B) show polynomial instability.

If (T, X, Q) satisfies exponential instability, then $\sup_{x\in X}\sigma(x) < 0$. On the other hand, subexponential instability implies there exists $x \in X$ such that $\sigma(x) = 0$. Following Pesin [**8**], we come to a weak notion of hyperbolic measures.

Definition. A Borel probability measure m is called *quasi-hyperbolic* if for m-a.e. $x \in X$ $\sigma(x) = \lim_{n\to\infty}(1/n)\log\sigma_n(x) < 0$.

We shall introduce instability of the system (T, X, Q) in terms of a (reference) measure m. We denote

$$\sigma_m(n) := \sup\left\{ m(X_{i_1\dots i_n}) \;\middle|\; X_{i_1\dots i_n} \in \bigvee_{h=0}^{n-1} T^{-h}Q, m(X_{i_1\dots i_n}) > 0 \right\}.$$

Definition. (Instability in measure) We say that *the instability of* $(T, X, Q; m)$ *is subexponential* if there exist $x \in X$ and a positive sequence $\{s_n\}_{n\ge1}$ with $\lim_{n\to\infty}(1/n)\log s_n = 0$ satisfying $\sigma_m(n, x) := m(X_{i_1\dots i_n}(x)) \ge s_n$ so that $\lim_{n\to\infty}(1/n)\log\sigma_m(n) = 0$. If $\lim_{n\to\infty}(1/n)\log\sigma_m(n) < 0$, then we say that *the instability of* $(T, X, Q; m)$ *is exponential*.

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Definition. A Borel probability measure m which is positive on open sets is called a *weak geometric measure* if there exist $s > 0$ and a positive sequence $\{W_n\}_{n\geq 1}$ satisfying $\lim_{n\to\infty}(1/n)\log W_n = 0$ such that for m-a.e. $x \in X$

$$W_n^{-1} \leq \frac{m(X_{i_1\dots i_n}(x))}{\operatorname{diam} X_{i_1\dots i_n}(x)^s} \leq W_n.$$

We call s an *exponent* of m.

Remark B. If $\sup_{n\geq 1} W_n < \infty$, then ν is called a *geometric measure*. Also, the property implies the *uniform mass distribution principle* (see [**8**]) which plays an important role in establishing a lower bound for $\dim_H X$.

As we will see in §2, the weak geometric measures reflect directly the nature of the instability of the system and play an important role in establishing a lower bound for $\dim_H X$.

Definition. [**17–22**] A non-atomic Borel probability measure ν is called a *weak Gibbs measure* for a function ϕ with a constant P if there exists a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n\to\infty}(1/n)\log K_n = 0$ such that for all $n \geq 1$, and all $X_{i_1\dots i_n} \in \bigvee_{h=0}^{n-1} T^{-h}Q$, ν-a.e. $x \in X_{i_1\dots i_n}$,

$$K_n^{-1} \leq \frac{\nu(X_{i_1\dots i_n})}{\exp\left(\sum_{h=0}^{n-1}\phi T^h(x) + nP\right)} \leq K_n.$$

Remark C. If the weak Gibbs sequence $\{K_n\}_{n\geq 1}$ satisfies $\sup_{k\geq 1} K_n < \infty$, then the weak Gibbs measure ν satisfies Bowen's Gibbs property.

In particular, we shall be interested in the case when P is connected with the topological pressure of ϕ.

Definition. We say that $\phi : X \to \mathbb{R}$ is a potential of *weak bounded variation* (WBV) if there exists a sequence of positive numbers $\{C_n\}$ satisfying $\lim_{n\to\infty}(1/n)\log C_n = 0$ and for all $n \geq 1$, and all $X_{i_1\dots i_n} \in \bigvee_{h=0}^{n-1} T^{-h}Q$,

$$\frac{\sup_{x\in X_{i_1\dots i_n}} \exp\left(\sum_{h=0}^{n-1}\phi(T^h x)\right)}{\inf_{x\in X_{i_1\dots i_n}} \exp\left(\sum_{h=0}^{n-1}\phi(T^h x)\right)} \leq C_n.$$

Definition. If (T, X, Q) satisfies the Markov property ($\operatorname{int}(\operatorname{cl}(\operatorname{int} X_i) \cap \operatorname{cl}(\operatorname{int} T X_j)) \neq \emptyset$ implies $\operatorname{cl}(\operatorname{int} T X_j) \supset \operatorname{cl}(\operatorname{int} X_i)$), then we say that (T, X, Q) is a *Markov system*. If $X_i \in Q$ satisfies $\operatorname{cl}(T(\operatorname{int} X_i)) = X$, then X_i is called a *full cylinder*. If Q consists of full cylinders, then (T, X, Q) is called a *Bernoulli system*.

Let (T, X, Q) be a Markov system. Then by the generating property for Q, for each $\underline{i} = (i_1 \dots i_n) \in I^n$ with $\operatorname{int}(T X_{i_n}) \supset \operatorname{int} X_{i_1}$, there is the unique point $x(\underline{i})$ satisfying $v_{\underline{i}} x(\underline{i}) = x(\underline{i}) \in \operatorname{cl}(\operatorname{int} X_{\underline{i}})$. For a potential ϕ of WBV, we define the partition function $Z_n(\phi)$ by

$$Z_n(\phi) := \sum_{\underline{i}:|\underline{i}|=n, \operatorname{int}(T X_{i_n})\supset \operatorname{int} X_{i_1}} \exp\left[\sum_{h=0}^{n-1}\phi T^h(x(\underline{i}))\right].$$

We recall that under the next two conditions there exists the limit $P_{\text{top}}(T, \phi) := \lim_{n\to\infty}(1/n)\log Z_n(\phi) \in (-\infty, \infty]$ which is called the *topological pressure* for ϕ (see [**20**]).

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The first condition gives a nice countable-state symbolic dynamics similar to sofic shifts [**15**].

(Finite range structure, FRS). $\mathcal{U} = \{\mathrm{int}(T^n X_{i_1 \dots i_n}) \mid \forall X_{i_1 \dots i_n}, \forall n > 0\}$ consists of finitely many open subsets $U_1, \dots, U_N$ of X.

(Transitivity). $\mathrm{int}\, X = \bigcup_{k=1}^{N} U_k$ and for all $l \in \{1, 2, \dots, N\}$, there exists $0 < s_l < \infty$ such that for each $k \in \{1, 2, \dots, N\}$, U_k contains an interior of a cylinder $X^{(k,l)}(s_l)$ of rank s_l such that $T^{s_l}(\mathrm{int}\, X^{(k,l)}(s_l)) = U_l$.

Definition. We say that a Borel probability measure ν on X is an *f-conformal measure* if $d(\nu T)|_{X_i}/d\nu|_{X_i} = f|_{X_i}$ and $\nu\big(\bigcup_{i \in I} \partial X_i\big) = 0$. In particular, if $f = \|DT\|^s$ then ν is called an *s-conformal measure*.

Remark D. If $-\log\|DT\|$ is of WBV, then s-conformal measures are weak Gibbs measures for $-s \log\|DT\|$ with 0.

Definition. We say that $\mathcal{S}_\phi := \{x \in X \mid (1/n)\sum_{k=0}^{n-1} \phi T^k(x) \to P_{\mathrm{top}}(T, \phi)\}$ is an *indifferent set with respect to ϕ*. If a periodic point x_0 is contained in $\mathcal{S}_\phi$ then we say that x_0 is a *generalized indifferent periodic point with respect to ϕ*.

Remark E. If $\phi = -s \log\|DT\|$ and s is a zero of (BE): $P_{\mathrm{top}}(T, -s \log\|DT\|) = 0$, then $\mathcal{S}_\phi$ is just the set of points with non-positive Lyapunov exponent (i.e., for all $x \in \mathcal{S}_\phi$, $\lim_{n\to\infty}(1/n)\log\|DT^n(x)\| \le 0$). In particular, if T is piecewise conformal then $\mathcal{S}_\phi$ is the set of points with zero Lyapunov exponent.

Let $B_1 \subset X$ be a union of cylinders X_i of rank one, where the index i belongs to a subset of I and define the stopping time over B_1, $R : X \to \mathbb{N} \cup \{\infty\}$ by $R(x) := \inf\{n \ge 0 \mid T^n x \in B_1\} + 1$. Then we define the jump transformation associated to B_1, $T^* : \bigcup_{n=1}^{\infty}\{R(x) = n\} \to X$ by $T^* x = T^{R(x)} x$. Define $X^* := X \setminus \bigcup_{m=0}^{\infty} T^{*-m}\big(\bigcap_{n\ge 0}\{R(x) > n\}\big)$ and $Q^* := \{X_{\underline{i}}\}_{\underline{i} \in I^*}$, where $I^* := \bigcup_{n=1}^{\infty}\{(i_1 \dots i_n) \in I^n \mid X_{i_1 \dots i_n} \subset \{R(x) = n\}\}$. Then it is easy to see that $(T^*, X^*, Q^* = \{X_{\underline{i}}\}_{\underline{i} \in I^*})$ is a piecewise C^0-invertible Markov system with FRS, where X^* is not necessarily compact.

Definition. We say that a piecewise C^0-invertible system satisfies *local exponential instability* with respect to B_1 if

$$\lim_{n\to\infty} \frac{1}{n} \log\left[\sup\left\{\mathrm{diam}\, Y \;\middle|\; Y \in \bigvee_{h=0}^{n-1} T^{*-h}(Q^*)\right\}\right] < 0.$$

Definition. We say that ϕ satisfies the *local bounded distortion* (LBD) with respect to B_1 if there exists $\theta > 0$ such that for all $\underline{i} = (i_1 \dots i_{|\underline{i}|}) \in I^*$, there exists $0 < L_\phi(\underline{i}) < \infty$ satisfying

$$|\phi(v_{\underline{i}}(x)) - \phi(v_{\underline{i}}(y))| \le L_\phi(\underline{i}) d(x, y)^\theta \quad (\text{for all } x, y \in X)$$

and

$$\sup_{\underline{i} \in I^*} \sum_{j=0}^{|\underline{i}|-1} L_\phi(i_{j+1} \dots i_{|\underline{i}|}) < \infty.$$

We define $\phi^* : \bigcup_{n=1}^{\infty}\{R(x) = n\} \to \mathbb{R}$ by $\phi^*(x) = \sum_{h=0}^{R(x)-1} \phi T^h(x)$. Then, the LBD property for ϕ allows one to establish equi-Hölder continuity of $\{\phi^* \circ v_{\underline{i}} \mid \underline{i} \in I^*\}$.

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If we assume further *local exponential instability of* (T, X, Q), then summability of variations of ϕ^* follows so that there exists

$$P_{\text{top}}(T^*, \phi^*) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi^*) \in (-\infty, \infty].$$

In particular, $P_{\text{top}}(T^*, \phi^*) < \infty$ holds if

$$\|\mathcal{L}_{\phi^*} 1\| := \sup_{x\in X} \sum_{\underline{i}\in I^*} \exp[\phi^* \circ v_{\underline{i}}(x)] < \infty.$$

Definition. We say that a piecewise C^1-invertible system is *locally uniformly expanding with respect to* B_1 if T^* associated to B_1 is piecewise uniformly expanding (i.e., $\sup_{\underline{i}\in I^*} \sup_{x\in T^* X_{\underline{i}}} \|Dv_{\underline{i}}(x)\| < 1$).

In the next section, we shall use piecewise uniformly expanding jump systems (T^*, X^*, Q^*) for which B_1 consists of full cylinders so that (T^*, X^*, Q^*) are Bernoulli.

2. *Main results*

The first theorem allows us to associate subexponential instability of the system (T, X, Q) with weak Gibbsianness of physical reference measures.

THEOREM 1. (Subexponential instability and weak Gibbsianness) *Let* (T, X, Q) *be a piecewise* C^1*-invertible transitive FRS Markov system. Suppose that* $-\log\|DT\|$ *is a potential of WBV and there exists a weak Gibbs measure* ν *for* $-s\log\|DT\|$ *(with* 0*) for a zero* $s > 0$ *of (BE):* $P_{\text{top}}(T, -s\log\|DT\|) = 0$. *Assume further either of the next two conditions.*

(1-1) *There exists* $x_0 \in X$ *with* $T^q x_0 = x_0$ *satisfying* $\sum_{h=0}^{q-1} \log\|DT(T^h x_0)\| = 0$.

(1-2) (T, X, Q) *is a piecewise conformal system and there exists* $x_0 \in X$ *which satisfies* $\lim_{n\to\infty}(1/n)\log\|DT^n(x_0)\| = 0$.

Then both instabilities of (T, X, Q) *and* $(T, X, Q; \nu)$ *are subexponential.*

Remark F. The periodic points in (1-1) are generalized indifferent periodic points with respect to $-s\log\|DT\|$ for all zeros $s > 0$ of (BE).

We need the next two lemmas for the proof of Theorem 1.

LEMMA 1.1. *If* ν *is a weak Gibbs measure for* ϕ *satisfying* $P_{\text{top}}(T, \phi) < \infty$ *with* $-P_{\text{top}}(T, \phi)$ *and* $\mathcal{S}_\phi \neq \emptyset$*, then the instability of* $(T, X, Q; \nu)$ *is subexponential.*

LEMMA 1.2. *Let* (T, X, Q) *be a piecewise conformal FRS Markov system. If* $-\log\|DT\|$ *satisfies the WBV property, then (1.2-a) and (1.2-b) are equivalent:*

(1.2-a) ν *is a weak geometric measure with exponent* $s > 0$,

(1.2-b) ν *is a weak Gibbs measure for* $-s\log\|DT\|$ $(s > 0)$ *with* 0.

Let (Σ, σ) be a countable Markov shift which is a symbolic dynamics of (T, X, Q), that is, there exists a uniformly continuous map $\pi : \Sigma \to X$ defined by

$$\pi(i_1 i_2 \dots) = \bigcap_{j=0}^{\infty} \text{cl}(T^{-j}(\text{int } X_{i_{j+1}})) \ (\neq \emptyset)$$

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which satisfies $T \circ \pi = \pi \circ \sigma$. $M_T(X)$ denotes the set of all T-invariant Borel probability measure on X and $M_\sigma(\Sigma)$ denotes the set of all σ-invariant Borel probability measures on Σ. We remark that for $\overline{m} \in M_\sigma(\Sigma)$ its pushforward $\pi_*\overline{m} := \overline{m}\pi^{-1}$ is contained in $M_T(X)$. Define $\overline{M}_T(X) := \{m \in M_T(X) \mid \text{there exists } \overline{m} \in M_\sigma(\Sigma) \text{ such that } m = \pi_*\overline{m}\}$. Now we come to state the second theorem, which gives fundamental results to solve our problems (P-1)–(P-3), that is, the existence of both an s-conformal measure ν for a unique zero of (BE)* and a T-invariant ergodic probability measure μ equivalent to ν, and a characterization of the smallest zero of (BE) in terms of thermodynamic formalism.

THEOREM 2. *Let (T, X, Q) be a piecewise C^1-invertible transitive FRS Markov system and let $-\log\|DT\|$ be of WBV. Suppose that there exists B_1 a union of full cylinders with respect to which (T, X, Q) is locally uniformly expanding and $-\log\|DT\|$ satisfies LBD. Then we have the following facts.*

(2-1) *If there exists $s_0 > 0$ satisfying $0 \le P_{\text{top}}(T^*, -s_0 \log\|DT^*\|) < \infty$, then there exist a unique zero s $(\ge s_0)$ of (BE)*: $P_{\text{top}}(T^*, -s \log\|DT^*\|) = 0$ and s is a zero of (BE): $P_{\text{top}}(T, -s \log\|DT\|) = 0$. There exists an s-conformal measure ν supported on X^* which is a weak Gibbs measure for $-s \log\|DT\|$ with 0.*

(2-2) *If ν satisfies $\log\|DT^*\| \in L^1(\nu)$, $R \in L^1(\nu)$ and $H_\nu(Q^*) < \infty$, then there exists a T-invariant ergodic probability measure μ which is equivalent to ν and satisfies $s = h_\mu(T)/\int_X \log\|DT\|\, d\mu (> 0)$.*

(2-3) *Assume further that the marginal set $\bigcap_{n\ge 0}\{R(x) > n\}$ consists of countably many periodic points satisfying (1-1). Then the unique zero $s > 0$ of (BE)* coincides with the smallest zero of (BE) and it attains the following supremum.*

$$s = \sup\left\{ \frac{h_m(T)}{\int_X \log\|DT\|\, dm} \;\middle|\; m \in \overline{M}_T(X), 0 < \int_X \log\|DT\|\, dm < \infty \right\}.$$

Remark G. Under the assumptions in (2-3), μ obtained in (2-2) is an equilibrium state for $-s \log\|DT\|$ (cf. [**19**]).

We define

$$\mathcal{W}(T) := \{f : X \to \mathbb{R} \mid f \text{ satisfies WBV and } P_{\text{top}}(T, f) < \infty\}.$$

For the proof of Theorem 2, we need a sequence of lemmas. The first lemma, which is used for proving (2-1), follows from [**20**, Theorem 4].

LEMMA 2.1. (cf. [**20**, Lemma 7]) *Let (T, X, Q) be a piecewise C^0-invertible transitive FRS Markov system and let $\phi \in \mathcal{W}(T)$. Suppose that there exists B_1, a union of full cylinders with respect to which (T, X, Q) satisfies local exponential instability, and ϕ satisfies local bounded distortion. If $\|\mathcal{L}_{\phi^*}1\| < \infty$ and $P_{\text{top}}(T^*, \phi^*) \in [0, \infty)$, then there exists a Borel probability measure ν on X supported on* int X^* *satisfying*

$$\left.\frac{d\nu T}{d\nu}\right|_{X_i} = \exp[P_{\text{top}}(T, \phi) - \phi] \quad (\textit{for all } i \in I)$$

and $\nu\big(\bigcup_{i\in I} \partial X_i\big) = 0$. In particular, if $P_{\text{top}}(T^*, \phi^*) = 0$ then $P_{\text{top}}(T, \phi) = 0$ and so ν is an $\exp[-\phi]$*-conformal measure.*

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LEMMA 2.2. *$s \to P_{\text{top}}(T^*, -s\log\|DT^*\|)$ is a strictly decreasing continuous function on (s_0, ∞) and there exists a unique zero of (BE)*.*

LEMMA 2.3. *If the s-conformal measure ν obtained in (2-1) satisfies $\log\|DT^*\| \in L^1(\nu)$ and $H_\nu(Q^*) < \infty$, then there exists a T^*-invariant exact probability measure μ^* equivalent to ν which is a unique equilibrium state for $-s\log\|DT^*\|$ satisfying $s = h_{\mu^*}(T^*)/\int_{X^*} \log\|DT^*\|\, d\mu^*$.*

The last lemma, which is used for proving (2-3), follows from [**3**, Theorem 2.4].

LEMMA 2.4. (Variational principle) *Let (T, X, Q) be a piecewise C^1-invertible transitive FRS Markov system. Let $-\log\|DT\|$ be a potential of WBV. Then*

$$P_{\text{top}}(T, -s\log\|DT\|) = \sup\left\{ h_{\overline{m}}(\sigma) - s\int_\Sigma \log\|DT\circ\pi\|\, d\overline{m} \mid \overline{m} \in M_\sigma(\Sigma) \right.$$
$$\left. \text{and either } h_{\overline{m}}(\sigma) < \infty \text{ or } \int_\Sigma \log\|DT\circ\pi\|\, d\overline{m} < \infty \text{ is satisfied} \right\}.$$

The third theorem gives a characterization of physical reference measures which directly reflect subexponential instability upon the statistical properties of the system. Under piecewise conformality it can be represented in terms of a geometric quantity. For a Borel probability measure m, we denote the distribution of the stopping time R associated to m by $F_m(n) := m(\{x \in X \mid R(x) \le n\})(\forall n \ge 1)$ and the second moment of R with respect to m by $E_m(R^2) := \int_{X^*} R^2\, dm$.

THEOREM 3. *Under the assumptions in Theorem 2, the periodic point x_0 satisfying (1-1) causes the following phenomena for the s-conformal measure ν with the smallest zero s of (BE) and*

$$s = \sup\left\{ \frac{h_m(T)}{\int_X \log\|DT\|\, dm} \;\middle|\; m \in \overline{M}_T(X), 0 < \int_X \log\|DT\|\, dm < \infty \right\}$$
$$= \frac{h_\mu(T)}{\int_X \log\|DT\|\, d\mu}$$

for an equilibrium state μ for $-s\log\|DT\|$ equivalent to ν:

(3-1) *non-Gibbsianness of ν in the sense of Bowen,*

(3-2) *subexponential instability of $(T, X, Q; \nu)$,*

(3-3) *a powerlike tail of the distribution of the stopping time R associated to ν, i.e., $1 - F_\nu(n) = \nu(\{x \in X \mid R(x) > n\}) \ge s_n$ with $\lim_{n\to\infty}(1/n)\log s_n = 0$.*

In particular, if (T, X, Q) is a piecewise conformal system, then $s = \dim_H X$.

By applying Theorem 3.1 in [**16, 17**] we have the next results.

COROLLARY 1. *Statistical properties for (T, μ) can be described in terms of statistics of the stopping time R as follows.*

(1) *(A criterion of finiteness of μ) $\sum_{n=1}^\infty \nu(\{R > n\}) < \infty$ if and only if $\mu(X) < \infty$.*

(2) *(Singularity of $d\mu/d\nu$) If $\nu(\{R > n\}) = n^{-\alpha}(\alpha > 1)$, then $d\mu/d\nu|_{\{R=n\}} \approx n$ and $d\mu/d\nu|_{\bigcap_{n\ge 0}\{R>n\}} = \infty$.*

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(3) *(The central limit theorem, CLT)* $E_\nu(R^2) = \sum_{n=1}^{\infty} \mu(\{R > n\})$. *If* $\nu(\{R > n\}) = n^{-\alpha}(\alpha > 1)$, *then* $\mu(\{R > n\}) \approx n^{-(\alpha-1)}$ *and so* $E_\nu(R^2) \approx \sum_{n=1}^{\infty} n^{-(\alpha-1)}$. *If* $E_\nu(R^2) < \infty$ *then the CLT holds for* $f \in L^\infty(\mu)$ *satisfying LBD.*

For the proof of Theorem 3, we need the next four lemmas. The first result follows immediately from the definition of Bowen's Gibbs property.

LEMMA 3.1. (Characterization of non-Gibbsianness)

(i) *Suppose that a potential* ϕ *with* $P_{\text{top}}(T, \phi) < \infty$ *admits a generalized indifferent periodic point. Then* $P_{\text{top}}(T, \phi) \leq \sup_{x \in X} \phi(x)$ *and there is no Borel probability measure which is Gibbs for* ϕ.

(ii) *Suppose that there exists a fixed point* x_0 *such that there exist* $a \geq 0$ *and* $b > 0$ *satisfying*

$$\nu(X_{i_1 \ldots i_n}(x_0)) = \exp[-(na + b \log n)].$$

Then there is no potential function ϕ *with respect to which* ν *is a Gibbs measure.*

LEMMA 3.2. *If* m *is a weak geometric measure with exponent* $s > 0$ *which is quasi-hyperbolic, then* m *is exact-dimensional and for* m*-a.e.* $x \in X$,

$$d_m(x) = d_m(x, Q) := \lim_{n \to \infty} \frac{\log m(X_{i_1 \ldots i_n}(x))}{\log \operatorname{diam} X_{i_1 \ldots i_n}(x)} = s.$$

LEMMA 3.3. *If* (T, X, Q) *is a piecewise conformal Markov system and* $-\log\|DT\|$ *is a potential of WBV, then* m *is quasi-hyperbolic if and only if* m *is hyperbolic (i.e.,* m*-a.e.,* $x \in X$ $\lim_{n \to \infty}(1/n) \log\|DT^n(x)\| > 0$).

LEMMA 3.4. *Let* (T, X, Q) *be a piecewise conformal Markov system satisfying the* locally uniformly expanding *property with respect to a subset* B_1 *which is a union of elements of* Q. *If* m *is a* T*-invariant ergodic probability measure which satisfies* $m\big(\bigcap_{n \geq 0}\{R > n\}\big) = 0$ *and* $\log\|DT\| \in L^1(m)$, *then* m *is a hyperbolic measure.*

Remark H. Recall the fact in Remark F. Then, by [**20**, Proposition 4] we see that a generalized indifferent periodic point x_0 is contained in the marginal set $\bigcap_{n \geq 0}\{R(x) > n\}$. Since $\bigcup_{j=0}^{\infty} T^{*-j}\{$ all generalized indifferent periodic points with respect to $\phi\} \subset S_\phi$, if the marginal set $\bigcap_{n \geq 0}\{R(x) > n\}$ consists of generalized indifferent periodic points with respect to ϕ, then $X \backslash X^* \subset S_\phi$. We can easily see that S_ϕ is a full measure set with respect to any ergodic equilibrium state m for ϕ with zero entropy satisfying $\phi \in L^1(m)$. For example, the Dirac measure supported on the generalized indifferent periodic orbit is such an equilibrium state.

Finally, we shall associate failure of differentiability of pressure functions to the existence of generalized indifferent periodic points with respect to potentials.

We denote

$$\mathcal{W}_1(T) := \{f \in \mathcal{W}(T) \mid \operatorname{Var}_n(f) \to 0 \ (n \to \infty)\}.$$

First we show that the well-known characterization of equilibrium states as tangent functionals still holds in our situation (cf. [**5**]).

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LEMMA 4.1. (Tangent property) *Let m be an equilibrium state for $\phi \in \mathcal{W}_1(T)$ satisfying $m \in \overline{M}_T(X)$. Then for all $\psi \in \mathcal{W}_1(T)$,*

$$P_{\text{top}}(T, \phi + \psi) - P_{\text{top}}(T, \phi) \geq \int_X \psi \, dm,$$

that is, m is a tangent functional of $P_{\text{top}}(T, \cdot) : \mathcal{W}_1(T) \to \mathbb{R}$ at ϕ.

By Remark G, μ obtained in Theorem 2 is an ergodic equilibrium state supported on X^* satisfying $\mu\big(\bigcup_{i\in I} \partial X_i\big) = 0$. On the other hand, by Remark H the Dirac measure $(1/q)\sum_{h=0}^{q-1} \delta_{T^h x_0}$ supported on the indifferent periodic orbit $\{x_0, Tx_0, \ldots, T^{q-1}x_0\} \subset \mathcal{S}_\phi$ gives another ergodic equilibrium state for ϕ. Both of them are contained in $\overline{M}_T(X)$.

LEMMA 4.2. *If $P_{\text{top}}(T, 0) < \infty$, then $C(X) \subset \mathcal{W}_1(T)$.*

Definition. We say that $P_{\text{top}}(T, \cdot) : \mathcal{W}_1(T) \to \mathbb{R}$ is differentiable at $\phi \in \mathcal{W}_1(T)$ if the limit

$$\lim_{t\to 0} \frac{P_{\text{top}}(T, \phi + tf) - P_{\text{top}}(T, \phi)}{t}$$

exists for all $f \in C(X)$.

THEOREM 4. (Non-differentiability of the pressure function) *Let (T, X, Q) be a piecewise C^0-invertible Markov system with $P_{\text{top}}(T, 0) < \infty$ and let $\phi \in \mathcal{W}_1(T)$ admit a generalized indifferent periodic point. Assume further that (T, X, Q) satisfies local exponential instability and ϕ satisfies LBD with respect to B_1 which is a union of full cylinders. If the $\exp[P_{\text{top}}(T, \phi) - \phi]$-conformal measure ν supported on X^* satisfies $\phi^*, R \in L^1(\nu)$ and $H_\nu(Q^*) < \infty$, then $P_{\text{top}}(T, \cdot) : \mathcal{W}_1(T) \to \mathbb{R}$ is non-differentiable at ϕ.*

In the end of this section, we shall discuss how the differentiability of pressure functions is related to the multifractal problem (cf. [**9**]). Let (T, X, Q) be a piecewise conformal transitive FRS Markov system which is locally uniformly expanding. Let $\phi \in \mathcal{W}_1(T)$ with $P_{\text{top}}(T, \phi) = 0$ be a non-positive function satisfying LBD with respect to a union of full cylinders B_1. If $P_{\text{top}}(T^*, \phi^*) \geq 0$, then by [**21**, Lemma 12] for all $0 \leq q \leq 1$ with $\|\mathcal{L}_{q\phi^*}1\| < \infty$, $\exists_1 t(q) \geq 0$ satisfying the equation $P_{\text{top}}(T^*, q\phi^* - t(q)\log\|DT^*\|) = 0$. By Lemma 2.1, $t(q)$ satisfies $P_{\text{top}}(T, q\phi - t(q)\log\|DT\|) = 0$. Since $P_{\text{top}}(T, .) : \mathcal{W}(T) \to \mathbb{R}$ is a convex function, $t(q)$ is a convex function so that there exist right and left derivatives $t'_-(q)$ and $t'_+(q)$ for $0 \leq q \leq 1$. Now, we specify first-order phase transition points at which $t(q)$ is non-differentiable for piecewise conformal finite Markov systems as follows.

PROPOSITION 5.1. *If there exist two different ergodic hyperbolic equilibrium states $\mu_1 \in \overline{M}_T(X)$ and $\mu_2 \in \overline{M}_T(X)$ for $q\phi - t(q)\log\|DT\| \in \mathcal{W}_1(T)$ with $\mu_1(\phi)/(\mu_1(\log\|DT\|)) \neq \mu_2(\phi)/(\mu_2(\log\|DT\|))$, then q is the first-order phase transition point.*

It follows from Theorem 4 that $P_{\text{top}}(T, \cdot)$ is not differentiable at $q\phi - t(q)\log\|DT\|$ with $q \in (0, 1)$ if $q\phi - t(q)\log\|DT\|$ admits an indifferent periodic point x_0.

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Remark I. Under $\inf_{x\in X}\|DT(x)\| \geq 1$, x_0 is an indifferent periodic point with respect to $q\phi - t(q)\log\|DT\|$ with $q \in (0,1)$ if and only if x_0 is an indifferent periodic point with respect to ϕ with zero Lyapunov exponent. In particular, if the marginal set $\bigcap_{n\geq 0}\{R(x) > n\}$ consists of periodic points with zero Lyapunov exponent, then any indifferent periodic point for ϕ is a periodic point with zero Lyapunov exponent.

By Remark I, although the Dirac measure $(1/p)\sum_{h=0}^{p-1}\delta_{T^h x_0}$ supported on the generalized indifferent periodic orbit with respect to ϕ satisfying (1-1) is an equilibrium state for $q\phi - t(q)\log\|DT\|$, it is not hyperbolic. The next result relates the first-order phase transition point to generalized indifferent periodic points with respect to $q\phi - t(q)\log\|DT\|$. Let $\{X_i\}_{i=1}^{M}(M \leq \infty)$ be a sequence of cylinders of rank one and for each $i = 1, \ldots, M$ we denote $A_{i+1} := \mathrm{cl}(\mathrm{int}\, X_{i+1})$. Let $\Gamma_0 := X$ and for each $i \geq 1$ define inductively $\Gamma_{i+1} = \bigcap_{n=0}^{\infty} T^{-n}(\Gamma_i \cap A_{i+1}^c)(\subset \Gamma_i)$.

LEMMA 5.1. (cf. [**20**, Theorem 8]) *Let (T, X, Q) satisfy local exponential instability and $q\phi - t(q)\log\|DT\|$ satisfy LBD with respect to $B_1 = \{X_i\}_{i=1}^{M}(M \leq \infty)$. Suppose that the Borel probability measure $\nu_{\Gamma_i\cap A_{i+1}}$ on $\Gamma_i \cap A_{i+1}$ obtained in Lemma 2.1 satisfies*

$$H_{\nu_{\Gamma_i\cap A_{i+1}}}(Q_{\Gamma_i\cap A_{i+1}}) < \infty, \quad \int_{\Gamma_i\cap A_{i+1}} R_{\Gamma_i\cap A_{i+1}}\, d\nu_{\Gamma_i\cap A_{i+1}} < \infty.$$

If $\bigcap_{i=1}^{M}\Gamma_i$ consists of indifferent periodic points with respect to $q\phi - t(q)\log\|DT\|$, then there exists a sequence of ergodic hyperbolic equilibrium states: $\{\mu_i\}_{i=1}^{M}$ which are mutually singular.

3. *Proofs*

Proof of Lemma 1.1. Let $x \in \mathcal{S}_\phi$. Then $\nu(X_{i_1\ldots i_n}(x)) \geq s_n$ (for all $n \geq 1$), where $s_n := K_n^{-1}\exp\left[\sum_{h=0}^{n-1}\phi T^h(x) - nP_{\mathrm{top}}(T,\phi)\right]$ and $\{K_n\}_{n\geq 1}$ is the weak Gibbs sequence. Since $\lim_{n\to\infty}(1/n)\log s_n = 0$, the subexponential instability of $(T, X, Q; \nu)$ is caused by x. □

Proof of Lemma 1.2. First, we note that

$$\inf_{w\in X_{i_1\ldots i_n}(x)}\|DT^n(w)\|^{-1}\min_{U\in\mathcal{U}}\{\mathrm{diam}\, U\} \leq \mathrm{diam}\, X_{i_1\ldots i_n}(x) \leq \sup_{w\in X_{i_1\ldots i_n}(x)}\|DT^n(w)\|^{-1}.$$

Then, by the WBV property for $-\log\|DT\|$, we have the following inequalities,

$$C_n^{-1}\exp\left[\sum_{h=0}^{n-1}(-\log\|DT(T^h x)\|)\right]\min_{U\in\mathcal{U}}\{\mathrm{diam}\, U\} \leq \mathrm{diam}\, X_{i_1\ldots i_n}(x) \leq C_n\exp\left[\sum_{h=0}^{n-1}(-\log\|DT(T^h x)\|)\right]. \tag{2}$$

Hence, we can obtain the desired fact immediately. □

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Proof of Theorem 1. Since $x_0 \in \mathcal{S}_{-s\log\|DT\|}$ in both cases (1-1) and (1-2), the subexponential instability of $(T, X, Q; \nu)$ follows from Lemma 1.1. Suppose (1-1). Then, it follows the inequalities (2) that

$$\operatorname{diam} X_{i_1\dots i_n}(x_0) \geq C_n^{-1} \min_{U\in\mathcal{U}}\{\operatorname{diam} U\},$$

which implies subexponential instability of (T, X, Q). Suppose (1-2). Then, the weak geometric property for ν obtained by Lemma 1.2 allows us to see that

$$\begin{aligned}(\operatorname{diam} X_{i_1\dots i_n}(x))^s &\geq W_n^{-1}\nu(X_{i_1\dots i_n}(x_0))\\ &\geq W_n^{-1}K_n^{-1}\exp\left[\sum_{h=0}^{n-1}(-s\log\|DT(T^h x_0)\|)\right] := s_n,\end{aligned}$$

where $\{K_n\}_{n\geq 1}$ is the weak Gibbs sequence for ν. Since $\lim_{n\to\infty}(1/n)\log s_n = 0$, subexponential instability of (T, X, Q) is valid. □

In order to prove Theorem 2, we recall basic results on jump transformations and induced transformations. We shall consider the induced map T_{B_1} over B_1 defined by $T_{B_1}x = T^{R_{B_1}(x)}x$, where $R_{B_1}(x) = \inf\{n \geq 1 \mid T^n x \in B_1\}$. Define $B_1{}^* := B_1\setminus\bigcup_{m=0}^{\infty}T_{B_1}^{-m}\big(\bigcap_{n=1}^{\infty}\{R_{B_1}(x) > n\}\big)$ and $Q_{B_1} := \{X_{\underline{i}}\}_{\underline{i}\in I_{B_1}}$, where $I_{B_1} := \bigcup_{n=1}^{\infty}\{(i_1\dots i_n) \in I^n \mid X_{i_1\dots i_n} \subset \{R_{B_1}(x) = n\}\}$. Then, $(T_{B_1}, B_1^*, Q_{B_1} = \{X_{\underline{i}}\}_{\underline{i}\in I_{B_1}})$ is a piecewise C^0-invertible Markov system with FRS, where B_1^* is not necessarily compact. For a given $\phi : X \to \mathbb{R}$, we define $\phi_{B_1} : \bigcup_{n=1}^{\infty}\{R_{B_1}(x) = n\} \to \mathbb{R}$ by $\phi_{B_1}(x) = \sum_{h=0}^{R_{B_1}(x)-1}\phi T^h(x)$. Then, we can easily verify the following facts.

SUBLEMMA A. [**20**] *Suppose that B_1 consists of full cylinders. Then*

(i) $R_{B_1} = R\circ T|_{B_1}$,

(ii) $T|_{B_1}\circ T_{B_1} = T^*\circ T|_{B_1}$,

(iii) $\phi_{B_1} - sR_{B_1} = (\phi^* - sR)\circ T|_{B_1} + (\phi - \phi\circ T_{B_1})$,

(iv) $P_{\text{top}}(T_{B_1}, \phi_{B_1}^{(s)}) = P_{\text{top}}(T^*, \phi^* - sR)$.

Let m be a T-invariant probability measure on X satisfying $m(B_1) > 0$. Then, it is well known that $B_1 \subset \bigcup_{i=0}^{\infty}T^{-i}B_1 \pmod m$ and $m(B_1) = m(B_1^*)$. Furthermore, $m|_{B_1}/m(B_1)$ is a T_{B_1}-invariant probability measure on B_1. In particular, if B_1 consists of full cylinders then by Sublemma A we have the next fact.

SUBLEMMA B. [**20**, Lemma 5] *$m^* := m_{B_1}(T|_{B_1})^{-1}$ is a T^*-invariant probability measure on X^*. If m is ergodic, then both m_{B_1} and m^* are ergodic and by Kac's formula*

$$\int_{B_1} R_{B_1}\,dm_{B_1} = \int_{X^*} R\,dm^* = m(B_1)^{-1} = m^*(X^*)^{-1}.$$

On the other hand, suppose that there exists a non-atomic non-singular probability measure ν on X which satisfies $d\nu T|_{X_i}/d\nu|_{X_i} > 0$ for $\nu(X_i) > 0$ and $\nu\big(\bigcap_{n\geq 0}\{R(x) > n\}\big) = 0$. If μ^* is a T^*-invariant ergodic probability measure μ^* on X^* equivalent to ν, then the next formula [**11**] gives a σ-finite T-invariant ergodic measure on X equivalent to ν. For a measurable subset E with respect to ν

$$\mu(E) = \sum_{n=0}^{\infty}\mu^*(\{R(x) > n\}\cap T^{-n}E). \tag{3}$$

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Proof of Lemma 2.2. We recall that equi-Hölder continuity of $\{-s\log\|DT^* \circ v_{\underline{i}}\| \mid \underline{i} \in I^*\}$ and exponential instability for (T^*, X^*, Q^*) allow us to establish $\sum_{n=1}^{\infty} \mathrm{Var}_n(T^*, -s\log\|DT^*\|) < \infty$ and hence $\|\mathcal{L}_{-s\log\|DT^*\|}1\| < \infty$ for $s \geq s_0$ (see [**20**, Theorem 3]). By applying Walters's argument in [**13**], we can show the existence of a unique equilibrium state for $-s\log\|DT^*\|$. Then, the strictly decreasing property of $P_{\mathrm{top}}(T^*, -s\log\|DT^*\|)$ on (s_0, ∞) follows from $\inf_{x\in X^*}\|DT^*(x)\| > 1$. By continuity of $s \to P_{\mathrm{top}}(T^*, -s\log\|DT^*\|)$, established in [**21**, Proposition 2], we have a unique zero of (BE)*. □

Proof of Lemma 2.3. As we have observed in the proof of Lemma 2.2, there exists a unique equilibrium state μ^* for $-s\log\|DT^*\|$ in the sense of P. Walters which is equivalent to ν. Then, our assumptions allow us to see that $0 = h_{\mu^*}(T^*) - s\int_{X^*}\log\|DT^*\|\,d\mu^*$. □

Proof of Lemma 2.4. First, we remark that $\mathrm{Var}_n(-\log\|DT\|) \to 0$ as $n \to \infty$. This fact allows us to see that $-s\log\|DT \circ \pi\|$ is a continuous function on Σ satisfying the WBV property and $P_{\mathrm{top}}(T, -s\log\|DT\|) = P_{\mathrm{top}}(\sigma, -s\log\|DT \circ \pi\|)$. Then, the desired result follows from the variational principle for countable Markov shifts proved in [**3**, Theorem 2.4]. □

Proof of Theorem 2. (2-1) follows from Lemmas 2.1–2.2 and (2-2) follows from Sublemma B, the formula (3) and Lemma 2.3. Define

$$\overline{\delta} := \sup\left\{\frac{h_{\overline{m}}(\sigma)}{\int_\Sigma \log\|DT \circ \pi\|\,d\overline{m}} \;\middle|\; \overline{m} \in M_\sigma(\Sigma), \int_\Sigma \log\|DT \circ \pi\|\,d\overline{m} > 0 \right.$$
$$\left. \text{and either } h_{\overline{m}}(\sigma) < \infty \text{ or } \int_\Sigma \log\|DT \circ \pi\|\,d\overline{m} < \infty \text{ is satisfied}\right\}.$$

For proving (2-3), first we show that for all $0 < s < \overline{\delta}$, $P_{\mathrm{top}}(T, -s\log\|DT\|) > 0$. For $\epsilon := \overline{\delta} - s > 0$, we have $\overline{m} \in M_\sigma(\Sigma)$, which satisfies $\int_\Sigma \log\|DT \circ \pi\|\,d\overline{m} > 0$, either $h_{\overline{m}}(\sigma) < \infty$ or $\int_\Sigma \log\|DT \circ \pi\|\,d\overline{m} < \infty$ and $h_{\overline{m}}(\sigma)/\int_\Sigma \log\|DT \circ \pi\|\,d\overline{m} > \overline{\delta} - \epsilon = s > 0$. Hence, $h_{\overline{m}}(\sigma) - s\int_\Sigma \log\|DT \circ \pi\|\,d\overline{m} > 0$ and so

$$P_{\mathrm{top}}(T, -s\log\|DT\|) \geq h_{\overline{m}}(\sigma) - s\int_\Sigma \log\|DT \circ \pi\|\,d\overline{m} > 0.$$

Next, show that for all $s \geq \overline{\delta}$, $P_{\mathrm{top}}(T, -s\log\|DT\|) = 0$. Suppose $P_{\mathrm{top}}(T, -s\log\|DT\|) \neq 0$. Since there exists x_0, a periodic point with period q satisfying (1-1), we have

$$P_{\mathrm{top}}(T, -s\log\|DT\|) \geq \frac{1}{q}\sum_{h=0}^{q-1}(-s\log\|DT(T^h x_0)\|) = 0,$$

which implies $P_{\mathrm{top}}(T, -s\log\|DT\|) > 0$. Then, by Lemma 2.4 we have that for all $\epsilon > 0$, there exists $\overline{m} \in M_\sigma(\Sigma)$ satisfying either $h_{\overline{m}}(\sigma) < \infty$ or $\int_\Sigma \log\|DT \circ \pi\|\,d\overline{m} < \infty$ and

$$h_{\overline{m}}(\sigma) - s\int_\Sigma \log\|DT \circ \pi\|\,d\overline{m} > P_{\mathrm{top}}(T, -s\log\|DT\|) - \epsilon.$$

In particular, for $0 < \epsilon_0 < P_{\mathrm{top}}(T, -s\log\|DT\|)$, we have $\overline{m} \in M_\sigma(\Sigma)$ satisfying

$$h_{\overline{m}}(\sigma) - s\int_\Sigma \log\|DT \circ \pi\|\,d\overline{m} > P_{\mathrm{top}}(T, -s\log\|DT\|) - \epsilon_0 > 0. \tag{4}$$

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We recall that $m := \pi_* \overline{m} (= \overline{m}\pi^{-1}) \in \overline{M}_T(X)$ is supported on $\pi(\Sigma)$. If $m(B_1) > 0$, then, by Sublemmas A and B,

$$\int_{X^*} \log\|DT^*\|\, dm^* = \int_{B_1} \log\|DT_{B_1}\|\, dm_{B_1} = \frac{\int_X \log\|DT\|\, dm}{m(B_1)} > 0,$$

so that $h_{\overline{m}}(\sigma)/\int_\Sigma \log\|DT \circ \pi\|\, d\overline{m} > s$ and hence $\overline{\delta} > s$. If $m(B_1) = 0$, then $m\big(\bigcup_{n=1}^\infty \{R(x) = n\}\big) = 0$ because $\{R(x) = n\} \subset T^{-(n-1)}B_1$ for $n \geq 2$. Since $X = \bigcup_{n=1}^\infty \{R(x) = n\} \cup \bigcap_{n\geq 0}\{R(x) > n\}$ we have $m\big(\bigcap_{n\geq 0}\{R(x) > n\}\big) = 1$, which implies that m is supported on the periodic orbits satisfying (1-1). From this fact, we see that $\int_\Sigma \log\|DT \circ \pi\|\, d\overline{m} = \int_X \log\|DT\|\, dm = 0$. Hence, if $h_{\overline{m}}(\sigma) > 0$ then for all $s' \in \mathbb{R}$,

$$P_{\text{top}}(T, -s' \log\|DT\|) \geq h_{\overline{m}}(\sigma) - s' \int_\Sigma \log\|DT \circ \pi\|\, d\overline{m} > 0,$$

which contradicts the existence of a zero of (BE) in (2-1). Hence, we have $h_{\overline{m}}(\sigma) = 0$ and $h_{\overline{m}}(\sigma) - s\int_\Sigma \log\|DT \circ \pi\|\, d\overline{m} = 0$, which gives a contradiction to the inequality (4). From these observations, $P_{\text{top}}(T, -s\log\|DT\|) > 0$ implies $s < \overline{\delta}$ and we can conclude that $\overline{\delta}$ is the smallest zero of (BE). Note that the T-invariant ergodic probability measure μ obtained in (2-2) equivalent to ν satisfies $\mu\big(\bigcup_{i\in I} \partial X_i\big) = 0$. Hence, $\pi^*\mu \in M_\sigma(\Sigma)$ and $\mu = \pi_*(\pi^*\mu) \in \overline{M}_T(X)$. Since $s = h_\mu(T)/\int_X \log\|DT\|\, d\mu = h_{\pi^*\mu}(\sigma)/\int_\Sigma \log\|DT \circ \pi\|\, d\pi^*\mu$ for the unique zero $s > 0$ of (BE)*, we have $s \leq \overline{\delta}$. On the other hand, since $\overline{\delta}$ is the smallest zero of (BE) and by (2-1) the unique zero $s > 0$ of (BE)* satisfies $P_{\text{top}}(T, -s\log\|DT\|) = 0$, we can establish $s \geq \overline{\delta}$. We denote

$$\delta := \sup\left\{ \frac{h_m(T)}{\int_X \log\|DT\|\, dm} \;\middle|\; m \in \overline{M}_T(X), 0 < \int_X \log\|DT\|\, dm < \infty \right\}.$$

Finally, it follows from the inequalities $s = h_\mu(T)/\int_X \log\|DT\|\, d\mu \leq \delta \leq \overline{\delta}$ that $s = \delta = \overline{\delta}$ for the unique zero $s > 0$ of (BE)*. We thus complete the proof. □

Proof of Lemma 3.2. $d_m(x, Q) = s$ follows immediately from the weak geometric property of m. By the generating property of Q, for each $n \geq 1$ we can choose $r_n > 0$ and $k > 0$ such that $X_{i_1\dots i_n}(x) \supset B(x, r_n) \supset X_{i_1\dots i_n i_{n+1}\dots i_{n+k}}(x)$. Since $\lim_{n\to\infty} \log \operatorname{diam} X_{i_1\dots i_{n+1}}(x)/\log \operatorname{diam} X_{i_1\dots i_n}(x) = 1$, if $d_m(x, Q)$ exists then $d_m(x)$ exists, so that $d_m(x, Q) = d_m(x)$. □

Proof of Lemma 3.3. By the inequalities (2) we have $\sigma(x) := \lim_{n\to\infty}(1/n)\log\sigma_n(x) = -\lim_{n\to\infty}(1/n)\log\|DT^n(x)\|$. □

Proof of Lemma 3.4. First, we recall that $m\big(\bigcap_{n\geq 0}\{R > n\}\big) = 0$ implies $m(B_1) > 0$ as $1 = m\big(\bigcup_{n=1}^\infty \{R = n\}\big) \leq \sum_{n=1}^\infty m(\{R > n\} \cap T^{-n}B_1)$. Hence, by Sublemmas A and B m-a.e. $x \in X$ $\lim_{n\to\infty}(1/n)\log\|DT^n(x)\| = \int_X \log\|DT\|\, dm = m(B_1)\int_{B_1} \log\|DT_{B_1}\|\, dm_{B_1} = m(B_1)\int_{X^*} \log\|DT^*\|\, dm^*$. The desired result follows from the uniformly expanding property of T^*. □

Proof of Theorem 3. (3-1) follows from Lemma 3.1 and (3-2) follows from Theorem 1. (3-3) follows from Remark H, that is,

$$K_n^{-1} \leq \nu(X_{i_1\dots i_n}(x_0)) \leq \nu(\{x \in X \mid R(x) > n\}),$$

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where $\{K_n\}_{n\geq 1}$ is the weak Gibbs sequence for ν. Hence, we can take $s_n = K_n$. Since μ satisfies the assumptions of Lemma 3.4, μ is hyperbolic and hence ν is a (quasi-)hyperbolic weak geometric measure with exponent $s = h_\mu(T)/\int_X \log\|DT\|\,d\mu > 0$. Hence, it follows from Lemma 3.2 that $d_\nu(x) = d_\nu(x, Q) = s$ ν-a.e. $x \in X$ and $s \leq \dim_H X$. For establishing an upper bound of $\dim_H X$, we first note that for each $n \geq 1$,

$$\bigcup_{(\underline{i}_1\ldots\underline{i}_n)\in I^{*n}} v_{(\underline{i}_1\ldots\underline{i}_n)}(X) = X\setminus\left(\bigcup_{m=0}^{n-1} T^{*-m}\left(\bigcap_{j=0}^{\infty}\{R > j\}\right)\right).$$

This equality shows that $\left\{\bigcup_{(\underline{i}_1\ldots\underline{i}_n)\in I^{*n}} v_{(\underline{i}_1\ldots\underline{i}_n)}(X)\right\}_{n\geq 1}$ is a nested sequence consisting of countable covers of X^*. Since

$$v_{(\underline{i}_1\ldots\underline{i}_n)}(X) \subset B(x, \operatorname{diam}\{v_{(\underline{i}_1\ldots\underline{i}_n)}(X)\})$$

for some $x \in v_{(\underline{i}_1\ldots\underline{i}_n)}(X)$ the s-dimensional Hausdorff measure of X^*, $H_s(X^*)$ satisfies

$$H_s(X^*) \leq \sum_{(\underline{i}_1\ldots\underline{i}_n)\in I^{*n}} (2\operatorname{diam}\{v_{\underline{i}_1\ldots\underline{i}_n}(X)\})^s.$$

Secondly, we recall that by summable variation of $-\log\|DT^*\|$ the s-conformal measure ν on X (supported on X^*) which is a weak geometric measure with exponent s is a geometric measure with exponent s associated to (T^*, X^*, Q^*). Hence, there exists $1 \leq W < \infty$ such that

$$H_s(X^*) \leq 2^s W \sum_{(\underline{i}_1\ldots\underline{i}_n)\in I^{*n}} \nu(v_{\underline{i}_1\ldots\underline{i}_n}(X)) < \infty,$$

which implies $s \geq \dim_H X^*$. Since the marginal set $\bigcap_{n\geq 0}\{R(x) > n\}$ is a countable set, $X\backslash X^*$ is also a countable set, so that $\dim_H X = \dim_H X^*$. We thus complete the proof. □

Proof of Lemma 4.1. We remark that for $\phi, \psi \in \mathcal{W}_1(T)$ both $\pi^*\phi$ and $\pi^*(\phi + \psi)$ are continuous functions on Σ satisfying the WBV property. Hence, it suffices to apply the variational principle for countable Markov shifts proved in [**3**, Theorem 2.4]. □

Proof of Theorem 4. Suppose that $m_1, m_2 \in \overline{M}_T(X)$ with $m_1 \neq m_2$ are equilibrium states for ϕ. Then, by Lemma 4.1, we see that for all $t > 0$ and all $f \in C(X)$

$$\frac{P_{\text{top}}(T, \phi + tf) - P_{\text{top}}(T, \phi)}{t} \geq m_i(f) \geq \frac{P_{\text{top}}(T, \phi + t(-f)) - P_{\text{top}}(T, \phi)}{-t} \quad (i = 1, 2)$$

and we can establish for all $f \in C(X)$ and $i = 1, 2$

$$\lim_{t\to+0}\frac{P_{\text{top}}(T, \phi + tf) - P_{\text{top}}(T, \phi)}{t} \geq m_i(f) \geq \lim_{t\to+0}\frac{P_{\text{top}}(T, \phi + t(-f)) - P_{\text{top}}(T, \phi)}{-t}.$$

If $P_{\text{top}}(T, .)$ is differentiable at ϕ, then we have a contradiction to $m_1 \neq m_2$. □

Proof of Proposition 5.1. Define $\mathbb{P}(q) := P_{\text{top}}(T, q\phi - t(q)\log\|DT\|)$. Then, $\mathbb{P}'_+(q) = \mathbb{P}'_-(q) = 0$ because of $\mathbb{P}(q) \equiv 0$. By Lemma 4.1, we can establish the next fact.

SUBLEMMA C. *Let $\mu \in \overline{M}_T(X)$ be an equilibrium state for $q\phi - t(q)\log\|DT\|$ with $\mu(\log\|DT\|) \neq 0$. Then we have $t'_-(q) \leq \mu(\phi)/\mu(\log\|DT\|) \leq t'_+(q)$.*

The desired result follows from Sublemma C. □

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