Macroeconomic Dynamics, **12**, 2008, 702–716. Printed in the United States of America.
doi:10.1017/S1365100508070375

COMPARATIVE STATICS AND DYNAMICS OF THE ROMER R&D GROWTH MODEL WITH QUALITY UPGRADING

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This paper replaces increasing product variety with quality upgrading in the Romer model [Romer, Paul M., *Journal of Political Economy* 98 (1990), S71–S102]. We show that the range of parameters for which a steady state exists can be divided into two subspaces with well-behaved comparative statics and saddle-point dynamics in one subspace, but with "perverse" comparative-statics properties and either equilibrium indeterminacy or instability in the other subspace. These results for the closed economy can also be used to characterize the dynamics of the M-country open-economy version of the model.

Keywords: Endogenous Growth, Transitional Dynamics, Indeterminacy

1. INTRODUCTION

Romer (1990) revolutionized growth theory with his article on "Endogenous Technological Change," which presented the first general equilibrium model in which sustained productivity growth was due to the innovative efforts of firms seeking to make profits in imperfectly competitive markets (rather than the constancy of returns to capital assumed in the earlier AK models). In this model and the subsequent literature, ongoing growth is the result of the introduction of new goods or processes. Subsequently, Grossman and Helpman (1991a, Ch. 4, 1991b), Aghion and Howitt (1992), and others developed a second branch of R&D growth models, in which innovation causes growth as it improves the quality of existing goods, thereby rendering previous innovations obsolete ("creative destruction"). "Schumpeterian" quality upgrading is commonly regarded as the more attractive link between innovation and growth. For instance, Aghion and Howitt (2006, p. 272) argue

> that the Schumpeterian paradigm holds the best promise of delivering a systematic, integrated, and yet operational framework for analyzing and developing context-dependent growth policies . . . , whereas the AK and the product variety paradigms fail to fully deliver on those promises.

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https://doi.org/10.1017/S1365100508070375 Published online by Cambridge University Press

This suggests that replacing increasing product variety with quality upgrading à la Grossman and Helpman (1991a, Ch. 4, 1991b) in Romer (1990) makes the model more attractive as a description of real-world growth. As both the Romer model and the Grossman–Helpman model possess steady states with "normal" comparative-statics properties (the steady-state growth rate rises in response to an increase in effective labor or a decrease in the consumers' discount rate) and a unique convergent equilibrium growth path, one might suspect that the same holds true for the Romer model with quality upgrading as well. This paper shows that this is not true. We introduce Grossman–Helpman quality upgrading into the Romer model and show that the range of parameters for which a steady state exists can be partitioned into two subspaces with well-behaved comparative statics and saddle-point dynamics in one subspace, but with "perverse" comparative-statics properties (the steady-state growth rate falls in response to an increase in effective labor or a decrease in the consumers' discount rate) and either equilibrium indeterminacy or instability in the other subspace. In the latter subspace, a parameter change possibly leads to a Hopf bifurcation. Using a theorem in Arnold (2007), we generalize our results to an M-country open economy.

Viewed from a different angle, this paper contributes to a strand of the literature that investigates the dynamics of Romer's path-breaking model. Romer's (1990) original paper is confined to steady-state analysis. Arnold (2000a, 2000b) shows that both the balanced-growth equilibrium of the model and the steady state of the optimal growth path are saddle points. Garcia–Castrillo and Sanso (2002) analyze the equilibrium dynamics globally. Benhabib et al. (1994) add complementarities between the intermediate goods to the Romer model and highlight the possibility of indeterminacy in equilibrium. Using numerical examples, Asada et al. (1998) demonstrate that the Benhabib-Perli-Xie (1994) extension also allows for Hopf bifurcations (as does a variant of the model with physical capital as an input in R&D). Slobodyan (2007) presents some analytical results on the possibility of Hopf bifurcations in this model. Arnold (2006) shows that the steady state is a saddle point if one adds population growth to the model and assumes diminishing returns to knowledge in R&D [i.e., in the Jones (1995) model]. Devereux and Lapham (1994) demonstrate that the steady state of the two-country open economy version of the Romer model [first studied by Rivera-Batiz and Romer (1991)] without international knowledge spillovers is unstable.

Section 2 presents the model. Section 3 derives the equations that determine the equilibrium growth path. Sections 4 and 5 are concerned with the steady state and dynamics, respectively; several derivations are relegated to the Appendix.[1] Section 6 concludes.

2. MODEL

Consider a closed economy inhabited by a continuum of mass one of identical individuals. Each individual supplies L units of labor and maximizes the intertemporal utility function $U(t) \equiv \int_t^\infty e^{-\rho(\tau - t)}[c(\tau)^{1-\sigma} - 1]/(1-\sigma)d\tau$,

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where t and τ denote time, $c(t)$ is consumption at time t, ρ (>0) is the subjective discount rate, and σ (>0) is the inverse of the intertemporal elasticity of substitution in consumption. There is a single final good, which is used both for consumption and for investment. Depreciation is neglected. Hence, $\dot{K} = Y - c$, where Y is aggregate production and K is the capital stock.[2] Output is produced using labor L_Y and an index of intermediate goods D_Y according to the Cobb–Douglas production function $Y = L_Y^{1-\alpha} D_Y^{\alpha}$ ($0 < \alpha < 1$). All this follows Romer (1990).

Whereas in Romer (1990) D_Y is composed of a range of intermediate goods that can be expanded via R&D, we follow the Grossman–Helpman (1991a, Ch. 4, p. 87, 1991b, p. 45) approach to quality upgrading and let

$$\ln D_Y = \int_0^1 \ln \left[\sum_{\omega=1}^{\Omega(j)} \lambda^{\omega} x_{\omega}(j) \right] dj,$$

where $x_{\omega}(j)$ is the input of quality ω of intermediate j, $\Omega(j)$ is the highest quality producible in sector j, and λ (>1) is the (exogenous) size of a quality jump. This formulation has three important properties. First, the "number" of intermediate goods is constant and equal to unity. However, whenever the highest quality producible in sector j, $\Omega(j)$, rises by one, the maximum productivity of the j-intermediates rises by a factor λ. Second, different qualities, ω, of a given intermediate, j, are perfect substitutes. This paves the way for "creative destruction": new quality leaders can push their competitors out of the market by charging a price no greater than λ times the unit cost of production. Third, although there are discrete quality jumps and creative destruction at the industry level, the fact that there is a continuum of different intermediates implies that the intermediates sector as a whole behaves "smoothly" [this is the main difference from Aghion and Howitt's (1992) quality upgrading model].

Quality jumps are brought about by R&D: if successful, R&D targeted at intermediate j raises $\Omega(j)$ by one. R&D is stochastic: letting $I(j)\,dt$ denote the probability of a quality jump in industry j in a short time interval dt and $L_A(j)$ R&D employment aimed at this industry during dt, we have $I(j) = L_A(j)/a$, where a (>0) parameterizes the productivity of R&D (scale effects prevail). We restrict attention to symmetric equilibria with $I(j) \equiv I$ and $L_A(j) = L_A$ uniform for all j and call I ($=L_A/a$) the rate of innovation. Due to the continuum-of-industries formulation, the "number" of markets with a quality improvement in dt is $I\,dt$. That is, $d[\int_0^1 \Omega(j)dj]/dt = I$. One unit of each quality of each intermediate good can be obtained from η (>0) units of physical capital. Innovators compete in prices and have a temporary monopoly for their respective innovations (either due to patent protection or because of technological leadership). All other agents behave competitively. There is a perfect capital market, in which idiosyncratic risk can be diversified away. Throughout the analysis, we restrict our attention to allocations with $I > 0$.

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3. EQUILIBRIUM

An equilibrium is a collection of prices and quantities such that consumers maximize utility, firms maximize profit, the value of an innovation is equal to the R&D cost, and all markets clear. In this section, we derive equilibrium conditions, which are used in Sections 4 and 5 to analyze the steady state and the dynamics of the model, respectively.

Let w denote the wage rate and $p_\omega(j)$ the price of quality ω of intermediate j. Perfect competition in the final-goods sector implies $wL_Y = (1-\alpha)Y$. Furthermore, because for any intermediate j, one unit of quality ω is a perfect substitute for λ units of quality $\omega - 1$, only the producer $\tilde{\omega}(j)$ with the lowest quality-adjusted price, $p_\omega(j)/\lambda^\omega$, faces a positive demand. This producer's demand is unit elastic, so profits increase as the price rises. In equilibrium, the producer of the maximum-quality intermediate, $\Omega(j)$, prices the lower-quality producers out of the market (so that $\tilde{\omega}(j) = \Omega(j)$) with the limit price $\lambda\eta r \equiv p$, where r is the interest rate and ηr is the price at which the lower-quality producers break even. Due to the symmetry of the demand functions, uniform prices, p, imply uniform quantities, $x_{\Omega(j)}(j) \equiv x$, so that $K = \eta x$. From the zero-profit condition for the final-goods sector, $px = Y - wL_Y = \alpha Y$, it follows that $x = \alpha Y/(\lambda\eta r)$. Letting $z \equiv Y/K$ gives $r = \alpha Y/(\lambda K) = \alpha z/\lambda$. Turning to the consumers, assuming bounded utility, an optimal consumption profile satisfies the Ramsey rule $\dot{c}/c = (r-\rho)/\sigma = (\alpha z/\lambda - \rho)/\sigma$ and the transversality condition $\lim_{t\to\infty} e^{-\rho t}\mu(t)B(t) = 0$, where B is the consumers' asset holdings and μ is the associated current-value multiplier. Letting $\chi \equiv c/K$, we have $\dot{\chi}/\chi = \dot{c}/c - \dot{K}/K$ or, using $\dot{K} = Y - c$, the definitions of z and χ, and $r = \alpha z/\lambda$,

$$\frac{\dot{\chi}}{\chi} = \chi - \left(1 - \frac{\alpha}{\lambda\sigma}\right)z - \frac{\rho}{\sigma}. \tag{1}$$

Because only maximum-quality intermediates are produced and because of symmetry, $\ln D_Y = \int_0^1 \ln[\lambda^{\Omega(j)}x]dj = \ln\Delta + \ln x$, where $\ln\Delta \equiv \ln\lambda \int_0^1 \Omega(j)dj$ is an indicator of the state of technology. Together with $x = K/\eta$, aggregate output becomes $Y = L_Y^{1-\alpha}(\Delta K/\eta)^\alpha$. Using the definition $z \equiv Y/K$, it follows that $K/L_Y = (\Delta/\eta)^{\alpha/(1-\alpha)}z^{-1/(1-\alpha)}$ and $Y/L_Y = [(\Delta/\eta)K/L_Y]^\alpha = [\Delta/(\eta z)]^{\alpha/(1-\alpha)}$. Let π denote current profit and $P_A(t) \equiv \int_t^\infty \exp\{-\int_t^\tau [r(s) + I(s)]ds\}\pi(\tau)d\tau$ the expected present value of a firm with a maximum-quality intermediate. The risk of losing the monopoly position due to an innovation acts like additional discounting. Because the capital market effectively diversifies this idiosyncratic risk, free entry into R&D requires equality of the value of an innovation and the cost of innovating: $P_A = wa$. Using $wL_Y = (1-\alpha)Y$, we get $P_A = (1-\alpha)aY/L_Y = (1-\alpha)a[\Delta/(\eta z)]^{\alpha/(1-\alpha)}$. Hence, $\dot{P}_A/P_A = \alpha(\dot{\Delta}/\Delta - \dot{z}/z)/(1-\alpha)$. From the definition of Δ, we have $\dot{\Delta}/\Delta = I\ln\lambda$. Therefore, $\dot{z}/z = I\ln\lambda - [(1-\alpha)/\alpha]\dot{P}_A/P_A$. A quality leader's profit is $\pi \equiv (p - \eta r)x = (1 - 1/\lambda)px = (1 - 1/\lambda)\alpha Y$. Differentiating the definition of the expected value of an innovation, P_A, with respect to time

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yields $\pi + \dot{P}_A = (r + I)P_A$. This can be interpreted as a no-arbitrage condition: the sum of current profit, π, and capital gains, $\dot{P}_A$, equals the sum of the opportunity cost of capital, rP_A, and the premium, IP_A, that compensates for the risk of losing the monopoly. Let $\nu \equiv Y/P_A$, so that the free-entry condition becomes $L_Y = a(1-\alpha)\nu$. Substituting $r = \alpha z/\lambda$, $I = (L - L_Y)/a = L/a - (1-\alpha)\nu$, $\pi = (1 - 1/\lambda)\alpha Y$, and $\nu \equiv Y/P_A$ into the no-arbitrage equation yields $\dot{P}_A/P_A = \alpha z/\lambda + L/a - (1 - \alpha/\lambda)\nu$. Inserting this equality and $I = L/a - (1-\alpha)\nu$ into the expression for $\dot{z}/z$ and simplifying terms gives

$$\frac{\dot{z}}{z} = \frac{1-\alpha}{\alpha}\left[\left(\frac{\alpha}{1-\alpha}\ln\lambda - 1\right)\frac{L}{a} + \left(1 - \frac{\alpha}{\lambda} - \alpha\ln\lambda\right)\nu - \frac{\alpha}{\lambda}z\right]. \qquad (2)$$

From $\nu = L_Y/[a(1-\alpha)]$, the definition of z, and $Y/L_Y = [\Delta/(\eta z)]^{\alpha/(1-\alpha)}$, we have $\nu = zK[\Delta/(\eta z)]^{-\alpha/(1-\alpha)}/[a(1-\alpha)]$. Differentiating with respect to time and using $\dot{K}/K = z - \chi$, $\dot{\Delta}/\Delta = I\ln\lambda$, $I = L/a - (1-\alpha)\nu$, and (2), we obtain

$$\frac{\dot{\nu}}{\nu} = \left(\ln\lambda - \frac{1}{\alpha}\right)\frac{L}{a} + \left[\frac{1}{\alpha} - \frac{1}{\lambda} - (1-\alpha)\ln\lambda\right]\nu + \left(1 - \frac{1}{\lambda}\right)z - \chi. \qquad (3)$$

Given the starting values $K(0)$ and $\Delta(0)$, equations (1)–(3) determine the evolution of the economy through time: (1) determines $c\ (= \chi K)$, (2) determines $Y\ (= zK)$, and (3) pins down $L_Y\ (= a(1-\alpha)\nu)$, $L_A\ (= L - L_Y)$, and $I\ (= L_A/a)$. This in turn determines the evolution of K (because $\dot{K} = Y - c$) and Δ (because $\dot{\Delta}/\Delta = I\ln\lambda$). Finally, the prices are determined by $r = \alpha z/\lambda$, $w = (1-\alpha)Y/L_Y$, and $P_A = wa$.

4. STEADY STATE

Let the steady-state value of any variable y be denoted y^*. $(\chi^*, z^*, \nu^*)'$ comprises a steady state of the model with positive growth if $(\chi, z, \nu)' = (\chi^*, z^*, \nu^*)'$ solves (1)–(3), $I^* > 0$, the transversality condition for the consumers' utility-maximization problem is satisfied, utility is bounded, and $(\chi^*, z^*, \nu^*)' > \mathbf{0}$. Let $l \equiv L/a$ and $\phi \equiv \rho/l$. It is convenient to define

$$\bar{\sigma} \equiv 1 - \frac{1 - \frac{\alpha}{\lambda}}{\alpha\ln\lambda}, \quad \bar{\phi} \equiv \frac{\alpha(\lambda - 1)}{\lambda(1-\alpha)}, \quad \phi(\sigma) \equiv \frac{1-\sigma}{1-\bar{\sigma}}\bar{\phi}. \qquad (4)$$

Notice that $\bar{\sigma} < 1$, $\bar{\phi} > 0$, and $\phi(\bar{\sigma}) = \bar{\phi}$. It will be essential to distinguish the cases $\bar{\sigma} < 0$ and $\bar{\sigma} > 0$. From (1)–(3),

$$\chi^* = \frac{l}{\sigma - \bar{\sigma}}\left[\frac{\lambda}{\alpha}(\bar{\phi}\sigma - \phi\bar{\sigma}) + \phi - \bar{\phi}\right] \qquad (5)$$

$$z^* = \frac{l}{\sigma - \bar{\sigma}}\frac{\lambda}{\alpha}(\bar{\phi}\sigma - \phi\bar{\sigma}) \qquad (6)$$

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$$\nu^* = \frac{l}{\alpha \ln \lambda}\left[\frac{\phi - \phi(\sigma)}{\sigma - \bar{\sigma}} + \frac{1}{1-\bar{\sigma}}\right]. \tag{7}$$

Using (7) and the definitions in (4) to rewrite $I^* = (L - L_Y^*)/a = l - (1-\alpha)\nu^*$, we obtain the steady-state rate of innovation:

$$I^* = \frac{\alpha\left(1 - \frac{1}{\lambda}\right) l - (1-\alpha)\rho}{(\sigma - \bar{\sigma})\alpha \ln \lambda} = l\,\frac{(1-\alpha)(\bar{\phi} - \phi)}{\alpha(\ln \lambda)(\sigma - \bar{\sigma})}. \tag{8}$$

From the latter equality in (8), it is obvious that $I^* > 0$ if, and only if, $\sigma > \bar{\sigma}$ and $\phi < \bar{\phi}$ or $\sigma < \bar{\sigma}$ and $\phi > \bar{\phi}$.

From $Y = L_Y^{1-\alpha}(\Delta K/\eta)^{\alpha}$, $\dot{\Delta}/\Delta = I \ln \lambda$, constancy of L_Y^*, and $(\dot{K}/K)^* = (\dot{Y}/Y)^*$, we have $(\dot{Y}/Y)^* = \alpha I^*(\ln \lambda)/(1-\alpha)$. Constancy of χ, z, and ν in the steady state implies that c, K, and P_A, respectively, grow at the same rate as Y, and so does the households' financial wealth, $B = K + P_A$. Differentiating the first-order condition for utility maximization yields $\dot{\mu}/\mu = -\sigma\dot{c}/c$. So the transversality condition, $\lim_{t\to\infty} e^{-\rho t}\mu(t)B(t) = 0$, boils down to $\rho > (1-\sigma)(\dot{Y}/Y)^* = (1-\sigma)\alpha I^*(\ln \lambda)/(1-\alpha)$. We note in passing that this condition is also necessary and sufficient for the boundedness of the utility integral. Using the second equality in (8), the transverality condition requires $\phi > \phi(\sigma)$ for $\sigma > \bar{\sigma}$ and $\phi < \phi(\sigma)$ for $\sigma < \bar{\sigma}$.

This proves that $\sigma > \bar{\sigma}$ and $\phi(\sigma) < \phi < \bar{\phi}$ (case 1) or $\sigma < \bar{\sigma}$ and $\bar{\phi} < \phi < \phi(\sigma)$ (case 2) is necessary for the existence of a steady state. In the Appendix, we prove that these conditions are also sufficient (the admissible parameter combinations are illustrated in Figure 1, where case 1 is subdivided into case 1.1 with $\bar{\sigma} < 0$ and case 1.2 with $0 < \bar{\sigma} < \sigma$). So we have

THEOREM 1. *A steady state exists if, and only if,* $\sigma > \bar{\sigma}$ *and* $\phi(\sigma) < \phi < \bar{\phi}$ (*case 1*) *or* $\sigma < \bar{\sigma}$ *and* $\bar{\phi} < \phi < \phi(\sigma)$ (*case 2*).

The steady states of both the Romer (1990) model and the Grossman–Helpman (1991a, Ch. 4, 1991b) model have "normal" comparative statics properties: the steady-state growth rate responds positively to an increase in effective labor, l, and to a decrease in the discount rate, ρ [see Romer (1990, equation (13), p. S92, and pp. S93–S96) and Grossman and Helpman (1991a, equation (4.18), p. 96, and pp. 97–98)].[3] Interestingly, the steady state of the Romer (1990) model with quality upgrading à la Grossman and Helpman (1991a, Ch. 4, 1991b) considered here possibly has "perverse" comparative-statics properties: from the first equality in (8), the steady-state rate of innovation, I^*, increases as l falls or ρ rises in case 2. To explain the model's comparative statics, we follow Barro and Sala-i-Martin (2004, Figure 7.3, p. 330) and construct a diagram with two curves that depict the steady-state rates of innovation, I^*, desired by the consumers (CG for "consumption growth") and brought forth by the production sector (PG for "production growth") as functions of the interest rate, r^* (see Figure 2). The steady-state equilibrium interest and innovation rates are found at the intersection of CG and PG. The results of the subsequent discussion are summarized in Table 1.

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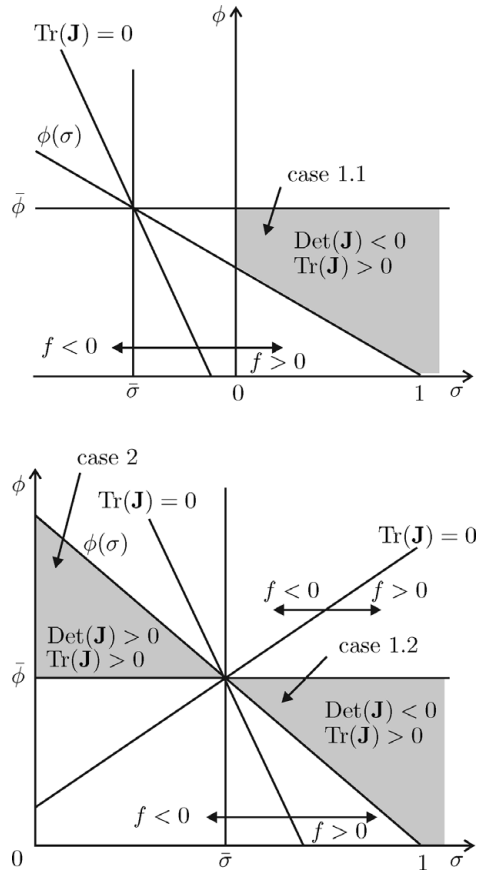


FIGURE 1. Signs of trace and determinant.

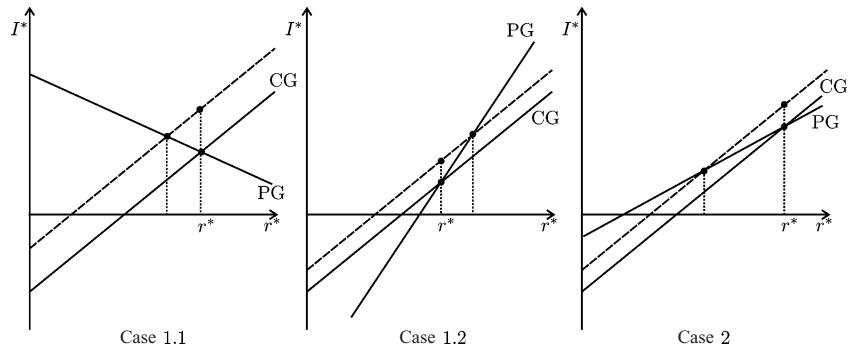


FIGURE 2. Comparative statics.

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TABLE 1. Summary of comparative statics effects

	Case 1.1	Case 1.2	Case 2
$I^* \uparrow$	Labor demand $\uparrow$	Labor demand $\downarrow$	Labor demand $\downarrow$
Slope of PG	< 0	$>$ slope of CG > 0	> 0, $<$ slope of CG
Interest rate $> r^*$	CG $>$ PG	CG $<$ PG	CG $>$ PG
$\rho \downarrow$	$r^* \downarrow, I^* \uparrow$	$r^* \uparrow, I^* \uparrow$	$r^* \downarrow, I^* \downarrow$
$l \uparrow$	$r^* \uparrow, I^* \uparrow$	$r^* \uparrow, I^* \uparrow$	$r^* \downarrow, I^* \downarrow$

As for the consumption side, the Ramsey rule, together with $(\dot{c}/c)^* = (\dot{Y}/Y)^* = \alpha I^*(\ln \lambda)/(1-\alpha)$, yields

$$I^* = \frac{1-\alpha}{\alpha \ln \lambda} \frac{r^* - \rho}{\sigma} \tag{9}$$

in a steady state (see the straight line CG in Figure 2). As usual, a higher interest rate induces households to choose a steeper consumption profile.

The source of the unusual comparative-statics properties is that the production-side relationship between the interest rate and the rate of innovation may be upward-sloping. That is, *firms may increase innovation in response to an increase in the interest rate* in equilibrium. To see this, notice that, using the equation for profits, $\pi = \alpha(1 - 1/\lambda)Y$, the value of a patent relative to Y in a steady state can be expressed as (see the Appendix)

$$\left(\frac{P_A}{Y}\right)^* = \frac{\alpha\left(1 - \frac{1}{\lambda}\right)}{r^* + I^* - \frac{\alpha}{1-\alpha}(\ln \lambda) I^*}. \tag{10}$$

Naturally, an increase in the interest rate reduces the value of an innovation. An increase in the rate of innovation, I^*, has a twofold effect on the value of an innovation. For one thing, because new quality leaders replace their predecessors, an increase in I^* acts like additional discounting (the creative destruction effect). For another, an increase in I^* implies that aggregate output, Y, and profits, $\pi = \alpha(1 - 1/\lambda)Y$, grow faster, thereby raising $(P_A/Y)^*$ (market size effect). Next, substituting for $(P_A/Y)^*$ from (10) into the labor market clearing condition $I^* = L/a - (1-\alpha)\upsilon^* = l - (1-\alpha)/(P_A/Y)^*$ and rearranging terms yields

$$l = I^* + \frac{1-\alpha}{\alpha} \frac{\lambda}{\lambda - 1} \left[r^* + \left(1 - \frac{\alpha}{1-\alpha} \ln \lambda\right) I^* \right]. \tag{11}$$

Equation (11) is depicted as the PG curve in Figure 2. An increase in r^* increases the demand for labor in production. This is because the value of an innovation and, given free entry into R&D, the wage rate fall. The crucial point is that, at a given interest rate, *an increase in the innovation rate causes a decline in the aggregate demand for labor if the market size effect is sufficiently strong,* namely if, and only if, the right-hand side of (11) is decreasing in I^*, which is the case if, and only if, $\bar{\sigma} > 0$ (i.e., in cases 1.2 and 2). Intuitively, an increase in I^* requires increasing

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R&D employment [see the first term in the sum on the right-hand side of (11)]. However, if $\bar{\sigma} > 0$, the market size effect is sufficiently pronounced such that the value of an innovation and, hence, the wage rate rise to such an extent that the ensuing cutback in employment in production (see the second term of the sum) more than outweighs higher R&D employment, and the total demand for labor falls.[4] Consequently, an increase in I^* is needed to eliminate the excess demand for labor brought about by an increase in r^*, and PG is upward-sloping. Moreover, comparing (9) and (11) shows that if PG is upward-sloping (i.e., if $\bar{\sigma} > 0$), it is steeper or flatter than CG depending on whether $\sigma > \bar{\sigma}$ (case 1.2) or $\sigma < \bar{\sigma}$ (case 2). The standard decreasing relationship between r^* and I^* arises only if $\bar{\sigma} < 0$ (case 1.1).

We can now infer the comparative statics from Figure 2. The left panel of Figure 2 illustrates case 1.1 ($\bar{\sigma} < 0$): PG is downward-sloping. If the interest rate exceeds its steady-state level, the households' desired rate of growth exceeds the rate of growth brought about by the entrepreneurs' innovation activity (CG lies above PG); a fall in the interest rate reduces consumption growth and strengthens the incentives to innovate, thereby equilibrating the system. Suppose, starting from a steady state (see the solid lines in the left panel of Figure 2), the discount rate, ρ, falls. From (9), CG shifts upward (see the dashed line in the left panel of Figure 2). As a consequence, CG exceeds PG at the former equilibrium interest rate. So, as one would expect, the interest rate, r^*, falls and innovation, I^*, soars until an equilibrium is restored. Similarly, an increase in l shifts PG to the right, thereby boosting both the interest rate and innovation.

Next, consider case 2 ($0 < \sigma < \bar{\sigma}$): PG is upward-sloping, but flatter than CG (see the right panel of Figure 2). As in case 1.1, CG exceeds PG for interest rates above the steady-state level, so that the interest rate has to fall to equilibrate the system. This means that the value of a patent and the wage rate rise, thereby reducing the demand for labor. Because decreasing R&D employment is needed to increase aggregate employment, contrary to case 1.1, the decline in the interest rate is accompanied by a fall in the rate of innovation. Even though PG is already smaller than CG, this leads the economy to a steady-state equilibrium, because the decline in the interest rate reduces CG even more. Suppose now that, starting from a steady state (see the solid lines in the right panel of Figure 2), ρ falls. As in case 1.1, CG shifts upward and exceeds PG at the former equilibrium interest rate (see the dashed line in the right panel of Figure 2). As explained above, this leads to a decrease in the rate of innovation. Similarly, an increase in l leads to decreases in the interest rate and in the rate of innovation. Evidently, it is the positive production-side relation between the interest rate and innovation, illustrated by the positive slope of the PG curve, that gives rise to "perverse" responses of I^* to changes in ρ and l in this case.

Finally, in case 1.2 ($0 < \bar{\sigma} < \sigma$), PG is upward-sloping and steeper than CG (see the middle panel of Figure 2). Starting from a steady state determined by the solid lines, let CG shift upward due to a fall in ρ (see the dashed line). Then CG exceeds PG at the former equilibrium interest rate, and in order to restore an

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equilibrium, r^* and I^* have to rise. Even though a quick glimpse at (8) suggests that the comparative statics is "normal" in this case, the production-side relationship between the interest rate and the rate of innovation is positive, and the response of the steady-state interest rate, r^*, to a change in the discount rate, ρ, is different from what one would suspect. An increase in l shifts PG to the right, thereby raising the interest rate and the rate of innovation.

Example 1. Let $\alpha = 0.4$ and $\lambda = 1.2$. This implies $\bar{\sigma} = -8.1414 < 0$. So $\partial I^*/\partial l > 0 > \partial I^*/\partial \rho$ if a steady state exists. Further, let $\sigma = 2$, $\rho = 0.02$, and $l = 2.005$. Because $\phi(\sigma) = -0.012 < \phi = 0.010 < \bar{\phi} = 0.111$, a steady state exists and case 1.1 applies. The innovation and growth rates are $I^* = 0.1645$ and $(\dot{Y}/Y)^* = 2\%$. More generally, experimentation with numerical values suggests that case 1.1 is a likely outcome of the model. However, it is not difficult to construct counterexamples.

Example 2. Let $\alpha = 0.8$, $\lambda = 2.5$, $l = 0.04$, $\sigma = 0.002$, and $\rho = 0.1$. We have $\bar{\sigma} = 0.0723 > 0$ and $\bar{\phi} = 2.4 < \phi = 2.5 < \phi(\sigma) = 2.5819$, so we are in case 2. In the steady state, $r^* = 0.1001$, $I^* = 0.0155$, and $(\dot{Y}/Y)^* = 5.686\%$. An increase in ρ to $\rho = 0.101$ increases r^*, I^*, and $(\dot{Y}/Y)^*$ to $r^* = 0.1011$, $I^* = 0.0194$, and $(\dot{Y}/Y)^* = 7.108\%$, respectively.

5. DYNAMICS

The market size effect tends to destabilize the model: as shown in the previous section, if it dominates the creative destruction effect (which it certainly does in cases 1.2 and 2), then an increase in the rate of innovation raises the value of an innovation, thereby strengthening the incentives to raising innovation further. So, turning to model dynamics, one might suspect that a steady state in case 1.2 or in case 2 is unstable, which would render the unusual comparative statics results for these cases uninteresting. Viewed from a different angle, a plausible guess is that a steady state is generally a saddle point, because steady states both in Romer (1990) and in Grossman and Helpman (1991a, Ch. 4, 1991b) are saddle points [see Theorem 1 in Arnold (2000b, p. 74) for the Romer model and Grossman and Helpman (1991a, p. 96) respectively].[5] Neither guess is right: although case 1 (including subcase 1.2) generally displays saddle-point dynamics, a steady state in case 2 can be unstable but can also be indeterminate (so that trajectories converging to a steady state with "perverse" comparative statics exist):

THEOREM 2. *A steady state in case 1 is a saddle point. A steady state in case 2 is unstable or indeterminate.*

Proof. See the Appendix. ■

Example 1. In this example, the eigenvalues are $q_0 = -0.0384$, $q_1 = 0.0887$, and $q_2 = 4.7966$. The steady state is a saddle point.

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Remark 1. Theorem 2 provides a complete analytical characterization of the model's dynamics. One may wonder if it is possible to rule out either instability or indeterminacy in case 2. But it is easy to construct examples for both kinds of dynamics.

Example 3. Let $\sigma = 0.01$, $\alpha = 0.8$, $\lambda = 2.5$, $l = 0.04$, and $\rho = 0.1$. Then $q_{0/1} = 0.0001 \pm 0.1039i$ and $q_2 = 0.2999$, so that the steady state is unstable. Starting from these parameters, a sufficiently strong decrease in σ or a sufficiently strong increase in ϕ ($\equiv \rho/l$) brings about indeterminacy. For instance, decreasing σ to $\sigma = 0.002$ leads to (Example 2 and) $q_{0/1} = -0.0460 \pm 0.2188i$ and $q_2 = 0.4046$, and raising ρ to $\rho = 0.101$ leads to $q_{0/1} = -0.0035 \pm 0.0946i$ and $q_2 = 0.2799$. (Case 1.2 applies throughout.)

Remark 2. Example 3 suggests that the model dynamics undergoes a Hopf bifurcation as σ grows from already small to even smaller values or as ϕ rises sufficiently far. In fact, one finds that, as one varies σ in Example 3, the real part of the complex eigenvalues, $q_{0/1}$, becomes negative as σ falls below the critical value $\sigma = 0.0099$.

Remark 3. The fact that indeterminacy can arise in case 2 proves that steady states with unusual comparative-statics properties are possibly approached by (multiple) equilibrium growth paths, as, for instance, in Example 2.

Remark 4. Arnold (2007) analyzes a class of growth models in which a world economy made up of several countries (with identical tastes and technologies, free trade, unrestricted mobility of financial capital, and international knowledge spillovers) replicates the equilibrium of a hypothetical integrated economy without restrictions on factor movements under certain conditions. The model considered here belongs to this class of growth models. So we can immediately infer from Theorem 1 in Arnold (2007, pp. 72–73) that replication is feasible if, and only if, physical capital is mobile internationally and/or multinational firms or international patent licensing are allowed for. The intuition is that the number of internationally mobile production activities has to be sufficiently large in relation to the number of internationally immobile factors of production. If one of the conditions for the feasibility of replication is satisfied and the integrated equilibrium is not unstable, then there is a convergent trajectory for the M-country world economy.

6. CONCLUSIONS

The integration of quality upgrading à la Grossman and Helpman (1991a, Ch. 4, 1991b) into the Romer (1990) model gives rise to unusual comparative-statics effects and interesting dynamics, which cannot occur in either of the two models. Moreover, if physical capital is mobile internationally and/or multinational firms or international patent licensing are allowed for, then these results carry over to a M-country world economy with international knowledge spillovers. These

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findings contribute to our understanding of the dynamics of one of the most prominent endogenous growth models.

NOTES

1. Further details of the calculations can be found in a technical appendix available at the authors' Web site.

2. The time argument is suppressed unless this might cause confusion.

3. Grossman and Helpman (1991a, Ch. 4, 1991b) find well-behaved comparative statics. This result does not hinge on the fact that they assume logarithmic utility. With the constant-elasticity-of-intertemporal-substitution utility function, the steady-state rate of innovation in their model becomes $I^* = [(\lambda - 1)l - \rho] / [\lambda - (1 - \sigma) \ln \lambda]$. Because the denominator is positive (as $\lambda > 1 + \ln \lambda > (1 - \sigma) \ln \lambda$), the numerator has to be positive too in a steady state with $I^* > 0$, so that I^* increases as l rises or ρ falls.

4. This can happen neither in Romer (1990) nor in Grossman and Helpman (1991a, Ch. 4, 1991b). In Romer (1990), the analogue of (11) is $l = r^*/\alpha + (\dot{A}/A)^*$, where A is the number of producible intermediates. Higher R&D employment [an increase in $(\dot{A}/A)^*$] increases aggregate employment one-to-one. In Grossman and Helpman (1991a, Ch. 4, 1991b), the analogue of (11) is $l = [r^* + (\lambda - \ln \lambda) I^*]/(\lambda - 1)$. The market size effect is present but (because $\lambda > \ln \lambda$) is never strong enough so as to make aggregate employment a decreasing function of I^*.

5. In the Grossman–Helpman (1991a, Ch. 4, 1991b) model, where Δ is the only state variable, this means that the economy jumps to its steady state. This result, proved by Grossman and Helpman using logarithmic utility, is easily generalized to a constant-elasticity-of-intertemporal-substitution utility function.

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APPENDIX: PROOFS

PROOF OF THEOREM 1

To prove the sufficiency of the conditions that define cases 1 and 2 for the existence of a steady state, we have to show that $(\chi^*, z^*, \upsilon^*)' > \mathbf{0}$ both in case 1 and in case 2. To do so, we notice that

$$\phi\bar{\sigma} - \bar{\phi}\sigma = \phi - \bar{\phi} - \frac{\lambda - \alpha}{\alpha\lambda \ln \lambda}[\phi - \phi(\sigma)]. \tag{A.1}$$

From (A.1), it follows immediately that

$$0 > \phi - \bar{\phi} > \phi\bar{\sigma} - \bar{\phi}\sigma \tag{A.2}$$

in case 1 and that the reverse inequalities hold in case 2. From (A.2),

$$\frac{\lambda}{\alpha}(\bar{\phi}\sigma - \phi\,\bar{\sigma}) + \phi - \bar{\phi} > \frac{\lambda}{\alpha}(\bar{\phi}\sigma - \phi\,\bar{\sigma}) + \phi\bar{\sigma} - \bar{\phi}\sigma = \left(\frac{\lambda}{\alpha} - 1\right)(\bar{\phi}\sigma - \phi\,\bar{\sigma}) > 0 \tag{A.3}$$

in case 1, and the reverse inequalities hold true in case 2. In case 1, $\sigma > \bar{\sigma}$. So, from (5) and (A.3), $\chi^* > 0$. From $\sigma < \bar{\sigma}$, (5), and (A.3) (with reversed inequalities), $\chi^* > 0$ also holds true in case 2. As for z^*, (6) and (A.2) imply $z^* > 0$. Finally, from (7), the fact $\phi - \phi(\sigma)$ and $\sigma - \bar{\sigma}$ have the same sign for the admissible parameter combinations, and $\bar{\sigma} < 1$, it is obvious that $\upsilon^* > 0$. ■

PROOF OF (10)

Inserting $\pi(\tau) = \alpha(1 - 1/\lambda)Y(\tau)$ and $Y(\tau) = Y(t)\exp\{\int_t^\tau [\dot{Y}(s)/Y(s)]ds\}$ into the definition of $P_A(t)$ gives

$$P_A(t) = \alpha\left(1 - \frac{1}{\lambda}\right)Y(t)\int_t^\infty \exp\left\{-\int_t^\tau \left[r(s) + I(s) - \frac{\dot{Y}(s)}{Y(s)}\right]ds\right\}d\tau.$$

Dividing through by $Y(t)$ and using the steady-state relationship $(\dot{Y}/Y)^* = \alpha I^*(\ln\lambda)/(1 - \alpha)$ gives (10). ■

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PROOF OF THEOREM 2

Linearizing the system (1)–(3) about the steady state yields

$$\begin{pmatrix} \dot{\chi} \\ \dot{z} \\ \dot{\nu} \end{pmatrix} = \begin{pmatrix} \chi^* & -\left(1-\frac{\alpha}{\lambda\sigma}\right)\chi^* & 0 \\ 0 & -\frac{1-\alpha}{\lambda}z^* & -\bar{\sigma}(1-\alpha)(\ln\lambda)z^* \\ -\nu^* & \left(1-\frac{1}{\lambda}\right)\nu^* & (\alpha-\bar{\sigma})(\ln\lambda)\nu^* \end{pmatrix} \begin{pmatrix} \chi-\chi^* \\ z-z^* \\ \nu-\nu^* \end{pmatrix}. \tag{A.4}$$

Let $\mathbf{J}$ denote the Jacobian matrix in (A.4). From (A.4) and (4)–(7), we obtain the determinant and trace of the linearized system,

$$\mathrm{Det}(\mathbf{J}) = -\frac{\alpha(1-\alpha)}{\lambda}\frac{\sigma-\bar{\sigma}}{\sigma}(\ln\lambda)\chi^* z^* \nu^* \tag{A.5}$$

$$\mathrm{Tr}(\mathbf{J}) = \frac{l}{\alpha}\frac{f(\sigma,\phi,\alpha,\lambda)}{\sigma-\bar{\sigma}}, \tag{A.6}$$

where

$$f(\sigma,\phi,\alpha,\lambda) \equiv \left(\alpha+\lambda-1+\frac{\alpha-\bar{\sigma}}{1-\bar{\sigma}}\right)[(1-\bar{\sigma})\phi-(1-\sigma)\bar{\phi}]+(\lambda-1)(\bar{\phi}-\phi)$$
$$+\frac{\alpha-\bar{\sigma}}{1-\bar{\sigma}}(\sigma-\bar{\sigma}). \tag{A.7}$$

$\mathrm{Det}(\mathbf{J}) < 0$ in case 1 (since $\sigma > \bar{\sigma}$) and $\mathrm{Det}(\mathbf{J}) > 0$ in case 2 (since $\sigma < \bar{\sigma}$). This is illustrated in Figure 1.

The trace, $\mathrm{Tr}(\mathbf{J})$, has the same sign as $f(\sigma,\phi,\alpha,\lambda)/(\sigma-\bar{\sigma})$. Differentiating (A.7) and using (4), we obtain

$$\frac{\partial f(\sigma,\phi,\alpha,\lambda)}{\partial\sigma} = \frac{\alpha^2\left(1-\frac{1}{\lambda}+\ln\lambda\right)+\alpha\left(\lambda-1-\ln\lambda+\frac{1}{\alpha}-1\right)}{1-\alpha} \equiv f_\sigma(\alpha,\lambda)$$

$$\frac{\partial f(\sigma,\phi,\alpha,\lambda)}{\partial\phi} = \frac{\lambda-\alpha}{\alpha\lambda\ln\lambda}(\alpha+\lambda-\alpha\lambda\ln\lambda) \equiv f_\phi(\alpha,\lambda).$$

Because $f_\sigma(\alpha,\lambda)$ and $f_\phi(\alpha,\lambda)$ do not depend on σ or ϕ, the equation $f(\sigma,\phi,\alpha,\lambda)=0$ determines a straight line in the (σ,ϕ)-plane. As, evidently, $f(\bar{\sigma},\bar{\phi},\alpha,\lambda)=0$, this line passes through the point $(\bar{\sigma},\bar{\phi})$. Its slope is $d\phi/d\sigma|_{f(\sigma,\phi,\alpha,\lambda)=0} = -f_\sigma(\alpha,\lambda)/f_\phi(\alpha,\lambda)$. The fact that $\lambda-1 > \ln\lambda$ implies $f_\sigma(\alpha,\lambda) > 0$. So the slope of the line is determined by the sign of $f_\phi(\alpha,\lambda)$. If $f_\phi(\alpha,\lambda) < 0$, then $d\phi/d\sigma|_{f(\sigma,\phi,\alpha,\lambda)=0} > 0$. Conversely, suppose $f_\phi(\alpha,\lambda) > 0$, so that the $f(\sigma,\phi,\alpha,\lambda)=0$-locus is downward-sloping. In this case, this locus is steeper than the line $\phi(\sigma)$ defined in (4) (as illustrated in Figure 1) if, and only if,

$$\frac{\alpha^2\left(1-\frac{1}{\lambda}+\ln\lambda\right)+\alpha\left(\lambda-1-\ln\lambda+\frac{1}{\alpha}-1\right)}{1-\alpha} > \frac{\alpha^2(\lambda-1)\ln\lambda}{(\lambda-\alpha)(1-\alpha)}.$$

Simplifying terms yields the equivalent condition

$$1-\alpha > \alpha(\ln\lambda)(1-\alpha\lambda). \tag{A.8}$$

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For $\alpha\lambda \geq 1$, the validity of (A.8) is obvious. For $\alpha\lambda < 1$, the term in parentheses is positive, so that, because of $\ln\lambda < \lambda - 1$, (A.8) is implied by $1 - \alpha > \alpha(\lambda - 1)(1 - \alpha\lambda)$, that is $1 > \alpha\lambda(1 + \alpha - \alpha\lambda)$. The validity of this inequality follows from the observation that

$$\alpha\lambda(1 + \alpha - \alpha\lambda) = \left(\frac{1+\alpha}{2}\right)^2 - \left(\alpha\lambda - \frac{1+\alpha}{2}\right)^2 < \left(\frac{1+\alpha}{2}\right)^2 < 1.$$

$\mathrm{Tr}(\mathbf{J}) = 0$ for all points on the locus $f(\sigma, \phi, \alpha, \lambda) = 0$. As $f_\sigma(\alpha, \lambda) > 0$, we have $f(\sigma, \phi, \alpha, \lambda) < 0$ to the left of the line and $f(\sigma, \phi, \alpha, \lambda) > 0$ to the right. It follows that for $\sigma > \bar{\sigma}$, the trace, $\mathrm{Tr}(\mathbf{J})$, is positive to the right and negative to the left of the line, and for $\sigma < \bar{\sigma}$, it is positive to the left of the line and negative to the right. We conclude from Figure 1 that the trace is positive both in case 1 and in case 2.

From the Routh–Hurwitz Theorem, the number of negative eigenvalues is equal to three minus the number of variations of sign in the scheme $-1 \,\|\, \mathrm{Tr}(\mathbf{J}) \,\|\, -\mathrm{B}(\mathbf{J}) + \mathrm{Det}(\mathbf{J})/\mathrm{Tr}(\mathbf{J}) \,\|\, \mathrm{Det}(\mathbf{J})$ (where $\mathrm{B}(\mathbf{J})$ is another term defined using the elements of the Jacobian). In case 1, we have $\mathrm{Det}(\mathbf{J}) < 0$ and $\mathrm{Tr}(\mathbf{J}) > 0$. So the Routh–Hurwitz sign scheme is $- \,\|\, + \,\|\, ? \,\|\, -$, and there is one negative eigenvalue. In case 2, we have $\mathrm{Det}(\mathbf{J}) > 0$ and $\mathrm{Tr}(\mathbf{J}) > 0$, so that the Routh–Hurwitz sign scheme is $- \,\|\, + \,\|\, ? \,\|\, +$ and the number of negative eigenvalues is zero or two.

None of the three variables in $(\chi, z, \nu)'$ is predetermined. But from $Y = L_Y^{1-\alpha}(\Delta K/\eta)^\alpha$, $L_Y = a(1-\alpha)\nu$, and the definition of z, we have

$$z = \frac{[a(1-\alpha)]^{1-\alpha}}{\eta^\alpha}\left(\frac{\Delta^{\alpha/(1-\alpha)}\nu}{K}\right)^{1-\alpha}.$$

So the initial values $K(0)$ and $\Delta(0)$ determine a relationship between $z(0)$ and $\nu(0)$. It follows that zero, one, or two negative eigenvalues correspond to instability, saddle-point stability, or indeterminacy, respectively. ■

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