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RESEARCH ARTICLE

Some rigidity results on non-gradient m-quasi-Einstein condition

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Abstract
The goal of this article is to find some geometric characterizations of Kenmotsu and almost Kenmotsu manifolds that admit a non-gradient m-quasi-Einstein structure. First, we prove that a Kenmotsu manifold admits a closed m-quasi-Einstein structure is an Einstein manifold. Next, we prove that if a three-dimensional Kenmotsu manifold admits a non-trivial m-quasi-Einstein structure (g, V, m, λ) with V as a conformal vector field, then it is of constant sectional curvature -1. Finally, we prove that if a non-Kenmotsu almost Kenmotsu $(k, \mu)'$-manifold admits a closed m-quasi-Einstein structure (g, V, m, λ) with $m \neq 1$, then it is locally isometric to the product space $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

1. Introduction and main results

A classical problem in Riemannian geometry is the search for Einstein metrics, which are often regarded as "best" metrics, since metrics of constant curvature belong to this class. So, Einstein metrics and their generalizations are very interesting in both differential geometry and mathematical physics. In recent literature, we find the study of Riemannian manifolds admitting several natural generalizations of Einstein metrics. These include (gradient) Ricci solitons [5], (gradient) almost Ricci solitons [2, 29], (gradient) m-quasi-Einstein metrics [3, 7, 24], etc.

A Riemannian manifold (M^n, g) of dimension $n \geq 3$ is said to be *m-quasi-Einstein manifold* if there exist a vector field $V \in \mathfrak{X}(M)$, and a real constant λ, satisfying

$$\mathrm{Ric}_g + \frac{1}{2}\pounds_V g - \frac{1}{m} V^\flat \otimes V^\flat = \lambda\, g, \tag{1.1}$$

where $0 < m \leq \infty$ and $\pounds_V$ denotes the Lie derivative operator along V, Ric_g is the Ricci tensor of g, and $V^\flat$ is the 1-form metrically dual to the vector field V. It was introduced and studied by Barros-Ribeiro [3] and Limoncu [24]. An m-quasi-Einstein structure is denoted by (g, V, m, λ). It is important to point out that if $m = \infty$, Equation (1.1) reduces a *Ricci soliton*. So, it is interesting to study the m-quasi-Einstein manifold as a generalization of the Ricci soliton. The Ricci soliton metrics play a very important role in Hamilton's Ricci flow as they correspond to the self-similar solutions and often arise as singularity models (see details [5]). Moreover, an m-quasi-Einstein manifold is called *trivial* if the vector field V is identically zero on M^n, and the triviality condition implies that M^n is an *Einstein manifold*: $\mathrm{Ric}_g = \lambda\, g$, otherwise it will be non-trivial. An m-quasi-Einstein structure (g, V, m, λ) is called a *closed m-quasi-Einstein structure* if the 1-form $V^\flat$ is closed: $d\,V^\flat = 0$. Further, Barros-Gomes [1] studied the Equation (1.1) and proved that *"if a compact m-quasi-Einstein manifold* (M^n, g), $n \geq 3$, *is Einstein, then V vanishes identically."* On the other hand, we also notice that m-quasi-Einstein structures were considered on the odd-dimensional Riemannian manifolds, in particular, contact metric manifolds by Ghosh [19] and almost cosymplectic manifolds by Chen [11].

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In particular, if V is the gradient of a smooth function f on M^n, then the m-quasi-Einstein manifold is called a *(gradient) m-quasi-Einstein manifold*, which was introduced and studied by Case et al. [7]. In this case, (1.1) reduces to

$$\mathrm{Ric}_g + \mathrm{Hess}_f - \frac{1}{m} df \otimes df = \lambda\, g, \tag{1.2}$$

where Hess_f denotes the *Hessian operator of* f. This class appears from the *m-Bakry-Emery-Ricci-tensor* Ric_f^m defined on (M^n, g) by $\mathrm{Ric}_f^m = \mathrm{Ric}_g + \mathrm{Hess}_f - \frac{1}{m} df \otimes df$, which is a natural extension of the Ricci tensor to smooth metric measure space and have natural geometric interpretations using warped product Einstein manifolds. It is important to point out that if $m = \infty$ and f constant, then Equation (1.2) reduces a *gradient Ricci soliton* (details in [5]) and an *Einstein equation,* respectively. On the other hand, when m is a positive integer, it corresponds to a warped product Einstein metrics (see [6]). We also remark that 1-quasi-Einstein metrics are more commonly called *static metrics,* and such metrics have connections to scalar curvature, the positive mass theorem, and general relativity. In recent years, (gradient) m-quasi-Einstein manifolds have been extensively studied by many mathematicians, for instance, [1, 3, 7–9, 14, 20, 26, 30].

The notion of a warped product is very popular in differential geometry as well as in general relativity, for example, [10]. Some solutions of Einstein field equations are warped products, and some space-time models, for example, Robertson–Walker space-time, Schwarzschild space-time, Reissner–Nordström space-time, and asymptotically flat space-time, are warped product manifolds (see [22] for more details). In [23], Kenmotsu first introduced and studied a special class of non-compact almost contact metric manifold, known as Kenmotsu manifold, which is characterized by warped product. The warped product $\mathbb{R} \times_\sigma N^{2n}$ (of the real line $\mathbb{R}$ and a Kähler manifold N^{2n}) with the warping function

$$\boldsymbol{\sigma}(t) = ce^t, \tag{1.3}$$

given on an open interval $J = (-\varepsilon, \varepsilon)$ admits Kenmotsu structure, and conversely, every point of a Kenmotsu manifold has a neighborhood, which is locally a warped product $J \times_\sigma N^{2n}$, where σ is given by equation (1.3).

Many authors studied Ricci solitons, almost Ricci solitons, and their generalizations in the framework of almost contact manifolds (see details [12, 13, 17, 18, 25–27]). First, Ghosh considered an almost contact metric, in particular, a Kenmotsu metric as Ricci soliton and proved that *"Kenmotsu 3-metric as a Ricci soliton is of constant negative curvature* -1 *in [18] and an* η*-Einstein Kenmotsu metric as a Ricci soliton is Einstein for higher dimension in [17].*" It is known that the warped product of real line $\mathbb{R}$ and an almost Kähler manifold N^{2n} carries almost Kenmotsu metric, see details in [16]. On the other hand, m-quasi-Einstein metric arises from the certain warped product of two Riemannian manifolds. In this regard, it is interesting to consider non-gradient m-quasi-Einstein structure on Kenmotsu and almost Kenmotsu manifolds.

On the other hand, contact metric manifolds admitting gradient m-quasi-Einstein structure, m-quasi-Einstein structure, and its generalizations have been studied by many researchers, and many achievements have been made, see [11, 19, 20]. Here, our goal is to consider m-quasi-Einstein Kenmotsu and almost Kenmotsu manifolds and to prove some fundamental results using local Riemannian geometry that characterize Einstein metrics in some of the broader classes of metrics mentioned above. The organization of the paper is as follows. In Section 2, we provide some preliminaries on Kenmotsu and almost Kenmotsu manifolds. Next, we consider an m-quasi-Einstein structure on Kenmotsu manifolds in Section 3 and on almost Kenmotsu manifolds in Section 4.

2. Preliminaries

In this section, we shall present some preliminaries which will be used during the paper ([4, 15, 16, 23]).

An *almost contact structure* on smooth manifold M (or M^{2n+1}) of dimension $(2n+1)$ is a triplet (φ, ξ, η), where φ is a $(1,1)$-tensor field, ξ a global vector field (called Reeb vector field), and η a 1-form, satisfying

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$$\varphi^2 = -id + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where *id* denotes the identity endomorphism. A smooth manifold M with such a structure is called an *almost contact manifold*. It follows from equation (2.1) that (see details [4]) $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$, $\text{rank}(\varphi) = 2n$. A Riemannian metric g on M is said to be *compatible* with an almost contact structure (φ, ξ, η) if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\,\eta(Y), \quad X, Y \in \mathfrak{X}(M),$$

where $\mathfrak{X}(M)$ is the Lie algebra of all vector fields on M. An almost contact manifold endowed with a compatible Riemannian metric g is said to be an *almost contact metric manifold* and is denoted by $M^{2n+1}(\varphi, \xi, \eta, g)$. The *fundamental 2-form* Φ on M is defined by $\Phi(X, Y) = g(X, \varphi Y)$ for any $X \in \mathfrak{X}(M)$. An almost contact metric manifold satisfying $d\eta = 0$ and $d\Phi = 2\,\eta \wedge \Phi$ is said to be an *almost Kenmotsu manifold* (e.g., [15, 16]). On almost Kenmotsu manifold, we define two tensor fields $h = \frac{1}{2}\pounds_\xi \varphi$ and $h' = h \circ \varphi$. They satisfy (see details [15, 16])

$$h\xi = 0 = h'\xi, \quad \text{Tr}_g(h) = \text{Tr}_g(h') = 0, \quad h\varphi + \varphi h = 0,$$

and for any $X \in \mathfrak{X}(M)$

$$\nabla_X \xi = X + h'X - \eta(X)\xi, \tag{2.2}$$

where ∇ is the *Levi-Civita connection* of g. An almost contact metric structure on M is said to be *normal* if the almost complex structure J on $M \times \mathbb{R}$ defined by

$$J(X, f\,d/dt) = (\varphi X - f\xi, \eta(X)\,d/dt), \quad X \in \mathfrak{X}(M),$$

where t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$; is *integrable*, or the tensor $N_\varphi = [\varphi, \varphi] + 2\,d\eta \otimes \xi$ vanishes on M, where $[\varphi, \varphi]$ denotes the *Nijenhuis tensor* of φ. A normal almost Kenmotsu manifold is said to be a *Kenmotsu manifold* (see [23]); equivalently, an almost contact metric manifold is said to be a *Kenmotsu manifold* (see [23]) if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\,\xi - \eta(Y)\,\varphi X, \quad X, Y \in \mathfrak{X}(M). \tag{2.3}$$

The following formulas hold on Kenmotsu manifolds, see [23]:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.5}$$

$$Q\xi = -2n\xi \tag{2.6}$$

for any $X, Y \in \mathfrak{X}(M)$, where R denotes the *curvature tensor* and Q is the *Ricci operator* associated with the Ricci tensor Ric_g given by $\text{Ric}_g(X, Y) = g(QX, Y)$ for $X, Y \in \mathfrak{X}(M)$. Taking g-trace of the formula (see [21]): $(\nabla_\xi Q)X = -2\,QX - 4nX$, we get $\xi(r) = -2(r + 2n(2n + 1))$.

Lemma 2.1. *The Ricci tensor for n-dimensional manifold M^n that admits a closed m-quasi-Einstein structure satisfies the following expression:*

$$\left(\frac{m-1}{m}\right)\text{Ric}_g(X, V) = \frac{1}{2}X(r) - \left\{\frac{r - (n-1)\lambda}{m}\right\}V^\flat(X), \quad X \in \mathfrak{X}(M^n). \tag{2.7}$$

Proof. As $V^\flat$ is closed, equation (1.1) can be exhibited as:

$$\nabla_X V + QX = \frac{1}{m}V^\flat(X)V + \lambda X, \quad X \in \mathfrak{X}(M^n). \tag{2.8}$$

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Applying this in the well-known formula $R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$ of curvature tensor, we can compute the following

$$R(X,Y)V=(\nabla_Y Q)X-(\nabla_X Q)Y-\frac{1}{m}\Big\{V^\flat(Y)QX-V^\flat(X)QY\Big\}$$
$$+\frac{\lambda}{m}\Big\{V^\flat(Y)X-V^\flat(X)Y\Big\},\quad X,Y\in\mathfrak{X}(M^n). \tag{2.9}$$

Now contracting equation (2.9) over Y with respect to the local orthonormal basis $\{E_i\}_{1\le i\le n}$ of tangent space at each point of M^n, we obtain

$$\mathrm{Ric}_g(X,V)=\sum_{i=1}^{n} g\Big((\nabla_X Q)E_i-(\nabla_{E_i}Q)X,E_i\Big)$$
$$+\frac{1}{m}\Big\{g(V,QX)-r\,V^\flat(X)\Big\}+\frac{(n-1)\lambda}{m}V^\flat(X) \tag{2.10}$$

for any $X\in\mathfrak{X}(M^n)$. Next contracting second Bianchi's identity, we acquire the formula $(\mathrm{div}\,Q)(X)=\frac{1}{2}X(r)$. Plugging it into equation (2.10), we conclude the required result. □

3. On non-gradient m-quasi-Einstein Kenmotsu manifolds

In this section, we study the non-gradient m-quasi-Einstein structure on Kenmotsu manifolds. Before presenting our main results, we recall a definition: a contact metric manifold M is said to be *η-Einstein*, if the Ricci tensor Ric_g has of the form

$$\mathrm{Ric}_g=a\,g+b\,\eta\otimes\eta,$$

where a and b are smooth functions on M. Now, we can easily verify that an η-Einstein Kenmotsu manifolds satisfies non-gradient m-quasi-Einstein structure with $V=\alpha\xi$ for some constant α, $\alpha(\alpha+m)=mb$ and a,b positive constants. Indeed, covariant derivative of $V=\alpha\xi$ along any $X\in\mathfrak{X}(M)$ and using equation (2.4) gives us

$$\nabla_X V=\alpha(X-\eta(X)\xi),$$

and therefore, $(\pounds_V g)(X,Y)=2\alpha\big\{g(X,Y)-\eta(X)\eta(Y)\big\}$. By virtue of this and η-Einstein condition, we directly write

$$\mathrm{Ric}_g(X,Y)+\frac{1}{2}\pounds_V g(X,Y)-\frac{1}{m}V^\flat(X)V^\flat(Y)$$
$$=(\alpha+a)\,g(X,Y)+b\,\eta(X)\eta(Y)-\alpha\left(1+\frac{\alpha}{m}\right)\eta(X)\eta(Y) \tag{3.1}$$

for all $X,Y\in\mathfrak{X}(M)$. This shows that if we take $\alpha(\alpha+m)=mb$, $b>0$, then we can say that M admits a non-gradient m-quasi-Einstein structure with $\lambda=(\alpha+a)$.

There is a natural question:

Under which geometric or analytical conditions does a Riemannian manifold, or an almost contact manifold, satisfying a given soliton equation necessarily become an Einstein manifold?

Several answers to this question for Riemannian and almost contact manifolds can be found in [17, 18, 25–27, 31]. An m-quasi-Einstein manifold is a natural generalization of Ricci soliton, and therefore, the following question arises:

Under what geometric conditions does an m-quasi-Einstein almost contact manifold necessarily become Einstein?

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Recently, Ghosh [19] found such conditions on contact metric manifolds. In this context, we are interested in characterizing those m-quasi-Einstein almost contact metrics that are Einstein. To this end, we prove the following result.

Theorem 3.1. *If (g, V, m, λ), $m \neq 1$, is a closed m-quasi-Einstein structure on Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ with $\eta(V) + m \neq 0$, then (M, g) is an Einstein manifold with Einstein constant $-2n$.*

Proof. Taking covariant derivative of equation (2.6) along $X \in \mathfrak{X}(M)$ and using equation (2.4), we acquire $(\nabla_X Q)\xi = -QX - 2nX$. Applying this and equation (2.5) in the scalar product of equation (2.9) with ξ yields

$$\frac{(\lambda + 2n + m)}{m}\Big\{V^\flat(X)\eta(Y) - V^\flat(Y)\eta(X)\Big\} = 0$$

for all $X, Y \in \mathfrak{X}(M)$. Now substituting ξ for Y in preceding equation, we have

$$\frac{(\lambda + 2n + m)}{m}\Big\{V^\flat(X) - \eta(V)\eta(X)\Big\} = 0, \quad X \in \mathfrak{X}(M). \tag{3.2}$$

As λ and m are constants, equation (3.2) implies that either (I) $V = \eta(V)\xi$ or (II) $\lambda + 2n + m = 0$.

Case I: In this case, first taking covariant derivative of $V = \eta(V)\xi$ along $X \in \mathfrak{X}(M)$ and using equation (2.4), we get

$$\nabla_X V = \eta(\nabla_X V)\xi + V^\flat(X)\xi + \eta(V)X - 2\eta(V)\eta(X)\xi. \tag{3.3}$$

Applying equation (2.8) into equation (3.3) yields

$$QX + \frac{1}{m}\Big\{V^\flat(X)\eta(V)\xi - V^\flat(X)V\Big\} + V^\flat(X)\xi + \eta(V)X$$
$$= \lambda X - (\lambda + 2n)\,\eta(X)\xi + 2\,\eta(V)\eta(X)\xi$$

for any $X \in \mathfrak{X}(M)$. Again recalling $V = \eta(V)\xi$, then last equation immediately reduces to

$$QX = (\lambda - \eta(V))X - (\lambda + 2n - \eta(V))\,\eta(X)\xi, \quad X \in \mathfrak{X}(M). \tag{3.4}$$

Next contracting this, we achieve

$$r + 2n\,\eta(V) = 2n\,(\lambda - 1). \tag{3.5}$$

In view of $\nabla_\xi \xi = 0$, which follows from equation (2.4), we deduce

$$\xi(r) + 2n\,\eta(\nabla_\xi V) = 0. \tag{3.6}$$

Thus, substituting ξ for X in equation (2.8) and then taking scalar product with ξ, we get

$$\xi(r) + 2n\Big\{\lambda + 2n + \frac{1}{m}\{\eta(V)\}^2\Big\} = 0. \tag{3.7}$$

On the other hand, recalling $\xi(r) = -2(r + 2n(2n + 1))$ and taking the covariant derivative of (3.7) along ξ, we have $\xi(r) = \frac{2n}{m}\,\eta(V)\eta(\nabla_\xi V)$, where we have used that $\nabla_\xi \xi = 0$. Obviously, comparing this with (3.6), we arrive at

$$\frac{2n}{m}\Big\{\eta(V) + m\Big\}\eta(\nabla_\xi V) = 0. \tag{3.8}$$

Since $m, n > 0$, by hypothesis, Equation (3.8) implies that $\eta(\nabla_\xi V) = 0$. Applying this into (3.6) yields $\xi(r) = 0$, and therefore, $r = -2n(2n + 1)$. It follows directly from (3.5) that $\eta(V) = \lambda + 2n$. Plugging this into (3.4), by a direct calculation, we conclude that (M, g) is an Einstein manifold with Einstein constant $-2n$.

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Case II: In this case, replacing X by ξ in equation (2.7) and using equation (2.6), we acquire

$$(r + 2n(2n+1))[\eta(V) + m] = 0, \tag{3.9}$$

where we have used $\xi(r) = -2(r + 2n(2n+1))$. By hypothesis, this implies that $r = -2n(2n+1)$. In view of this, Equation (2.7) becomes $(m-1)(QV + 2n\,V) = 0$, where we have used the fact that r is constant. For $m \neq 1$, this directly implies that $QV + 2n\,V = 0$. Taking its covariant derivative along $X \in \mathfrak{X}(M)$ and recalling equation (2.8), we obtain

$$(\nabla_X Q)V - Q^2X + \frac{1}{m}\,V^\flat(X)QV + (\lambda - 2n)\,QX + \frac{2n}{m}\,V^\flat(X)V + 2n\,\lambda X = 0.$$

By virtue of $QV = -2nV$, the foregoing equation becomes

$$(\nabla_X Q)V - Q^2X + (\lambda - 2n)\,QX + 2n\lambda\,X = 0, \quad X \in \mathfrak{X}(M). \tag{3.10}$$

At this point, contracting equation (3.10) over X with respect to the orthonormal basis $\{E_i\}_{1\leq i\leq 2n+1}$ of tangent space at each point of M and using $r = -2n(2n+1)$, we obtain

$$\sum_{i=1}^{2n+1} g((\nabla_{E_i} Q)V, E_i) - ||Q||^2 - 2n\,r = 0. \tag{3.11}$$

Since the scalar curvature $r = -2n(2n+1)$ is constant, it follows from the formula: $\mathrm{Tr}_g\{Y \to (\nabla_Y Q)X\} = \frac{1}{2}\,X(r)$ and equation (3.11) that $||Q||^2 = -2n\,r$. By virtue of this and $r = -2n(2n+1)$, we compute

$$||Q + 2n\,I||^2 = ||Q||^2 + 4n\,r + 4n^2(2n+1) = 0.$$

Since the length of the symmetric tensor $Q + 2n\,I$ vanishes, we must have $\mathrm{Ric}_g = -2n\,g$. Hence, (M, g) is Einstein with Einstein constant $-2n$, as required. □

Corollary 3.1. *If a Kenmotsu manifold admitting a closed m-quasi-Einstein structure (g, V, m, λ) with $V = \eta(V)\xi$, then g is an Einstein manifold with Einstein constant $-2n$.*

Remark 3.1. *If the 1-form $V^\flat$ is closed, then equation (1.1) reduces to the form of equation (1.2) with $df = V^\flat$, and therefore, closed m-quasi-Einstein structure is an important generalization of gradient Ricci soliton.*

Next, we consider the potential vector field V being collinear with Reeb vector field ξ and prove the following.

Theorem 3.2. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a Kenmotsu manifold admitting a non-gradient m-quasi-Einstein structure (g, V, m, λ), where the vector field V is collinear with the Reeb vector field ξ. Then M is an η-Einstein manifold with constant scalar curvature $2n(\lambda - 1)$.*

Proof. As V is collinear with ξ, that is, $V = f\xi$ for some smooth function f on M. Making use of equation (2.4) in the covariant derivative of $V = f\xi$ along X yields $\nabla_X V = X(f)\xi + f(X - \eta(X)\xi)$. This directly implies that

$$(\pounds_V g)(X, Y) = X(f)\,\eta(Y) + Y(f)\,\eta(X) - 2f\,\eta(X)\eta(Y)$$

for all $X, Y \in \mathfrak{X}(M)$. Thus, Equation (1.1) reduces to

$$\mathrm{Ric}_g(X, Y) + \frac{1}{2}\Big\{X(f)\,\eta(Y) + Y(f)\,\eta(X)\Big\} = f\left(1 + \frac{f}{m}\right)\eta(X)\eta(Y) + \lambda\,g(X, Y) \tag{3.12}$$

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for all $X, Y \in \mathfrak{X}(M)$. Now replacing Y by ξ in equation (3.12) and noting that equation (2.6), we get

$$X(f) + \xi(f)\,\eta(X) = 2\left\{\lambda + 2n + f\left(1 + \frac{f}{m}\right)\right\}\eta(X) \tag{3.13}$$

for any $X \in \mathfrak{X}(M)$. Again, replacing X by ξ in (3.13) yields $\xi(f) = \lambda + 2n + f(1 + \frac{f}{m})$. Plugging it into (3.13), we acquire $X(f) = 2\left\{\lambda + 2n + f(1 + \frac{f}{m})\right\}\eta(X)$. By virtue of this, Equation (3.12) becomes

$$\mathrm{Ric}_g = \lambda\, g - (\lambda + 2n)\,\eta \otimes \eta, \tag{3.14}$$

which follows that (M, g) is an η-Einstein manifold with scalar curvature $2n(\lambda - 1)$ – a constant. This finishes the proof. □

Corollary 3.2. *If a Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ admitting a non-gradient m-quasi-Einstein structure (g, V, m, λ) with $V = \eta(V)\xi$, then M is an η-Einstein manifold with constant scalar curvature $2n(\lambda - 1)$.*

In the following, we study the three-dimensional m-quasi-Einstein Kenmotsu manifold with V as a conformal vector field.

Theorem 3.3. *If a Kenmotsu manifold $M^3(\varphi, \xi, \eta, g)$ admits a non-gradient m-quasi-Einstein structure with V as a conformal vector field, then M has constant sectional curvature -1, provided that $r \neq 2(\lambda - \rho - 1)$.*

Proof. Taking into account that V is a conformal vector field, we have $(\pounds_V g)(X, Y) = 2\rho\, g(X, Y)$ for any $X, Y \in \mathfrak{X}(M)$ and some smooth function ρ on M. This reduces Equation (1.1) to the following form:

$$QX = (\lambda - \rho)X + \frac{1}{m}V^\flat(X)V, \quad X \in \mathfrak{X}(M). \tag{3.15}$$

Recalling the well-known curvature tensor expression on 3-dimensional Riemannian manifold:

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y \\ & - \frac{r}{2}\Big\{g(Y, Z)X - g(X, Z)Y\Big\} \end{aligned} \tag{3.16}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. By virtue of equation (3.15), Equation (3.16) can be written as

$$\begin{aligned} R(X, Y)Z = {} & \frac{1}{m}\Big\{V^\flat(X)g(Y, Z) - V^\flat(Y)g(X, Z)\Big\}V + \frac{1}{m}V^\flat(Z)\Big\{V^\flat(Y)X - V^\flat(X)Y\Big\} \\ & + \frac{(4(\lambda - \rho) - r)}{2}\Big\{g(Y, Z)X - g(X, Z)Y\Big\}. \end{aligned} \tag{3.17}$$

Now replacing Z by ξ in equation (3.17) and using equation (2.5), we get

$$\begin{aligned} & \frac{1}{m}\Big\{V^\flat(X)\eta(Y) - V^\flat(Y)\eta(X)\Big\}V + \frac{1}{m}\eta(V)\Big\{V^\flat(Y)X - V^\flat(X)Y\Big\} \\ & \qquad = \frac{1}{2}(r - 4(\lambda - \rho) - 2)\Big\{\eta(Y)X - \eta(X)Y\Big\} \end{aligned} \tag{3.18}$$

At this point, tracing equation (3.15) gives $\frac{||V||^2}{m} = r - 3(\lambda - \rho)$. In view of this, inserting $X = V$ and replacing Y by ξ in equation (3.18), we obtain

$$\big(r - 2(\lambda - \rho - 1)\big)\Big\{V - \eta(V)\xi\Big\} = 0. \tag{3.19}$$

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Let us assume $V = \eta(V)\xi = f\xi$, where $f = \eta(V)$. Conveniently differentiating this along $X \in \mathfrak{X}(M)$ and recalling equation (2.4) yields $\nabla_X V = (Xf)\xi + f(X - \eta(X)\xi)$. As V is conformal, this entails that

$$X(f)\,\eta(Y) + Y(f)\,\eta(X) - 2f\,\eta(X)\eta(Y) = 2\rho\,g(X, Y). \tag{3.20}$$

Choosing X and Y orthogonal to ξ in the foregoing equation, we acquire $\rho = 0$. Thus, by virtue of $V = \eta(V)\xi$, Equation (3.15) immediately reduces to

$$QX = \lambda X + \frac{1}{m}\{\eta(V)\}^2\,\eta(X)\xi, \quad X \in \mathfrak{X}(M). \tag{3.21}$$

Contracting this yields $r = 3\lambda + \frac{1}{m}\{\eta(V)\}^2$. Differentiating this along ξ and noting that $\nabla_\xi \xi = 0$, we achieve

$$\xi(r) = \frac{2}{m}\eta(V)\eta(\nabla_\xi V). \tag{3.22}$$

Since V is conformal, it follows that

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) = 2\rho\,g(X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$. This implies that $\eta(\nabla_\xi V) = \rho = 0$. Thus, Equation (3.22) follows $\xi(r) = 0$. On the other hand, recalling $\xi(r) = -2(r + 2n(2n + 1))$, and therefore, $\xi(r) = -2(r + 6)$ for dimension 3. Hence, the scalar curvature $r = -6$. Further, inserting $Y = Z = \xi$ in equation (3.16) and using equation (2.6), we have

$$QX = \left(1 + \frac{r}{2}\right)X - \left(3 + \frac{r}{2}\right)\eta(X)\xi, \quad X \in \mathfrak{X}(M). \tag{3.23}$$

Now using $r = -6$ in equation (3.23), we acquire $QX = -2X$. Plugging it into equation (3.16) implies that M is of constant negative curvature -1. This concludes the proof. □

Corollary 3.3. *If a Kenmotsu manifold $M^3(\varphi, \xi, \eta, g)$ admits a non-gradient m-quasi-Einstein structure with V a conformal vector field with $V = \eta(V)\xi$, then M has constant sectional curvature -1.*

4. On non-gradient m-quasi-Einstein almost Kenmotsu manifolds

In this section, we study the geometry of non-gradient m-quasi-Einstein structure on almost Kenmotsu $(k, \mu)'$-manifold, which yields a wider class. We recall some preliminaries. An almost Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be an almost Kenmotsu $(k, \mu)'$-manifold if ξ belongs to the (k, μ)-nullity distribution, that is,

$$R(X, Y)\xi = k\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)h'X - \eta(X)h'Y\} \tag{4.1}$$

for any $X, Y \in \mathfrak{X}(M)$, where $h' = ho\varphi$ and k and μ are real constants. Classifications of almost Kenmotsu manifolds with ξ belonging to $(k, \mu)'$-nullity distribution have been carried out by several authors. For more details, we refer the reader to [15, 16, 32]. On an almost Kenmotsu $(k, \mu)'$-manifold the following formulas hold, see [16]:

$$h'^2 = (k + 1)\varphi^2 \;\Leftrightarrow\; h^2 = (k + 1)\varphi^2, \tag{4.2}$$

$$Q\xi = 2nk\xi. \tag{4.3}$$

According to the relation equation (4.2), we have $k \leq -1$ and the equality holds if and only if $h = 0$. In Section 3, we studied m-quasi-Einstein structure on Kenmotsu manifolds (on which $h = 0$); therefore, in this section, we mainly consider the non-Kenmotsu case for an almost Kenmotsu $(k, \mu)'$-manifold, that is, $k < -1$, or equivalently, $h \neq 0$ everywhere. Now recalling a lemma where Wang-Liu [32] obtained the expression of Ricci operator on non-Kenmotsu almost Kenmotsu $(k, \mu)'$-manifold.

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Lemma 4.1. *(Lemma 3.3 of [32]) Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a non-Kenmotsu almost Kenmotsu $(k, \mu)'$-manifold. Then $\mu = -2$ and the Ricci operator Q of M is given by*

$$QX = -2nX + 2n(k+1)\eta(X)\xi - 2nh'X \tag{4.4}$$

for any $X \in \mathfrak{X}(M)$. In this case, the scalar curvature of M is $2n(k-2n)$.

Remark 4.1. *We note that an Einstein almost Kenmotsu manifold satisfying the hypothesis of Lemma 4.1 do not exist. Indeed, if M is an Einstein manifold, then $\mathrm{Ric}_g = \frac{r}{2n+1} g$. Since $Q\xi = 2nk\xi$ and the scalar curvature is $2n(k-2n)$, we get $k = -1$, and hence, $h' = 0$. This contradicts the hypothesis $h' \neq 0$. Thus, Einstein almost Kenmotsu manifold satisfying a nullity condition with $h' \neq 0$ do not exist.*

Theorem 4.1. *If a non-Kenmotsu almost Kenmotsu $(k, \mu)'$-manifold admitting a closed m-quasi-Einstein structure (g, V, m, λ) with $m \neq 1$, then the manifold is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 with a flat n-dimensional manifold and $\lambda = -4n$.*

Proof. First taking inner product of equation (2.9) with ξ and using equation (4.3) gives that

$$\begin{aligned} g(R(X,Y)V,\xi) =& g((\nabla_Y Q)X - (\nabla_X Q)Y, \xi) \\ & - \left(\frac{\lambda - 2nk}{m}\right)\left\{V^\flat(X)\eta(Y) - V^\flat(Y)(\eta(X)\right\} \end{aligned} \tag{4.5}$$

for any $X, Y \in \mathfrak{X}(M)$. Taking covariant derivative of equation (4.3) along any $X \in \mathfrak{X}(M)$ and using equation (2.2), we have $(\nabla_X Q)\xi = 2nk\,(X + h'X) - Q(X + h'X)$. Thus, using equation (4.4), we get

$$(\nabla_X Q)\xi = 2n\Big\{k+1)X + (k+2)h'X + h'^2X - (k+1)\eta(X)\xi\Big\}, \tag{4.6}$$

where we have used $h'\xi = 0$. Applying equations (4.2) and (2.1) in equation (4.6), we obtain $(\nabla_X Q)\xi = 2n(k+2)\,h'X$. This transform the equation (4.5) into the form

$$g(R(X,Y)V,\xi) = \left(\frac{2nk - \lambda}{m}\right)\left\{V^\flat(X)\eta(Y) - V^\flat(Y)\eta(X)\right\}, \tag{4.7}$$

by using the symmetric property of Q and h'. Now, using equation (4.1) in equation (4.7), we get

$$\left(\frac{\lambda - 2nk - mk}{m}\right)\left\{V^\flat(X)\eta(Y) - V^\flat(Y)\eta(X)\right\} = \mu\left\{V^\flat(h'X)\eta(Y) - V^\flat(h'Y)\eta(X)\right\} \tag{4.8}$$

for any $X, Y \in \mathfrak{X}(M)$. Next substituting ξ for Y in equation (4.8) and using $h'\xi = 0$, we acquire

$$\mu V^\flat(h'X) = \sigma\left\{V^\flat(X) - \eta(V)(\eta(X)\right\}, \tag{4.9}$$

where $\sigma = \left(\frac{\lambda - 2nk - mk}{m}\right)$. Then inserting $X = h'X$ in equation (4.9) and noting that equations (4.2) and (2.1) yields

$$\left(\mu^2(k+1) + \sigma^2\right)\left\{V^\flat(X) - \eta(V)(\eta(X)\right\} = 0, \quad X \in \mathfrak{X}(M). \tag{4.10}$$

Since λ, k, and m are constants, Equation (4.10) shows that either $(I)\ V^\flat(X) = \eta(V)(\eta(X)$ or $(II)\ \mu^2 (k+1) + \sigma^2 = 0$.

Case I: In this case, we have $V = \eta(V)\xi$, and therefore, with the aid of equation (2.2), we get

$$\begin{aligned} \nabla_X V =& \eta(\nabla_X V)\xi + V^\flat(X)\xi \\ & + V^\flat(h'X)\xi + \eta(V)\,(X + h'X) - 2\,\eta(V)\eta(X)\xi\,. \end{aligned}$$

In view of equations (2.8) and (4.3), the foregoing equation reduces to

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$$\mathrm{Ric}_g(X,Y)-\frac{1}{m}V^\flat(X)\Big\{V^\flat(Y)-\eta(V)\eta(Y)\Big\}+V^\flat(X+h'X)\,\eta(Y)$$
$$+\eta(V)\,g(X+h'X,Y)=\lambda\,g(X,Y)-(\lambda-2nk-2\,\eta(V))\,\eta(X)\eta(Y) \tag{4.11}$$

for any $X, Y \in \mathfrak{X}(M)$. Contracting this and using $r = 2n(k-2n)$ (from Lemma 4.1), we get $\eta(V) = (\lambda + 2n)$, and therefore, $V = (\lambda + 2n)\xi$, which reduces the equality equation (4.11) to the following form:

$$\mathrm{Ric}_g(X,Y)+(\lambda+2n)\,g(h'X,Y)=-2n\,g(X,Y)+2n(k+1)\,\eta(X)\eta(Y), \tag{4.12}$$

where we have used $h'\xi = 0$. Plugging the value of QX from equation (4.4) in equation (4.12) gives $\lambda h'X = 0$ for any $X \in \mathfrak{X}(M)$. Since M is a non-Kenmotsu almost Kenmotsu $(k,\mu)'$-manifold, this implies that $\lambda = 0$, and therefore, $V = 2n\xi$. Differentiating this conveniently along any $X \in \mathfrak{X}(M)$ and using equation (2.2) provides $\nabla_X V = 2n(X + h'X - \eta(X)\xi)$. This gives $\mathrm{div}\, V = 4n^2$, as $\mathrm{Tr}_g(h') = 0$. Further, contracting equation (2.8), we get

$$\mathrm{div}\, V = -r + \frac{||V||^2}{m} + \lambda(2n+1) = -r + \frac{4n^2}{m} + \lambda(2n+1),$$

by using $V = 2n\xi$. Comparing the two values of $\mathrm{div}\, V$ and using $r = 2n(k-2n)$ and $\lambda = 0$, we get $2n\left(k - \frac{2n}{m}\right) = 0$. Since $m, n > 0$, this shows that $k = \frac{2n}{m} > 0$, which contradicts the notion $k < -1$.

Case II: This case gives us

$$\mu^2(k+1) + \sigma^2 = 0. \tag{4.13}$$

Since scalar curvature $r = 2n(k-2n)$ on almost Kenmotsu $(k,\mu)'$-manifold is constant, so $X(r) = 0$. This directly reduces equation (2.7) to the form:

$$(m-1)\,\mathrm{Ric}_g(V,X) = 2n(\lambda - k + 2n)\,V^\flat(X), \quad X \in \mathfrak{X}(M). \tag{4.14}$$

Now plugging the value of QX from equation (4.4) in previous equality, we get

$$(m-1)\Big\{(k+1)\eta(V)\eta(X) - V^\flat(h'X)\Big\} = (\lambda - k + m + 2n - 1)\,V^\flat(X) \tag{4.15}$$

for any $X \in \mathfrak{X}(M)$. Next substituting ξ for X in equation (4.15) and using $h'\xi = 0$, we achieve

$$(mk - \lambda - 2n)\,\eta(V) = 0. \tag{4.16}$$

If possible, let $mk - \lambda - 2n = 0$. Then Equation (4.15) reduces to

$$V^\flat(h'X) + (k+1)\Big\{V^\flat(X) - \eta(V)\eta(X)\Big\} = 0, \tag{4.17}$$

as $m \neq 1$. Combining this with equation (4.9) and recalling $mk - \lambda - 2n = 0$, we acquire

$$(m+n)(k+1)\Big\{V^\flat(X) - \eta(V)\eta(X)\Big\} = 0. \tag{4.18}$$

As M is non-Kenmotsu and $m > 0$, it follows that $V^\flat(X) = \eta(V)\eta(X)$, which leads to a contradiction, by **Case (I)**. Thus, Equation (4.16) implies that $\eta(V) = 0$. It directly follows from equation (2.2) that $\eta(\nabla_\xi V) = 0$, and therefore, results of equations (2.8) and (4.3) give $\lambda = 2nk$. Using this and $\sigma = \left(\frac{\lambda - 2nk - mk}{m}\right)$ in equation (4.13), we obtain $m^2(k+2)^2 = 0$, because $\mu = -2$ and M^{2n+1} is a non-Kenmotsu manifold, consequently, $k = -2$ and $\lambda = -4n$. Thus, Equation (4.2) becomes $h'^2 = -\varphi^2$. Hence, by Proposition 4.1 and Corollary 4.2 of [16], we can conclude that M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold. □

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In particular, we can state the following corollary.

Corollary 4.1. *If a non-Kenmotsu almost Kenmotsu* $(k, \mu)'$*-manifold admitting a gradient m-quasi-Einstein structure* (g, Df, m, λ) *with* $m \neq 1$*, then* $\lambda = -4n$ *and the manifold is locally isometric to the product space* $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Note that Ricci operator on a generalized almost Kenmotsu $(k, \mu)'$-manifold $M^{2n+1}(\varphi, \xi, \eta, g)$, $n > 1$, with $h' \neq 0$, can be expressed as (Lemma 3.4 of [32]):

$$QX = -2nX + 2n(k+1)\eta(X)\xi - 2(n-1)h'X + \mu hX \tag{4.19}$$

for any $X \in \mathfrak{X}(M)$ and the scalar curvature is $2n(k-2n)$. Thus, generalized almost Kenmotsu $(k, \mu)'$-manifold with constant scalar curvature shows that k is constant. Further, Proposition 3.2 of [28] gives us the relation $\xi(k) = -2(k+1)(\mu+2)$, and therefore, $\mu = -2$ for non-Kenmotsu generalized almost Kenmotsu $(k, \mu)'$-manifold. Thus, by Theorem 4.1, we can deduce the following.

Theorem 4.2. *Let a non-Kenmotsu generalized almost Kenmotsu* $(k, \mu)'$*-manifold admitting a closed m-quasi-Einstein structure* (g, V, m, λ) *with* $m \neq 1$*. If g have constant scalar curvature, then* $\lambda = -4n$ *and the manifold is locally isometric to the Riemannian product space* $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

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