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LIQUIDITY EXTERNALITIES AND THE WALLACE CONJECTURE

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This paper presents a pure currency economy with a nondegenerate distribution of money holdings in which, as conjectured by Wallace (*Quarterly Journal of Economics* 129, 259–274, 2014), there are transfer schemes financed by money creation that improve ex ante welfare relative to no-intervention. Differently from what was advocated by Wallace, pecuniary-like externalities, rather than the need to share liquidity risks, are responsible for the result.

Keywords: Money, Monetary Policy, Wallace Conjecture

1. INTRODUCTION

In a recent paper, Wallace (2014) has conjectured that in pure currency economies in which the distribution of money and trades mutually affect each other and the equilibrium without intervention is not efficient, there are transfer schemes financed by money creation that improve ex ante welfare relative to no-intervention. An economy qualifies as pure currency, according to Wallace, if the only durable object available is money, that helps expand trade possibilities among strangers whose histories are private information, consistent with the notion that the most general function of money is to substitute for a missing record-keeping technology.[1]

To date, the conjecture has neither been proved nor refuted. Wallace's informal argument in support of its validity is that money creation is bound to be helpful, since, in pure currency economies, "trades at t are doing double duty: they must be a compromise between those that are best for production and consumption at t and those that are geared solely toward producing a desirable distribution of money holdings entering the next date".[2] The idea is that money creation serves to compensate traders for being unable to rebalance their cash portfolios between

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consecutive trading sessions, as in Levine (1991), where inflationary policy helps compensate unlucky traders who run out of cash.[3]

Should this be the only beneficial role of money creation, one would have a fairly strong case to move beyond pure currency economies that allow agents to rebalance cash portfolios when needed, thanks to the risk-sharing opportunities afforded either by a large family (Shi (1997)) or a recurring trading session with transferable utility (Lagos and Wright (2005)). Indeed, these risk-sharing opportunities, which were introduced simply with the intention of making pure currency economies with uncertain trading prospects analytically tractable, would end up precluding by construction any possibility for money creation to play a beneficial role. Admittedly, this seems a steep price to be paid just to achieve tractability of a monetary framework.

This paper shows that the suboptimality of no-intervention obtains even if the traders have the option to rebalance their cash portfolios between trading sessions, as money creation can still be helpful to correct liquidity externalities working through relative price effects. We convey the point in a competitive version of the pure currency economy of Lagos and Wright (2005) adapted to fit Wallace's assumptions, manufacturing a nondegenerate but tractable distribution of money through heterogeneous time-discounts.[4] Conveniently, the model features both a market that allows traders to rebalance their cash portfolios, which obviates the need to compensate them, and multiple goods, which opens up the possibility of having nontrivial relative price effects.

We show that money creation induces two conflicting effects: (i) it lowers the return of cash, as pointed out by Friedman (1969);[5] (ii) it lowers the price of the good acquired with cash, as its demand shrinks. The first effect operates negatively but asymmetrically on agents with different degrees of impatience, affecting relatively more the patient ones. The second effect has a positive and symmetric effect on all the agents. The net effect is never exactly nil, except possibly for nongeneric payoffs configurations, hence, no-intervention is generically suboptimal. While in economies with a single good or a degenerate distribution of cash, the latter effect never dominates the former, being either absent or too weak, in multiple goods economies with heterogeneous cash holdings, the latter effect can overturn the former, when the supply of the good is not too elastic.

In sum, this paper points out that liquidity externalities, similar to the pecuniary externalities that beset credit-constrained economies even absent uncertainty,[6] can be a reason to consider active monetary intervention, apart from any need to insure liquidity risks. Given the pervasiveness of pecuniary externalities in second best economies, this argument would seem to apply quite generally to monetary economies. The generic second best inefficiency of equilibrium allocations when markets are incomplete was first pointed out by Hart (1975) and subsequently formalized by Geanakoplos and Polemarchakis (1986). These are economies in which there is only a limited number of instruments to transfer purchasing power across states of uncertainty. Our model is one where, instead, there is no uncertainty, but only one instrument that is used to finance expenditures in the presence of active liquidity constraints. A similar effect has been pointed out to play a role

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in nonmonetary, credit-constrained economies, by Lorenzoni (2008). Although we have silenced the liquidity risk-sharing channel, pecuniary externalities and missing risk-sharing opportunities are not mutually exclusive. An economy in which both were present would presumably exhibit a combination of the effects highlighted in this paper and in Wallace (2014).

The general message of this paper is that, if one is interested in questions relating to optimal monetary policy, uninsured liquidity risks may well be important, as stressed by Wallace (2014), but also general equilibrium, relative price effects should be carefully taken into account. As for the former effects to play a significant role, economies with a degenerate distribution of liquidity are not well suited, for these latter effects to play a significant role, economies with a single consumption good are not appropriate. This calls for the adoption of monetary models that are not too aggregative.

The rest of the paper is organized as follows. Section 2 presents the pure currency economy. Section 3 proves the result. Section 4 concludes. The proofs are in the Appendix.

2. A PURE CURRENCY ECONOMY

Model. Time is discrete and continues forever. Each time period is divided into two subperiods, day and night, in which two different goods are produced, traded, and consumed by a continuum of infinitely lived, anonymous agents who cannot commit to future actions. During the day, half of the agents are randomly assigned to be either buyers or sellers of a perishable consumption good, whose consumption x yields to buyers utility $u(x)$, a twice continuously differentiable real-valued function with $u' > 0 > u''$, and production y entails for sellers a cost $c(y)$, a twice continuously differentiable real-valued function with $c' > 0$, $c'' \geq 0$, together with standard Inada conditions. During the night, agents can produce and consume another perishable good, which serves as the numeraire of the economy. All agents can consume and produce the nighttime good deriving linear utility of consumption X and linear cost of effort e. Markets are Walrasian with competitive price p of the day good in units of the numeraire good, whose price is normalized to one. So far, the model is just a competitive version of Lagos and Wright (2005). The model is suitable for our purposes, since it has multiple commodity markets, which is necessary for relative price effects to play a role. We twist it assuming that there are equal masses of two types of agents, who discount future payoffs at different rates, $\beta_i < 1$ for $i = 1, 2$. The agents are equally likely to begin life as either type. There is an intrinsically worthless, perfectly divisible and storable object called money, whose value in units of numeraire is ϕ. Its supply, M, is controlled by the government as described below.

Efficiency. The efficient allocation in this economy populated by agents with different time-discount rates is nonstationary and type-specific, assigning to the impatient types a profile of daytime consumption which is decreasing over time, while the profile of production is increasing, and viceversa for the patient

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types. This allocation cannot be decentralized, absent commitment. Following Wallace (2014), we adopt a more restricted notion of efficiency, requiring a time-independent allocation that equates marginal utility and cost for both types, $u'(x_i) = c'(y_i)$ for each i, and satisfies feasibility, $\sum_i x_i = \sum_i y_i$.[7]

Trade. The sequence of trades within a period is as follows. During the day, buyers of type i spend their beginning-of-period money holdings, m_i, to purchase the consumption good, x_i, to solve $V_i^b(m_i) = Max\ u(x_i) + W_i^b\left(\overline{m}_i^b\right)$, subject to the constraint,

$$px_i \leq \phi m_i, \tag{1}$$

with multiplier $\lambda_i \geq 0$. End-of-day money holdings are $\overline{m}_i^b \equiv m_i - \frac{px_i}{\phi}$. Sellers i choose production y_i to solve $V_i^s(m_i) = Max\ \ -c(y_i) + W_i^s(\overline{m}_i^s)$. End-of-day money holdings are $\overline{m}_i^s \equiv m_i + \frac{py_i}{\phi}$. The function $W_i^j(\overline{m}_i^j)$ represents the value of operating in the night market with money holdings $\overline{m}_i^j$ for an agent $j = b, s$. The expected value for a type i at the beginning of the period is $V_i(m_i) = \frac{1}{2}\sum_j V_i^j(m_i)$. At night, an agent j chooses consumption X_i^j, effort e_i^j, and next period money $m_{i,+1}$, to solve $W_i^j(\overline{m}_i^j) = Max\ X_i^j - e_i^j + \beta_i V_i(m_{i,+1})$ subject to the budget constraint, $X_i^j + \phi m_{i,+1} = e_i^j + \phi\overline{m}_i^j + \tau$, where τ is a transfer from the government, and $V_i(m_{i,+1})$ the expected value of operating next period with $m_{i,+1}$, which is uniform across agents of the same type, due to the linearity of the nighttime payoff.

Policy. We consider the same transfer scheme of Wallace (2014), which is linear in end-of-period money holdings, $\tau = a + b(e_i^j - X_i^j + \phi\overline{m}_i^j)$, with $b > 0$ and $\tau \geq 0$ for all types, to insure that all the agents are willing to participate in the scheme truthfully revealing their money holdings. Summing over types, the government budget constraint is $\phi M_{+1} = \phi M(1 + b) + a$. To simplify the notation, let $\gamma(1 + b) \equiv \frac{M_{+1}}{M}$, which gives $a = (1 + b)\phi M(\gamma - 1)$. In Wallace's terminology, neutral or no-intervention corresponds to $\gamma = 1$, progressive intervention to $\gamma > 1$, and regressive intervention to $\gamma < 1$.

Equilibrium. Under stationarity, the agents' optimization, taking government policy and prices as given, leads to the type i demand of the daytime good as a function of its price,

$$x_i = u'^{-1}\left(\frac{2\gamma - \beta_i}{\beta_i}p\right). \tag{2}$$

Optimality in production requires, by perfect competition, equality of the marginal cost of production with the price of the daytime good, giving the supply of the daytime good as a function of its price, $y = c'^{-1}(p)$. The equilibrium price of the daytime good is determined by the market-clearing condition, $\sum_i \frac{x_i}{2} = y$. Substituting (2) and $y = c'^{-1}(p)$ into it, we obtain

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$$\sum_i u'^{-1}\left(\frac{2\gamma-\beta_i}{\beta_i}p\right)-2c'^{-1}(p)=0. \tag{3}$$

The multiplier of constraint (1) is

$$\lambda_i=\frac{u'(x_i)}{c'(y)}-1=\frac{2(\gamma-\beta_i)}{2\gamma-\beta_i}. \tag{4}$$

Only values of $\gamma\geq\overline{\beta}\equiv\max_i\beta_i$ are compatible with the existence of a monetary equilibrium, since (4) needs to be nonnegative. For our purposes, it is enough to consider policies $\gamma>\overline{\beta}$, which we call admissible. Thus, the real demand for money is determined by the binding constraint (1) and the value of money by money market clearing. The market for the nighttime good clears by Walras Law.

A stationary monetary equilibrium (SME) is a positive p that satisfies (3) for any admissible γ. Once p is determined, the remaining variables are recursively determined. The first Proposition establishes existence and uniqueness of the SME.

PROPOSITION 1. *An SME exists and is unique, for any admissible* γ.

Distribution. In general, at the equilibrium, the choice of cash holdings differs across types. The next Proposition shows that the heterogeneity in discounting is necessary and sufficient to have a nondegenerate equilibrium distribution of money holdings and determines which type holds more cash.

PROPOSITION 2. *At the SME, the money distribution is nondegenerate iff* $\beta_1\neq\beta_2$. *The patient type holds more cash than the impatient type.*

When agents are homogeneous, the model reduces to the Lagos and Wright (2005) environment, which gives rise to a degenerate equilibrium distribution of money holdings. With heterogeneous agents, the distribution of money holdings has two mass points, at equilibrium. Aggregate variables remain constant at equilibrium and the distribution of money holdings remains stationary although nondegenerate. The patient agents hold more cash than the impatient ones, thus, financing higher consumption.

Elasticities. Since the equilibrium is unique and varies smoothly in the policy parameter, we can perform comparative static exercises with respect to policy. Let the relative risk aversion functions of utility be $\rho(x_i)\equiv-\frac{u''(x_i)x_i}{u'(x_i)}$ and cost $\eta(y)\equiv\frac{c''(y)y}{c'(y)}$, and the elasticity of a variable z to changes in policy be $\varepsilon_z\equiv\frac{dz}{d\gamma}\frac{\gamma}{z}$. Variables with a tilde denote equilibrium values. Differentiating (2) with respect to policy, at the SME, we obtain the elasticity of consumption,

$$\varepsilon_{x_i}=-\frac{\mu_i(\gamma)+\varepsilon_p}{\rho(\widetilde{x}_i)}, \tag{5}$$

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where $\mu_i(\gamma) \equiv \frac{2\gamma}{2\gamma-\beta_i}$. Money creation has a direct and an indirect effect on consumption. The direct effect reduces the return of money, operating asymmetrically on traders, with the cash-rich being affected more than the cash-poor, as $\mu_i(\gamma)$ is positive for both agents but larger for the patient ones. The indirect effect works through the change in the price of the daytime good, due to the lower demand of the good that has to be acquired with cash. This effect is symmetric across traders and its direction depends on the sign of the price elasticity. The overall effect may be positive or negative and is inversely related to the curvature of utility. Differentiating $y = c'^{-1}(p)$ with respect to the policy parameter, at the SME, we obtain the elasticity of the supply of the daytime good to changes in policy, $\varepsilon_y = \frac{\varepsilon_p}{\eta(\widetilde{y})}$, which is unambiguously determined by the sign of the price elasticity and is inversely related to the curvature of the cost. Finally, the price elasticity is obtained differentiating (3) with respect to the policy parameter,

$$\varepsilon_p = -\frac{\eta(\widetilde{y}) \sum_i \sigma(\widetilde{x}_i)\, \mu_i(\gamma)}{\eta(\widetilde{y}) \sum_i \sigma(\widetilde{x}_i) + 2\widetilde{y}}, \tag{6}$$

where $\sigma(\widetilde{x}_i) \equiv \frac{\widetilde{x}_i}{\rho(\widetilde{x}_i)}$. The price elasticity is always negative and its size is directly related to the curvature of the cost. The next Proposition characterizes the elasticity of consumption to changes in policy.

PROPOSITION 3. *At the SME, if* $\eta(\widetilde{y}) = 0$ *or* $\beta_1 = \beta_2$, $\varepsilon_{x_i} < 0$ *for both types; if* $\eta(\widetilde{y}) > 0$ *and* $\beta_1 \neq \beta_2$, *there exists a unique* $\overline{\eta} > 0$, *such that, for the cash-poor,* $\varepsilon_{x_i} \gtreqless 0$ *iff* $\eta(\widetilde{y}) \gtreqless \overline{\eta}$, *while for the cash-rich,* $\varepsilon_{x_i} < 0$.

The elasticity of consumption is unambiguously negative for both types when the marginal cost of daytime production is constant, thus, the price elasticity is nil, or when there is no heterogeneity. In these cases, the change in the price of the good can never overturn the change in the value of cash, either because it is absent or because it is present but the latter is symmetric across agents. Therefore, for the price effect to matter, two features are necessary: (i) a less-than-perfectly elastic supply of the good; and (ii) the presence of heterogeneous traders who react differently to changes in the return of cash. When the agents are heterogeneous and the price elasticity is sufficiently large, the effect of a drop in price induced by a more progressive policy overturns the negative effect of lowering the value of cash for the cash-poor type, while the price effect can never reverse the effect of the change in the value of money for the cash-rich type.

Inefficiency of SME. With homogeneous agents, the multiplier of the liquidity constraint can be made nil for all agents, adopting the Friedman rule, $\gamma = \beta_1 = \beta_2$. With such a policy, the liquidity constraint is slack and the equilibrium allocation is efficient. When the agents are heterogeneous, instead, no single value of $\gamma \geq \overline{\beta}$ can make (4) nil for both types. Thus, at the SME, the liquidity constraint (1) is binding for at least the impatient type, and the equilibrium allocation is inefficient for any policy choice. The next Proposition proves the point.

PROPOSITION 4. *If* $\beta_1 \neq \beta_2$, *the SME is inefficient, for any* $\gamma \geq \overline{\beta}$.

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3. THE RESULT

The analysis above implies that, in this pure currency economy with agents' heterogeneity, all the assumptions in Wallace (2014) are fulfilled. By Proposition 2, the distribution of money holdings is nondegenerate. Moreover, the nighttime trades affect the distribution of money for the next day and the daytime distribution affects the daytime trades. Therefore, Assumption 1 holds. By Proposition 4, the equilibrium is inefficient for any policy, including no-intervention. Moreover, by Proposition 1, money is valuable at such equilibrium. Therefore, Assumption 2 holds. The conjecture of Wallace is that, under these assumptions, no-intervention is not welfare maximizing.

Suboptimality of No-Intervention. Agents who are equally likely to begin life as either type, look forward to the average discounted equilibrium payoff $U(\gamma) = \sum_i \left[u(\widetilde{x}_i) - c(\widetilde{y})\right]$. Given the aim of the government of maximizing $U(\gamma)$ through the choice of γ in the admissible range, the first-order condition for the optimality of no-intervention, $U'(1) = 0$, gives

$$\sum_i \widetilde{\lambda}_i \widetilde{x}_i \varepsilon_{x_i} \Big|_{\gamma=1} = 0. \tag{7}$$

Since (7) is a necessary condition for the optimality of no-intervention, to prove its suboptimality, it is enough to show that (7) fails to hold. Notice that, at the SME without intervention, consumption is strictly positive for both types and the multipliers of the cash constraints are strictly positive for both types. Moreover, by Proposition 3, the elasticities of consumption are nonzero for both types, except possibly for the impatient type if the risk aversion of the cost function is equal to $\overline{\eta}$. Hence, the terms for the two types separately are never both simultaneously nil. However, the two terms could still exactly cancel each other out, since we know, by Proposition 3, that the consumption elasticities of the two types sometimes move in opposite directions. The following Proposition establishes that the terms do not cancel each other out exactly, generically in the space of cost functions.

PROPOSITION 5. *No-intervention is suboptimal, generically in the space of cost functions.*

The crucial elements responsible for the generic suboptimality of no-intervention are the strictly positive multipliers of the cash constraints and the non-offsetting effects of policy changes on the equilibrium consumption of the agents. In this environment with multiple goods, these are immediate consequences of the assumptions of Wallace (2014), since: (i) the inefficiency of the equilibrium without intervention implies the divergence of marginal benefit and cost for the good acquired with cash, and, hence, positive shadow values of the cash constraints; and (ii) the heterogeneity of money holdings implies different marginal effects of cash transfers across agents, giving rise to nontrivial liquidity externalities.

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Comparison with Wallace. In his paper, Wallace examines two examples, a search economy with random bilateral meetings, for which no clear-cut results are obtained, and a Bewley economy with competitive markets, for which the conjecture is shown to hold true. In the latter pure currency economy, taken from Kehoe et al. (1992), there are two types of agents with high and low endowment switching overtime with probability π. If every period types switch deterministically, that is $\pi = 1$, Townsend (1980) has shown that the best policy is a regressive scheme that corresponds to the so-called Friedman rule, which prescribes for monetary authorities to offset the effect of time-discount on the agents' intertemporal decision to accumulate cash.[8] For the parametrization chosen by Wallace, there exist a positive cutoff for π so that above the cutoff the best policy is still regressive, while below the cutoff the best policy is progressive. No-intervention is optimal only at the cutoff, hence, non-generically in the space of the parameters of the model. A progressive scheme discourages currency savings of the high-endowment type at the time of the switch, since it distorts the return on currency, but encourages consumption of the low-endowment type when there is no switch, since it redistributes currency in their favor. The probability of a switch determines which of the two effects dominates. If the probability is large, a switch is more likely to occur, and the traditional effect that distorts the return of currency prevails, while if it is small the effect of insuring the low-endowment types prevails.

The logic of our model, which is a search economy but with competitive markets, hence, a blend of the two examples considered by Wallace, is different. First, with heterogeneous time-discounts, the Friedman rule is not well defined, since there is no unified way of measuring impatience for all the agents. Second, while the traditional effect of progressive intervention, namely the distortion of the rate of return of currency, is still present, here, this affects traders asymmetrically, with the patient ones being affected more than the impatient ones; the countervailing effect, on the other hand, here, involves the change in the price of the daytime good, due to the lower demand of the good that has to be acquired with cash. This latter effect works in the opposite direction and is symmetric across traders as it depends on the common price elasticity. Third, our non-genericity argument is based on a perturbation of payoffs, rather than the cutoff for the switching probability argument used by Wallace in his example.

Optimal Policy. In his conjecture, Wallace is concerned with the suboptimality of no-intervention, explicitly acknowledging that the optimal policy may be regressive or progressive under different specifications of the underlying environment. This is true in our model as well. From the analysis above, it immediately follows that the presence of the price effect is necessary for progressive policy to be optimal. To see this, suppose the marginal cost of production $c'(y)$ is constant. This makes $\eta = 0$, silencing the effect of the pecuniary externality since the supply becomes perfectly elastic and changes in aggregate demand do not alter the price, p. This leads to (6) being nil, and, hence, (5) negative for every agent. This, in turn, implies a negative (7), calling for regressive intervention. On the other

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hand, by Proposition 3, the time-discount heterogeneity is crucial to generate asymmetric effects of the decrease in the return of money on cash-rich and cash-poor types. Without such asymmetries, progressive policy would never have a chance to dominate regressive intervention. A general characterization of the optimal policy is not obvious as the objective function may not be globally concave in the policy parameter. In general, the larger the curvature of the cost as measured by η, the stronger the price effect turns out to be, and this, in turn, makes progressive policies more likely to be optimal. From the analysis above, we immediately have that the optimal policy is strictly in the interior of the policy space if the curvature of the cost function is larger than the value $\overline{\eta}$, identified in Proposition 3. This is because at the boundary of the policy space, the multiplier of the patient agent is nil and both the multiplier and the elasticity of the impatient agent are strictly positive. As shown in Lemma 1 in the Appendix, with the highest time-discount sufficiently close to one, progressive intervention turns out to be optimal in this case, being the price effect on impatience types sufficiently strong to overturn the negative effect of the monetary expansion on the value of money, while the effect on patient types is nil at no-intervention, their shadow value being nil.

4. CONCLUDING COMMENTS

We presented an argument for the suboptimality of no-intervention in a pure currency economy satisfying the criteria of the Wallace conjecture, that hinges on the presence of liquidity externalities. This argument should apply fairly generally to monetary economies with multiple goods and heterogeneous cash holdings, as, in such economies, changes in policy have opposite—and typically not offsetting—effects on the value of cash and the relative price of goods.

Some final remarks are in order. First, the competitive nature of trade plays an important role in the analysis, as the general equilibrium effect of a change in relative prices requires market prices to be operative. Second, and related, given the competitive nature of the environment, we have ignored questions connected to mechanism design. Finally, the heterogeneity in time-discounts is crucial, as they allow price effects to have their full impact on allocations. Other types of heterogeneity of preferences would not suffice for our purposes.

NOTES

1. Money is memory, as argued by Kocherlakota (1998).
2. Wallace (2014), p. 265.
3. The point is examined further in Kehoe et al. (1992).
4. This is similar to Boel and Waller (2019). Other papers considering the effect of policy with a nondegenerate distribution of money, include Berentsen et al. (2005) and Bhattacharya et al. (2005).
5. This well-known effect, formalized by Bewley (1980) and Townsend (1980), appears in Wallace (2014).
6. As shown by Moore (2013). In general economies with financial frictions, Davila and Korinek (2018) have disentangled two types of pecuniary externalities, which they call distributive and collateral externalities. Our liquidity externality is similar to the latter, as it arises from price changes affecting the agents through binding liquidity constraints.

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7. The same is true in Boel and Waller (2019), where, due to heterogeneous time-discounts, the unrestricted planner solution is time-inconsistent, hence, their focus on the discretionary planner solution, which is the natural benchmark in economies without commitment, such as these.

8. For recent work on the optimality of regressive schemes, see Bajaj et al. (2017).

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APPENDIX

Optimality Conditions. The f.o.c. for x_i is $u'(x_i) - p\lambda_i - p\frac{W_i^{b\prime}(\overline{m}_i^b)}{\phi} = 0$. Combining it with the envelope condition, $W_i^{b\prime}\left(\overline{m}_i^b\right) = \phi$, obtain $\lambda_i = \frac{u'(x_i)}{p} - 1$ for each i. The f.o.c. for y_i is $-c'(y_i) + p\frac{W_i^{s\prime}(\overline{m}_i^s)}{\phi} = 0$. Combining it with the envelope condition, $W_i^{s\prime}\left(\overline{m}_i^s\right) = \phi$, obtain $c'(y_i) = p$ for both i. Using τ and a in the text, rewrite the nighttime budget constraint

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as $X_i^j - e_i^j = \phi \overline{m}_i^j - \frac{\phi m_{i,+1}}{1+b} + \phi M\,(\gamma - 1)$ and substitute it into the objective function. The f.o.c. for $m_{i,+1}$ is $-\frac{\phi}{1+b} + \beta_i V_i'\left(m_{i,+1}\right) = 0$. The envelope condition is $V_i'\,(m_i) = \frac{\phi(\lambda_i+2)}{2}$. Substitute λ_i and delay it one period, inserting it into the f.o.c., to obtain the Euler equation. At a stationary equilibrium with $\phi_{+1}M_{+1} = \phi M$ and γ as defined in the text, this gives (2). Using $c'\,(y) = p$ into λ_i, obtain (4).

Proof of Proposition 1. Let $\Phi\,(p, \gamma)$ be the LHS of (3). The function $\Phi\,(p, \gamma)$ is continuous in p for all admissible γ. By the Inada conditions, $\Phi\,(0, \gamma) = +\infty$, $\Phi\,(\infty, \gamma) = -\infty$, for all admissible γ. By the Intermediate Value Theorem, $\exists \widetilde{p} > 0$ s.t. $\Phi\,(\widetilde{p}, \gamma) = 0$ for all admissible γ. Since $\frac{\partial \Phi(p,\gamma)}{\partial p} < 0$ for all admissible γ, $\widetilde{p}$ is unique. ■

Proof of Proposition 2. Since $\gamma > \overline{\beta}$, (4) is strictly positive for $\forall i$. Thus, (1) is binding for $\forall i$, implying $m_i = \frac{p}{\phi} x_i$ for $\forall i$. Hence, $m_1 \neq m_2$ iff $x_1 \neq x_2$. By (2), $x_1 \neq x_2$ iff $\beta_1 \neq \beta_2$. By the binding (1), m_i is increasing in x_i. By (2), x_i is increasing in β_i. ■

Proof of Proposition 3. Substitute (6) into (5), rearrange terms and obtain

$$\varepsilon_{x_i} = -\frac{\frac{\mu_i(\gamma)}{\rho(\widetilde{x}_i)}\left[2\widetilde{y} - \eta\,(\widetilde{y})\,\frac{(\beta_{-i}-\beta_i)\sigma\,(\widetilde{x}_{-i})}{(2\gamma-\beta_{-i})}\right]}{\eta\,(\widetilde{y})\sum_i \sigma\,(\widetilde{x}_i) + 2\widetilde{y}}, \tag{A1}$$

where $-i$ denotes the other agent wrt i. When $\eta\,(\widetilde{y}) = 0$ or $\beta_1 = \beta_2$, $\varepsilon_{x_i} < 0$ for $\forall i$. When $\eta\,(\widetilde{y}) > 0$ and $\beta_1 \neq \beta_2$, for the impatient type $\varepsilon_{x_i} \gtreqless 0$ iff $\eta\,(\widetilde{y}) \gtreqless \overline{\eta} \equiv \frac{2\widetilde{y}(2\gamma-\beta_{-i})}{(\beta_{-i}-\beta_i)\sigma(\widetilde{x}_{-i})}$; for the patient type, $\varepsilon_{x_i} < 0$. ■

Proof of Proposition 4. The necessary conditions for efficiency are $u'\,(x_i) = c'\,(y_i)$ for $\forall i$, together with feasibility. By (4) and $\gamma \geq \overline{\beta}$, $u'\,(\widetilde{x}_i) \geq c'\,(\widetilde{y})$ for $\forall i$, with a strict inequality for at least one i. Hence, at least one necessary condition for efficiency is violated by the SME allocation. ■

Proof of Proposition 5. Substitute (4) and (A1) into (7), rearrange terms and obtain

$$\eta\,(\widetilde{y}) = \frac{2\widetilde{y}\sum_i \left(1-\beta_i\right)\sigma\,(\widetilde{x}_i)\,\mu_i\,(1)\prod_i\left(2-\beta_i\right)}{\left(\beta_1-\beta_2\right)^2\prod_i \sigma\,(\widetilde{x}_i)\,\mu_i\,(1)}. \tag{A2}$$

If $\eta\,(\widetilde{y}) = 0$, (A2) is violated. Consider $\eta\,(\widetilde{y}) > 0$. Perturb the cost, $c_\epsilon\,(y) = c\,(y) + \epsilon\left[c\,(y) - c'\,(\widetilde{y})\,y\right]$, with $\epsilon \geq 0$. This does not disturb the equilibrium, since $c_\epsilon'\,(\widetilde{y}) = c'\,(\widetilde{y})$, but changes the cost curvature to $\eta_\epsilon\,(\widetilde{y}) = (1+\epsilon)\,\eta\,(\widetilde{y})$. Take any $\epsilon > 0$. The RHS of (A2) is unaffected, but the LHS is larger, thus, equality cannot be maintained. Given the uniqueness of the SME, (A2) holds for one value of η, and the cost function is uniquely identified. Thus, the set of cost functions such that (A2) holds is closed with empty interior, and its complement is open and dense. ■

LEMMA 1. *For $\overline{\beta} \to 1$ and $\eta\,(\widetilde{y}) > \overline{\eta}$, the optimal policy has $\gamma > 1$.*

Proof. By (4) and (A1), for $\overline{\beta} \to 1$ and $\eta\,(\widetilde{y}) > \overline{\eta}$, with $\gamma = 1$, the LHS of (7) is strictly positive. Hence, the optimal policy must be interior. ■

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