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Equivariant homology

BY DAVID M. AUSTIN
Mathematics Department, University of British Columbia, Vancouver BC V6T 1Z2, Canada

AND PETER J. BRAAM
Mathematical Institute, 23–29 St. Giles, Oxford OX1 3LB

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Abstract

This paper studies the homology of the homotopy quotient of a G-manifold. We start by formulating a complex based on differential forms to compute this homology. This leads to a topological version of integration formulas over fibred products of G-manifolds. In particular the TQFT formulas studied by Witten and Atiyah–Jeffrey now appear as a pairing of homology and cohomology classes. We compare our construction with both the double complex of Axelrod and Witten and with the distributional complex of Duflo and Vergne.

1. *Introduction*

In recent years equivariant cohomology has played an important role in various mathematical problems related to quantum field theory. Examples include the cohomology of moduli spaces (see Atiyah–Bott[**1**] and Kirwan[**12**]), gluing constructions for Donaldson invariants (see Taubes[**16**] and Austin-Braam[**4**]) and in the computations of Verlinde algebras (see Donaldson[**8**], Witten[**18**] and Jeffrey-Kirwan[**11**]).

In this paper we study the homology of the homotopy quotient of a G-manifold, which we call the equivariant homology of the G-manifold.[1] We present a deRham model for the theory, prove an integration formula, which is important in gauge theory applications, and relate it to a variety of other constructions made recently.

If M is a G-manifold, the homotopy quotient is formed by $M_G = EG \times_G M$. Equivariant cohomology is defined to be the cohomology of M_G. It has been known for a while how to integrate equivariant cohomology classes over M through the 'localisation theorem' and also how to use differential forms to compute equivariant cohomology (for an overview, see Atiyah–Bott[**2**]). A more complete pairing arises by considering integration not just over cycles in M but also cycles in M_G. Hence one is naturally led to consider the homology of M_G.

We begin in Section 2 with the construction of a Cartan model for equivariant homology, and prove a fundamental theorem which states that its homology is equal to that of the homotopy quotient in the case that M is closed and oriented. A similar construction has been made independently by Taubes[**16**].

[1] We will not explore whether this definition of equivariant homology agrees with the definition using spectra.

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If G acts freely on M with closed, oriented quotient, there is a fundamental class $[M]_G$ with dimension equal to $\dim M/G$. This class has a simple expression (see Theorem 2) which has information about all of the characteristic classes of the principal fibration $M \to M/G$ (see Corollary 3). A crucial formula for integration over G-manifolds which are fibred products, a formula needed in many gauge theory applications, is derived using this property in Section 2·3.

Axelrod and Witten [**5**] proposed an interesting geometrical construction for a variety of equivariant (co)homology groups. Their construction is based on mixing the deRham theory of M with Hodge theory of $\mathbf{g}_{\mathbf{C}}$. Duflo-Vergne [**9**] made another variation on equivariant cohomology using distributions. In Section 3, we state several theorems relating these constructions to either equivariant cohomology or equivariant homology. Together with the results in Section 2, this answers a question of Axelrod and Witten [**5**] to determine the topological significance of these constructions.

The fundamental class for free G-manifolds has played an important role in Witten's work on topological quantum field theory [**17**]. Atiyah–Jeffrey [**3**] and Getzler [**10**] identified Witten's integrals as expressions for the Euler class of certain bundles using equivariant cohomology. In Section 4, we investigate Witten's integral. With the equivariant homology machinery at our disposal it is almost immediate to prove that the integral is a pairing of an equivariant cohomology class with the fundamental class in equivariant homology. More interesting is that one can identify the novel part of the complicated formula of Witten and Atiyah–Jeffrey as the fundamental class of the moduli space, expressed as an equivariant homology cycle (Theorem 13). This formulation will be helpful in identifying other topological Lagrangeans.

Among the issues we leave unexplored are the further study of the pairing between equivariant homology and cohomology. In particular, one could wonder if there is a generalisation of the localisation formula, which pairs a specific equivariant homology cycle with arbitrary equivariant cohomology classes, for pairing arbitrary homology and cohomology classes. The L^2 equivariant cohomology groups introduced by Axelrod and Witten, and various other generalisations of his double complex, should also prove most interesting.

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2. *Equivariant (co)homology*

2·1. *Definitions*. Let G be a compact, connected Lie group acting smoothly on a manifold M. The equivariant cohomology $H^*_G(M)$ is defined as the ordinary cohomology[2] of the homotopy quotient $M_G \equiv EG \times_G M$, where $EG \to BG$ is the classifying space for the group G. Recall that for free G-actions, M_G is homotopy equivalent to M/G and the projection $M_G \to BG$ is the classifying map for $M \to M/G$. In general, $H^*_G(M)$ is a module over $H^*_G(pt) = H^*(BG)$, and there is a natural map $H^*_G(M) \to H^*(M)$. For general background, see Berline–Getzler–Vergne [**6**].

For our purposes, the Cartan model gives an especially convenient means of

[2] We work with real coefficients throughout.

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computing equivariant cohomology. Remember that the algebra $H^*(BG)$ is naturally isomorphic to the symmetric algebra $S^*(\mathbf{g}^*)^G$ of polynomials on $\mathbf{g}$, the Lie algebra of G; linear functions have degree 2. More generally, let

$$\Omega^*_G(M) = (S^*(\mathbf{g}^*) \otimes \Omega^*(M))^G$$

and define a differential d_G as $\mathrm{id} \otimes d - 2\pi C$ where C is the map induced by the universal element $V \in \mathbf{g}^* \otimes \mathrm{Vect}(M)$ through tensor product of multiplication on $S^*(\mathbf{g}^*)$ and contraction on $\Omega^*(M)$. It is a well known theorem (see Cartan [7]) that this complex computes the equivariant cohomology over the real numbers.

For a free G-action, the isomorphism $H^*(M/G)$ to $H^*_G(M)$ is given in terms of this complex by the pullback of differential forms under the quotient map $M \to M/G$. The inverse is defined using a connection on the G-bundle $M \to M/G$ and its curvature which is an element of $(\mathbf{g} \otimes \Omega^*(M))^G$. Viewing the curvature as a G-equivariant map $\mathbf{g}^* \to \Omega^*(M)$, we have a map $\Omega^*_G(M) \to \Omega^*(M)$ which sends an equivariant differential form to a G-invariant form on M. To obtain a form on M/G, restrict to the horizontal subspaces on TM defined by the connection.

Next we turn to equivariant homology. This we define to be the homology of the homotopy quotient:

$$H_{G,*}(M) = H_*(M_G).$$

This time there is a cap product $H_{G,j}(M) \otimes H^i_G(M) \to H_{G,j-i}(M)$, and a natural map

$$i_*: H_j(M) \to H_{G,j}(M),$$

which will be of interest to us.

Recall that the homology of a closed, oriented manifold M can be computed through a complex of differential forms $\Omega^{\dim M - *}(M)$ on which the exterior derivative serves as a differential of degree -1. The natural properties of homology, such as induced morphisms under smooth maps, are recovered when one extends the complex to include currents. When M is not closed or oriented, one may similarly work with compactly supported forms twisted by the orientation sheaf.

In this framework equivariant homology can be computed through a Cartan model similar to that for equivariant cohomology. Let

$$\Omega_{G,*}(M) = (S^*(\mathbf{g}) \otimes \Omega^{\dim M - *}(M))^G,$$

where we grade a differential form a on M as $\dim(M) - \deg(a)$ and elements of $\mathbf{g}$ with $+2$. We can define a boundary map ∂_G as $\mathrm{id} \otimes d - 2\pi c$, where c is obtained by contracting by the universal element V. The boundary operator hence has degree -1 and square 0. In Section 2·2, we will prove the following:

THEOREM 1. *When M is closed and oriented,*

$$H_{G,*}(M) = H_*(\Omega_{G,*}(M), \partial_G).$$

In terms of differential forms the cap product $\cap : H^i_G(M) \otimes H_{G,j}(M) \to H_{G,j-i}(M)$, is realized by combining contraction on $\mathbf{g}$ and $\mathbf{g}^*$ with wedge product. Finally, as a special case, the pairing

$$\langle , \rangle_M : H_{G,j}(M) \otimes H^j_G(M) \to \mathbb{R}$$

is obtained by tensoring the pairing $S^*(\mathbf{g}) \otimes S^*(\mathbf{g}^*) \to \mathbb{R}$ with the wedge product and integration:

$$\langle X \otimes a, \phi \otimes b \rangle_M = \langle X, \phi \rangle \int_M a \wedge b.$$

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In the case of a free G-action on a closed, oriented manifold M, the quotient M/G is orientable and so there is a fundamental class $[M]_G \equiv [M/G] \in H_{\dim M/G}(M_G)$, of particular interest. The differential form expression for $[M]_G$ has been the focus of recent interest appearing first in the work of Witten [**17**] and Atiyah–Jeffrey [**3**], and then later in a different form in Kumar–Vergne [**13**]. For a basis $\{X_\alpha\}_{\alpha=1}^n$ of $\mathbf{g}$, write a connection 1-form on M as $X_\alpha \otimes \omega_\alpha$ with curvature $X_\alpha \otimes \Omega_\alpha$. The following theorem, the proof of which will be given in section 2·2, gives an expression for the fundamental class.

THEOREM 2.

$$[M]_G = \gamma \equiv \frac{1}{v}\omega_1 \wedge \ldots \wedge \omega_n \wedge \exp\left(\frac{1}{2\pi}\Omega_\alpha \otimes X_\alpha\right),$$

where v is the volume of G, associated to the basis dual to X_α.

This shows that the leading order term of $[M]_G$ – that is, the term which is constant in X_α – is the unit volume form of the fibres in $M \to M/G$. The other terms involve the characteristic classes of this bundle. As such the fundamental class should be regarded as a Thom form for principal fibrations.

The case of a circle action shows the essential features: let A be a connection form on M, and X the generator of the Lie algebra s^1

$$[M]_G = \frac{1}{2\pi}A \wedge \left(1 + X\frac{1}{2\pi}F^A + \frac{1}{2}X^2\left(\frac{1}{2\pi}\right)^2 (F^A)^2 + \frac{1}{6}X^3\left(\frac{1}{2\pi}\right)^3 (F^A)^3 + \ldots\right).$$

As the following corollary shows, the expression in Theorem 2 encodes all of the characteristic classes of the principal G-bundle into the fundamental class.

COROLLARY 3. *For a universal characteristic class $\phi \in S^*(\mathbf{g}^*)$ of dimension* $\dim M/G$,

$$\langle [M]_G, \phi \rangle = \phi(M \to M/G)$$

where $\phi(M \to M/G)$ denotes the characteristic number of the bundle $M \to M/G$.

The importance of this formula is that conversion of invariant polynomials into characteristic classes – that is, the Chern–Weil homomorphism – is accomplished just through pairing with the fundamental class.

2·2. *Equivariant homology and the homotopy quotient.* Here we will show that the equivariant homology, as defined in Section 2·1, can be computed in terms of a complex of equivariant differential forms when the manifold M is closed and oriented. This is the content of Theorem 1. Our proof will consist of two steps, the first of which is the following lemma proven by Kumar–Vergne [**13**] in a slightly different context.

LEMMA 4. *Let M be a free G-manifold of dimension m with projection $\pi: M \to M/G$. Then there is an isomorphism*

$$H^{m-n-*}(M/G) \to H_*(\Omega_{G,*}(M))$$

given by $\eta \mapsto \pi^*(\eta) \cap \gamma$.

Proof. Our proof proceeds in three steps: first, we show that the γ class is

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independent of the connection used in its definition. Next, we prove the result for a trivial bundle. Finally, we patch the general situation together using a Mayer–Vietoris argument.

To show that γ is independent of the connection chosen, we suppose that ω^0 and ω^1 are two connections. Call the difference $\eta = \sum_i \eta_i \otimes X_i = \omega^1 - \omega^0$, a $\mathbf{g}$-valued form on M. Define the family of connections $\omega^t = \omega^0 + t\eta$ having curvature Ω^t and defining a family of classes γ^t. First notice that

$$\frac{d}{dt}\exp\left(\frac{1}{2\pi}X_i \otimes \Omega_i^t\right) = \left(\frac{1}{2\pi}X_i \otimes d_{\omega^t}\eta_i\right)\exp\left(\frac{1}{2\pi}X_i \otimes \Omega_i^t\right),$$

where $d_{\omega^t}\eta$ is the covariant derivative of η in the connection ω^t. The volume is differentiated similarly:

$$\frac{d}{dt}\omega_1^t \dots \omega_n^t = \sum_j \omega_1^t \dots \eta_j \dots \omega_n^t.$$

Let ϕ_j be a basis for $\mathbf{g}^*$ dual to X_j. We now compute $d/dt\,\gamma^t$ using the facts $\partial_G(\gamma_t) = 0$ and $i_{X_j}\eta = 0$ together with the Bianchi identity.

$$\begin{aligned}
\frac{d}{dt}\gamma^t &= \left(\frac{1}{2\pi}X_i \otimes d_{\omega^t}\eta_i\right)\gamma^t + \omega_1^t \dots \eta_j \dots \omega_n^t \exp\left(\frac{1}{2\pi}X_i \otimes \Omega_i^t\right) \\
&= \left(\frac{1}{2\pi}X_i \otimes d\eta_i\right)\omega_1^t \dots \omega_n^t \exp\left(\frac{1}{2\pi}X_i \otimes \Omega_i^t\right) \\
&\quad -(-1)^j \eta_j \omega_1^t \dots \widehat{\omega_j^t} \dots \omega_n^t \exp\left(\frac{1}{2\pi}X_i \otimes \Omega_i^t\right) \\
&= (d - 2\pi i_{X_j}\phi_j)\left(\left(\frac{1}{2\pi}X_i \otimes \eta_i\right)\gamma_t\right) \\
&= \partial_G\left(\left(\frac{1}{2\pi}X_i \otimes \eta_i\right)\gamma_t\right).
\end{aligned}$$

This says that γ^0 and γ^1 differ by a ∂_G-exact form and so define the same homology class.

For a trivial bundle $U \times G$, the complex $\Omega_{G,*}(U \times G)$ splits as a tensor product of the de Rham complex on U and the Cartan complex $\Omega_{G,*}(G)$. Using the product connection shows that γ may be represented by the volume on the fibre G. The homology of the Cartan complex on G is generated by this volume form and so the statement of the theorem follows immediately for $U \times G$.

Two observations will help finish the proof. First notice that if U and V are two G-invariant open submanifolds, a Mayer–Vietoris sequence relates the homology of the Cartan complex for $U \cap V$, U, V and $U \cup V$. This is seen, in the usual way, by utilizing a G-invariant partition of unity. Secondly, the γ class of a free submanifold of a free manifold is given by restricting the γ class of the larger manifold. Hence if U and V are two open free G-submanifolds of a free G-manifold M, there is a map from the Mayer–Vietoris sequence for the de Rham cohomology of U/G and V/G into the homology of the equivariant complexes defined by U and V given by cap product with the respective γ classes.

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Finally, cover M/G by finitely many open sets U_k restricted to which the bundle $M \to M/G$ is trivial. By induction we may assume that the statement of the theorem is true for M restricted to $V = \bigcup_{k \leqslant n} U_k$. It is also true on U_{n+1} and on $U_{n+1} \cap V$ because the bundle $M \to M/G$ is trivial restricted to each of these. We obtain that the statement is true for $\bigcup_{k \leqslant n+1} U_k$ by using the five lemma in conjunction with the Mayer–Vietoris sequences for the pair (U_{n+1}, V). ∎

The following lemma shows that the homology of the equivariant complex need only be identified for a free G-space. We denote finite-dimensional approximations to the universal G-bundle by $EG_p \to BG_p$ where EG_p is p-connected.

LEMMA 5. *Let $j \geqslant 0$. For p sufficiently large, there is a natural isomorphism $H_j(\Omega_{G,*}(EG_p \times M), \partial_G) \to H_j(\Omega_{G,*}(M), \partial_G)$.*

Proof. Notice first that both complexes are filtered by the degree in $S^*(\mathbf{g})$. The E^1 term of the induced spectral sequences given by the complex $(\Omega_{G,*}(EG_p \times M), \partial_G)$ is

$$E^1_{2k,l} = S^k(\mathbf{g})^G \otimes H_l(EG_p \times M).$$

For the complex $(\Omega_{G,*}(M), \partial_G)$, the E^1 term is

$$E^1_{2k,l} = S^k(\mathbf{g})^G \otimes H_l(M).$$

The projection $EG_p \times M \to M$ induces the usual map of filtered complexes $I: \Omega_{G,*}(EG_p \times M) \to \Omega_{G,*}(M)$ by integrating over EG_p; that is, $I(f \otimes \eta) = f \otimes \int_{EG_p} \eta$. This map induces a map on the E^1 terms of the spectral sequences given by tensoring the projection $H_l(EG_p \times M) \to H_l(M)$ with the identity on $S^k(\mathbf{g})^G$. Notice that for values of $l \leqslant p$, this projection is an isomorphism since EG_p is p-connected.

Then I induces an isomorphism on the $E^1_{2k,l}$ terms of the spectral sequences when the total degree $2k+l \leqslant p$ and hence on each later term of the sequence for which $2k+l < p$. It then follows that I induces an isomorphism on homology when $j < p$. ∎

We will now give the proof of Theorem 1 which says that our deRham model computes equivariant homology:

$$H_{G,*}(M) \cong H_*(\Omega_{G,*}(M), \partial_G)$$

when M is closed and oriented.

Proof. This follows from the two lemmas above using Poincaré duality on the finite dimensional approximations to the homotopy quotient. For j small compared to p, the definition of equivariant homology tells us that

$$\begin{aligned} H_{G,j}(M) = H_{G,j}(EG \times M) &\cong H_j((EG_p \times M)_G) \\ &\cong H_j(EG_p \times_G M) \\ &\cong H^{\dim EG_p + m - n - j}(EG_p \times_G M), \end{aligned}$$

where the last isomorphism follows from Poincaré duality. We now apply Lemmas 4 and 5 to see that

$$H^{\dim EG_p + m - n - j}(EG_p \times_G M) \cong H_j(\Omega_{G,*}(EG_p \times M), \partial_G) \cong H_j(\Omega_{G,*}(M), \partial_G).$$ ∎

Notice that Theorem 2 now follows immediately from Theorem 1 and Lemma 4.

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A few words about the functoriality of equivariant homology are now in order. First of all, G-equivariant maps $f: M \to N$ induce maps between the homotopy quotients $f: M_G \to N_G$ and hence induces a map $f_*: H_{G,*}(M) \to H_{G,*}(N)$. However, this map is not at all clear from the Cartan model for equivariant homology since differential forms behave contravariantly.[3] One should think of this as being similar to the 'umkehrungs' homomorphism which pushes forward cohomology (compare Atiyah–Bott [2]). For instance, if f is a G-equivariant fibration, the map f_* is induced by integrating differential forms over the fibration. Similarly, G-equivariant inclusions will factor through the Thom isomorphism relating the homology of a submanifold to the homology of its normal bundle. It is enough to understand these two cases since any map is the composition of an inclusion and a fibration.

Of course, there is another map $f^*: H_{G,*}(N) \to H_{G,\dim M - \dim N + *}(M)$ given by pulling back differential forms. This map intertwines the induced map in cohomology with Poincaré duality. For proper G-maps, it is the dual of the 'umkehrungs' homomorphism in equivariant cohomology; that is,

$$\langle f^*\eta, \omega \rangle = \langle \eta, f_* \omega \rangle.$$

2·3. *Fibred products and examples.* Fibred products arise naturally in many applications so we would like to understand how to pair equivariant cohomology and homology classes in fibred products. Suppose that a G-bundle M can be written as the fibred product

$$M \equiv M_1 \times_F M_2,$$

where M_1 is a principal G-bundle, M_2 and F are G-manifolds, and the fibred product is defined by G-equivariant maps $e^i: M_i \to F$. Given classes $a_i \in H^*_G(M_i)$, the Künneth map together with restriction define an equivariant cohomology class $a_1 \boxtimes a_2$ on the free G-space M. Denoting the projections by $p_i: M \to M_i$, this can be written as $p_1^* a_1 \cup p_2^* a_2$, restricted to M_G.

THEOREM 6. *Let*

$$\psi_1: H^*_G(M_1) \to H_{G,*}(F): a_1 \to (e^1)_* ([M_1]_G \cap a_1)$$

and denote by Ψ_2 *integration over the fibre* $H^*_G(M_2) \to H^*_G(F)$. *Then*

$$\langle [M]_G, a_1 \boxtimes a_2 \rangle_M = \langle \psi_1(a_1), \Psi_2(a_2) \rangle_F. \tag{1}$$

Proof. We have the following diagram of spaces:

$$\begin{array}{ccc} M_G & \xrightarrow{p_2} & (M_2)_G \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle e^2} \\ (M_1)_G & \xrightarrow{e^1} & F_G. \end{array}$$

Notice that the fibre for e^2 agrees with that for p_1 so that

$$(p_1)_* (p_2)^* (b) = (e^1)^* (e^2)_* b.$$

[3] We note here that when implementing a map, such as pull back or push forward, between G-manifolds on equivariant differential forms, we apply it to the form part of the equivariant differential form, and leave the Lie algebra part untouched.

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Now:

$$\begin{aligned}\langle [M]_G, p_1^* a_1 \cup p_2^* a_2 \rangle_M &= \langle [M_1]_G, a_1 \cup p_{1*} p_2^* a_2 \rangle_{M_1} \\ &= \langle [M_1]_G, a_1 \cup e_1^* e_{2*} a_2 \rangle_{M_1} \\ &= \langle (e^1)_* ([M_1]_G \cap a_1), (e^2)_* a_2 \rangle_F\end{aligned}$$

which proves the proposition. ∎

We end this section with an example which illuminates the usefulness of equivariant homology. Suppose that $G = SU(2)$ acts freely on M_1, a $(2d+1)$ dimensional manifold, with quotient map $\pi: M_1 \to M_1/SU(2)$ and first Pontrjagin class p_1. Suppose, furthermore, that $M_2 = S^2 = SU(2)/S^1$ and that F is a point. The equivariant homology and cohomology groups are easily described: $H_G^*(M_2) = \mathbb{R}[v]$ where v has degree 2, $H_{G,*}(F) = \mathbb{R}[\hat{u}]$ and $H_G^*(F) = \mathbb{R}[u]$ where $\hat{u}$ and u are dual generators of degree 4. In this case, the fibred product is the usual product $M = M_1 \times M_2$ and M/G is identified with M_1/S^1.

We wish to evaluate $\langle [M]_G, 1 \boxtimes a_2 \rangle_M$ where $1 \in H_G^*(M_1)$ and $a_2 = v^d \in H_G^{2d}(S^2)$. To apply Theorem 6, we must evaluate $\psi(1)$ and $\Psi(a_2)$.

LEMMA 7.

$$\psi(1) = \int_{M_1} \gamma = p_1^{(d-1)/2} \hat{u}^{(d-1)/2} \in H_{G,2d-2}(F),$$

where $\hat{u} \in S^2(\mathfrak{g})^G$ *pairs with the generator* $u \in H_G^4(F)$ *to give* 1.

Proof. The proof follows easily from Corollary 3. Notice that $\langle \psi(1), u^{(d-1)/2} \rangle_F = \langle \gamma, u^{(d-1)/2} \rangle_{M_1}$ since integration of differentials forms and contraction with invariant polynomials commute with one another. But Corollary 3 shows that $\langle \gamma, u^{(d-1)/2} \rangle_{M_1}$ equals the characteristic number of the bundle which is one half the Pontrjagin class raised to the appropriate power. ∎

The evaluation of $\Psi(a_2)$ is more straightforward. The generating class, v, may be written as $v = vol_{S^2} + \phi_i \otimes h_i$ where h_i satisfies $dh_i = i_{X_i} vol_{S^2}$, i.e. it is the height function on S^2 defined by the axis of the rotation determined by X_i. It is then clear that $\Psi(v) = 1$ and more generally

$$\Psi(v^d) = \begin{cases} u^{(d-1)/2} & \text{for odd } d \\ 0 & \text{otherwise.} \end{cases}$$

The example concludes by applying the proposition to obtain

$$\langle [M]_G, 1 \boxtimes a_2 \rangle = \langle \psi(1), \Psi(v^d) \rangle_F = p_1^{(d-1)/2}.$$

More generally, one sees that characteristic classes of the free G-space will pair with the equivariant cohomology on the space with isotropy. For instance, if the fibred product is taken over $F = S^2$, one will detect the first Chern class of the circle subbundle in M_1 defined by a fibre of the map $e^1: M_1 \to S^2$.

3. *The Axelrod–Witten complex and the Duflo–Vergne construction*

In this section we discuss a complex, due to Axelrod–Witten [5], which arises naturally from topological quantum field theory. It replaces the auxiliary algebraic constructions with similar geometric constructions, and the resulting complex is similar to the deRham complex on the G-manifold times the Lie algebra. At first sight, the complex may seem somewhat unwieldy. However, we will see that it comes

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in many guises which not only cover the known cohomology and homology theory and pairing, but additionally give some new theories which remain largely unexplored.

3·1. *The Axelrod–Witten double complex.* Consider once more the G-manifold M and let $\mathbf{g}_{\mathbb{C}}$ be the complexified Lie algebra of G. Axelrod and Witten introduced the complex

$$CW^*(M) \equiv (\Omega^*(M;\mathbb{C}) \otimes \Omega^{0,*}(\mathbf{g}_{\mathbb{C}}))^G$$

with the following filtration and boundary operator. An element lies in the subgroup $\mathscr{F}^j$ defined by the filtration if it is composed of a differential form on M and a form $f(\phi, \bar{\phi})\, d\bar{\phi}_I$ on $\mathbf{g}_{\mathbb{C}}$ which vanishes to order o at the origin and $2o + |I| = j$ where $|I|$ is the usual degree of a $(0, *)$ form. The boundary operator equals:

$$d + \Sigma_\alpha \phi_\alpha i_{X_\alpha} + \bar{\partial},$$

where ϕ_α are now considered as complex linear functions on $\mathbf{g}_{\mathbb{C}}$. Note that in the M direction we have the usual equivariant coboundary operator and vector fields from the action of G on its Lie algebra are not part of this coboundary operator.

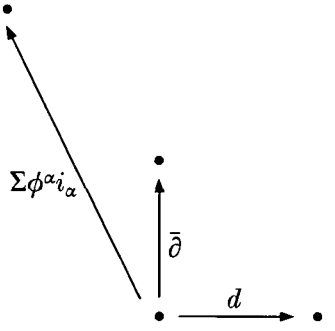


Fig. 1.

There is a map from the Cartan model for equivariant cohomology into the Axelrod–Witten complex: it is the identity on $\Omega^*(M)$ tensored with the inclusion of real polynomials into holomorphic functions, considered as elements of $\Omega^{0,0}(\mathbf{g}_{\mathbb{C}})$. This map preserves gradings. We see that we can extend this to a map from a 'completed' version of the Cartan complex to Axelrod–Witten's complex:

$$(\Omega^*(M) \otimes H^{0,0}(\mathbf{g}_{\mathbb{C}}))^G \to CW^*(M).$$

THEOREM 8. *This map induces an isomorphism on cohomology.*

Proof. This is easily proven using the fact that a chain map of filtered complexes induces an isomorphism on cohomology if it induces an isomorphism on one of the E_j terms of the spectral sequence.

The filtration under consideration is by the degree of elements in $H^{0,0}(\mathbf{g}_{\mathbb{C}})$ and $\Omega^{0,*}(\mathbf{g}_{\mathbb{C}})$ respectively. We see that the E_2^{ij} terms of both complexes equal $(H^i(M) \otimes (H^{0,0}(\mathbf{g}_{\mathbb{C}}))^j)^G$.

3·2. *Homology.* Axelrod and Witten indicated that there are many small variations on the complex and here we present one which computes equivariant homology. This modification lies in replacing the forms on $\mathbf{g}_{\mathbb{C}}$ with distributional $(\dim \mathbf{g}, *)$-forms[4] on

[4] A more symmetrical relation between the cohomology and homology case can be acquired by giving each complex coefficients in the polarization $K^{\frac{1}{2}}$ where K is the canonical bundle of $\mathbf{g}_{\mathbb{C}}$. The Killing form trivialises this polarization and we could use $\Omega^{0,*}$ in both the cohomological and homological situation.

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$\mathbf{g}_{\mathbb{C}}$ with compact support. Denoting the distributions with compact support on a space X with values in a topological vector space E by $\mathscr{E}'(X,E)$ we can define the filtered complex as:

$$CW_*(M) = \mathscr{E}'^{0,*}(\mathbf{g}_{\mathbb{C}}, \Omega^*(M)).$$

We use the distributional form $f(\phi,\bar{\phi})\,dV_\phi\,d\bar{\phi}_I$ to define the filtration. The Fourier transform $\mathscr{F}(f)$ of the compactly supported distribution f is a real analytic function on the vector space $\mathbf{g}_{\mathbb{C}}^*$. If $(X,\bar{X})$ is a pair of a holomorphic coordinate and its conjugate on $\mathbf{g}_{\mathbb{C}}^*$ then we can determine the order $O(f)$ of vanishing of $\mathscr{F}(f)$ as a function of X, i.e. the smallest m such that $(\nabla_\partial)^m \mathscr{F}(f)(0)$ is not zero. We then declare that this element lies in the filtered subgroup indexed by $2O(f)+\dim \mathbf{g}-|I|$. The filtration could be paraphrased as giving its holomorphic order as a distribution. The boundary operator will still be:

$$\mathscr{D} = d+\bar{\partial}+\sum \phi_\alpha\, i_{X_\alpha}.$$

The boundary operator defines a filtered complex as seen by noting the effect of multiplication by a ϕ_α after Fourier transform. It changes to $1/i\,\partial/\partial X_\alpha$ so the order of vanishing decreases.

We will now explore the relation between this complex and the deRham model of equivariant homology. Under Fourier transform

$$S^*(\mathbf{g}) \to \mathscr{E}'^{n,n}(\mathbf{g}_{\mathbb{C}}), \quad \text{with } n = \dim \mathbf{g},$$

by mapping a polynomial $p(X_\alpha)$ to a complex derivative $p(1/i\,\partial/\partial\phi_\alpha)\,\delta$ of the δ function. Alternatively one can interpret $S^*(\mathbf{g})$ as a space of holomorphic polynomial functions on the vector space $\mathbf{g}_{\mathbb{C}}^*$ and apply the Fourier transform. Note that the degree of a polynomial corresponds with the order of the distribution which we obtain. Tensoring this with the identity on forms we have a map of complexes which again preserves the filtration, as illustrated in figure 2.

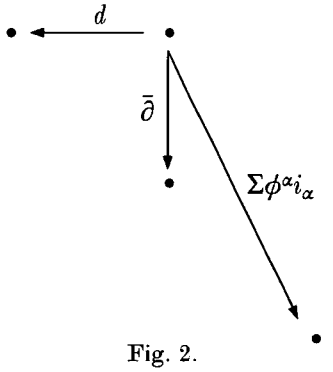


Fig. 2.

Following the reasoning of the cohomological situation we need to determine the space of holomorphic distributional forms with compact support. This is most easily seen through Serre duality:

LEMMA 9. *Let V be a complex vector space of dimension k. The cohomology of $\mathscr{E}^{k,*}(V)$ is zero in all dimensions but k. It has a completion isomorphic to the space of hyperfunctions on V, which is the dual to $H^{0,0}(V)$, the space of holomorphic functions on V, through an extension of the integral pairing.*

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Proof. See Serre[**15**]. ∎

We now want to consider the complex:

$$(\Omega^*(M) \otimes H^{n,n}(\mathscr{E}'^{*,*}(\mathbf{g}_\mathbf{C})))^G$$

on which the cohomological boundary operator acts. We note that under Fourier transform this becomes a completion of the homological complex. Now we have:

THEOREM 10. *There is a chain map*:

$$(\Omega^{\dim M-*}(M) \otimes H^{n,n}(\mathscr{E}'^{*,*}(\mathbf{g}_\mathbf{C})))^G \to CW_*(M)$$

which induces an isomorphism on homology.

3·3. *Pairings and L^2-equivariant (co)homology.* Axelrod and Witten went further with their complex and also pointed out that there is an obvious pairing between the cycles in CW_* and CW^*:

$$\langle a, \beta \rangle = \int_M \int_{\mathbf{g}_\mathbf{C}} a \wedge \beta. \tag{2}$$

We see the following proposition:

PROPOSITION 11. *The pairing in the Axelrod–Witten complex is the natural pairing between equivariant homology and cohomology.*

A more delicate complex can also be built. Instead of taking all forms on $\mathbf{g}_\mathbf{C}$ we could work with the complex of forms on $M \times \mathbf{g}_\mathbf{C}$ which are of type $(0, *)$ along $\mathbf{g}_\mathbf{C}$ and have finite L^2-norm. For compact M this is essentially a condition on the behaviour of the form in the $\mathbf{g}_\mathbf{C}$ direction. This complex has a bilinear pairing on it by integrating the wedge product of forms over $M \times \mathbf{g}_\mathbf{C}$. The resulting equivariant cohomology group could be called L^2-equivariant cohomology and probably satisfies Poincaré duality.

3·4. *Comparison with the Duflo–Vergne construction.* Another equivariant cohomology complex closely related to ours was invented by Duflo–Vergne[**9**], and studied further by Kumar and Vergne[**13**]. They consider the complex:

$$DF_* \equiv \mathscr{D}'(\mathbf{g}, \Omega^{\dim M-*}(M))^G$$

of G-invariant distributions on the *real* Lie algebra with values in differential forms on M. This complex is supplied with the usual cohomological boundary operator $d - 2\pi \sum_\alpha \phi_\alpha i_{X_\alpha}$. This is an interesting variation on a formulation of equivariant cohomology, which has been around for some time, using the complex:

$$C^\infty(\mathbf{g}, \Omega^*(M))^G$$

which is a completion of the Cartan model.

Fourier transformation maps $S^*(\mathbf{g})$, regarded as the space of polynomial functions on $\mathbf{g}^*$, to distributions on $\mathbf{g}$: a polynomial in $S^*(\mathbf{g})$ defines a derivative of the delta function on $\mathbf{g}$. There is then a well defined chain map:

$$\Omega_{G,*}(M) \to DF_*$$

where in fact we are mapping into distributions with compact support. In passing we note that it is now clear how to filter the complex DF_*.

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In order to relate the construction to ours we note that for any G invariant subset $S \subset \mathbf{g}$ we can consider a complex DF_*^S of distributions on $\mathbf{g}$ supported in S and with values in differential forms on M. The above discussion shows:

THEOREM 12. *There is a natural isomorphism between* $H_{G,*}(M)$ *and* $H_*(DF_*^{\{0\}})$.

It seems as if all classes studied by Duflo–Vergne and Kumar–Vergne are in fact classes represented by compactly supported distributions. It will be interesting to see where the distributions without compact support play a role.

4. *Poincaré duality and Witten's class*

In this section, we will show how equivariant homology allows one to write useful formulas for the Euler class of a bundle. For instance, we recover Witten's formal expression for the Donaldson invariant and Jones polynomial following the framework of Atiyah–Jeffrey [3]. This formula is an instance of the following remarkable metaprinciple from physics: it is possible to evaluate integrals over the solution space of some equation (expectation values) without explicitly solving the equation itself.

4·1. *The fundamental class.* In various applications it is more useful to write the fundamental class as an integral (this was also discussed by Kumar and Vergne [13]). We use n to denote the dimension of $\mathbf{g}$ and begin by using the Berezin integral to write:

$$\gamma = \int_{\mathbf{g}} e^{(\omega,\eta)+\frac{1}{2\pi}\Omega_\alpha \otimes X_\alpha} \mathscr{D}\eta, \tag{3}$$

where the integration is defined as a functional on $\Lambda^*(\mathbf{g})$ mapping forms of degree less than $\dim \mathbf{g}$ to zero, and an oriented volume form $\eta_1 \wedge \ldots \wedge \eta_n$, constructed using the Killing metric $(\ ,\ )$, to one.

Furthermore, when we want to regard $\gamma \in \Omega^*(M) \otimes S^*(\mathbf{g})$ as a distribution on $\mathbf{g}$ with values in $\Omega^*(M)$, we use the Fourier transform to write:

$$\gamma(\phi) = \frac{1}{(2\pi)^n} \int_{\mathbf{g}} \int_{\mathbf{g}} e^{i(X,\phi)}\, e^{(\omega,\eta)+\frac{1}{2\pi}(\Omega, X)} \mathscr{D}\eta\, dX. \tag{4}$$

Here $\gamma(\phi)$ is expressed as a generalised function on $\mathbf{g}$ in the variable ϕ. It follows that for a function $f(\phi)$ on $\mathbf{g}$, the differential form on M resulting from the distributional pairing of γ and f is

$$\langle \gamma, f \rangle = \int_{\mathbf{g}} \gamma(\phi) f(\phi)\, d\phi = \frac{1}{(2\pi)^n} \int_{\mathbf{g}} \int_{\mathbf{g}} \int_{\mathbf{g}} e^{i(X,\phi)}\, e^{(\omega,\eta)+\frac{1}{2\pi}(\Omega, X)} f(\phi)\, \mathscr{D}\eta\, dX\, d\phi. \tag{5}$$

4·2. *Witten's class.* In his important paper on topological field theory [17], Witten derived integral formulas for cohomological computations that occur in gauge theories. The starting point is a free Riemannian G-space P and an equivariant map $s_G : P \to V$ where V is a representation of G. The question now is to take a cohomology class $\Theta \in H^*(P/G)$ and compute

$$\int_{s_G^{-1}(0)/G} \Theta, \tag{6}$$

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in terms of an integral over P, V and various auxiliary spaces. One (infinite-dimensional) application is to the case when P is the space of connections over a 4-manifold, G the gauge group, V the space of self-dual 2-forms on P with values in $\mathbf{g}$ and s_G assigns to a connection the self-dual part of its curvature. The space $s_G^{-1}(0)/G$ functions as a model for the moduli space of instantons and the integral (6) is the Donaldson invariant of the 4-manifold. Perhaps the key point of the procedure is that Witten's integral expression avoids the solution of the classical problem $s_G(p) = 0$ by integrating over unconstrained fields.

Let $V_G = P \times_G V$ be the vector bundle associated to P and $e(V_G)$ its Euler class. Then (6) equals

$$\int_{P/G} e(V_G)\,\Theta.$$

Next we use the Thom class U_V of V_G and the section s of V_G defined by s_G to transform the integral to:

$$\int_{P/G} s^*(U_V)\,\Theta.$$

At this time it becomes useful to realise that both Θ and U_V arise from equivariant cohomology classes $\tilde{\Theta}$ and $\tilde{U}_V$ on P and therefore we can use (5) to write the integral as:

$$\int_P \int_{\mathbf{g}} \gamma(\phi)\,\tilde{U}_V(\phi)\,\tilde{\Theta}(\phi)\,d\phi,$$

where we have identified $S^*(\mathbf{g})$ with distributions on $\mathbf{g}$, $S^*(\mathbf{g}^*)$ with polynomial functions on $\mathbf{g}$ and used the integral notation to denote the pairing.

Witten's integral is now easily obtained using Mathai–Quillen's formula for the Thom class (see Mathai–Quillen[**14**]). This gives a different perspective on the computations of Atiyah–Jeffrey[**3**] which accomplish a similar task. Mathai–Quillen's formula for the Thom form, as an equivariant cohomology class, is

$$\tilde{U}_V(\phi, x) = \pi^{-m} e^{-|s_G(x)|^2} \int_V e^{\chi^t \phi \chi + i d s_G(x)^t \chi} \mathscr{D}\chi \in \Omega^*(P) \otimes S^*(\mathbf{g}^*),$$

where $m = \dim V$, $\mathscr{D}\chi$ indicates Berezin integration and x is a point of P. Furthermore, we use (5) to obtain

$$\int_{P/G} e(V_G)\,\Theta = \frac{1}{(2\pi)^n \pi^m} \int_P \int_{\mathbf{g}} \int_{\mathbf{g}} \int_{\mathbf{g}} \int_V \exp\Big\{ -|s_G(x)|^2 + i(X, \phi) + (\omega, \eta) + \frac{1}{2\pi}(\Omega, X) + \chi^t \phi \chi + i d s_G(x)^t \chi \Big\} \tilde{\Theta}(\phi)\,\mathscr{D}\chi \mathscr{D}\eta\,dX\,d\phi.$$

What remains is to make use of the Riemannian metric on P to write an expression for a connection on P. Following Atiyah–Jeffrey[**3**], we express the infinitesimal action of the group at a point through the isomorphism of the Lie algebra with the vertical tangent space at a point $p \in P$ as $C: \mathbf{g} \to T_p P$. Using the inner product we define a 1-form ν on P with values in $\mathbf{g}^*$ by $\nu(v)(\xi) = \langle v, C\xi \rangle$, where $v \in T_p P$ and $\xi \in \mathbf{g}$. Notice that the kernel of ν is the orthogonal complement to the fibre. The transformation $R = C^*C$ relates the Killing metric on $\mathbf{g}$ with the metric induced from

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T_pP through the map C; that is, $\langle C\xi, C\zeta\rangle = (R\xi, \zeta)$. The connection on P defined by orthogonal projection in the metric has connection 1-form

$$w = R^{-1}\nu$$

and hence the curvature on the horizontal subspace of T_pP is $\Omega = dw = R^{-1}d\nu$.

We may now write

$$\int_{P/G} e(V_G)\,\Theta = \frac{1}{(2\pi)^n\pi^m}\int_P\int_{\mathfrak{g}}\int_{\mathfrak{g}}\int_{\mathfrak{g}}\int_V \exp\Big\{-|s_G(x)|^2 + i(X,\phi) + (R^{-1}\nu,\eta) + \frac{1}{2\pi}(R^{-1}d\nu, X) + \chi^t\phi\chi + ids_G(x)^t\chi\Big\}\tilde{\Theta}(\phi)\,\mathscr{D}\chi\mathscr{D}\eta\,dX\,d\phi.$$

Making the substitution $X = R\lambda$ gives

$$\int_{P/G} e(V_G)\,\Theta = \frac{1}{(2\pi)^n\pi^m}\int_P\int_{\mathfrak{g}}\int_{\mathfrak{g}}\int_{\mathfrak{g}}\int_V \exp\Big\{-|s_G(x)|^2 + i(R\lambda,\phi) + \langle dA,\eta\rangle + \frac{1}{2\pi}(d\nu,\lambda) + \chi^t\phi\chi + ids_G(x)^t\chi\Big\}\tilde{\Theta}(\phi)\,\mathscr{D}\chi\mathscr{D}\eta\,d\lambda\,d\phi$$

where we have also used coordinates A_i on P to express the integral $\exp(\omega,\eta)\,\mathscr{D}\eta$ in terms of the metric on P. This agrees with Atiyah and Jeffrey [3, equation (2·13)].

At this point we see that the formula can be interpreted as follows:

THEOREM 13. *The fundamental class γ of the moduli space $s_G^{-1}(0) \subset P$ in the equivariant homology of P can be expressed as an equivariant differential form (closed for the homological boundary operator)*:

$$\gamma = \frac{1}{(2\pi)^n\pi^m}\int_{\mathfrak{g}}\int_{\mathfrak{g}}\int_V \exp\{-|s_G(x)|^2 + i(R\lambda,\phi) + \langle dA,\eta\rangle + \frac{1}{2\pi}(d\nu,\lambda) + \chi^t\phi\chi + ids_G(x)^t\chi\}\,\mathscr{D}\chi\mathscr{D}\eta\,d\lambda.$$

The integral formula of Witten and Atiyah–Jeffrey equals the pairing.

$$\langle [s_G^{-1}(0)/G], \Theta\rangle = \langle \gamma, \tilde{\Theta}\rangle$$

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