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A NEW QUADRILATERAL MINI-ELEMENT FOR STOKES EQUATIONS

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Abstract. We introduce a new stable MINI-element pair for incompressible Stokes equations on quadrilateral meshes, which uses the smallest number of bubbles for the velocity. The pressure is discretized with the P_1-midpoint-edge-continuous elements and each component of the velocity field is done with the standard $\mathbf{Q}_1$-conforming elements enriched by one bubble a quadrilateral. The superconvergence in the pressure of the proposed pair is analyzed on uniform rectangular meshes, and tested numerically on uniform and non-uniform meshes.

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1. Introduction

In the finite element methods for incompressible Stokes problems, the pair of discrete velocity and pressure should fulfill the inf-sup condition for the stability, which is attributed to the analysis of Babuška [2] and Brezzi [4]. On triangular meshes, the MINI-element pair [1] of "$[P_1$-conforming $+$ bubble$]^2 \times P_1$-continuous" is one of the stable pairs of the lowest order, since the simplest pair of "$[P_1$-conforming $]^2 \times P_0$" is not stable.

The several pairs of "$\big([\mathbf{Q}_1$-conforming $+$ bubble$]^2 +$ bubble$\big) \times \mathbf{Q}_1$-continuous" have been suggested as the MINI-element on quadrilateral meshes [3,7]. Their discrete velocities are the standard conforming spaces enriched by at least three bubbles a quadrilateral, while the triangular MINI-element is done by two bubbles a triangle.

In this paper, we will introduce a new stable MINI-element pair on quadrilateral meshes which is "$[\mathbf{Q}_1$-conforming $+$ bubble$]^2 \times P_1$-midpoint-edge-continuous". Compared to other quadrilateral MINI-elements, the discrete pressure has continuous averages over edges instead of the pointwise continuity. The number of bubbles for the velocity is then reduced to two a quadrilateral, similar to the triangular MINI-element. The proposed MINI-element pair is interpreted as a subspace of the stable "$[\mathbf{Q}_2$-conforming$]^2 \times$ piecewise P_1" on quadrilateral meshes.

Regarding the finite element solutions with the MINI-elements, although the order of convergence in pressure is analyzed to be $O(h)$ by the standard error analysis, the superconvergence of order $O(h^{3/2})$ is fully understood on three-directional triangular meshes with the aid of the stabilized formulation [6]. We will analyze the similar superconvergence for the pressure of the proposed MINI-element on uniform rectangular meshes, as well as numerical tests on both uniform and non-uniform meshes.

In the following section, our MINI-element will be defined and proven to be stable through the inf-sup condition. Section 3 will be assigned for the superconvergence in pressure, which consists of three subsections

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for the stabilized formulation of the finite element method, the supercloseness of the bilinear interpolation and the main proof of the superconvergence, respectively. The paper will be closed with the numerical results in the final section.

Throughout the paper, standard notations for Sobolev spaces are employed and $L^2_0(\Omega)$ denotes the space of all functions in $L^2(\Omega)$ whose average over Ω vanish. For a bounded set $S \subset \mathbb{R}^2$, with its boundary ∂S, $\|\cdots\|_{m,S}$ and $|\cdots|_{m,S}$ will mean the norm and seminorm for $H^m(S)$. We will denote by $(\cdot,\cdot)_S$ and $<\cdot,\cdot>_{\partial S}$ the $L^2(S)$ and $L^2(\partial S)$ inner products, respectively. If $S = \Omega$ or $m = 0$, they may be omitted in the indices.

2. A new quadrilateral MINI-element

Let Ω be a simply connected polygonal domain and $\{\mathcal{T}_h\}_{h>0}$ a regular family of triangulations of Ω which consist of convex quadrilaterals. In $\mathcal{T}_h$, h is proportional to the maximum of diameters of quadrilaterals in $\mathcal{T}_h$.

2.1. P_1-midpoint-edge-continuous quadrilateral pressure

Let P_h be a space of piecewise linear functions as follows:

$$P_h = \left\{ q_h \in L^2(\Omega) \;\; q_h|_Q \in \text{span}\{1, x, y\} \text{ for all quadrilaterals } Q \in \mathcal{T}_h \right\},$$

then the P_1-midpoint-edge-continuous quadrilateral finite element space [9] is defined by

$$\mathcal{NC}^h(\Omega) = \{ q_h \in P_h \;\; q_h \text{ is continuous at every midpoints of edges in } \mathcal{T}_h \}.$$

For each vertex $\mathbf{V}$ in $\mathcal{T}_h$, a function $\psi^{\mathbf{V}} \in \mathcal{NC}^h(\Omega)$ is defined by its values at all midpoints m of edges in $\mathcal{T}_h$ such that

$$\psi^{\mathbf{V}}(m) = \begin{cases} 1, & \text{if } m \text{ belongs to an edge which meets } \mathbf{V}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, if we fix an arbitrary vertex $\mathbf{V}_0$ in $\mathcal{T}_h$, the following set is a basis for $\mathcal{NC}^h(\Omega)$,

$$\{ \psi^{\mathbf{V}} \in \mathcal{NC}^h(\Omega) \;\; \mathbf{V} \text{ is a vertex in } \mathcal{T}_h, \;\; \mathbf{V} \neq \mathbf{V}_0 \}.$$

Hence, the dimension of $\mathcal{NC}^h(\Omega)$ is the number of vertices in $\mathcal{T}_h$ subtracted by one.

Define an interpolation $\pi_h : H^1(\Omega) \cap C(\Omega) \to \mathcal{NC}^h(\Omega)$ as

$$\pi_h v = \sum_{\mathbf{V}} \frac{v(\mathbf{V})}{2} \psi^{\mathbf{V}}, \quad \forall v \in H^1(\Omega) \cap C(\Omega), \tag{2.1}$$

where the summation runs over all interior vertices $\mathbf{V}$ in $\mathcal{T}_h$. Then, the interpolation error is estimated by the following, for $v \in H^2(\Omega)$,

$$\|v - \pi_h v\|_0 + h \left(\sum_{Q \in \mathcal{T}_h} |v - \pi_h v|^2_{1,Q} \right)^{1/2} \leq C\, h^2\, |v|_2 . \tag{2.2}$$

It is noted that $\mathcal{NC}^h(\Omega)$ is a common subspace of other low order nonconforming spaces such as the Rotated Q_1 or DSSY finite element spaces [5, 10].

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2.2. A new MINI-element pair

Let $\widehat{Q} = [-1,1]^2$ be a reference rectangle and $\widehat{\mathfrak{Q}}_1, \widehat{\mathfrak{Q}}_2$ bilinear and biquadratic polynomial spaces on $\widehat{Q}$, respectively, that is,

$$\widehat{\mathfrak{Q}}_1 = \text{span}\left\{1, \widehat{x}, \widehat{y}, \widehat{x}\widehat{y}\right\}, \quad \widehat{\mathfrak{Q}}_2 = \widehat{\mathfrak{Q}}_1 \bigoplus \text{span}\left\{\widehat{x}^2, \widehat{y}^2, \widehat{x}^2\widehat{y}, \widehat{x}\widehat{y}^2, \widehat{x}^2\widehat{y}^2\right\}.$$

For each quadrilateral $Q \in \mathcal{T}_h$, the canonical bijective bilinear map from $\widehat{Q}$ to Q is denoted by F_Q and function spaces $\mathfrak{Q}_1(Q), \mathfrak{Q}_2(Q)$ on Q are defined as follows:

$$\mathfrak{Q}_k(Q) = \left\{\widehat{v} \circ F_Q^{-1} \;\; \widehat{v} \in \widehat{\mathfrak{Q}}_k\right\}, \quad k = 1, 2.$$

Then, the standard conforming finite element spaces $\mathfrak{Q}_{1,h}, \mathfrak{Q}_{2,h}$ over $\mathcal{T}_h$ are defined by

$$\mathfrak{Q}_{k,h} = \{v_h \in C(\Omega) \;\; v_h|_Q \in \mathfrak{Q}_k(Q) \text{ for all quadrilaterals } Q \in \mathcal{T}_h\},$$

as well as $\mathfrak{Q}_{k,h,0} = \mathfrak{Q}_{k,h} \cap H_0^1(\Omega)$ for $k = 1, 2$.

It is well-known that the pair of $[\mathfrak{Q}_{2,h,0}]^2$ for velocity and $P_h \cap L_0^2(\Omega)$ for pressure satisfy the inf-sup condition for Stokes equations in the following proposition, whose proof is referred to Section 3.2, Chapter II in [8].

Proposition 2.1. *Let $p_h \in P_h \cap L_0^2(\Omega)$. There exists $\mathbf{v}_h \in [\mathfrak{Q}_{2,h,0}]^2$ such that*

$$\int_\Omega p_h \text{ div } \mathbf{v}_h \text{ d}s = \; ||p_h||_{0,\Omega}^2, \quad |\mathbf{v}_h|_{1,\Omega} \leq \frac{1}{\beta}||p_h||_{0,\Omega},$$

for a positive constant β which depends only on Ω.

In order to introduce a MINI-element pair, we choose a bubble function $b_Q \in \mathfrak{Q}_2(Q)$ for each quadrilateral $Q \in \mathcal{T}_h$ such that

$$b_Q \circ F_Q = \left(\widehat{x}^2 - 1\right)\left(\widehat{y}^2 - 1\right) \in \widehat{\mathfrak{Q}}_2, \tag{2.3}$$

which vanishes on the boundary of Q. Then, identifying b_Q with its trivial extension into $C(\Omega)$, we propose a new MINI-element pair for Stokes equations as follows:

$$X_h = \left[\mathfrak{Q}_{1,h,0} \bigoplus_{Q \in \mathcal{T}_h} \text{span}\{b_Q\}\right]^2, \quad M_h = \mathcal{NC}^h(\Omega) \cap L_0^2(\Omega).$$

The pair $X_h \times M_h$ turns out to be stable to solve Stokes equations in the following theorem.

Theorem 2.2 (Inf-sup condition)**.** *Let $\mu_h \in M_h$. There exists $\mathbf{v}_h \in X_h$ such that*

$$\int_\Omega \mu_h \text{ div}\, \mathbf{v}_h \text{ d}s = \; ||\mu_h||_{0,\Omega}^2, \quad |\mathbf{v}_h|_{1,\Omega} \leq \frac{1}{\beta}||\mu_h||_{0,\Omega},$$

for a positive constant β which depends only on Ω.

Proof. Define an interpolation operator $\Pi_h : \mathfrak{Q}_{2,h,0} \to \mathfrak{Q}_{1,h,0} \bigoplus_{Q \in \mathcal{T}_h} \text{span}\{b_Q\}$ such that

$$\Pi_h v(\mathbf{V}) = v(\mathbf{V}), \quad \int_Q \Pi_h v \text{ d}s = \int_Q v \text{ d}s, \quad \forall v \in \mathfrak{Q}_{2,h,0}, \tag{2.4}$$

for all vertices $\mathbf{V}$ and quadrilaterals Q in $\mathcal{T}_h$. Then, by Bramble lemma, we have

$$|v - \Pi_h v|_{1,Q} \leq Ch|v|_{2,Q}, \quad \text{for all quadrilaterals } Q \in \mathcal{T}_h, \tag{2.5}$$

with some constant C regardless of v, Q and h.

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Applying the argument of Lemma A.6, Appendix A in [8] for triangles into quadrilaterals, the following inverse inequality is established with some constant C,

$$|v|_{2,Q} \leq Ch^{-1}|v|_{1,Q}. \tag{2.6}$$

From (2.5) and (2.6), the operator Π_h is bounded by

$$|\Pi_h v|_{1,\Omega} \leq C|v|_{1,\Omega}, \quad \forall v \in \mathfrak{Q}_{2,h,0}. \tag{2.7}$$

Now, if $\mu_h \in M_h$, by Proposition 2.1, there exists $\mathbf{w}_h \in [\mathfrak{Q}_{2,h,0}]^2$ such that

$$\int_\Omega \mu_h \ \mathrm{div}\, \mathbf{w}_h \ \mathrm{d}s = \ ||\mu_h||^2_{0,\Omega}, \quad |\mathbf{w}_h|_{1,\Omega} \leq \frac{1}{\beta}||\mu_h||_{0,\Omega}, \tag{2.8}$$

for a positive constant β which depends only on Ω. For $\mathbf{w}_h = (w_1, w_2)$, define $\mathbf{v}_h \in X_h$ by

$$\mathbf{v}_h = (\Pi_h w_1, \Pi_h w_2).$$

Then, it will be shown

$$\int_\Omega \mu_h \ \mathrm{div}\, \mathbf{w}_h \ \mathrm{d}s = \int_\Omega \mu_h \ \mathrm{div}\, \mathbf{v}_h \ \mathrm{d}s. \tag{2.9}$$

We have, by integration by parts,

$$\int_\Omega \mu_h \ \mathrm{div}\,(\mathbf{w}_h - \mathbf{v}_h) \ \mathrm{d}s = \sum_{Q\in\mathcal{T}_h} \int_{\partial Q} \mu_h(\mathbf{w}_h - \mathbf{v}_h)\cdot \mathbf{n}\, \mathrm{d}l - \sum_{Q\in\mathcal{T}_h} \int_Q \nabla\mu_h \cdot (\mathbf{w}_h - \mathbf{v}_h)\, \mathrm{d}s, \tag{2.10}$$

where $\mathbf{n}$ is the unit vector which is outward normal to ∂Q. Since $\nabla\mu_h$ is a constant vector, by definition of Π_h in (2.4), the second term in the right hand side of (2.10) is null.

Denote by m_E the midpoint of edge $E \in \mathcal{T}_h$. Since $\mathbf{w}_h - \mathbf{v}_h$ is continuous on interior edges and vanishes on boundary edges, the first term is rewritten as:

$$\sum_{Q\in\mathcal{T}_h} \int_{\partial Q} \mu_h(\mathbf{w}_h - \mathbf{v}_h)\cdot \mathbf{n}\, \mathrm{d}l = \sum_{Q\in\mathcal{T}_h} \sum_{E\in\partial Q} \int_E \big(\mu_h - \mu_h(m_E)\big)(\mathbf{w}_h - \mathbf{v}_h)\cdot \mathbf{n}\, \mathrm{d}l. \tag{2.11}$$

We note that $\mu_h - \mu_h(m_E)$ and $\mathbf{w}_h - \mathbf{v}_h$ are the linear and quadratic functions on E vanishing at the midpoint and two endpoints of E, respectively. Thus, all integrals over edges in (2.11) vanish, since the integrands are odd and cubic.

This means (2.9) and completes the proof with (2.7), (2.8). □

Let $(\mathbf{u}, p) \in [H^1_0(\Omega)]^2 \times L^2_0(\Omega)$ be the solution of the variational form of a Stokes equation such that, for a source function $\mathbf{f} \in [L^2(\Omega)]^2$,

$$(\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \mathrm{div}\,\mathbf{v}) + (q, \mathrm{div}\,\mathbf{u}) = (\mathbf{f}, \mathbf{v}), \quad \forall(\mathbf{v}, q) \in \big[H^1_0(\Omega)\big]^2 \times L^2_0(\Omega), \tag{2.12}$$

and $(\mathbf{u}_h, p_h) \in X_h \times M_h$ be the solution of the following discrete variational problem,

$$(\nabla\mathbf{u}_h, \nabla\mathbf{v}_h) - (p_h, \mathrm{div}\,\mathbf{v}_h) + (q_h, \mathrm{div}\,\mathbf{u}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall(\mathbf{v}_h, q_h) \in X_h \times M_h. \tag{2.13}$$

If $(\mathbf{u}, p)$ belongs to $[H^2(\Omega)]^2 \times H^1(\Omega)$, through the inf-sup condition in Theorem 2.2, the approximation error is estimated by

$$|\mathbf{v} - \mathbf{v}_h|_1 + \|p - p_h\|_0 \leq Ch(|\mathbf{v}|_2 + |p|_1), \tag{2.14}$$

for a constant C which depends only on Ω and the regularity of $\mathcal{T}_h$.

While the order of convergence in pressure is $O(h)$ as in (2.14) on general quadrilateral meshes, the remaining of the paper will be devoted to the analysis of the superconvergence of order $O(h^{3/2})$ on uniform rectangular meshes.

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3. Superconvergence in pressure

The superconvergence in pressure is well understood [6] for the MINI-element of Arnold *et al.* [1] on three-directional triangular meshes. In this section, following the arguments in [6] with the attention on the midpoint-edge-continuous pressure instead of the continuous one, we will prove that the proposed MINI-element has the same property. Throughout the section, we assume that $\mathcal{T}_h$ consists of uniform rectangles, whose edges are all parallel to the coordinate axes.

For a function $\mathbf{v}_h$ in X_h, the bilinear and bubble parts of $\mathbf{v}_h$ are denoted by $\mathbf{v}_L, \mathbf{v}_b$, respectively, so that:

$$\mathbf{v}_h = \mathbf{v}_L + \mathbf{v}_b, \quad \mathbf{v}_L \in [\mathbf{Q}_{1,h,0}]^2, \quad \mathbf{v}_b|_Q \in [\mathrm{span}\{b_Q\}]^2 \ \text{ for all } Q \in \mathcal{T}_h.$$

We note that they are orthogonal to each other in H^1 inner product since:

$$(\nabla \mathbf{v}_L, \nabla \mathbf{v}_b)_Q = 0, \qquad \text{for each rectangle } Q \in \mathcal{T}_h. \tag{3.1}$$

3.1. Stabilized variational form

We will show that if $(\mathbf{u}_h, p_h) \in X_h \times M_h$ satisfy (2.13), the pair of the bilinear part $\mathbf{u}_L$ of $\mathbf{u}_h$ and the pressure p_h is a solution of a stabilized variational problem.

Set a bilinear form B_h on $H_0^1(\Omega) \times L_0^2(\Omega)$ as:

$$B_h(\mathbf{u}, p; \mathbf{v}, q) = (\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) + (q, \operatorname{div} \mathbf{u}) + \sum_{Q \in \mathcal{T}_h} (-\Delta \mathbf{u} + \nabla p, \tau_Q \nabla q)_Q, \tag{3.2}$$

where τ_Q is the bubble function b_Q in (2.3) multiplied by a constant for each rectangle $Q \in \mathcal{T}_h$ such that

$$\tau_Q = \frac{\int_Q b_Q \, \mathrm{d}s}{\int_Q |\nabla b_Q|^2 \, \mathrm{d}s} \, b_Q.$$

We note that τ_Q is a nonnegative function which satisfies,

$$(\nabla \tau_Q, \nabla \tau_Q)_Q = (\tau_Q, 1)_Q = \alpha^2 h^2 (1,1)_Q, \quad \|\tau_Q\|_{\infty,Q} = \gamma^2 h^2, \tag{3.3}$$

with some fixed positive constants α and γ. From (3.3), we have an identity,

$$\left\| \tau_Q^{1/2} \nabla q_h \right\|_{0,Q} = \alpha h \, \|\nabla q_h\|_{0,Q}, \tag{3.4}$$

for all rectangles $Q \in \mathcal{T}_h$ and piecewise linear pressures $q_h \in M_h$.

Lemma 3.1. *If $(\mathbf{u}_h, p_h) \in X_h \times M_h$ satisfies (2.13), then $(\mathbf{u}_L, p_h) \in [\mathbf{Q}_{1,h,0}]^2 \times M_h$ does*

$$B_h(\mathbf{u}_L, p_h; \mathbf{v}_L, q_h) = (\mathbf{f}, \mathbf{v}_L) + \sum_{Q \in \mathcal{T}_h} (\mathbf{f}, \tau_Q \nabla q_h)_Q, \quad \forall (\mathbf{v}_L, q_h) \in [\mathbf{Q}_{1,h,0}]^2 \times M_h. \tag{3.5}$$

Proof. Let $(\mathbf{v}_L, q_h) \in [\mathbf{Q}_{1,h,0}]^2 \times M_h$, then for any bubble $\mathbf{v}_b$, (2.13) establishes that, from the orthogonality in (3.1),

$$\begin{aligned} (\nabla \mathbf{u}_L, \nabla \mathbf{v}_L) - (p_h, \operatorname{div} \mathbf{v}_L) + (q_h, \operatorname{div} \mathbf{u}_L) &= (\mathbf{f}, \mathbf{v}_L + \mathbf{v}_b) - (\nabla \mathbf{u}_b, \nabla \mathbf{v}_b) + (p_h, \operatorname{div} \mathbf{v}_b) - (q_h, \operatorname{div} \mathbf{u}_b) \\ &= (\mathbf{f}, \mathbf{v}_L) + \sum_{Q \in \mathcal{T}_h} (\mathbf{f} - \nabla p_h, \mathbf{v}_b)_Q - (\nabla \mathbf{u}_b, \nabla \mathbf{v}_b) + \sum_{Q \in \mathcal{T}_h} (\nabla q_h, \mathbf{u}_b)_Q. \end{aligned} \tag{3.6}$$

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We note that ∇q_h is piecewisely constant and $\mathbf{u}_b|_Q = \tau_Q(c_1, c_2)$ for some constants c_1, c_2. If we choose a bubble $\mathbf{v}_b$ such that $\mathbf{v}_b|_Q = \tau_Q \nabla q_h$ for each rectangle $Q \in \mathcal{T}_h$, then the following identity comes from (3.3),

$$(\nabla \mathbf{u}_b, \nabla \mathbf{v}_b) = \sum_{Q \in \mathcal{T}_h} (\nabla q_h, \mathbf{u}_b).$$

Thus, the last two terms in (3.6) vanish and (3.6) implies (3.5), since $\Delta \mathbf{u}_L = 0$. □

Although the bilinear form B_h in (3.2) is not uniformly coercive on $[\mathbf{\Omega}_{1,h,0}]^2 \times M_h$, we have a stability for B_h in the following lemma. Define a triple norm as

$$|||(\mathbf{v}, q)|||_B = \left(|\mathbf{v}|_1^2 + ||q||_0^2 + \sum_{Q \in \mathcal{T}_h} \left\| \tau_Q^{1/2} \nabla q \right\|_{0,Q}^2 \right)^{1/2}. \tag{3.7}$$

Lemma 3.2. *If $(\mathbf{v}_L, q_h) \in [\mathbf{\Omega}_{1,h,0}]^2 \times M_h$, there exists $\mathbf{w}_L \in [\mathbf{\Omega}_{1,h,0}]^2$ such that*

$$B_h(\mathbf{v}_L, q_h; \mathbf{w}_L, q_h) \geq \beta_B \; |||(\mathbf{v}_L, q_h)|||_B \; |||(\mathbf{w}_L, q_h)|||_B, \tag{3.8}$$

where β_B is a constant which depends only on Ω.

Proof. By the inf-sup condition in Theorem 2.2, there exists $\mathbf{z}_h \in X_h$ such that

$$(-q_h, \operatorname{div} \mathbf{z}_h) = ||q_h||_0^2, \quad |\mathbf{z}_h|_1 \leq \frac{1}{\beta} ||q_h||_0, \tag{3.9}$$

with some constant β regardless of h.

For the bubble part $\mathbf{z}_b$, we expand, from (3.4),

$$\begin{aligned} |(q_h, \operatorname{div} \mathbf{z}_b)| &= \sum_{Q \in \mathcal{T}_h} (\nabla q_h, \mathbf{z}_b)_Q \leq \sum_{Q \in \mathcal{T}_h} h \, \|\nabla q_h\|_{0,Q} \, |\mathbf{z}_b|_{1,Q} \\ &= \alpha^{-1} \sum_{Q \in \mathcal{T}_h} \left\| \tau_Q^{1/2} \nabla q_h \right\|_{0,Q} |\mathbf{z}_b|_{1,Q} \\ &\leq \alpha^{-1} \left(\sum_{Q \in \mathcal{T}_h} \left\| \tau_Q^{1/2} \nabla q_h \right\|_{0,Q}^2 \right)^{1/2} \left(\sum_{Q \in \mathcal{T}_h} |\mathbf{z}_b|_{1,Q}^2 \right)^{1/2} \\ &\leq \frac{\beta^2}{2} |\mathbf{z}_b|_1^2 + \frac{1}{2\alpha^2 \beta^2} \left(\sum_{Q \in \mathcal{T}_h} \left\| \tau_Q^{1/2} \nabla q_h \right\|_{0,Q}^2 \right), \end{aligned} \tag{3.10}$$

as well as for the bilinear part $\mathbf{z}_L$,

$$|(\nabla \mathbf{v}_L, \nabla \mathbf{z}_L)| \leq |\mathbf{v}_L|_1 |\mathbf{z}_L|_1 \leq \frac{\beta^2}{2} |\mathbf{z}_L|_1^2 + \frac{1}{2\beta^2} |\mathbf{v}_L|_1^2. \tag{3.11}$$

Then, by the orthogonality (3.1), we combine (3.9)–(3.11) to get the following:

$$B_h(\mathbf{v}_L, q_h; \mathbf{z}_L, 0) \geq \frac{1}{2} ||q_h||_0^2 - \sigma \left(|\mathbf{v}_L|_1^2 + \sum_{Q \in \mathcal{T}_h} \left\| \tau_Q^{1/2} \nabla q_h \right\|_{0,Q}^2 \right), \tag{3.12}$$

for a constant $\sigma = \max\left(1/(2\alpha^2\beta^2), 1/(2\beta^2)\right)$.

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If we set $\mathbf{w}_L = \mathbf{v}_L + 2/(1+2\sigma)\mathbf{z}_L$, then, from (3.12), $\mathbf{w}_L$ satisfies that

$$\begin{aligned} B_h(\mathbf{v}_L, q_h; \mathbf{w}_L, q_h) &= B_h(\mathbf{v}_L, q_h; \mathbf{v}_L, q_h) + 2/(1+2\sigma) B_h(\mathbf{v}_L, q_h; \mathbf{z}_L, 0) \\ &\geq 1/(1+2\sigma) |||(\mathbf{v}_L, q_h)|||_B^2. \end{aligned}$$

From (3.9), the triple norm of $(\mathbf{w}_L, q_h)$ is bounded by

$$|||(\mathbf{w}_L, q_h)|||_B \leq \big(1 + 2/(\beta + 2\beta\sigma)\big) |||(\mathbf{v}_L, q_h)|||_B.$$

Thus, $\mathbf{w}_L$ satisfies (3.8) for the constant $\beta_B = 1/(1+2\sigma)\big(1+2/(\beta+2\beta\sigma)\big)^{-1}$ which depends only on Ω. □

3.2. Supercloseness

Let $\widehat{Q} = [-1,1]^2$ be the reference rectangle and $\widehat{E} = \{(x,y) \in \widehat{Q} \;\; x = 1\}$ an edge of $\widehat{Q}$. If $\widehat{g} \in H^1(\widehat{Q})$, for a vector field $V = (x+1, 0)$ on $\widehat{Q}$, we have the following:

$$2\int_{\widehat{E}} \widehat{g}^2 \;\mathrm{d}l = \int_{\widehat{Q}} \mathrm{div}\,(\widehat{g}^2 V)\;\mathrm{d}s = \int_{\widehat{Q}} \big(\widehat{g}^2 + 2\widehat{g}\;\nabla\widehat{g}\cdot V\big)\,\mathrm{d}s.$$

Thus, if E is an edge of a rectangle Q in $\mathcal{T}_h$ and $g \in H^1(Q)$, then the following local trace theorem holds, with a constant C which depends only the shape of Q,

$$\|g\|_{0,E} \leq C\big(h^{-1}\|g\|_{0,Q}^2 + \|g\|_{0,Q}\,|g|_{1,Q}\big)^{1/2}. \tag{3.13}$$

Especially, if r_h is a linear function on Q in $\mathcal{T}_h$, since $|r_h|_{1,Q} \leq Ch^{-1}\|r_h\|_{0,Q}$, we have

$$\|r_h\|_{0,E} \leq Ch^{-1/2}\|r_h\|_{0,Q}. \tag{3.14}$$

We will establish two superclosenesses in the following two lemmas, which are the principal ingredients for the proof of the superconvergence of the pressure p_h in (2.13). For any continuous function $v \in C(\Omega)$, denote by v_I the standard bilinear interpolation of v into $\mathbf{\Omega}_{1,h}$ such that

$$v_I(\mathbf{V}) = v(\mathbf{V}) \quad \text{for all vertices } \mathbf{V} \text{ in } \mathcal{T}_h.$$

If $\mathbf{v} = (v_1, v_2) \in [C(\Omega)]^2$, then $\mathbf{v}_I = (v_{1,I}, v_{2,I})$.

Lemma 3.3. *Let $\mathbf{u} \in [H^3(\Omega)]^2$. Then, for all $\mathbf{w}_h \in [\mathbf{\Omega}_{1,h}]^2$,*

$$\int_\Omega \nabla(\mathbf{u} - \mathbf{u}_I)\cdot\nabla\mathbf{w}_h\;\mathrm{d}s \leq\; Ch^2|\mathbf{u}|_3|\mathbf{w}_h|_1,$$

where C is a constant which depends only on Ω.

Proof. Let u, u_I, w be the first components of $\mathbf{u}, \mathbf{u}_I$ and $\mathbf{w}_h$, respectively. It is sufficient to prove

$$\int_\Omega (u - u_I)_x w_x\;\mathrm{d}s \leq\; Ch^2|u|_3\;|w|_1. \tag{3.15}$$

For each rectangle Q in $\mathcal{T}_h$, define a linear functional Ψ on $H^3(Q)$ by the following:

$$\Psi(v) = \int_Q (v - v_I)_x w_x\;\mathrm{d}s, \quad \forall v \in H^3(Q).$$

Then, we have

$$|\Psi(v)| \leq Ch\,|v|_{2,Q}\,|w|_{1,Q}. \tag{3.16}$$

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Since w_x is a function of y, by simple calculation with the definition of v_I, we have

$$\Psi\left(x^2\right) = \Psi\left(y^2\right) = \Psi(xy) = 0. \tag{3.17}$$

Let $\overline{u_{xx}}, \overline{u_{xy}}, \overline{u_{yy}}$ be the respective averages of u_{xx}, u_{xy}, u_{yy} over Q and φ a quadratic function in $P_2(Q)$ such that

$$\varphi = \frac{1}{2}\left(\overline{u_{xx}}\ x^2 + \overline{u_{yy}}\ y^2\right) + \overline{u_{xy}}\ xy. \tag{3.18}$$

Then, by Poincaré Lemma, we establish

$$|u-\varphi|_{2,Q} = \left(\|u_{xx} - \overline{u_{xx}}\|_{0,Q}^2 + 2\,\|u_{xy} - \overline{u_{xy}}\|_{0,Q}^2 + \|u_{yy} - \overline{u_{yy}}\|_{0,Q}^2 \right)^{1/2} \le Ch\,|u|_{3,Q}\,. \tag{3.19}$$

From (3.16), (3.17), (3.19), we have

$$|\Psi(u)| = |\Psi(u-\varphi)| \le Ch\,|u-\varphi|_{2,Q}\,|w|_{1,Q} \le Ch^2\,|u|_{3,Q}\,|w|_{1,Q}\,,$$

which means (3.15). □

The following proposition is a frequently used characteristic for the jump of a midpoint-edge-continuous function across an edge in $\mathcal{T}_h$.

Proposition 3.4. *Let $r_h \in \mathcal{NC}^h(\Omega)$ and Q_1, Q_2 be two adjacent rectangles in $\mathcal{T}_h$ which share an edge E. If $g \in H^1(Q_1 \cup Q_2)$, then*

$$\left| \int_E g(r_h|_{Q_1} - r_h|_{Q_2})\ \mathrm{d}l \right| \ \le\ C \sum_{k=1}^{2} |g|_{1,Q_k} \left\| \tau_Q^{1/2} \nabla r_h \right\|_{0,Q_k}, \tag{3.20}$$

where C is a constant which depends only the shape of rectangles in $\mathcal{T}_h$.

Proof. Since r_h is continuous at the midpoint of E, there is an average $\overline{r_h}$ of r_h over E such that

$$\int_E r_h|_{Q_1}\ \mathrm{d}l = \int_E r_h|_{Q_2}\ \mathrm{d}l = \int_E \overline{r_h}\ \mathrm{d}l.$$

We split the left hand side of (3.20) into

$$\left| \int_E g\,(r_h|_{Q_1} - r_h|_{Q_2})\ \mathrm{d}l \right| \le \left| \int_E g\,(r_h|_{Q_1} - \overline{r_h})\ \mathrm{d}l \right| + \left| \int_E g\,(r_h|_{Q_2} - \overline{r_h})\ \mathrm{d}l \right|.$$

For each $k = 1, 2$, denote by $\overline{g_k}$ the average of g over Q_k, then we have, from (3.4) and the local trace Theorem (3.13),

$$\begin{aligned} \left| \int_E g(r_h|_{Q_k} - \overline{r_h})\ \mathrm{d}l \right| = \left| \int_E (g - \overline{g_k})(r_h|_{Q_k} - \overline{r_h})\ \mathrm{d}l \right| \ &\le\ \|g - \overline{g_k}\|_{0,E}\ \|r_h|_{Q_k} - \overline{r_h}\|_{0,E} \\ &\le Ch^{1/2}\,|g|_{1,Q_k}\,h^{1/2}\,|r_h|_{1,Q_k} \ \le\ C\,|g|_{1,Q_k} \left\| \tau_Q^{1/2} \nabla r_h \right\|_{0,Q_k}, \end{aligned}$$

where C is a constant which depends only on the shape of Q_k. It means (3.20). □

Lemma 3.5. *Let $\mathbf{u} \in [H^3(\Omega)]^2$. Then, for all $r_h \in M_h$,*

$$\int_\Omega r_h\ \mathrm{div}\,(\mathbf{u} - \mathbf{u}_I)\ \mathrm{d}s \ \le\ Ch^{3/2}||\mathbf{u}||_3 \left(||r_h||_0^2 + \sum_{Q \in \mathcal{T}_h} \left\| \tau_Q^{1/2} \nabla r_h \right\|_{0,Q}^2 \right)^{1/2},$$

where C is a constant which depends only on Ω.

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Proof. Let u, u_I be the first components of $\mathbf{u}, \mathbf{u}_I$, respectively. It is enough to show that

$$\int_\Omega r_h\ (u-u_I)_x\ \mathrm{d}s\ \le\ Ch^{3/2}||u||_3\ \left(||r_h||_0^2+\sum_{Q\in\mathcal{T}_h}\left\|\tau_Q^{1/2}\nabla r_h\right\|_{0,Q}^2\right)^{1/2}. \tag{3.21}$$

For each rectangle Q in $\mathcal{T}_h$ of width h, define a linear functional Ψ on $H^3(Q)$ by

$$\Psi(v)=\int_Q\left(r_h(v-v_I)_x-\frac{h^2}{12}(v_{xx}r_h)_x\right)\mathrm{d}s,\quad \forall v\in H^3(Q). \tag{3.22}$$

Since $|r_h|_{1,Q}\le Ch^{-1}\|r_h\|_{0,Q}$, we estimate $|\Psi(v)|$ as

$$|\Psi(v)|\le Ch\,\|r_h\|_{0,Q}\,|v|_{2,Q}+\frac{h^2}{12}\left(|v|_{3,Q}\,\|r_h\|_{0,Q}+|v|_{2,Q}\,|r_h|_{1,Q}\right) \tag{3.23a}$$

$$\le Ch\,\|r_h\|_{0,Q}\left(|v|_{2,Q}+h\,|v|_{3,Q}\right). \tag{3.23b}$$

We note that the integrand in (3.22) vanishes, if v belongs to $\mathrm{span}\{1,x,y,xy,y^2\}$. By definition of v_I, we have

$$\Psi(x^2)=2\int_Q\left(r_h(x-\alpha)-\frac{h^2}{12}(r_h)_x\right)\mathrm{d}s, \tag{3.24}$$

where α is the x-coordinate of the center of Q. Since the integral in (3.24) vanishes for the linear function r_h, $\Psi(\varphi)=0$ for all quadratic functions $\varphi\in P_2(Q)$. Then, by similar arguments in (3.18) and (3.19), the following estimation is obtained from (3.23a):

$$|\Psi(u)|=|\Psi(u-\varphi)|\le Ch\,\|r_h\|_{0,Q}\left(\,|u-\varphi|_{2,Q}+h\,|u|_{3,Q}\,\right)\le Ch^2\,\|r_h\|_{0,Q}\,|u|_{3,Q}\,. \tag{3.25}$$

Next, the second term in the integrand for $\Psi(u)$ in (3.22) satisfies:

$$\sum_{Q\in\mathcal{T}_h}\int_Q(u_{xx}r_h)_x\ \mathrm{d}s=\sum_{Q\in\mathcal{T}_h}\int_{\partial Q}u_{xx}r_h\mathbf{n}_x\ \mathrm{d}l \tag{3.26a}$$

$$=\sum_E\int_E u_{xx}(r_h|_{Q_L}-r_h|_{Q_R})\ \mathrm{d}l+\int_{\partial\Omega}u_{xx}r_h\mathbf{n}_x\ \mathrm{d}l, \tag{3.26b}$$

where E runs over all interior vertical edges and Q_L shares E with Q_R in its right side.

By Proposition 3.4 above, the first term in (3.26a) and (3.26b) is estimated by

$$\left|\sum_E\int_E u_{xx}(r_h|_{Q_L}-r_h|_{Q_R})\ \mathrm{d}l\right|\le C\,|u|_{3,\Omega}\left(\sum_{Q\in\mathcal{T}_h}\left\|\tau_Q^{1/2}\nabla r_h\right\|_{0,Q}^2\right)^{1/2}. \tag{3.27}$$

From the global trace theorem for u_{xx} and the local one for r_h in (3.14), the second term is done as

$$\left|\int_{\partial\Omega}u_{xx}r_h\mathbf{n}_x\ \mathrm{d}l\right|\le\|u_{xx}\|_{0,\partial\Omega}\,\|r_h\|_{0,\partial\Omega}\le Ch^{-1/2}\,\|u\|_{3,\Omega}\,\|r_h\|_{0,\Omega}\,. \tag{3.28}$$

We establish (3.21) from (3.22) and (3.25)–(3.28). □

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3.3. Superconvergence

If the solution $(\mathbf{u}, p)$ in (2.12) has more regularity $(\mathbf{u}, p) \in [H^2(\Omega)]^2 \times H^1(\Omega)$, then, since $-\Delta\mathbf{u} + \nabla p = \mathbf{f}$, $(\mathbf{u}, p)$ satisfies the following:

$$B_h(\mathbf{u}, p; \mathbf{v}, q) = (\mathbf{f}, \mathbf{v}) + \sum_{Q\in\mathcal{T}_h} (\mathbf{f}, \tau_Q \nabla q)_Q, \quad \forall (\mathbf{v}, q) \in [H_0^1(\Omega)]^2 \times M_h. \tag{3.29}$$

For the finite element solution $(\mathbf{u}_h, p_h) \in X_h \times M_h$ in (2.13), we have the following orthogonality, from (3.29) and Lemma 3.1,

$$B_h(\mathbf{u} - \mathbf{u}_L, p - p_h; \mathbf{v}_L, q_h) = 0, \quad \forall (\mathbf{v}_L, q_h) \in [\mathfrak{Q}_{1,h,0}]^2 \times M_h, \tag{3.30}$$

where $\mathbf{u}_L$ is the bilinear part of $\mathbf{u}_h$.

We reach at the superconvergence of the pressure $p_h \in M_h$ in the following theorem.

Theorem 3.6. *Let $(\mathbf{u}, p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$, $(\mathbf{u}_h, p_h) \in X_h \times M_h$ be the solutions of (2.12), (2.13), respectively. If the solution $(\mathbf{u}, p)$ belongs to $[H^3(\Omega)]^2 \times H^2(\Omega)$, then*

$$||p - p_h||_0 \ \le \ Ch^{3/2}(||\mathbf{u}||_3 + |p|_2),$$

for some constant C which depends only on Ω.

Proof. Let $p_i \in \mathcal{NC}^h(\Omega)$ be the standard interpolation of $p \in H^2(\Omega)$ by (2.1) and define $p_I \in M_h$ as

$$p_I = p_i - \fint_\Omega p_i \, \mathrm{d}s.$$

Then, from (2.2), the interpolation error is estimated by

$$\|p - p_I\|_0 + h \left(\sum_{Q\in\mathcal{T}_h} |p - p_I|_{1,Q}^2 \right)^{1/2} \ \le \ C h^2 \, |p|_2 \, , \tag{3.31}$$

for some constant C which depends only on Ω.

For p_I and the bilinear interpolant $\mathbf{u}_I$ of $\mathbf{u}$, the orthogonality in (3.30) is rewritten as

$$B_h(\mathbf{u} - \mathbf{u}_I, p - p_I; \mathbf{v}_L, q_h) = B_h(\mathbf{u}_L - \mathbf{u}_I, p_h - p_I; \mathbf{v}_L, q_h). \tag{3.32}$$

Then, there exists $(\mathbf{w}_L, r_h) \in [\mathfrak{Q}_{1,h,0}]^2 \times M_h$ such that, by (3.32) and Lemma 3.2,

$$|||(\mathbf{u}_L - \mathbf{u}_I, p_h - p_I)|||_B |||(\mathbf{w}_L, r_h)|||_B \le \frac{1}{\beta} \, B_h(\mathbf{u}_L - \mathbf{u}_I, p_h - p_I; \mathbf{w}_L, r_h) \tag{3.33a}$$

$$= \frac{1}{\beta} \, B_h(\mathbf{u} - \mathbf{u}_I, p - p_I; \mathbf{w}_L, r_h), \tag{3.33b}$$

with some constant β, since $(\mathbf{u}_L - \mathbf{u}_I, p_h - p_I) \in [\mathfrak{Q}_{1,h,0}]^2 \times M_h$.

From the definition of B_h, we have

$$\begin{aligned} B_h(\mathbf{u} - \mathbf{u}_I, p - p_I; \mathbf{w}_L, r_h) &= (\nabla(\mathbf{u} - \mathbf{u}_I) \cdot \nabla \mathbf{w}_L) + \big(r_h, \operatorname{div}(\mathbf{u} - \mathbf{u}_I)\big) \\ &\quad - (p - p_I, \operatorname{div} \mathbf{w}_L) + \sum_{Q\in\mathcal{T}_h} (\nabla(p - p_I), \tau_Q \nabla r_h)_Q - \sum_{Q\in\mathcal{T}_h} (\Delta(\mathbf{u} - \mathbf{u}_I), \tau_Q \nabla r_h)_Q \, . \end{aligned} \tag{3.34}$$

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We apply the superclosenesses in Lemmas 3.3 and 3.5 for the first two terms in the right hand side of (3.34) to get,

$$\left(\nabla(\mathbf{u}-\mathbf{u}_I)\cdot\nabla\mathbf{w}_L\right) \le Ch^2\,|\mathbf{u}|_3\;\;|||(\mathbf{w}_L,r_h)|||_B, \tag{3.35a}$$

$$\left(r_h,\operatorname{div}(\mathbf{u}-\mathbf{u}_I)\right) \le Ch^{3/2}\,\|\mathbf{u}\|_3\;\;|||(\mathbf{w}_L,r_h)|||_B. \tag{3.35b}$$

The third term is simply done by (3.31) into

$$|(p-p_I,\operatorname{div}\mathbf{w}_L)| \le Ch^2\,|p|_2\;\;|||(\mathbf{w}_L,r_h)|||_B. \tag{3.36}$$

With (3.3) and (3.31), the fourth term is estimated by

$$\begin{aligned}\left|\sum_{Q\in\mathcal{T}_h}\left(\nabla(p-p_I),\tau_Q\nabla r_h\right)_Q\right| &= \left|\sum_{Q\in\mathcal{T}_h}\left(\tau_Q^{1/2}\nabla(p-p_I),\tau_Q^{1/2}\nabla r_h\right)_Q\right| \\ &\le \gamma Ch^2|p|_2\;|||(\mathbf{w}_L,r_h)|||_B.\end{aligned} \tag{3.37}$$

For the last term in the right hand side of (3.34), let $\overline{\Delta\mathbf{u}}\in[P_h]^2$ be a piecewise constant interpolation of $\Delta\mathbf{u}$ such that

$$\int_Q \Delta\mathbf{u}\;\mathrm{d}s = \int_Q \overline{\Delta\mathbf{u}}\;\mathrm{d}s, \qquad \text{for all } Q \text{ in } \mathcal{T}_h, \tag{3.38}$$

whose interpolation error is estimated by

$$\left\|\Delta\mathbf{u}-\overline{\Delta\mathbf{u}}\right\|_0 \le Ch\,|\mathbf{u}|_3\,. \tag{3.39}$$

Then, by (3.3) and (3.38), we have,

$$(\Delta(\mathbf{u}-\mathbf{u}_I),\tau_Q\nabla r_h)_Q = \left(\Delta\mathbf{u}-\overline{\Delta\mathbf{u}},\tau_Q\nabla r_h\right)_Q + \left(\overline{\Delta\mathbf{u}},\tau_Q\nabla r_h\right)_Q \tag{3.40a}$$

$$= \left(\tau_Q^{1/2}\left(\Delta\mathbf{u}-\overline{\Delta\mathbf{u}}\right),\tau_Q^{1/2}\nabla r_h\right)_Q + \alpha^2h^2\left(\Delta\mathbf{u},\nabla r_h\right)_Q, \tag{3.40b}$$

since $\Delta\mathbf{u}_I$ vanishes and ∇r_h is a constant vector.

By (3.3) and (3.39), the sum of the first terms in (3.40a) and (3.40b) for all $Q\in\mathcal{T}_h$ is estimated by

$$\left|\sum_{Q\in\mathcal{T}_h}\left(\tau_Q^{1/2}\left(\Delta\mathbf{u}-\overline{\Delta\mathbf{u}}\right),\tau_Q^{1/2}\nabla r_h\right)_Q\right| \le \gamma Ch^2|\mathbf{u}|_3\;|||(\mathbf{w}_L,r_h)|||_B. \tag{3.41}$$

For the second term, through integration by parts and since $\operatorname{div}\mathbf{u}$ vanishes, we have:

$$\begin{aligned}\sum_{Q\in\mathcal{T}_h}(\Delta\mathbf{u},\nabla r_h)_Q &= \sum_{Q\in\mathcal{T}_h} <\Delta\mathbf{u}\cdot\mathbf{n},r_h>_{\partial Q} - \sum_{Q\in\mathcal{T}_h}(\operatorname{div}\Delta\mathbf{u},r_h)_Q \\ &= \sum_{Q\in\mathcal{T}_h} <\Delta\mathbf{u}\cdot\mathbf{n},r_h>_{\partial Q}\,.\end{aligned}$$

Applying Proposition 3.4 for the edges in the interior of Ω, we obtain

$$\left|\sum_{Q\in\mathcal{T}_h}(\Delta\mathbf{u},\nabla r_h)_Q\right| \le C\,|\Delta\mathbf{u}|_1\,|||(\mathbf{w}_L,r_h)|||_B + |<\Delta\mathbf{u}\cdot\mathbf{n},r_h>_{\partial\Omega}|\,. \tag{3.42}$$

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TABLE 1. Error table for uniform rectangular meshes (M1).

Mesh	$\|\|\mathbf{u}-\mathbf{u}_h\|\|_0$		$\|\|\nabla\mathbf{u}-\nabla\mathbf{u}_h\|\|_0$		$\|\|p-p_h\|\|_0$	
	Value	Order	Value	Order	Value	Order
16×16	2.2187E-3		1.8359E-1		5.7334E-2	
32×32	5.2254E-4	2.0861	8.8997E-2	1.0447	2.3560E-2	1.2830
64×64	1.2736E-4	2.0367	4.4001E-2	1.0162	9.6639E-3	1.2857
128×128	3.1456E-5	2.0175	2.1889E-2	1.0074	3.5357E-3	1.4506
256×256	7.8178E-6	2.0085	1.0917E-2	1.0035	1.1435E-3	1.6285
512×512	1.9491E-6	2.0039	5.4530E-3	1.0015	3.5626E-4	1.6825
1024×1024	4.8663E-7	2.0019	2.7253E-3	1.0007	1.1460E-4	1.6363

TABLE 2. Error table for non-uniform bisection mesh (M2).

mesh	$\|\|\mathbf{u}-\mathbf{u}_h\|\|_0$		$\|\|\nabla\mathbf{u}-\nabla\mathbf{u}_h\|\|_0$		$\|\|p-p_h\|\|_0$	
	value	order	value	order	value	order
16×16	2.6092E-3		1.9955E-1		6.1404E-2	
32×32	6.1353E-4	2.0884	9.6495E-2	1.0482	2.4388E-2	1.3322
64×64	1.4974E-4	2.0347	4.7694E-2	1.0167	9.8453E-3	1.3087
128×128	3.7029E-5	2.0157	2.3729E-2	1.0072	3.5883E-3	1.4561
256×256	9.2103E-6	2.0073	1.1837E-2	1.0033	1.1704E-3	1.6164
512×512	2.2979E-6	2.0029	5.9127E-3	1.0014	3.7041E-4	1.6598
1024×1024	5.7649E-7	1.9949	2.9551E-3	1.0006	1.2099E-4	1.6142

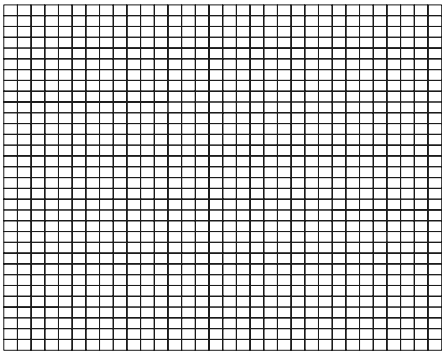


FIGURE 1. 32×32 Uniform rectangular mesh (M1).

Then, the global trace theorem for $\Delta\mathbf{u}$ and the local one in (3.14) for r_h establish

$$\left| < \Delta\mathbf{u}\cdot\mathbf{n}, r_h >_{\partial\Omega} \right| \le Ch^{-1/2}\,\|\Delta\mathbf{u}\|_1\ \ |||(\mathbf{w}_L, r_h)|||_B. \tag{3.43}$$

Combining (3.33a)–(3.37) and (3.40a)–(3.43), we have

$$|||(\mathbf{u}_L-\mathbf{u}_I, p_h-p_I)|||_B \le Ch^{3/2}(||\mathbf{u}||_3+|p|_2),$$

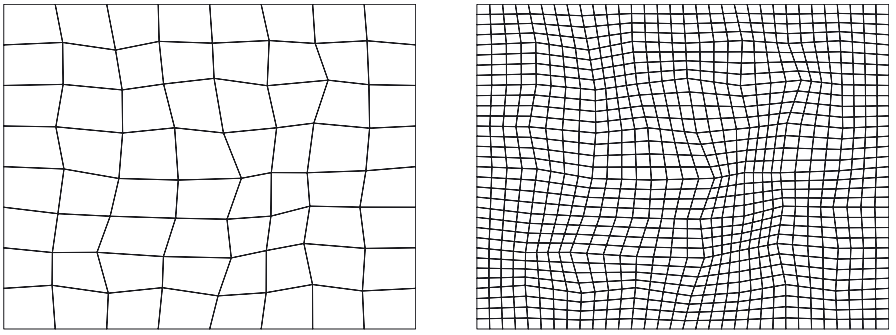
which completes the proof through (3.31) and the definition of the triple norm in (3.7). □

4. NUMERICAL RESULTS

For the test problem, we chose the stream function ϕ on $\Omega=(0,1)^2$ such that

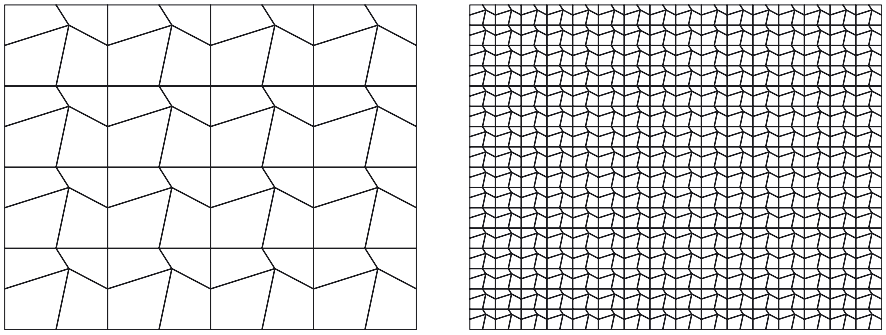
$$\phi(x,y)=s(x)s(y), \qquad \text{for } s(t)=\sin(2\pi t)\left(t^2-t\right),$$

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(a) 8×8 initial non-uniform mesh (b) 32×32 non-uniform mesh by bisecting

FIGURE 2. Non-uniform meshes by recursive bisecting from 8×8 mesh (M2).



(a) 8×8 zigzag mesh (b) 32×32 zigzag mesh

FIGURE 3. Zigzag meshes (M3).

TABLE 3. Error table for zigzag meshes (M3).

Mesh	$\|\|\mathbf{u}-\mathbf{u}_h\|\|_0$		$\|\|\nabla\mathbf{u}-\nabla\mathbf{u}_h\|\|_0$		$\|\|p-p_h\|\|_0$	
	Value	Order	Value	Order	Value	Order
16×16	5.0339E-3		2.8134E-1		1.4795E-1	
32×32	1.2634E-3	1.9944	1.4068E-1	0.9999	7.5569E-2	0.9692
64×64	3.1487E-4	2.0044	7.0199E-2	1.0029	3.7558E-2	1.0087
128×128	7.8388E-5	2.0061	3.5027E-2	1.0030	1.8444E-2	1.0260
256×256	1.9551E-5	2.0034	1.7489E-2	1.0020	9.0836E-3	1.0218
512×512	4.8839E-6	2.0011	8.7357E-3	1.0014	4.4880E-3	1.0172
1024×1024	1.1883E-6	2.0391	4.3573E-3	1.0035	2.1831E-3	1.0397

and the velocity $\mathbf{u}$ and pressure p as

$$\mathbf{u}(x,y) = \mathbf{curl}\ \phi, \quad p(x,y) = \sin(2\pi x)\left(\frac{1}{25-10\tan^2 y}+\frac{3}{10}\right).$$

Solving (2.13) with the source function $\mathbf{f} = -\Delta\mathbf{u} + \nabla p$, we obtained $(\mathbf{u}_h, p_h) \in X_h \times M_h$.

The numerical results are shown in Tables 1 and 2 for uniform rectangular(M1) and non-uniform bisection(M2) meshes as depicted in Figures 1 and 2, respectively. The non-uniform bisection meshes are obtained by recursive bisecting from the 8×8 initial non-uniform mesh in Figure 2a.

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We can observe the expected order of convergence for the velocity in (2.14). For the pressure, the superconvergence of more order than 3/2 analyzed in Theorem 3.6 appears in uniform rectangular and even non-uniform bisection meshes.

For the zigzag meshes (M3) as in Figure 3, the numerical results in Table 3 does not show superconvergence in pressure any more.

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