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DYNAMICS ON THE PERFECT KERNEL OF NON-AMENABLE HIGHER-RANK GENERALIZED BAUMSLAG-SOLITAR GROUPS

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Abstract In this article, we study the space of subgroups of non-amenable generalized Baumslag-Solitar groups (GBS groups) of rank d, that is, groups acting cocompactly on an oriented tree with vertex and edge stabilizers isomorphic to $\mathbb{Z}^d$. Our results generalize the study of Baumslag-Solitar groups, and of GBS groups of rank 1. We give an explicit description of the perfect kernel of a non-amenable GBS group G of rank d and show the existence of a partition of the perfect kernel into a countably infinite set of pieces that are invariant under the action by conjugation of G, and such that each piece contains a dense orbit.

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1. Introduction

A generalized Baumslag-Solitar group (GBS group) of rank d is a group that acts cocompactly on an oriented tree such that the vertex and edge stabilizers are isomorphic to $\mathbb{Z}^d$. As a consequence of Bass-Serre theory, a generalized Baumslag-Solitar group is defined by a finite iteration of HNN extensions and amalgamated free products of $\mathbb{Z}^d$ over $\mathbb{Z}^d$.

GBS groups of rank $d > 1$ are a generalization of GBS groups of rank 1, which arise as a natural generalization of Baumslag-Solitar groups $\mathrm{BS}(m,n) = \langle b,t \mid tb^m t^{-1} = b^n \rangle$. Baumslag-Solitar groups were introduced in [BS62] to give the first examples of two-generated finitely presented non-Hopfian groups. GBS groups have been widely studied in relation to various properties. In [Lev15], Levitt computed the minimal number of generators of a GBS group of rank 1. He studied their automorphism groups in [Lev07]. The classification up to quasi-isometry of GBS groups of rank 1 is known (see [Why01]) and the classification up to measure equivalence of Baumslag-Solitar groups has been announced by the authors of [GPT$^+$]. In [LdGZS25], the authors determined which GBS groups (of arbitrary rank) are residually finite and which are LERF.

To any GBS group G of rank d is associated its **modular homomorphism** $\Delta_G : G \to \mathrm{GL}_d(\mathbb{Q})$, which was introduced in [But25]. Starting from a vertex v of the Bass-Serre tree, the conjugation by any element g of G induces a homomorphism $\mathrm{Stab}(v) \to \mathrm{Stab}(g \cdot v)$. As all vertex stabilizers are commensurable, this homomorphism can be seen as an element of $\mathrm{GL}_d(\mathbb{Q})$. This defines a homomorphism $\Delta_G^{(v)} : G \to \mathrm{GL}_d(\mathbb{Q})$. Two such homomorphisms $\Delta_G^{(v)}$, $\Delta_G^{(w)}$ are conjugate. Up to conjugation, this defines the modular homomorphism $\Delta_G : G \to \mathrm{GL}_d(\mathbb{Q})$. A GBS group G is called **unimodular** if $\mathrm{Im}(\det \circ \Delta_G) \subseteq \{1, -1\}$. As an example, one characterization of unimodularity for a GBS group of rank 1 is the existence of an infinite cyclic normal subgroup (see [Lev07][Section 2] for instance).

In this article, we will focus on the set of subgroups $\mathrm{Sub}(G)$ of a GBS group G from a topological point of view. This means that we are more interested in the topological structure of $\mathrm{Sub}(G)$, seen as a closed subset of the Cantor set $\{0,1\}^G$, than in the algebraic properties of the subgroups of G. Cantor-Bendixson theory (see [Kec95]) leads to a unique decomposition $\mathrm{Sub}(G) = \mathcal{K}(G) \sqcup C$ into a closed subspace without isolated points called the **perfect kernel** of G and a countable set C. As the action by conjugation of G induces a homeomorphism of $\mathrm{Sub}(G)$, the perfect kernel is G-invariant. We are interested in the computation of the perfect kernel and the dynamics induced by the action by conjugation of G on it.

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The perfect kernel of any finitely generated abelian group is empty. In [CGP10], the authors classified the set of subgroups of all countable abelian groups up to homeomorphism. In [BGK12], the authors proved that the perfect kernel of the lamplighter group $(\mathbb{Z}/p\mathbb{Z})^n \wr \mathbb{Z}$ (where p is a prime number) is the set of subgroups of $\bigoplus_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})^n$.

If G is a finitely generated group, then finite-index subgroups are isolated. Thus the perfect kernel of G is included in the set of infinite-index subgroups of G. The authors of [CGLM23] observed that equality holds for the non-abelian finitely generated free group $\mathbb{F}_r$ on r generators, and that the action of $\mathbb{F}_r$ on its perfect kernel is topologically transitive. Recall that an action of a group G on a topological space X is **topologically transitive** if for all non-empty open subsets $U,V \subseteq X$, there exists $g \in G$ such that $gU \cap V \neq \emptyset$. In the case where X is Polish, this is equivalent to the existence of a dense orbit. The authors of [AG24] extended this result to a large class of groups acting on trees. They proved that the perfect kernel of a finitely generated group G with infinitely many ends is also equal to the set of infinite-index subgroups, and that the action by conjugation is topologically transitive on the perfect kernel as soon as G does not contain any non-trivial finite normal subgroup. More generally, they proved that for any finitely generated group G that acts (minimally and irreducibly) on a tree $\mathcal{T}$ such that the action of G on $\mathcal{T}$ is acylindrical, any subgroup H of G such that the quotient graph $H\backslash\mathcal{T}$ is infinite belongs to the perfect kernel of G. In this case, they also proved that the action by conjugation of G on the closure of the set of subgroups H acting on $\mathcal{T}$ with infinitely many orbits of edges is topologically transitive. Recall that an action of a group G on a tree $\mathcal{T}$ is **acylindrical** if there exists $R > 0$ such that the stabilizer of any path of length larger than R is trivial.

GBS groups of rank d are typical examples of groups whose action on their Bass-Serre tree is *not* acylindrical, because the stabilizer of any finite subtree of the Bass-Serre tree is isomorphic to $\mathbb{Z}^d$. The authors of [CGLMS25] (who studied Baumslag-Solitar groups) and of [Bon26] (who extended some results obtained by the aforementioned authors to GBS groups of rank 1) observed that this leads to very different dynamics for the action by conjugation of a non-amenable GBS group G of rank 1 on its perfect kernel. More precisely, they showed that $\mathcal{K}(G) = \mathrm{Sub}_{[\infty]}(G)$ if and only if G is not unimodular. They also described a countably infinite G-invariant partition of $\mathcal{K}(G)$, such that G acts topologically transitively on each piece. One piece is closed and all the other ones are open (and also closed if and only if G is unimodular). To obtain this decomposition, the authors of [CGLMS25] introduced the **phenotype**, which is a G-invariant function $\mathrm{Sub}(G) \to \mathbb{N}^* \cup \{\infty\}$, and which was generalized in [Bon26]. This function is explicit and encodes the decomposition of the perfect kernel.

In this article, we will extend these results to non-amenable GBS groups of an arbitrary rank d, that is to say, those that are neither isomorphic to $\mathbb{Z}[\mathrm{A},\mathrm{A}^{-1}](\mathbb{Z}^d) \rtimes \mathbb{Z}$ for some $\mathrm{A} \in \mathrm{M}_d(\mathbb{Z}) \cap \mathrm{GL}_d(\mathbb{Q})$ (where $\mathbb{Z}[\mathrm{A},\mathrm{A}^{-1}](\mathbb{Z}^d)$ is the subgroup of $\mathbb{Q}^d$ defined by $\{\mathrm{A}^k u, k \in \mathbb{Z}, u \in \mathbb{Z}^d\}$ and $\mathbb{Z}$ acts on $\mathbb{Z}[\mathrm{A},\mathrm{A}^{-1}](\mathbb{Z}^d)$ by multiplication by A) nor to any amalgamated free product $\mathbb{Z}^d *_{\mathbb{Z}^d} \mathbb{Z}^d$ where both injections are defined by matrices of determinant ± 2 (*cf.* Proposition 3.19). Understanding the space of subgroups of amenable GBS groups is a different problem: it requires an understanding of the space of subgroups of the abelian group $\mathbb{Z}\left[\mathrm{A},\mathrm{A}^{-1}\right](\mathbb{Z}^d)$. We think that this should be tackled using the results of Cornulier,

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Guyot and Pitsch about the space of subgroups of an abelian group (*cf.* [CGP10]), but this lies beyond the scope of our methods, which crucially make use of the action on the Bass-Serre tree.

More precisely, we prove the following result (*cf.* Theorem 5.2):

Theorem 1. *Let G be a non-amenable GBS group of rank d defined by a reduced graph of groups $\mathscr{H}$ and let $\mathcal{T}$ be the Bass-Serre tree of $\mathscr{H}$. Then*

$$\mathcal{K}(G) = \{H \leq G \mid H\backslash\mathcal{T} \text{ is infinite}\}.$$

We also give some sufficient conditions that depend on the modular homomorphism (see Section 3.2) for the perfect kernel to be equal to $\mathrm{Sub}_{[\infty]}(G)$. Namely, we prove the following (*cf.* Corollary 5.5):

Theorem 2. *Let G be a non-amenable and non-unimodular GBS group of rank d such that the finite-index subgroups of $\mathrm{Im}\left(\Delta_G^{(v)}\right)$ are $\mathbb{Q}$-irreducible. Then*

$$\mathcal{K}(G) = \mathrm{Sub}_{[\infty]}(G).$$

Recall that a subgroup Γ of $\mathrm{GL}_d(\mathbb{K})$ is called **$\mathbb{K}$-irreducible** if it does not stabilize any non-trivial proper vector subspace of $\mathbb{K}^d$.

We also obtain a generalization of the main results of [CGLMS25] and [Bon26] (*cf.* Equation 6.4 and Theorem 6.5 if G is not a semidirect product, and Theorem 6.15 otherwise):

Theorem 3. *Let G be a non-amenable GBS group of rank d. There exists a countably infinite partition of the perfect kernel of G into G-invariant pieces that contain dense orbits.*

This implies in particular that the action is topologically transitive on each piece.

We also investigate the topology of the pieces that appear in this decomposition, which is slightly different depending on whether G is a semidirect product $\mathbb{Z}^d \rtimes \mathbb{F}_r$ (*cf.* Theorem 6.15 – keeping the notation of this theorem, the pieces will be the sets $\mathcal{P}_{\mathscr{C}} \setminus \mathcal{D}_{\mathscr{C}}$ and the orbits in the countable sets $\mathcal{D}_{\mathscr{C}}$ under the G-conjugation, for $\mathscr{C}$ varying in $Conj(G_v)$) or not (*cf.* Proposition 6.6 – keeping the notation of this proposition, the pieces of the decomposition of the perfect kernel will be given by the sets $\mathcal{K}(G) \cap \mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda))$, for Λ varying in $\mathrm{Sub}(\mathbb{Z}^d)$). However, as all these pieces of this partition are F_σ subsets, they turn out to be G_δ subsets (hence Polish).

The main challenge raised by the generalization of the results of [Bon26] to GBS groups of arbitrary rank is that, while injections of the edge groups into the vertex groups were simply encoded by non-zero integers in the rank 1 case, they are now encoded by matrices. Understanding the space of subgroups of a higher-rank GBS group thus requires a more subtle analysis of the action of the group *via* the modular homomorphism on the set of vector subspaces of $\mathbb{Q}^d$, and a more careful analysis of partial actions. In particular, while the decomposition of Theorem 3 was explicit in the case of GBS groups of rank 1 (and encoded by the "phenotype", see [Bon26, Definition 5.2 and 5.5, Theorem 6.6]), it is now encoded by an abstract equivalence relation on the set of subgroups of $\mathbb{Z}^d$.

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Moreover, we do not know whether the topological transitivity can be strengthened into high topological transitivity (while the action on each piece was proved to be highly topologically transitive in [Bon26, Theorem 2]). Also notice that the need to provide a separate analysis for semidirect products $\mathbb{Z}^d \rtimes \mathbb{F}_r$ is a conceptual difference with the case of GBS groups of rank 1 (see Section 6.2).

The paper is organized as follows. Given a graph of groups $\mathscr{H}$ of fundamental group G, we extend the notion of $\mathscr{H}$-preactions and of $\mathscr{H}$-graphs in Section 3.1. They were introduced in [FLMMS22] in the case where G is an amalgamated free product or an HNN-extension (*i.e.* if $\mathscr{H}$ consists of a single edge), and adapted in [CGLMS25] in the case of Baumslag-Solitar groups and in [Bon26] in the case of GBS groups of rank 1. These graphs can be thought of as a generalization of Schreier graphs in the case of free groups: any subgroup H of a free group $\mathbb{F}_r$ can be encoded by a covering of the bouquet of r circles (which is the Schreier graph of the action $(H\backslash\mathbb{F}_r, H) \curvearrowleft \mathbb{F}_r$, with respect to the standard generating set). When H has infinite index, this covering is infinite, and perturbing this graph very far from the origin amounts to building a subgroup of $\mathbb{F}_r$ that is close to H. In Section 5, we prove Theorem 1 and Theorem 2. Finally, in Section 6, we show the existence of the decomposition of Theorem 3 and investigate the topology of the pieces. This gives rise to a natural generalization of the phenotype defined in [CGLMS25] and in [Bon26]. However, we do not know if this decomposition is still explicit in this wider context, and we do not know if our arguments can be used to prove *high* topological transitivity results as in [GLMS24] for Baumslag-Solitar groups and in [Bon26] for GBS groups of rank 1.

2. Preliminaries and notation

We denote by $\mathcal{P}$ the set of prime numbers in $\mathbb{N}$. For every integer N and $p \in \mathcal{P}$, we denote by $|N|_p$ the p-adic valuation of N, that is, the largest $n \in \mathbb{N}$ such that p^n divides N. By "countable" we mean finite or in bijection with $\mathbb{N}$. Given a group G, we denote by $\mathrm{Sub}(G)$ the set of subgroups of G and by $\mathrm{Sub}_{[\infty]}(G)$ the subset of $\mathrm{Sub}(G)$ that consists of infinite-index subgroups. If $H \leq G$ is a subgroup, we denote by $[H]_G$ the G-conjugacy class of H. For any $d \in \mathbb{N}^*$, we denote by $\mathcal{L}(\mathbb{Z}^d)$ the set of lattices of $\mathbb{Z}^d$, *i.e.* the set of finite-index subgroups of $\mathbb{Z}^d$. If $\mathbb{K}$ is a field and $\mathrm{M} \in \mathrm{M}_d(\mathbb{K})$, we denote by $\mathrm{Spec}_{\mathbb{K}}(\mathrm{M})$ the spectrum of M in $\mathbb{K}$, *i.e.* the set of elements $\lambda \in \mathbb{K}$ such that $\det(\mathrm{M} - \lambda I_d) = 0$. If Γ is a subgroup of $\mathrm{GL}_d(\mathbb{K})$, we say that Γ is $\mathbb{K}$-irreducible if Γ does not stabilize any non-trivial vector subspace of $\mathbb{K}^d$. Likewise, an element $\gamma \in \mathrm{GL}_d(\mathbb{K})$ is $\mathbb{K}$-irreducible if it stabilizes no non-trivial vector subspace of $\mathbb{K}^d$.

2.1. Graphs

We refer to [Bon26], Section 2 for the notation and definitions around graphs and Schreier graphs. We add the following terminology: given a graph $\mathscr{H}$, an element of $\mathcal{E}(\mathscr{H}) \times \{\mathbf{s}, \mathbf{t}\}$ is called a **half-edge**. The **inferior** half-edge of $e \in \mathcal{E}(\mathscr{H})$ is $(e, \mathbf{s})$ and its **superior** half-edge is $(e, \mathbf{t})$. Given a graph $\mathscr{H}$ and a spanning tree $\mathscr{T}$ of $\mathscr{H}$, for any vertices $u, v \in \mathcal{V}(\mathscr{H})$, we denote by $[u; v]_{\mathscr{T}}$ the unique edge path in $\mathscr{T}$ that connects u to v.

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2.2. Space of subgroups of a countable group

Let G be an infinite countable group. Endowed with the **Chabauty topology**, the set of subgroups $\mathrm{Sub}(G)$ of G is a closed subspace of the Cantor set $\{0,1\}^G$. An explicit basis of open sets is given by the following clopen sets:

$$\mathcal{V}(O,I) = \{H \in \mathrm{Sub}(G), H \cap O = \emptyset \text{ and } I \subseteq H\}$$

for any finite subsets $O, I \subseteq G$.

We will make use of the following lemma that describes the topology of some subsets of $\mathrm{Sub}(G)$.

Lemma 2.1. *Let G be a countable group and let G_0 be any subgroup of G. Then, for any subgroup $H_0 \leq G_0$, the set*

$$\{H \leq G \mid H \cap G_0 = H_0\}$$

is closed in $\mathrm{Sub}(G)$. If moreover the group H_0 is finitely generated and has finite index in G_0, then it is also open.

Proof. One has

$$\{H \leq G \mid H \cap G_0 = H_0\} = \bigcap_{(h,g) \in H_0 \times (G_0 \setminus H_0)} \{H \leq G \mid h \in H \text{ and } g \notin H\},$$

which is closed as an intersection of basic clopen subsets of $\mathrm{Sub}(G)$.

If H_0 is finitely generated and has finite index in G_0, let $\{h_1, \ldots, h_n\}$ be a finite generating set of H_0 and let us write $G_0/H_0 = \{H_0, g_1 H_0, \ldots, g_m H_0\}$. We then have

$$\{H \leq G \mid H \cap G_0 = H_0\} = \bigcap_{(i,j) \in [\![1,n]\!] \times [\![1,m]\!]} \{H \leq G \mid h_i \in H \text{ and } g_j \notin H\},$$

which is open as a finite intersection of basic clopen sets. □

Applying the Cantor-Bendixson Theorem (see [Kec95, Section 6, Chapter 1] for instance) to the Polish space $\mathrm{Sub}(G)$ leads to a unique decomposition $\mathrm{Sub}(G) = K \bigsqcup C$ where C is countable and K is a closed subspace of $\mathrm{Sub}(G)$ without isolated points. The set K is called the **perfect kernel** of G and denoted by $\mathcal{K}(G)$. It is the largest closed subset of $\mathrm{Sub}(G)$ without isolated points, or equivalently, the set of subgroups all of whose neighbourhoods are uncountable.

If G is finitely generated, then finite-index subgroups are isolated. In particular, we get the following inclusion $\mathcal{K}(G) \subseteq \mathrm{Sub}_{[\infty]}(G)$. The converse inclusion is true in the case of finitely generated free groups (see [CGLM23][Proposition 2.1] and [AG24][Corollary 5.17]):

Proposition 2.2. *Let $\mathbb{F}_r$ be the free group on r generators ($2 \leq r \leq \infty$). Then*

- *if $r < \infty$, then $\mathcal{K}(\mathbb{F}_r) = \mathrm{Sub}_{[\infty]}(\mathbb{F}_r)$;*
- *$\mathcal{K}(\mathbb{F}_\infty) = \mathrm{Sub}(\mathbb{F}_\infty)$.*

Moreover, there exists a dense orbit for the action by conjugation of $\mathbb{F}_r$ on $\mathcal{K}(\mathbb{F}_r)$.

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Remark 2.3. The key point of the proof of Proposition 2.2 is the identification of subgroups of $\mathbb{F}_r = \langle (a_i)_{i\in[\![1,r]\!]}\rangle$ with coverings of the bouquet B_r of r circles (which are exactly the Schreier graphs of subgroups of $\mathbb{F}_r$ with respect to the generating set $\{a_i, i\in[\![1,r]\!]\}$). More precisely, this relies on the two following facts. Let B be a (possibly infinite) graph. Then:

- for every covering $E\to B$ and every connected finite subgraph $K\subsetneq E$, there exists a covering $p: E'\to B$ whose degree is infinite and such that E' contains a subgraph K' which is isomorphic to K as a labelled graph (this allows one to obtain the aforementioned explicit description of the perfect kernel);
- for all coverings $E_i\to B$ ($i\in\mathbb{N}$), given any connected finite subgraph $K_i\subsetneq E_i$ (for every $i\in\mathbb{N}$), there exists a covering $p: E\to B$ whose degree is infinite and such that E contains disjoint subgraphs $(K_i')_{i\in\mathbb{N}}$ such that K_i' and K_i are isomorphic as labelled graphs for every $i\in\mathbb{N}$ (this allows one to build a dense orbit).

This point of view will be useful in the study of the action by conjugation of a semidirect product $\mathbb{Z}^d\rtimes\mathbb{F}_r$ on its perfect kernel (*cf.* Section 6.2).

2.3. Graphs of groups

In this section, we recall the fundamentals of Bass-Serre theory. We refer to [Ser83] for more details. A **graph of groups** is an oriented graph $\mathscr{H}$ equipped with a collection of **vertex groups** $G_v, v\in\mathcal{V}(\mathscr{H})$, a collection of **edge groups** $G_e, e\in\mathcal{E}(\mathscr{H})$ such that $G_e = G_{\overline{e}}$ for every edge $e\in\mathcal{E}(\mathscr{H})$ and, for $\mathbf{u}\in\{\mathbf{s},\mathbf{t}\}$, injective homomorphisms $\iota_{e,\mathbf{u}}: G_e\hookrightarrow G_{\mathbf{u}(e)}$ such that $\iota_{e,\mathbf{s}} = \iota_{\overline{e},\mathbf{t}}$ for every edge e.

The **fundamental group** of a graph of groups $\mathscr{H}$ is defined by the following presentation: let us fix a spanning tree $\mathscr{T}$ in $\mathscr{H}$. Denote by $\{t_e, e\in\mathcal{E}(\mathscr{H})\}$ a generating set of the free group $\mathbb{F}_{|\mathcal{E}(\mathscr{H})|}$ of rank $|\mathcal{E}(\mathscr{H})|$ and define

$$G = \left(*_{v\in\mathcal{V}(\mathscr{H})} G_v * \mathbb{F}_{|\mathcal{E}(\mathscr{H})|}\right) / \left\langle\!\!\left\langle \left(t_e\iota_{e,\mathbf{t}}(x)t_e^{-1}\iota_{e,\mathbf{s}}(x)^{-1}\right)_{(e,x)\in\mathcal{E}(\mathscr{H})\times G_e}, (t_e t_{\overline{e}})_{e\in\mathcal{E}(\mathscr{H})}, (t_e)_{e\in\mathcal{E}(\mathscr{T})}\right\rangle\!\!\right\rangle \tag{2.4}$$

The isomorphism class of the group G defined as above does not depend on the choice of the spanning tree (*cf.* Proposition 20 in [Ser83], Section 5.1).

There exists a (unique up to unique isomorphism) oriented tree $\mathcal{T}$, called the **Bass-Serre tree** of $\mathscr{H}$ on which G acts without inversion with quotient $\mathscr{H}$ and such that there exist sections $\mathcal{V}(\mathscr{H})\to\mathcal{V}(\mathcal{T})$ and $\mathcal{E}(\mathscr{H})\to\mathcal{E}(\mathcal{T})$ (which we denote by $v\to\tilde{v}$ and $e\to\tilde{e}$ respectively) of the projection $\pi:\mathcal{T}\to\mathscr{H}$ satisfying the following conditions:

$$\mathrm{Stab}(\tilde{v}) = G_v \ \forall v\in\mathcal{V}(\mathscr{H}). \tag{2.5}$$

$$\mathrm{Stab}(\tilde{e}) = G_e \ \forall e\in\mathcal{E}(\mathscr{H}). \tag{2.6}$$

More precisely (*cf.* [Ser83][Section 5.3]), the set of vertices of $\mathcal{T}$ is

$$\mathcal{V}(\mathcal{T}) = \bigsqcup_{v\in\mathcal{V}(\mathscr{H})} G/G_v,$$

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and its set of edges is

$$\mathcal{E}(\mathcal{T}) = \bigsqcup_{e \in \mathcal{E}(\mathscr{H})} G/G_e.$$

Conversely, any group action $G \curvearrowright \mathcal{T}$ without inversion is obtained by this construction (*cf.* [Ser83, Section 5]).

3. $\mathscr{H}$-preactions and $\mathscr{H}$-graphs

In this section we generalize the interpretation of graphs of subgroups as "blown up and shrunk" Schreier graphs obtained for HNN-extensions or amalgamated free products in [FLMMS22], for Baumslag-Solitar groups in [CGLMS25] and for rank 1 GBS groups in [Bon26].

This gives us a tool to approximate some subgroups of iterated HNN-extensions and amalgamated free products that we will apply to GBS groups.

3.1. General setting

In this section, we fix a graph of groups $\mathscr{H}$ endowed with a spanning tree $\mathscr{T}$ and we denote by G the fundamental group of $\mathscr{H}$, defined by Presentation (2.4). To any subgroup of G, we will associate a "$\mathscr{H}$-graph", which is a labelled graph that satisfies some combinatorial conditions. It will reduce the problem of approximating a subgroup of G to that of approximating its $\mathscr{H}$-graph.

First, we introduce the notion of **$\mathscr{H}$-preaction** of G. Informally, this is a collection of partial bijections (each of these corresponding to an edge generator or an element of a vertex group in the presentation (2.4)) such that the partial bijections associated to generators of a vertex group G_v define a genuine G_v-action. The following definition generalizes the notion of an $\mathcal{H}$-preaction introduced for GBS groups of rank 1 in [Bon26, Definition 3.1].

Definition 3.1. An $\mathscr{H}$-preaction on a countable set X is a collection of (possibly non-transitive) right G_v-actions α_v defined on subsets D_v of X for every $v \in \mathcal{V}(\mathscr{H})$ (*i.e.* homomorphisms $\alpha_v : G_v \to \mathrm{Sym}(D_v)$) and of partial bijections $\beta_e : \mathrm{dom}(\beta_e) \to \mathrm{rng}(\beta_e)$ for every $e \notin \mathcal{E}(\mathscr{T})$ satisfying the following conditions:

1. for every $e \in \mathcal{E}(\mathscr{H}) \setminus \mathcal{E}(\mathscr{T})$,
   - $\mathrm{dom}(\beta_e) \subseteq D_{\mathbf{s}(e)}$;
   - $\mathrm{dom}(\beta_e)$ is $\alpha_{\mathbf{s}(e)}(\iota_{e,\mathbf{s}}(G_e))$-stable;
   - $\mathrm{dom}(\beta_{\overline{e}}) = \mathrm{rng}(\beta_e)$ and $\beta_{\overline{e}} = \beta_e^{-1}$;
2. for every $e \in \mathcal{E}(\mathscr{T})$, for every $g \in G_e$ and $x \in D_{\mathbf{s}(e)} \cap D_{\mathbf{t}(e)}$, one has
$$x \cdot \alpha_{\mathbf{s}(e)}(\iota_{e,\mathbf{s}}(g)) = x \cdot \alpha_{\mathbf{t}(e)}(\iota_{e,\mathbf{t}}(g));$$
3. for every $e \in \mathcal{E}(\mathscr{H}) \setminus \mathcal{E}(\mathscr{T})$, for every $g \in G_e$ and $x \in \mathrm{dom}(\beta_e)$, one has
$$x \cdot \alpha_{\mathbf{s}(e)}(\iota_{e,\mathbf{s}}(g)) \cdot \beta_e = x \cdot \beta_e \cdot \alpha_{\mathbf{t}(e)}(\iota_{e,\mathbf{t}}(g));$$

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4. for all vertices $v,w \in \mathcal{V}(\mathscr{H})$, for every vertex $u \in [v;w]_{\mathscr{T}}$, one has

$$D_v \cap D_w \subseteq D_u.$$

As an example, any G-action is an $\mathscr{H}$-preaction.

We say that an $\mathscr{H}$-preaction $\alpha' = \{D'_v \curvearrowleft^{\alpha'_v} G_v, \beta'_e : v \in \mathcal{V}(\mathscr{H}), e \in \mathcal{E}(\mathscr{H}) \setminus \mathcal{E}(\mathscr{T})\}$ on a countable set X' **extends** an $\mathscr{H}$-preaction $\alpha = \{D_v \curvearrowleft^{\alpha_v} G_v, \beta_e : v \in \mathcal{V}(\mathscr{H}), e \in \mathcal{E}(\mathscr{H}) \setminus \mathcal{E}(\mathscr{T})\}$ on a countable set X if there exists an injection $\iota : X \hookrightarrow X'$ such that

- for every $v \in \mathcal{V}(\mathscr{H})$, $\iota(D_v) \subseteq D'_v$, and on D_v one has $\alpha'_v(g) \circ \iota = \iota \circ \alpha_v(g)$ for every $g \in G_v$;
- for every edge $e \in \mathcal{E}(\mathscr{H}) \setminus \mathcal{E}(\mathscr{T})$, one has $\iota(\mathrm{dom}(\beta_e)) \subseteq \mathrm{dom}(\beta'_e)$, and on $\mathrm{dom}(\beta_e)$ one has $\beta'_e \circ \iota = \iota \circ \beta_e$.

We then call α a **sub-$\mathscr{H}$-preaction** of α'.

To alleviate notations, given an $\mathscr{H}$-preaction α defined on a countable set X, we will simply denote by $x \cdot g$ the element $x \cdot \alpha_v(g)$ if $x \in D_v$ and $g \in G_v$, and by $x \cdot t_e$ the element $x \cdot \beta_e$ if $x \in \mathrm{dom}(\beta_e)$. Likewise, we will write $\mathrm{dom}(t_e)$ and $\mathrm{dom}(\beta_e)$ interchangeably.

Given an $\mathscr{H}$-preaction α defined on a countable set X, one can define its **Schreier graph** $\mathrm{Sch}(\alpha)$ in a similar way to that for genuine actions: its set of vertices is X and

- for every $v \in \mathcal{V}(\mathscr{H})$ and every $x \in D_v$, there is an edge labelled g with source x and target $x \cdot g$ for every $g \in G_v$;
- for every $e \in \mathcal{E}(\mathscr{H}) \setminus \mathcal{E}(\mathscr{T})$ and every $x \in \mathrm{dom}(t_e)$, there is an edge labelled t_e with source x and target $x \cdot t_e$.

To any $\mathscr{H}$-preaction α on a pointed countable set (X,x) is associated its **stabilizer** $\mathrm{Stab}_\alpha(x)$: any cycle c of $\mathrm{Sch}(\alpha)$ based at x defines an element $\varphi(c)$ of G, obtained by multiplying the labels of the edges of c. The stabilizer $\mathrm{Stab}_\alpha(x)$ is the subgroup of G generated by all the elements $\varphi(c)$, for c varying in the set of cycles of $\mathrm{Sch}(\alpha)$ based at x.

Now we turn to the definition of $\mathscr{H}$-graphs. Informally, the $\mathscr{H}$-graph of an $\mathscr{H}$-preaction α is obtained from the Schreier graph of α in the following way: for every vertex $v \in \mathcal{V}(\mathscr{H})$, shrink every G_v-orbit $x \cdot G_v$ (with $x \in D_v$) to one vertex, and then label the obtained vertex by the action $x \cdot G_v \curvearrowleft G_v$. As the isomorphism class of a transitive right action of a countable group on a countable set X is uniquely determined by the stabilizer of any point of X, these labels will be G_v-conjugacy classes of subgroups of G_v. For any two vertices $x \cdot G_v$, $x' \cdot G_w$, we put some edges labelled $e \in \mathcal{E}(\mathscr{H})$ between $x \cdot G_v$ and $x' \cdot G_w$ if and only if $(\mathbf{s}(e),\mathbf{t}(e)) = (v,w)$ and

- either $e \in \mathscr{T}$ and $x \cdot G_v \cap x' \cdot G_w \neq \emptyset$;
- or $e \notin \mathscr{T}$ and $x \in \mathrm{dom}(t_e)$ and $x \cdot G_v t_e \cap x' \cdot G_w \neq \emptyset$.

More specifically, denoting by s_e the element of G equal to t_e if $e \notin \mathcal{E}(\mathscr{T})$ and to id otherwise, the number of such edges is precisely the number of G_e-orbits of $x \cdot G_v$ which are sent by s_e onto $x' \cdot G_w$. Because of the fact that vertex groups need not be commutative for the moment, we also add a label to the inferior and the superior half-edge in order to remember "where" the two orbits $x \cdot G_v s_e$ and $x' \cdot G_w$ intersect.

We now give the formal definition, which is a generalization of the notion of $\mathcal{H}$-graph introduced in [Bon26, Definition 3.4]:

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Definition 3.2. Let α be an $\mathscr{H}$-preaction of G on a countable set X. One defines the $\mathscr{H}$-graph $\mathcal{G}$ of α as follows:

1. its vertex set is the set of G_v-orbits for v varying in $\mathcal{V}(\mathscr{H})$:
$$\mathcal{V}(\mathcal{G}) = \bigsqcup_{v\in\mathcal{V}(\mathscr{H})} D_v/G_v;$$
2. its edge set is $\mathcal{E}(\mathcal{G}) = \mathcal{E}^+(\mathcal{G}) \sqcup \mathcal{E}^-(\mathcal{G})$ where
$$\mathcal{E}^+(\mathcal{G}) = \bigsqcup_{e\in\mathcal{E}^+(\mathscr{T})} \left(D_{\mathbf{s}(e)} \cap D_{\mathbf{t}(e)}\right)/\iota_{e,\mathbf{s}}(G_e) \bigsqcup \bigsqcup_{e\in\mathcal{E}^+(\mathscr{H})\setminus\mathcal{E}(\mathscr{T})} \mathrm{dom}(t_e)/\iota_{e,\mathbf{s}}(G_e)$$
and
$$\mathcal{E}^-(\mathcal{G}) = \bigsqcup_{e\in\mathcal{E}^+(\mathscr{T})} \left(D_{\mathbf{s}(e)} \cap D_{\mathbf{t}(e)}\right)/\iota_{e,\mathbf{t}}(G_e) \bigsqcup \bigsqcup_{e\in\mathcal{E}^+(\mathscr{H})\setminus\mathcal{E}(\mathscr{T})} \mathrm{rng}(t_e)/\iota_{e,\mathbf{t}}(G_e)$$
with
   - for every $e \in \mathcal{E}^+(\mathscr{T})$, for every $x \in D_{\mathbf{s}(e)} \cap D_{\mathbf{t}(e)}$:
   $$\mathbf{s}\left(x\cdot\iota_{e,\mathbf{s}}(G_e)\right) = x\cdot G_{\mathbf{s}(e)}$$
   and
   $$\mathbf{t}\left(x\cdot\iota_{e,\mathbf{t}}(G_e)\right) = x\cdot G_{\mathbf{t}(e)}.$$
   Moreover
   $$\overline{x\cdot\iota_{e,\mathbf{s}}(G_e)} = x\cdot\iota_{e,\mathbf{t}}(G_e);$$
   - for every $e \in \mathcal{E}^+(\mathscr{H})\setminus\mathcal{E}(\mathscr{T})$, for every $x \in D_{\mathbf{s}(e)} \cap \mathrm{dom}(t_e)\left(D_{\mathbf{t}(e)}\right)$:
   $$\mathbf{s}\left(x\cdot\iota_{e,\mathbf{s}}(G_e)\right) = x\cdot G_{\mathbf{s}(e)}$$
   and
   $$\mathbf{t}\left(x\cdot\iota_{e,\mathbf{t}}(G_e)\right) = xt_e\cdot G_{\mathbf{t}(e)}.$$
   Moreover
   $$\overline{x\cdot\iota_{e,\mathbf{s}}(G_e)} = xt_e\cdot\iota_{e,\mathbf{t}}(G_e);$$
3. each vertex $x\cdot G_v$ is labelled $([\mathrm{Stab}_{G_v}(x)]_{G_v}, v)$ (where $[\mathrm{Stab}_{G_v}(x)]_{G_v}$ denotes the conjugacy class of the subgroup $\mathrm{Stab}_{G_v}(x)$ in G_v);
4. for every vertex $x\cdot G_v$, we fix an identification between $x\cdot G_v$ and the quotient set $\mathrm{Stab}_{G_v}(x)\backslash G_v$ (in a G_v-equivariant way). Each edge $(\mathrm{Stab}_{G_v}(x)g)\iota_{e,\mathbf{s}}(G_e)$ is labelled e and its inferior half-edge is labelled $\underline{(\mathrm{Stab}_{G_v}(x)g)\iota_{e,\mathbf{s}}(G_e)}$; in particular (applying this last condition to $\overline{e}$), it is labelled $\overline{(\mathrm{Stab}_{G_v}(x)g)\iota_{e,\mathbf{s}}(G_e)}$ at its target.

Remark 3.3. In Item 3, the data of $[\mathrm{Stab}_{G_v}(x)]_{G_v}$ is equivalent to the data of the isomorphism class of the G_v-action $\mathrm{Stab}_{G_v}(x)\backslash G_v \curvearrowleft G_v$, or equivalently, of the isomorphism class of the G_v-action on $x\cdot G_v$.

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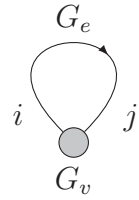


Figure 1. Graph of groups $\mathscr{H}$ defining an HNN-extension.

Example 3.4. Let us consider the HNN-extension G of some group G_v over a group G_e defined by the two inclusions $i : G_e \hookrightarrow G_v$ and $j : G_e \hookrightarrow G_v$. It is the fundamental group of the graph of groups defined in Figure 1.

The group G inherits the following presentation

$$G \simeq \langle G_v, t_e \mid t_e^{-1} i(g) t_e = j(g) \rangle. \tag{3.5}$$

Let us consider an $\mathscr{H}$-preaction that consists of two G_v-orbits $x \cdot G_v$, $y \cdot G_v$, and such that t_e sends two points of $x \cdot G_v$ (say x,x') that lie in two different $i(G_e)$-orbits to two other points in $y \cdot G_v$ (say y,y') that lie in two different $j(G_e)$-orbits (see Figure 2).

The $\mathscr{H}$-graph of the above action consists of

- two vertices that correspond to the two G_v-orbits;
- two edges between those vertices that correspond to the $i(G_e)$,$j(G_e)$-orbits.

This graph is represented in Figure 3.

Notice that the chosen identifications required in Item 4 send x (*resp.* y) to the coset $\mathrm{Stab}_{G_v}(x)1$ (*resp.* $\mathrm{Stab}_{G_v}(y)1$) of $\mathrm{Stab}_{G_v}(x)\backslash G_v$ (*resp.* $\mathrm{Stab}_{G_v}(y)\backslash G_v$).

Similarly, one defines the $\mathscr{H}$-graph of a subgroup H using the correspondence between subgroups and right actions:

Definition 3.6. The $\mathscr{H}$-graph of a subgroup H of G is the $\mathscr{H}$-graph of the right action of G on $H\backslash G$.

Remark 3.7. As observed in [Bon26, Section 3] in the case of infinite cyclic vertex and edge groups, the data of the graph of groups of H is equivalent to its $\mathscr{H}$-graph: recall that the set of vertices of the Bass-Serre tree $\mathcal{T}$ associated to a graph of groups $\mathscr{H}$ of fundamental group G is $\bigsqcup_{v \in \mathcal{V}(\mathscr{H})} G/G_v$, and its set of edges is $\bigsqcup_{e \in \mathcal{E}(\mathscr{H})} G/G_e$. Thus, for any subgroup H of G, the set of vertices (*resp.* of edges) of $H\backslash\mathcal{T}$ is exactly

$$\bigsqcup_{v \in \mathcal{V}(\mathscr{H})} H\backslash(G/G_v) = \bigsqcup_{v \in \mathcal{V}(\mathscr{H})} (H\backslash G)/G_v,$$

(*resp.* $\bigsqcup_{e \in \mathcal{E}(\mathscr{H})} H\backslash(G/G_e) = \bigsqcup_{e \in \mathcal{E}(\mathscr{H})} (H\backslash G)/G_e$), which is exactly the set of vertices (*resp.* of edges) of the $\mathscr{H}$-graph of H (taking the quotient $H\backslash G$ amounts to taking the Schreier graph of H, then taking the disjoint union over the vertices of $\mathscr{H}$ amounts to "blowing it up", and taking the quotient by every vertex (*resp.* edge) group amounts to

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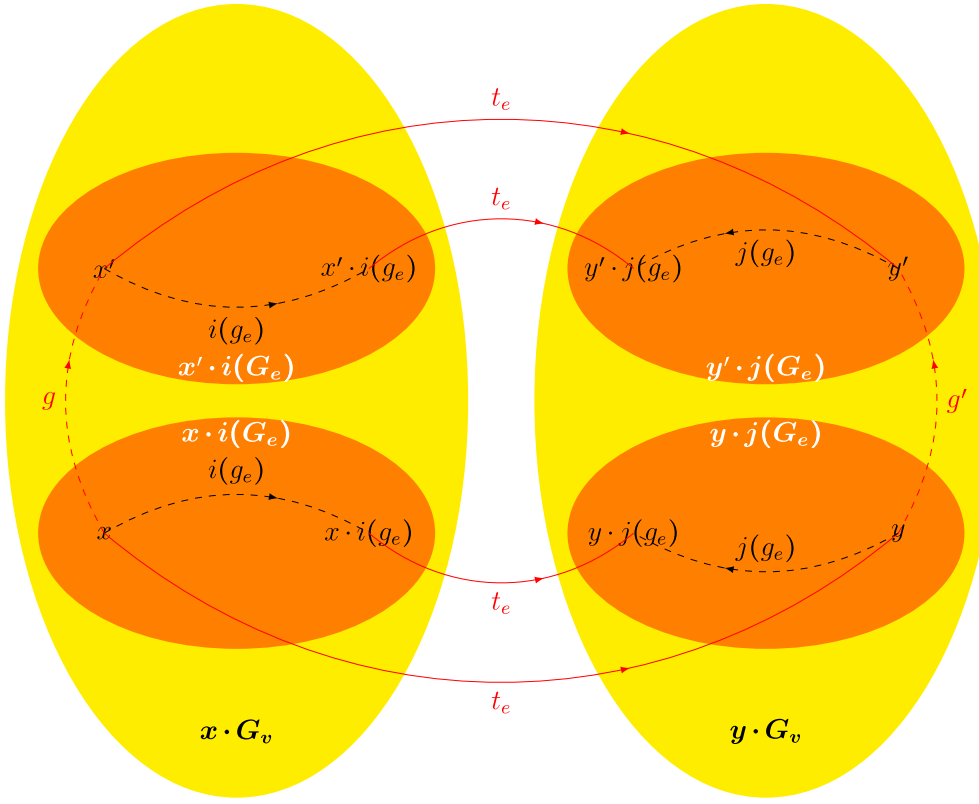


Figure 2. Schreier graph of an $\mathscr{H}$-preaction (here $g, g' \in G_v$ and $g_e \in G_e$).

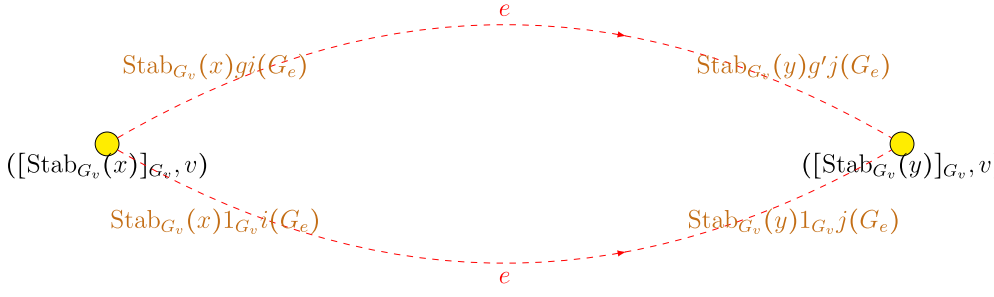


Figure 3. $\mathscr{H}$-graph of the $\mathscr{H}$-preaction represented in Figure 2.

shrinking the orbits of vertex groups (*resp.* edge groups), which amounts to constructing the vertices (*resp.* edges) of the $\mathscr{H}$-graph of H).

Now we want to define an abstract notion of $\mathscr{H}$-graph. To achieve this, we first prove the following lemma, which gives a combinatorial condition on the labels of the vertices of the $\mathscr{H}$-graph of a preaction.

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Lemma 3.8. *Let $\mathcal{G}$ be the $\mathscr{H}$-graph of an $\mathscr{H}$-preaction α defined on a set X_0. For every $E \in \mathcal{E}(\mathcal{G})$, denoting by e the label of E, by $([\Lambda_0]_{G_{\mathbf{s}(e)}}, \mathbf{s}(e))$ (resp. $([\Lambda_1]_{G_{\mathbf{t}(e)}}, \mathbf{t}(e))$) the label of $\mathbf{s}(E)$ (resp. $\mathbf{t}(E)$), by $\Lambda_0 g_0 \iota_{e,\mathbf{s}}(G_e)$ (resp. $\Lambda_1 g_1 \iota_{e,\mathbf{t}}(G_e)$) the label of the inferior (resp. superior) half-edge of E, one has*

$$\left[\iota_{e,\mathbf{s}}^{-1}\left(g_0^{-1}\Lambda_0 g_0\right)\right]_{G_e} = \left[\iota_{e,\mathbf{t}}^{-1}\left(g_1^{-1}\Lambda_1 g_1\right)\right]_{G_e}.$$

Proof. For every $e \in \mathcal{E}(\mathscr{H})$, let us define $s_e = \begin{cases} t_e & \text{if } e \notin \mathcal{E}(\mathscr{T}) \\ id & \text{otherwise} \end{cases}$. By construction, there exist $x, y \in X_0$ and $g, g' \in G_e$ such that

- $\mathrm{Stab}_{G_{\mathbf{s}(e)}}(x) = \Lambda_0$;
- $\mathrm{Stab}_{G_{\mathbf{t}(e)}}(y) = \Lambda_1$;
- $x \cdot g_0 \iota_{e,\mathbf{s}}(g) s_e = y \cdot g_1 \iota_{e,\mathbf{t}}(g')$.

Hence, denoting by $X = x \cdot g_0 \iota_{e,\mathbf{s}}(g)$ and $Y = y \cdot g_1 \iota_{e,\mathbf{t}}(g')$, one has $Y = X s_e$; thus,

$$\begin{aligned}
\mathrm{Stab}_{G_{\mathbf{t}(e)}}(Y) \cap \iota_{e,\mathbf{t}}(G_e) &= \mathrm{Stab}_{\iota_{e,\mathbf{t}}(G_e)}(Y) \\
&= \mathrm{Stab}_{s_e^{-1}\iota_{e,\mathbf{s}}(G_e)s_e}(X s_e) \\
&= s_e^{-1}\left(\mathrm{Stab}_{\iota_{e,\mathbf{s}}(G_e)}(X)\right) s_e \\
&= s_e^{-1}\left(\mathrm{Stab}_{G_{\mathbf{s}(e)}}(X) \cap \iota_{e,\mathbf{s}}(G_e)\right) s_e \\
&= \iota_{e,\mathbf{t}}\left(\iota_{e,\mathbf{s}}^{-1}\left(\mathrm{Stab}_{G_{\mathbf{s}(e)}}(X)\right)\right)
\end{aligned}$$

so, taking the preimage under $\iota_{e,\mathbf{t}}$, we get

$$\iota_{e,\mathbf{s}}^{-1}(\mathrm{Stab}_{G_{\mathbf{s}(e)}}(X)) = \iota_{e,\mathbf{t}}^{-1}(\mathrm{Stab}_{G_{\mathbf{t}(e)}}(Y)),$$

which amounts to saying that

$$g^{-1}\iota_{e,\mathbf{s}}^{-1}\left(g_0^{-1}\Lambda_0 g_0\right) g = g'^{-1}\iota_{e,\mathbf{t}}^{-1}\left(g_1^{-1}\Lambda_1 g_1\right) g',$$

which proves the statement. □

Using this lemma, one can extend the notion of $\mathscr{H}$-graph as follows:

Definition 3.9. An $\mathscr{H}$-graph $\mathcal{G}$ is a labelled graph satisfying the following conditions:

- every vertex is labelled $([\Lambda]_{G_v}, v)$ for some $v \in \mathcal{V}(\mathscr{H})$ and some subgroup Λ of G_v;
- every edge $E \in \mathcal{E}(\mathcal{G})$ is labelled by an edge $e \in \mathcal{E}(\mathscr{H})$ such that $\mathbf{s}(E)$ is labelled $(C, \mathbf{s}(e))$ (for some $G_{\mathbf{s}(e)}$-conjugacy class C of subgroups of $G_{\mathbf{s}(e)}$) and $\mathbf{t}(E)$ is labelled $(C', \mathbf{t}(e))$ (for some $G_{\mathbf{t}(e)}$-conjugacy class C' of subgroups of $G_{\mathbf{t}(e)}$);
- the inferior half-edge of an edge labelled e with source labelled $([\Lambda]_{G_v}, v)$ is labelled by an element of $\Lambda \backslash G_v / \iota_{e,\mathbf{s}}(G_e)$;
- every edge labelled e whose inferior half-edge is labelled $\Lambda_0 g_0 \iota_{e,\mathbf{s}}(G_e)$ (with $\Lambda_0 \leq G_{\mathbf{s}(e)}$) and whose superior half-edge is labelled $\Lambda_1 g_1 \iota_{e,\mathbf{t}}(G_e)$ (with $\Lambda_1 \leq G_{\mathbf{t}(e)}$) satisfies

$$\left[\iota_{e,\mathbf{s}}^{-1}\left(g_0^{-1}\Lambda_0 g_0\right)\right]_{G_e} = \left[\iota_{e,\mathbf{t}}^{-1}\left(g_1^{-1}\Lambda_1 g_1\right)\right]_{G_e};$$

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- inferior (*resp.* superior) half-edges of different edges labelled e (for some $e \in \mathcal{E}(\mathscr{H})$) sharing the same source (*resp.* target) cannot share the same label.

A vertex V of an $\mathscr{H}$-graph labelled $([\Lambda]_{G_v}, v)$ is called **saturated relative to an edge** $e \in \mathcal{E}(\mathscr{H})$ if and only if $\mathbf{s}(e) = v$ and the set of labels of the inferior half-edges of edges labelled e with source V is the whole $\Lambda \backslash G_v / \iota_{e,\mathbf{s}}(G_e)$. The vertex V is called **saturated** if it is saturated relative to every edge with source v. An $\mathscr{H}$-graph is called **saturated** if all its vertices are saturated. In the case where edge groups have finite index in vertex groups, notice that this holds if and only if every vertex labelled $([\Lambda]_{G_v}, v)$ has exactly $|\Lambda \backslash G_v / \iota_{e,\mathbf{s}}(G_e)|$ outgoing edges labelled e for every edge $e \in \mathcal{E}(\mathscr{H})$ satisfying $\mathbf{s}(e) = v$. Finally, an $\mathscr{H}$-action is called saturated if its $\mathscr{H}$-graph is saturated, namely if it is a genuine G-action.

In the formalism of Bass, who introduced the right notion of coverings in the setting of graphs of groups in [Bas93], an $\mathscr{H}$-graph $\mathcal{G}$ is the base space of an *immersion* $\mathcal{G} \to \mathscr{H}$ of the graph of groups $\mathscr{H}$. This immersion is a *covering* if and only if $\mathcal{G}$ is saturated, that is to say, $\mathcal{G}$ is the $\mathscr{H}$-graph of a subgroup of G. Notice, however, that the notion of coverings of graphs of groups introduced by Bass is finer than our notion of $\mathscr{H}$-graphs, as he establishes a correspondence between $\mathrm{Sub}(G)$ and coverings of the graph of groups $\mathscr{H}$. Here, different subgroups can share the same $\mathscr{H}$-graph: consider for instance a graph of groups $\mathscr{H}$ defined by a loop based at a vertex labelled H (for some abstract group H), and take the edge group to be equal to H, and both inclusions to be the identity. The fundamental group of $\mathscr{H}$ is $H \times \mathbb{Z}$ and for any element h of H, the $\mathscr{H}$-graph of the cyclic subgroup $\langle (h,1) \rangle$ consists of a single loop based at a vertex labelled $H \cap \langle (h,1) \rangle = \{\mathrm{id}\}$. However, the subgroups $\langle (h,1) \rangle$ are pairwise different when h varies in H.

The following three Lemmas (3.10, 3.11 and 3.14) give rise to a useful tool to approximate subgroups of G. We show that, under some conditions, an approximation of the $\mathscr{H}$-graph of a subgroup gives rise to an approximation of the subgroup itself. More specifically, if we start from a subgroup H of G and any non-saturated sub-$\mathscr{H}$-preaction α_1 of the corresponding $\mathscr{H}$-preaction, Lemma 3.14 and Lemma 3.10 will allow one to extend α_1 into a saturated $\mathscr{H}$-preaction β_1 whose $\mathscr{H}$-graph $\mathcal{G}$ contains the $\mathscr{H}$-graph $\mathcal{G}_1$ of α_1, and such that $\mathcal{G}/\mathcal{G}_1$ is a tree. If moreover we were able to build a finite connected $\mathscr{H}$-graph that contains both a cycle $\mathcal{L}$ and $\mathcal{G}_1$ as disjoint subgraphs, then Lemma 3.11, together with Lemma 3.14 and Lemma 3.10 would yield a saturated $\mathscr{H}$-preaction β_2 that extends α_1 and whose $\mathscr{H}$-graph $\mathcal{G}'$ contains $\mathcal{G}_1$, such that $\mathcal{G}'/\mathcal{G}_1$ contains the single cycle $\mathcal{L}$. Those two preactions β_1 and β_2 will extend α_1. Moreover, they will be non-isomorphic because the underlying graphs of their $\mathscr{H}$-graphs do not share the same homotopy type.

The proofs of these are very similar to the ones given in the case of GBS groups of rank 1 in [Bon26] (Lemma 3.3, Lemma 3.4 and Lemma 3.5). For the convenience of the reader, we adapt the main ingredients of the proofs to our wider setting.

Lemma 3.10. *Let $(\alpha_i)_{i\in\mathbb{N}}$ be a collection of $\mathscr{H}$-preactions whose $\mathscr{H}$-graphs $(\mathcal{G}_i)_{i\in\mathbb{N}}$ are contained in a saturated $\mathscr{H}$-graph $\mathcal{G}$ as pairwise disjoint subgraphs such that the quotient $\mathcal{G}/\left(\bigsqcup_{i\in\mathbb{N}} \mathcal{G}_i\right)$ is a tree. There exists a G-action α whose $\mathscr{H}$-graph is $\mathcal{G}$ and that extends α_i for every $i \in \mathbb{N}$.*

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Lemma 3.11. *Let $\mathcal{F}$ be a finite $\mathscr{H}$-graph. There exists an $\mathscr{H}$-preaction whose $\mathscr{H}$-graph is $\mathcal{F}$.*

The proofs of Lemma 3.10 and Lemma 3.11 rely on straightforward inductions based on the following proposition:

Proposition 3.12. *Let α_0 be an $\mathscr{H}$-preaction defined on a countable set X_0 whose $\mathscr{H}$-graph $\mathcal{G}_0$ is contained in an $\mathscr{H}$-graph $\mathcal{G}$ such that:*

- $\mathcal{V}(\mathcal{G}) = \mathcal{V}(\mathcal{G}_0)$;
- $\mathcal{E}(\mathcal{G}) = \mathcal{E}(\mathcal{G}_0) \sqcup \{E\}$ *for some edge E such that* $\mathbf{s}(E), \mathbf{t}(E) \in \mathcal{V}(\mathcal{G}_0)$.

Then, there exists an $\mathscr{H}$-preaction α whose $\mathscr{H}$-graph is $\mathcal{G}$. Moreover, if $\mathbf{s}(E)$ and $\mathbf{t}(E)$ belong to two different connected components of $\mathcal{G}_0$ (that is to say, $\alpha_0 = \alpha_1 \sqcup \alpha_2$ for some sub-$\mathscr{H}$-preactions α_1 and α_2, and $\mathbf{s}(E)$ (resp. $\mathbf{t}(E)$) corresponds to a vertex orbit for α_1 (resp. α_2)), the constructed preaction α extends both α_1 and α_2.

Proof. We adapt Constructions A and B defined in [Bon26, Section 3]. Let e be the label of E and let us denote by $(V_1, V_2) := (\mathbf{s}(E), \mathbf{t}(E))$, and by $(v_1, v_2) := (\mathbf{s}(e), \mathbf{t}(e))$. Let $([\Lambda_i]_{G_{v_i}}, v_i)$ be the label of V_i (for $i \in \{1,2\}$) and let $\Lambda_1 g_1 \iota_{e,\mathbf{s}}(G_e)$ and $\Lambda_2 g_2 \iota_{e,\mathbf{t}}(G_e)$ be the label of the inferior half-edge and of the superior half-edge of E, respectively. One has $[\iota_{e,\mathbf{s}}^{-1}(g_1^{-1}\Lambda_1 g_1)]_{G_e} = [\iota_{e,\mathbf{t}}^{-1}(g_2^{-1}\Lambda_2 g_2)]_{G_e}$, *i.e.* there exists $h \in G_e$ such that

$$\iota_{e,\mathbf{s}}^{-1}(g_1^{-1}\Lambda_1 g_1) = h^{-1}\iota_{e,\mathbf{t}}^{-1}(g_2^{-1}\Lambda_2 g_2)h. \tag{3.13}$$

We distinguish two cases:

Construction A: If $e \notin \mathcal{E}(\mathscr{T})$, then there exist $x_1, x_2 \in X_0$ such that:

- $x_i \in \mathrm{dom}(G_{v_i})$ for $i \in \{1,2\}$;
- $\mathrm{Stab}_{G_{v_i}}(x_i) = \Lambda_i$ for $i \in \{1,2\}$;
- $x_1 \cdot g_1 \notin \mathrm{dom}(t_e)$ and $x_2 \cdot g_2 \notin \mathrm{rng}(t_e)$.

We extend t_e and t_e^{-1} on a subset of $X := X_0$ by defining, for every $g \in G_e$:

$$(x_1 \cdot g_1(\iota_{e,\mathbf{s}}(g))) \cdot t_e = x_2 \cdot g_2 \iota_{e,\mathbf{t}}(hgh^{-1}),$$

which is well-defined by Equation 3.13. The resulting $\mathscr{H}$-preaction is suitable.

Construction B: If $e \in \mathcal{E}(\mathscr{T})$, then there exist $x_1, x_2 \in X_0$ such that:

- $x_i \in \mathrm{dom}(G_{v_i})$ for $i \in \{1,2\}$;
- $\mathrm{Stab}_{G_{v_i}}(x_i) = \Lambda_i$ for $i \in \{1,2\}$;
- $x_1 \cdot g_1 \notin \mathrm{dom}(G_{v_2})$ and $x_2 \cdot g_2 \notin \mathrm{dom}(G_{v_1})$.

We let

$$X := X_0 / \left(x_1 \cdot g_1 \iota_{e,\mathbf{s}}(g) \sim x_2 \cdot g_2 \iota_{e,\mathbf{t}}(hgh^{-1}) \ \forall g \in G_e\right)$$

and we let α be the $\mathscr{H}$-preaction induced by α_0 on X. As previously, it is well-defined by Equation 3.13. By construction, the $\mathscr{H}$-graph of α is $\mathcal{G}$. □

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Finally we explain how to saturate an $\mathscr{H}$-graph in our new setting.

Lemma 3.14. *For every (non-saturated) $\mathscr{H}$-graph $\mathcal{G}$, there exists a saturated $\mathscr{H}$-graph $\tilde{\mathcal{G}}$ that contains $\mathcal{G}$ and such that the quotient $\tilde{\mathcal{G}}/\mathcal{G}$ is an infinite forest.*

Proof. We build an increasing sequence of $\mathscr{H}$-graphs $(\mathcal{G}_n)_{n\mathbb{N}}$ such that the union $\bigcup_{n\in\mathbb{N}}\mathcal{G}_n$ is the desired $\mathscr{H}$-graph. This works by induction using the following construction: if V is a non-saturated vertex labelled $([\Lambda_0]_{G_v},v)$, whose non-saturation is witnessed by

- an edge $e\in\mathcal{E}(\mathscr{H})$ with source v;
- an element $g_0\in G_v$ such that there is no inferior half-edge labelled $\Lambda_0 g_0\iota_{e,\mathbf{s}}(G_e)$ with source V

then, denoting by $w=\mathbf{t}(e)$, one defines

$$\Lambda_1=\iota_{e,\mathbf{t}}(\iota_{e,\mathbf{s}}^{-1}(g_0^{-1}\Lambda_0 g_0))$$

and one builds a new vertex W labelled $([\Lambda_1]_{G_w},w)$ and a new edge E labelled e with source V and target W such that

- the inferior half-edge $(E,\mathbf{s})$ is labelled $\Lambda_0 g_0\iota_{e,\mathbf{s}}(G_e)$;
- the superior half-edge $(E,\mathbf{t})$ is labelled $\Lambda_1\iota_{e,\mathbf{t}}(G_e)$.

□

3.2. Generalized Baumslag-Solitar groups

In this section, we will apply the tools we introduced in the previous section to GBS groups.

A GBS group of rank d is the fundamental group of a finite graph of groups whose vertex and edge stabilizers are isomorphic to $\mathbb{Z}^d$. As any injective homomorphism $\mathbb{Z}^d\to\mathbb{Z}^d$ is represented by an element of $\mathrm{M}_d(\mathbb{Z})\cap\mathrm{GL}_d(\mathbb{Q})$, a GBS group can be represented by an oriented graph $\mathscr{H}$ endowed with a function that associates an integer matrix whose determinant is non-zero to each half-edge:

$$\begin{array}{ccccc}\mathrm{M} & : & \mathcal{E}(\mathscr{H})\times\{\mathbf{s},\mathbf{t}\} & \to & \mathrm{M}_d(\mathbb{Z})\cap\mathrm{GL}_d(\mathbb{Q})\\ & & (e,\mathbf{u}) & \mapsto & \mathrm{M}_{e,\mathbf{u}}\end{array}$$

and that satisfies $\mathrm{M}_{\overline{e},\mathbf{t}}=\mathrm{M}_{e,\mathbf{s}}$ for every $e\in\mathcal{E}(\mathscr{H})$. Up to shrinking some edges, we can assume that the graph $\mathscr{H}$ is **reduced**, that is to say, the only edges $e\in\mathcal{E}(\mathscr{H})$ one of whose labels is in $\mathrm{GL}_d(\mathbb{Z})$ are loops.

Given a GBS group G defined by a graph of groups $\mathscr{H}$, let us denote by $\mathcal{T}$ its Bass-Serre tree and let us define the **modular homomorphism** as follows (*cf.* [LdGZS25]): fix a vertex $v\in\mathcal{V}(\mathcal{T})$. Observe that for any $g\in G$, the group $G_{gv}\cap G_v$ has finite index in G_v, hence g belongs to the abstract commensurator of G_v. Define $\Delta_G^{(v)}(g)$ as the equivalence class of the homomorphism $\begin{array}{ccc}G_v\cap G_{g^{-1}v} & \to & G_v\cap G_{gv}\\ h & \mapsto & ghg^{-1}\end{array}$. As G_v is isomorphic to $\mathbb{Z}^d$, the homomorphism $\Delta_G^{(v)}$ can be identified with a homomorphism $\Delta_G^{(v)}:G\to\mathrm{GL}_d(\mathbb{Q})$. The definition of the modular homomorphism does not depend on the choice of the vertex v up to conjugation by an element of $\mathrm{GL}_d(\mathbb{Q})$. Practically, the image of the modular

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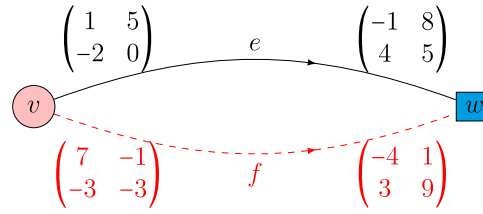


Figure 4. A graph of a GBS group.

homomorphism (based at some vertex v) is generated by the matrices $\mathrm{A}_1\mathrm{B}_1^{-1}\cdots\mathrm{A}_n\mathrm{B}_n^{-1}$ for every edge path $e_1,\dots,e_n$ labelled $(\mathrm{A}_1,\mathrm{B}_1),\dots,(\mathrm{A}_n,\mathrm{B}_n)$ and based at v. In particular, the modular homomorphism of a GBS group defined by a tree of groups is trivial. A GBS group G of rank d is **unimodular** if $\mathrm{Im}(\det\circ\Delta_G)\subseteq\{1,-1\}$.

Example 3.15. Let us consider the GBS group of rank 2 defined in Figure 4.

Let us choose the edge f as a spanning tree in the graph represented in Figure 4. Then, the presentation of G associated to this choice is the following:

$$\begin{aligned}G\simeq\langle x_v,y_v,x_w,y_w,t_e \mid{}& [x_v,y_v]=[x_w,y_w]=1,\\ & x_v^7y_v^{-3}=x_w^{-4}y_w^3,x_v^{-1}y_v^{-3}=x_wy_w^9,\\ & t_e^{-1}(x_vy_v^{-2})t_e=x_w^{-1}y_w^4,t_e^{-1}x_v^5t_e=x_w^8y_w^5\rangle.\end{aligned}$$

The modular homomorphism is trivial on the vertex generators x_v,y_v,x_w,y_w and sends t_e to the product $\begin{pmatrix}1 & 5\\ -2 & 0\end{pmatrix}\cdot\begin{pmatrix}-1 & 8\\ 4 & 5\end{pmatrix}^{-1}\cdot\begin{pmatrix}-4 & 1\\ 3 & 9\end{pmatrix}\cdot\begin{pmatrix}7 & -1\\ -3 & -3\end{pmatrix}^{-1}$:

$$\Delta_G^{(v)}:\begin{array}{ccc}G & \to & \mathrm{GL}_2(\mathbb{Q})\\ x_v,y_v,x_w,y_w & \mapsto & I_2\\ t_e & \mapsto & \frac{1}{296}\begin{pmatrix}-153 & -301\\ 46 & 342\end{pmatrix}\end{array}$$

In the case where the vertex stabilizers are abelian, the definition of an $\mathscr{H}$-graph simplifies as follows. An $\mathscr{H}$-graph $\mathcal{G}$ is then a labelled graph satisfying the following conditions:

1. every vertex is labelled (Λ,v) for some $v\in\mathcal{V}(\mathscr{H})$ and some subgroup Λ of G_v;
2. every edge $E\in\mathcal{E}(\mathcal{G})$ is labelled e for some $e\in\mathcal{E}(\mathscr{H})$ such that $\mathbf{s}(E)$ is labelled $(\Lambda_0,\mathbf{s}(e))$ (for some $\Lambda_0\leq G_{\mathbf{s}(e)}$) and $\mathbf{t}(E)$ is labelled $(\Lambda_1,\mathbf{t}(e))$ (for some $\Lambda_1\leq G_{\mathbf{t}(e)}$);
3. every edge labelled e whose source (*resp.* target) is labelled $(\Lambda_0,\mathbf{s}(e))$ (*resp.* $(\Lambda_1,\mathbf{t}(e))$) satisfies
$$\iota_{e,\mathbf{s}}^{-1}(\Lambda_0)=\iota_{e,\mathbf{t}}^{-1}(\Lambda_1);$$
4. every vertex labelled (Λ,v) has at most $|\Lambda\backslash G_v/\iota_{e,\mathbf{s}}(G_e)|$ outgoing edges labelled e (with $\mathbf{s}(e)=v$) and at most $|\Lambda\backslash G_v/\iota_{e,\mathbf{t}}(G_e)|$ incoming edges labelled e (with $\mathbf{t}(e)=v$).

Hence, for a GBS group of rank d, the definition of an $\mathscr{H}$-graph becomes the following:

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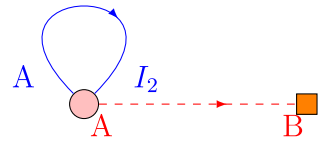


Figure 5. The graph of groups $\mathscr{H}_0$.

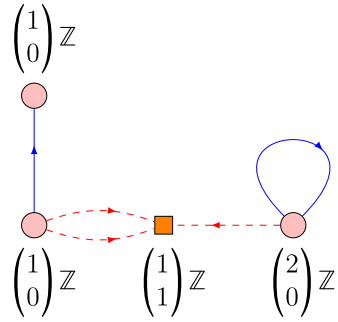


Figure 6. An example of an $\mathscr{H}_0$-graph.

Definition 3.16. Let $d \geq 1$ and let $\mathscr{H}$ be a finite graph of groups all of whose vertex and edge groups are isomorphic to $\mathbb{Z}^d$. An $\mathscr{H}$-graph is a labelled graph that satisfies the following four conditions:

1. every vertex is labelled (Λ, v) for some $v \in \mathcal{V}(\mathscr{H})$ and some subgroup Λ of $\mathbb{Z}^d$;
2. every edge $E \in \mathcal{E}(\mathcal{G})$ is labelled e for some $e \in \mathcal{E}(\mathscr{H})$ such that $\mathbf{s}(E)$ is labelled $(\Lambda_0, \mathbf{s}(e))$ (for some $\Lambda_0 \leq G_{\mathbf{s}(e)}$) and $\mathbf{t}(E)$ is labelled $(\Lambda_1, \mathbf{t}(e))$ (for some $\Lambda_1 \leq G_{\mathbf{t}(e)}$);
3. **Transfer Equation** every edge labelled e whose source (*resp.* target) is labelled $(\Lambda_0, \mathbf{s}(e))$ (*resp.* $(\Lambda_1, \mathbf{t}(e))$) satisfies
$$\left(\mathrm{M}_{e,\mathbf{s}}^{-1}\Lambda_0\right) \cap \mathbb{Z}^d = \left(\mathrm{M}_{e,\mathbf{t}}^{-1}\Lambda_1\right) \cap \mathbb{Z}^d;$$
4. every vertex labelled (Λ, v) has at most $\left|\mathbb{Z}^d / \langle \Lambda, \mathrm{M}_{e,\mathbf{s}}\mathbb{Z}^d \rangle\right|$ incident edges labelled e (with $\mathbf{s}(e) = v$) and at most $\left|\mathbb{Z}^d / \langle \Lambda, \mathrm{M}_{e,\mathbf{t}}\mathbb{Z}^d \rangle\right|$ incident edges labelled e (with $\mathbf{t}(e) = v$);

It is saturated if and only if equality holds for every vertex and edge in the last item.

Definition 3.17. An **$\mathscr{H}$-path** is an $\mathscr{H}$-graph whose underlying graph $E_1, \ldots, E_r$ is an edge path. It is said to be **reduced** if the underlying graph is reduced, namely, there is no $i \in [\![1, r-1]\!]$ such that $E_i = \overline{E_{i+1}}$.

Example 3.18. Let $\mathrm{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathrm{B} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$. Let us define the GBS group G_0 as the fundamental group $\pi_1(\mathscr{H}_0)$ of the following graph of groups defined in Figure 5.

The labelled graph represented in Figure 6 is a (non-saturated) $\mathscr{H}_0$-graph.

We will focus on non-amenable GBS groups. They are characterized by the following proposition:

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Proposition 3.19. *Let G be a GBS group of rank d defined by a reduced graph of groups $\mathscr{H}$. Then G is amenable if and only if $\mathscr{H}$ is a single loop one of whose labels is in $\mathrm{GL}_d(\mathbb{Z})$, or a single edge e with $\mathbf{s}(e) \neq \mathbf{t}(e)$, both of whose labels have determinant ± 2.*

Proof. Let us assume that $\mathscr{H}$ consists of a single loop e labelled (A,B) with $\mathrm{B} \in \mathrm{GL}_d(\mathbb{Z})$. Denoting by $\mathrm{M} := \mathrm{AB}^{-1}$, the group G is isomorphic to $\mathbb{Z}[\mathrm{M},\mathrm{M}^{-1}]\mathbb{Z}^d \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{Z}[\mathrm{M},\mathrm{M}^{-1}]\mathbb{Z}^d$ by multiplication by M. As an extension of an abelian group by an abelian group, G is amenable.

Now let us assume that $\mathscr{H}$ consists of an edge e which is not a loop such that the labels A and B of e have determinant ± 2. The Bass-Serre tree $\mathcal{T}$ of $\mathscr{H}$ is a bi-infinite line on which G acts with kernel N isomorphic to $\mathbb{Z}^d$. The action of G/N on $\mathcal{T}$ also has a single orbit of edges. The stabilizer of any edge is trivial, and, as A and B have determinant ± 2, the stabilizer of any vertex is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Thus, Bass-Serre theory tells us that G/N is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, which is virtually isomorphic to $\mathbb{Z}$. Hence, N and G/N are amenable, so G is amenable.

Conversely, let us assume that $\mathscr{H}$ is neither a single loop one of whose labels is in $\mathrm{GL}_d(\mathbb{Z})$, nor a single edge e with $\mathbf{s}(e) \neq \mathbf{t}(e)$, both of whose labels have determinant ± 2. Then the action of G on its Bass-Serre tree $\mathcal{T}$ is of general type; thus, G contains a free group on two generators. In particular, G is non-amenable. □

4. An equivalence relation on $\mathbf{Sub}(\mathbb{Z}^d)$

In this section, we assume that G is a non-amenable GBS group that is *not* the fundamental group of any graph of groups defined by a single vertex and a collection of loops labelled by invertible integer matrices. In other words, G is neither amenable nor isomorphic to a semidirect product $\mathbb{Z}^d \rtimes \mathbb{F}_r$.

First, we introduce an equivalence relation that will give the decomposition of Theorem 3 (the fact that it is indeed an equivalence relation will be proved in Corollary 4.3).

Definition 4.1. Let Λ_0 and Λ_1 be two subgroups of $\mathbb{Z}^d$. We say that Λ_0 and Λ_1 are $\mathscr{H}$-equivalent with respect to a vertex $v \in \mathcal{V}(\mathscr{H})$ (denoted w.r.t. v) if there exists a connected $\mathscr{H}$-graph that contains two vertices labelled (Λ_0, v) and (Λ_1, v), respectively.

Lemma 4.2. *Suppose that there exists an $\mathscr{H}$-path $E_1,\dots,E_r$ that connects two vertices labelled (Λ_0, v) and (Λ_1, w), respectively. Then, for all edges $e, f \in \mathcal{E}(\mathscr{H})$ satisfying $\mathbf{s}(e) = v$ and $\mathbf{t}(f) = w$, there exists an $\mathscr{H}$-path of type $e,\dots,e_1,\dots,e_r,\dots,f$ that connects two vertices labelled (Λ_0, v) and (Λ_1, w) and that contains $E_1,\dots,E_r$ as a subpath.*

Proof. Let $E_1,\dots,E_r$ be an $\mathscr{H}$-path labelled $e_1,\dots,e_r$ (with $e_i \in \mathcal{E}(\mathscr{H})$ for every $i \in [\![1,r]\!]$) such that

- $\mathbf{s}(E_1)$ is labelled (Λ_0, v);
- $\mathbf{t}(E_r)$ is labelled (Λ_1, w).

Without loss of generality, one can assume this $\mathscr{H}$-path to be reduced. We first prove that there exists a reduced $\mathscr{H}$-path labelled $e,\dots,e_1,\dots,e_r$ that connects a vertex labelled

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(Λ_0, v) to a vertex labelled (Λ_1, w), and that contains $E_1, \dots, E_r$ as a subpath. If $e = e_1$, then the edge path $E_1, \dots, E_r$ is suitable.

Otherwise, let us denote by $(\mathrm{A}, \mathrm{B}) = (\mathrm{M}_{e,\mathbf{s}}, \mathrm{M}_{e,\mathbf{t}})$ and $v' = \mathbf{t}(e)$. Let us define

$$\Lambda_0' = \mathrm{B}(\mathrm{A}^{-1}\Lambda_0 \cap \mathbb{Z}^d).$$

By construction, (Λ_0, v) and (Λ_0', v') satisfy the Transfer Equation 3

$$\mathrm{A}^{-1}\Lambda_0 \cap \mathbb{Z}^d = \mathrm{B}^{-1}\Lambda_0' \cap \mathbb{Z}^d.$$

Hence there exists an $\mathscr{H}$-graph that consists of a single edge E_0' labelled e connecting a vertex V_0 labelled (Λ_0, v) to a vertex V_1 labelled (Λ_0', v').

Case 1: let us first assume that $|\det(\mathrm{B})| \geq 2$. As $|\det(B)| \geq 2$ and $\Lambda_0' \subseteq \mathrm{B}\mathbb{Z}^d$, one has

$$\left|\mathbb{Z}^d / \langle \mathrm{B}\mathbb{Z}^d, \Lambda_0' \rangle\right| = |\det(\mathrm{B})| \geq 2.$$

Hence the labelled graph which consists of

- the edge E_0';
- an edge E_1' labelled $\overline{e}$ connecting V_1 to a vertex V_2 labelled (Λ_0, v)

is an $\mathscr{H}$-graph of type $e, \overline{e}$ that connects a vertex labelled (Λ_0, v) to a vertex labelled (Λ_0, v). Finally, as $e \neq e_1$ by assumption, the labelled graph obtained by the concatenation of the $\mathscr{H}$-paths E_0', E_1' and $E_1, \dots, E_r$ is an $\mathscr{H}$-graph of type $e, \overline{e}, e_1, \dots, e_r$ that connects a vertex labelled (Λ_0, v) to a vertex labelled (Λ_1, w) as required.

Case 2: otherwise, $|\det(\mathrm{B})| = 1$. In particular, the graph $\mathscr{H}$ being reduced, e is a loop. We distinguish two subcases:

Subcase 2.1: There exists an edge $f \neq \overline{e}$ with source v such that $|\det(\mathrm{M}_{f,\mathbf{t}})| \geq 2$. Hence we can apply Case 1 to obtain an $\mathscr{H}$-path E_1', E_2' of type $f, \overline{f}$ that connects a vertex labelled (Λ_0', v) to a vertex labelled (Λ_0', v). As $f \neq \overline{e}$ and (Λ_0, v) and (Λ_0', v') satisfy the Transfer Equation 3, the labelled graph which consists of

- the concatenation of the $\mathscr{H}$-path E_0' with the $\mathscr{H}$-path E_1', E_2';
- an edge E_3' labelled $\overline{e}$ with source $\mathbf{t}(E_2')$ and target a new vertex labelled (Λ_0, v)

is an $\mathscr{H}$-graph. As $e \neq e_1$, we can concatenate the $\mathscr{H}$-path E_0', E_1', E_2', E_3' and the $\mathscr{H}$-path $E_1, \dots, E_r$ to obtain an $\mathscr{H}$-path labelled $e, f, \overline{f}, \overline{e}, e_1, \dots, e_r$ that connects a vertex labelled (Λ_0, v) to a vertex labelled (Λ_1, w).

Subcase 2.2: Otherwise, as G is non-amenable, Proposition 3.19 implies that $\mathscr{H}$ consists of a collection of at least two loops based at a single vertex such that every label (except possibly A) is in $\mathrm{GL}_d(\mathbb{Z})$. As G is not a semidirect product $\mathbb{Z}^d \rtimes \mathbb{F}_r$, one necessarily has $|\det(A)| \geq 2$. Let $f \in \mathscr{H} \setminus \{e, \overline{e}\}$. Subcase 2.1 delivers an $\mathscr{H}$-path E_1', E_2', E_3', E_4' of type $f, \overline{e}, e, \overline{f}$ that connects a vertex labelled (Λ_0', v) to a vertex labelled (Λ_0', v). As $f \neq \overline{e}$, the $\mathscr{H}$-graph which consists of

- the concatenation of the $\mathscr{H}$-path E_0' with the $\mathscr{H}$-graph E_1', E_2', E_3', E_4';
- an edge E_5' of type $\overline{e}$ that connects $\mathbf{t}(E_4')$ to a new vertex labelled (Λ_0, v)

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is an $\mathscr{H}$-path. As $e \neq e_1$, we can concatenate the $\mathscr{H}$-path $E'_0, E'_1, E'_2, E'_3, E'_4, E'_5$ and the $\mathscr{H}$-path $E_1, \dots, E_r$ to obtain an $\mathscr{H}$-path labelled $e, f, \overline{e}, e, \overline{f}, \overline{e}, e_1, \dots, e_r$ that connects a vertex labelled (Λ_0, v) to a vertex labelled (Λ_1, w).

Hence we have proved that there exists an $\mathscr{H}$-path labelled $e, \dots, e_1, \dots, e_r$ that connects a vertex labelled (Λ_0, v) to a vertex labelled (Λ_1, w). Using this result on the reverse path (that connects a vertex labelled (Λ_1, w) to a vertex labelled (Λ_0, v)) leads to an $\mathscr{H}$-path of type $e, \dots, e_1, \dots, e_r, \dots, f$ that connects a vertex labelled (Λ_0, v) to a vertex labelled (Λ_1, w), which leads to the conclusion. □

Corollary 4.3. *$\mathscr{H}$-equivalence with respect to a prescribed vertex v is an equivalence relation.*

More precisely, the following holds: if $E_1, \dots, E_r$ (resp. $F_1, \dots, F_s$) is an $\mathscr{H}$-path that connects a vertex labelled (Λ_0, v) (resp. (Λ_1, v)) to a vertex labelled (Λ_1, v) (resp. (Λ_2, v)), then there exists an $\mathscr{H}$-path that contains $E_1, \dots, E_{r-1}$ and $F_2, \dots, F_s$ as disjoint (labelled) subpaths.

Proof. Let $\Lambda_0, \Lambda_1, \Lambda_2 \leq \mathbb{Z}^d$.

As the empty path is an $\mathscr{H}$-path that connects any vertex labelled (Λ_0, v) to itself, Λ_0 is $\mathscr{H}$-equivalent to itself (w.r.t. v).

If Λ_0 is $\mathscr{H}$-equivalent to Λ_1 w.r.t. v, then there exists a reduced $\mathscr{H}$-path $E_1, \dots, E_r$ with source labelled (Λ_0, v) and target labelled (Λ_1, v). The reversed $\mathscr{H}$-path $\overline{E_r}, \dots, \overline{E_1}$ has its source labelled (Λ_1, v) and its target labelled (Λ_0, v). Thus, Λ_1 is $\mathscr{H}$-equivalent to Λ_0 w.r.t. v.

Let us assume that Λ_0 is $\mathscr{H}$-equivalent to Λ_1 w.r.t. v and that Λ_1 is $\mathscr{H}$-equivalent to Λ_2 w.r.t. v. Let $E_1, \dots, E_r$ be a reduced $\mathscr{H}$-path of type $e_1, \dots, e_r$ with source labelled (Λ_0, v) and target labelled (Λ_1, v) and let $F_1, \dots, F_s$ be a reduced $\mathscr{H}$-path of type $f_1, \dots, f_s$ with source labelled (Λ_1, v) and target labelled (Λ_2, v).

Case 1: let us assume that $f_1 \neq \overline{e_r}$. Then, the concatenation of these $\mathscr{H}$-paths delivers an $\mathscr{H}$-path with source labelled (Λ_0, v) and target labelled (Λ_2, v).

Case 2: otherwise, let us denote by $e = e_r = \overline{f_1}$.

Subcase 2.1: first, we assume that there exists an edge $g \neq e_r$ **such that** $\mathbf{t}(g) = v$. By Lemma 4.2, there exists an $\mathscr{H}$-path $E'_1, \dots, E'_t$ of type $e_1, \dots, e_r, \dots, g$ that contains $E_1, \dots, E_r$ and that connects a vertex labelled (Λ_0, v) to a vertex labelled (Λ_1, v). As $g \neq \overline{f_1}$, the concatenation of the $\mathscr{H}$-paths $E'_1, \dots, E'_t$ and $F_1, \dots, F_s$ delivers an $\mathscr{H}$-path with source labelled (Λ_0, v) and target labelled (Λ_2, v).

Subcase 2.2: now we assume that e **is the unique edge with target** v. Notice that in this case, e cannot be a loop. If $r = 0$ or $s = 0$, we are done. Hence we assume that $r, s \geq 1$. Let $u = \mathbf{s}(e)$ and $(\mathrm{A}, \mathrm{B}) := (\mathrm{M}_{e,\mathbf{s}}, \mathrm{M}_{e,\mathbf{t}})$. Let $\left(\widetilde{\Lambda_1}, u\right)$ and $\left(\widetilde{\Lambda_2}, u\right)$ be the labels of $\mathbf{s}(E_r)$ and $\mathbf{t}(F_1)$, respectively. By the Transfer Equation 3 we have

$$\begin{aligned} \mathrm{A}^{-1}\widetilde{\Lambda_1} \cap \mathbb{Z}^d &= \mathrm{B}^{-1}\Lambda_1 \cap \mathbb{Z}^d \\ &= \mathrm{A}^{-1}\widetilde{\Lambda_2} \cap \mathbb{Z}^d. \end{aligned}$$

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Let us define

$$\Lambda_1' = \mathrm{B}\left(\mathrm{A}^{-1}\widetilde{\Lambda_1} \cap \mathbb{Z}^d\right).$$

As e is not a loop and $\mathscr{H}$ is reduced, one has $|\det(B)| \geq 2$. Hence the labelled graph which consists of

- an edge E_0' labelled e with source labelled $\left(\widetilde{\Lambda_1}, u\right)$ and target V labelled (Λ_1', v);
- an edge E_1' labelled $\overline{e}$ with source V and target a vertex labelled $\left(\widetilde{\Lambda_2}, u\right)$

is an $\mathscr{H}$-graph. Hence the concatenation of the $\mathscr{H}$-path $E_1, \dots, E_{r-1}$, the $\mathscr{H}$-path E_0', E_1' and the $\mathscr{H}$-path $F_2, \dots, F_s$ delivers an $\mathscr{H}$-path that connects a vertex labelled (Λ_0, v) to a vertex labelled (Λ_2, v).

In any case, we have proved that Λ_0 is $\mathscr{H}$-equivalent to Λ_2 w.r.t. v. □

Lemma 4.4. *Let Λ_0 be a subgroup of $\mathbb{Z}^d$ and $v \in \mathcal{V}(\mathscr{H})$. There exists a finite $\mathscr{H}$-graph*

- *that is not simply connected;*
- *that contains a vertex labelled (Λ_0, v);*
- *that contains at least two non-saturated vertices.*

Proof. We distinguish two cases:

Case 1: assume that there exists a loop $e_0 \in \mathcal{E}(\mathscr{H})$ based at v. Let us consider an $\mathscr{H}$-path E_1, E_2 labelled e_0, e_0 such that

- the vertex $\mathbf{t}(E_1) = \mathbf{s}(E_2)$ is labelled (Λ_0, v);
- the vertex $\mathbf{s}(E_1)$ is labelled (Λ_1, v), where $\Lambda_1 = \mathrm{M}_{e_0,\mathbf{s}}\left(\mathrm{M}_{e_0,\mathbf{t}}^{-1}\Lambda_0 \cap \mathbb{Z}^d\right)$ (so that Λ_1, Λ_0 and e_0 satisfy the Transfer Equation (3));
- the vertex $\mathbf{t}(E_2)$ is labelled (Λ_2, v), where $\Lambda_2 = \mathrm{M}_{e_0,\mathbf{t}}\left(\mathrm{M}_{e_0,\mathbf{s}}^{-1}\Lambda_0 \cap \mathbb{Z}^d\right)$ (so that Λ_0, Λ_2 and e_0 satisfy the Transfer Equation (3)).

We apply Lemma 4.2 to get an $\mathscr{H}$-path labelled $e_0, \dots, e_0$ with source $\mathbf{t}(E_2)$ and target $\mathbf{s}(E_1)$.

Subcase 1.a: If there exists $f_0 \in \mathcal{E}(\mathscr{H}) \setminus \{e_0, \overline{e_0}\}$, then the vertices $\mathbf{t}(E_2)$ and $\mathbf{s}(E_1)$ are neither saturated relative to f_0 nor to $\overline{f_0}$, which leads to the conclusion in this particular case.

Subcase 1.b: Otherwise, as G is non-amenable, one has $|\det(\mathrm{M}_{e_0,\mathbf{s}})| \geq 2$ and $|\det(\mathrm{M}_{e_0,\mathbf{t}})| \geq 2$ by Proposition 3.19. In particular:

- as $\mathbf{s}(E_1)$ is labelled by a subgroup Λ_1 of $\mathrm{M}_{e_0,\mathbf{s}}\mathbb{Z}^d$, one has
$$\left|\mathbb{Z}^d / \left\langle \Lambda_1, \mathrm{M}_{e_0,\mathbf{s}}\mathbb{Z}^d \right\rangle\right| = |\det(\mathrm{M}_{e_0,\mathbf{s}})| \geq 2,$$
so, as $\mathbf{s}(E_1)$ has a single outgoing edge labelled e_0, the vertex $\mathbf{s}(E_1)$ is non-saturated relative to e_0;
- likewise, the vertex $\mathbf{t}(E_2)$ is non-saturated relative to $\overline{e_0}$.

Case 2: otherwise, we fix an edge $e_0 \in \mathcal{E}(\mathscr{H})$ with source v such that $w := \mathbf{t}(e_0) \neq v$. In particular, the graph of groups $\mathscr{H}$ being reduced, denoting by

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$(\mathrm{A},\mathrm{B}) := (\mathrm{M}_{e_0,\mathbf{s}},\mathrm{M}_{e_0,\mathbf{t}})$, one has $|\det(A)| \geq 2$ and $|\det(B)| \geq 2$. Let us define an $\mathscr{H}$-path $E_1,E_2,E_3,E_4,E_5,E_6,E_7,E_8$ as follows:

- E_1, E_3, E_5, E_7 are labelled e_0 and E_2, E_4, E_6, E_8 are labelled $\overline{e_0}$;
- $\mathbf{s}(E_1)$ is labelled (Λ_0,v);
- the vertices $\mathbf{t}(E_1)$, $\mathbf{t}(E_3)$, $\mathbf{t}(E_5)$ and $\mathbf{t}(E_7)$ are all labelled $(\mathrm{B}(\mathrm{A}^{-1}\Lambda_0 \cap \mathbb{Z}^d),v)$;
- the vertices $\mathbf{t}(E_2)$, $\mathbf{t}(E_4)$, $\mathbf{t}(E_6)$ and $\mathbf{t}(E_8)$ are labelled $(\Lambda_0 \cap \mathrm{A}\mathbb{Z}^d,v)$.

Notice that at least four vertices are non-saturated relative to some edge $\widehat{e} \in \mathcal{E}(\mathscr{H})$. Indeed, by Proposition 3.19, as G is non-amenable and $\mathscr{H}$ is reduced:

- either $|\det(\mathrm{A})| \geq 3$. In this case, the vertices $\mathbf{t}(E_2)$, $\mathbf{t}(E_4)$, $\mathbf{t}(E_6)$ and $\mathbf{t}(E_8)$ are non-saturated relative to e_0;
- or $|\det(\mathrm{B})| \geq 3$. In this case, $\mathbf{t}(E_1)$, $\mathbf{t}(E_3)$, $\mathbf{t}(E_5)$ and $\mathbf{t}(E_7)$ are non-saturated relative to $\overline{e_0}$;
- or there exists an edge $f_0 \neq e_0$ with source v. In this case, the vertices $\mathbf{t}(E_2)$, $\mathbf{t}(E_4)$, $\mathbf{t}(E_6)$ and $\mathbf{t}(E_8)$ are non-saturated relative to f_0;
- or there exists an edge $f_0 \neq \overline{e_0}$ with target w. In this case, $\mathbf{t}(E_1)$, $\mathbf{t}(E_3)$, $\mathbf{t}(E_5)$ and $\mathbf{t}(E_7)$ are non-saturated relative to f_0.

In any case, denoting by V_1,V_2,V_3,V_4 four non-saturated vertices relative to some edge $\widehat{e}$, Lemma 4.2 delivers a reduced $\mathscr{H}$-path $F_1,\dots,F_r$ of type $\widehat{e},\dots,\overline{\widehat{e}}$ with source and target labelled by the label of V_1 and V_2, respectively. Gluing its source to V_1 and its target to V_2 leads to an $\mathscr{H}$-graph $\mathcal{C}$ in which the vertices V_3 and V_4 are non-saturated relative to $\widehat{e}$. Hence $\mathcal{C}$ is suitable. □

5. Perfect kernel of non-amenable GBS groups

The goal of this section is to give an explicit description of the perfect kernel in the case where the GBS group G is non-amenable, *i.e.* is defined neither by a single loop with at least one invertible matrix nor by a segment with two matrices having determinant ± 2.

We start with the following lemma that gives an inclusion in a more general setting:

Lemma 5.1. *Let G be the fundamental group of a finite graph of groups $\mathscr{H}$ such that, for every $v \in \mathcal{V}(\mathscr{H})$, the group G_v is Noetherian (i.e. every subgroup of G_v is finitely generated (e.g. the groups G_v are finitely generated abelian groups)). Let us denote by $\mathcal{T}$ the Bass-Serre tree of $\mathscr{H}$. Then*

$$\mathcal{K}(G) \subseteq \{H \leq G \mid H\backslash\mathcal{T} \text{ is infinite}\}.$$

Proof. Let H be a subgroup of G whose graph of groups $H\backslash\mathcal{T}$ is finite. Let us show that under the assumptions of the lemma, the set

$$\Omega = \{H' \leq G, H \leq H'\}$$

is a countable neighbourhood of H in $\mathrm{Sub}(G)$.

First notice that every element of Ω has a finite graph of groups. In particular, any element of Ω is finitely generated: denoting by $\mathscr{K}$ its graph of groups, Bass-Serre theory tells us that it is generated by a finite number of subgroups of the groups G_v (one for

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each vertex of $\mathscr{K}$, each of these being finitely generated by noetherianity of the groups G_v), and by one element per edge of $\mathscr{K}$. In particular:

- as H belongs to Ω, it is finitely generated, so Ω is an open neighbourhood of H;
- Ω is included in the subset of finitely generated subgroups, hence is countable.

This proves that H has a countable neighbourhood, thus $H \notin \mathcal{K}(G)$. □

Now we prove Theorem 1:

Theorem 5.2. *Let G be a non-amenable GBS group defined by a reduced graph of groups $\mathscr{H}$ and let $\mathcal{T}$ be the associated Bass-Serre tree. Then*

$$\mathcal{K}(G) = \{H \leq G \mid H\backslash\mathcal{T} \text{ is infinite}\}.$$

Proof. The inclusion $\mathcal{K}(G) \subseteq \{H \leq G \mid H\backslash\mathcal{T}$ is infinite$\}$ follows from Lemma 5.1. By the correspondence between $\mathscr{H}$-graphs and quotients of $\mathcal{T}$ by subgroups of G established in Remark 3.7, it suffices to show that any subgroup H of G whose $\mathscr{H}$-graph $\mathcal{G}$ is infinite belongs to $\mathcal{K}(G)$. Let us fix $v \in \mathcal{V}(\mathscr{H})$. Let us denote by α the associated $\mathscr{H}$-preaction. Let β be a sub-$\mathscr{H}$-preaction of α whose $\mathscr{H}$-graph is a finite subgraph K of $\mathcal{G}$. By assumption, K has a vertex V_0 labelled (Λ_0, v_0) for some $\Lambda_0 \leq \mathbb{Z}^d$ and some $v_0 \in \mathcal{V}(\mathscr{H})$, which is not saturated relative to some edge e_0 with source v_0. By Lemma 3.14, there exists a saturated $\mathscr{H}$-graph $\mathcal{G}_0$ that contains K and such that the quotient $\mathcal{G}_0/K$ is an infinite forest. Hence, by Lemma 3.10, there exists a saturated $\mathscr{H}$-preaction α_0 that extends β and whose $\mathscr{H}$-graph is $\mathcal{G}_0$.

Let us first assume that $\mathscr{H}$ does not consist of a single vertex and a collection of loops labelled by matrices in $\mathrm{GL}_d(\mathbb{Z})$. By Lemma 4.4, there exists a non-simply connected finite $\mathscr{H}$-graph $\mathcal{C}$ that contains

- a vertex labelled (Λ_0, v);
- two vertices V and W labelled (Λ_1, v_1) and (Λ_2, v_2) that are non-saturated relative to some edges denoted by e, f, respectively.

By Lemma 3.11, $\mathcal{C}$ is the $\mathscr{H}$-graph of an $\mathscr{H}$-preaction γ. Lemma 4.2 delivers an $\mathscr{H}$-path $\mathcal{P}$ of type $e_0, \ldots, \bar{e}$ that connects V_0 to V. The vertex W is non-saturated relative to f in the $\mathscr{H}$-graph $K' := K \cup \mathcal{P} \cup \mathcal{C}$. By Lemma 3.14, there exists a saturated $\mathscr{H}$-graph $\mathcal{G}_1$ that contains K' and such that the quotient $\mathcal{G}_1/K'$ is a forest. Hence, by Lemma 3.10, there exists a saturated $\mathscr{H}$-preaction α_1 that extends both β and γ and whose $\mathscr{H}$-graph is $\mathcal{G}_1$. As $\mathcal{G}_0$ and $\mathcal{G}_1$ are non-isomorphic (because they do not share the same homotopy type), the (saturated) $\mathscr{H}$-preactions α_0 and α_1 both extend β and their stabilizers in G are not conjugate. We proved that any neighbourhood of α (or equivalently, of H) contains two non-isomorphic G-actions α_0, α_1 (equivalently, two non-conjugate subgroups H_0, H_1 of G) whose $\mathscr{H}$-graphs $\mathcal{G}_0$, $\mathcal{G}_1$ are infinite (*i.e.*, the quotient graphs $H_0\backslash\mathcal{T}$, $H_1\backslash\mathcal{T}$ are infinite). This proves that the space $\{H \leq G \mid H\backslash\mathcal{T}$ is infinite$\}$ has no isolated point. Thus, this space is included in $\mathcal{K}(G)$.

Otherwise, our group G is of the form $\mathbb{Z}^d \rtimes \mathbb{F}_r$ (where $r \geq 2$ denotes the number of loops) and each generator of $\mathbb{F}_r$ acts on $\mathbb{Z}^d$ by multiplication by an invertible integer matrix. Denoting by $\pi : G = \mathbb{Z}^d \rtimes \mathbb{F}_r \to \mathbb{F}_r$ the canonical surjection, every subgroup H of

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G is fully determined by its intersection $H_0 = H \cap \mathbb{Z}^d$ with $\mathbb{Z}^d$, its image $\pi(H) \leq \mathbb{F}_r$ under π satisfying

$$x \cdot H_0 = H_0 \ \forall x \in \pi(H)$$

and, given a basis $(a_i)_{i\in I}$ of $\pi(H)$ (the set I being countable), elements $(u_i, a_i) \in H$ for every $i \in I$. Notice that in this case, the Bass-Serre tree of G is the Cayley graph of $\mathbb{F}_r$ with respect to the standard generating set, and that for any subgroup H of G, the quotient graph $H\backslash\mathcal{T}$ is infinite if and only if $\pi(H) \in \mathrm{Sub}_{[\infty]}(\mathbb{F}_r)$. We distinguish two cases:

1. Let us first assume that $\mathrm{rk}(H_0) = d$. Let us denote by $(\mathrm{A},\mathrm{B}) = (\mathrm{M}_{e_0,\mathbf{s}}, \mathrm{M}_{e_0,\mathbf{t}})$, let us consider an edge $f_0 \neq e_0$ and let us write $(\mathrm{C},\mathrm{D}) = (\mathrm{M}_{f_0,\mathbf{s}}, \mathrm{M}_{f_0,\mathbf{t}})$. As the subgroup of $\mathrm{GL}_d(\mathbb{Z})$ generated by DC^{-1} acts on the (finite) set of lattices of determinant $\pm\det(H_0)$, there exists an integer $k \in \mathbb{N}^*$ satisfying

$$\left(\mathrm{DC}^{-1}\right)^k H_0 = H_0.$$

   In particular, there exists an $\mathscr{H}$-cycle $\mathcal{C} = E_1, \dots, E_k$ all of whose edges are of type f_0 such that $\mathbf{s}(E_i)$ is labelled $\left(\left(\mathrm{DC}^{-1}\right)^i H_0, \mathbf{s}(e_0)\right)$ for every $i \in [\![1,k]\!]$. As no vertex of $\mathcal{C}$ is saturated relative to e_0, Lemma 4.2 delivers an $\mathscr{H}$-path $\mathcal{P}$ of type $e_0, \dots, \overline{e_0}$ that connects V_0 to a vertex V of $\mathcal{C}$. Any other vertex of $\mathcal{C}$ is not saturated relative to e_0 in $K' := K \cup \mathcal{P} \cup \mathcal{C}$. Hence we can conclude as in the previous case.

2. If $\mathrm{rk}(H_0) < d$, let us write H_0 in its Smith normal form:

$$H_0 = \mathrm{P}\,\mathrm{diag}(d_1, \dots, d_r, 0, \dots, 0)\mathbb{Z}^d$$

   (with $\mathrm{P} \in \mathrm{GL}_d(\mathbb{Z})$, $r < d$ and $d_i | d_{i+1}$ for every $i \in [\![1, r-1]\!]$). For every $N \in \mathbb{N}$, define

$$H_0^{(N)} = \mathrm{P}\,\mathrm{diag}\left(d_1, \dots, d_r, N\prod_{i=1}^r d_i, \dots, N\prod_{i=1}^r d_i\right)\mathbb{Z}^d.$$

   Claim. *Every matrix stabilizing the subgroup H_0 also stabilizes $H_0^{(N)}$.*

   Proof of the claim. Denoting by M a matrix stabilizing H_0, and by $e_1, \dots, e_d$ the canonical basis of $\mathbb{Z}^d$, the matrix $\mathrm{P}^{-1}\mathrm{MP}$ stabilizes $\langle d_1 e_1, \dots, d_r e_r\rangle$. In particular:

$$\mathrm{P}^{-1}\mathrm{MP}\left\langle d_1 e_1, \dots, d_r e_r, \left(N\prod_{i=1}^r d_i\right) e_{r+1}, \dots, \left(N\prod_{i=1}^r d_i\right) e_d\right\rangle$$
$$\leq \left\langle d_1 e_1, \dots, d_r e_r, \left(N\prod_{i=1}^r d_i\right)\mathbb{Z}^d\right\rangle.$$

   Now, $\left(N\prod_{i=1}^r d_i\right)\mathbb{Z}^d \leq \langle d_1 e_1, \dots, d_r e_r, \left(N\prod_{i=1}^r d_i\right) e_{r+1}, \dots, \left(N\prod_{i=1}^r d_i\right) e_d\rangle$: if $j \leq r$, then

$$\left(N\prod_{i=1}^r d_i\right) e_j = \left(N\prod_{i\neq j} d_i\right)(d_j e_j) \in d_j e_j \mathbb{Z}.$$

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Thus, $\mathrm{P}^{-1}\mathrm{MP}$ stabilizes $\langle d_1 e_1, \dots, d_r e_r, (N\prod_{i=1}^r d_i)\, e_{r+1}, \dots, (N\prod_{i=1}^r d_i)\, e_d\rangle$, hence M stabilizes $H_0^{(N)}$. □

In particular, $H_0^{(N)}$ is $\pi(H)$-stable. Moreover, as $(a_i)_{i\in I}$ is a free basis of $\langle\, a_i,\ i\ \in\ I\,\rangle$, one has

$$\big(\mathbb{Z}^d \times 1\big) \cap \langle (u_i, a_i), i \in I\rangle = \{1\}.$$

This implies that

$$\begin{aligned}\widetilde{H}_N &:= \left\langle H_0^{(N)}, (u_i, a_i)_{i\in I}\right\rangle \\ &= H_0^{(N)} \rtimes \left\langle (u_i, a_i)_{i\in I}\right\rangle.\end{aligned}$$

As moreover $H_0^{(N)}$ tends to H_0 in $\mathrm{Sub}(\mathbb{Z}^d)$ as N tends to $+\infty$, we deduce that the sequence of subgroups $\widetilde{H}_N$ converges to H (non-trivially, because $\widetilde{H}_N \cap \mathbb{Z}^d = H_0^{(N)}$). As $\pi\left(\widetilde{H}_N\right) = \pi(H)$, the $\mathscr{H}$-graphs of $\widetilde{H}_N$ and H have isomorphic skeletons (in particular, the subgroups $\widetilde{H}_N$ have infinite $\mathscr{H}$-graphs). □

Remark 5.4. Notice that we did not make use of the cocompactness of the action $G \curvearrowright \mathcal{T}$. Thus, Theorem 5.2 extends to a larger class of groups that includes GBS groups of rank d, *i.e.* the class of non-amenable groups acting (not necessarily cocompactly) on an oriented tree with vertex and edge stabilizers isomorphic to $\mathbb{Z}^d$. In particular, in the case of a non-cocompact action on the Bass-Serre tree, the perfect kernel consists of the whole set of subgroups $\mathrm{Sub}(G)$.

The following corollary gives a class of GBS groups that satisfy the equality $\mathcal{K}(G) = \mathrm{Sub}_{[\infty]}(G)$. Recall that in rank 1, the authors of [CGLMS25] and [Bon26] proved that this equality is true for non-unimodular GBS groups only. Let v be a vertex of $\mathscr{H}$ and recall that the modular homomorphism $\Delta_G^{(v)}$ based at v is defined by the following data:

- $\Delta_G^{(v)}$ is trivial on the vertex groups;
- for every edge generator t_e, denoting by $e_1, \dots, e_r$ the unique reduced edge path in $\mathscr{T}$ with source v and target $\mathbf{s}(e)$, and by $e_{r+1}, \dots, e_s$ the unique reduced edge path in $\mathscr{T}$ with source $\mathbf{t}(e)$ and target v, one has

$$\Delta_G^{(v)}(t_e) = \mathrm{M}_{e_1,\mathbf{s}}\mathrm{M}_{e_1,\mathbf{t}}^{-1} \dots \mathrm{M}_{e_r,\mathbf{s}}\mathrm{M}_{e_r,\mathbf{t}}^{-1}\mathrm{M}_{e,\mathbf{s}}\mathrm{M}_{e,\mathbf{t}}^{-1}\mathrm{M}_{e_{r+1},\mathbf{s}}\mathrm{M}_{e_{r+1},\mathbf{t}}^{-1} \dots \mathrm{M}_{e_s,\mathbf{s}}\mathrm{M}_{e_s,\mathbf{t}}^{-1}.$$

Corollary 5.5. *Let G be a non-amenable and non-unimodular GBS group of rank d such that the finite-index subgroups of $\mathrm{Im}\left(\Delta_G^{(v)}\right)$ are $\mathbb{Q}$-irreducible. Then*

$$\mathcal{K}(G) = \mathrm{Sub}_{[\infty]}(G).$$

Before proving Corollary 5.5, we give an explicit example of a GBS group G of rank 2 whose perfect kernel consists of the set of infinite-index subgroups of G. Let us define $\mathrm{A} = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$ and $\mathrm{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and let us define $G = \pi_1(\mathscr{H})$ as the fundamental group of

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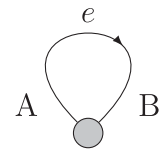


Figure 7. The graph of groups $\mathscr{H}$.

the graph of groups defined in Figure 7. In other words, G is defined by the following presentation:

$$G \simeq \langle x,y,t \mid xy = yx, t^{-1}x^2y^2t = x, t^{-1}x^2y^4t = y^2 \rangle.$$

The image of the modular homomorphism Δ_G is the subgroup of $\mathrm{GL}_2(\mathbb{Q})$ generated by $\mathrm{AB}^{-1} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$. Hence $\mathrm{Im}(\det \circ \Delta_G) = 2^{\mathbb{Z}}$; thus, G is not unimodular. Finite-index subgroups of $\mathrm{Im}(\Delta_G)$ are of the form $\langle (\mathrm{AB}^{-1})^n \rangle$ (where $n \in \mathbb{Z} \setminus \{0\}$). Notice that $\mathrm{Spec}_{\mathbb{R}}(\mathrm{AB}^{-1}) = \{2+\sqrt{2}, 2-\sqrt{2}\}$, so $\mathrm{Spec}_{\mathbb{R}}((\mathrm{AB}^{-1})^n) = \left\{ (2+\sqrt{2})^n, (2-\sqrt{2})^n \right\}$ for every $n \in \mathbb{Z}$. Thus, for any $n \in \mathbb{Z} \setminus \{0\}$, one has $\mathrm{Spec}_{\mathbb{Q}}((\mathrm{AB}^{-1})^n) = \emptyset$, *i.e.* the subgroup $\langle (\mathrm{AB}^{-1})^n \rangle$ does not stabilize any non-trivial proper vector subspace of $\mathbb{Q}^2$. Hence, by Corollary 5.5, one has $\mathcal{K}(G) = \mathrm{Sub}_{[\infty]}(G)$.

Now let us prove Corollary 5.5. To begin with, we prove the following lemma:

Lemma 5.6. *Let H be a subgroup of G and let us denote by $\mathcal{G}$ its $\mathscr{H}$-graph. Let $v \in \mathcal{V}(\mathscr{H})$ and let (Λ_0, v) be the label of a vertex V_0 of $\mathcal{G}$. Then:*

$$\left\{ \Delta_G^{(v)}(g) \cdot \Lambda_0 \otimes \mathbb{Q} \mid g \in G \right\} = \{ \Lambda_0' \otimes \mathbb{Q} \mid \exists a \text{ vertex } V \in \mathcal{V}(\mathcal{G}) \text{ labelled } (\Lambda_0', v) \}$$
$$= \{ \Lambda_0' \otimes \mathbb{Q} \mid \Lambda_0' \text{ is } \mathscr{H}\text{-equivalent to } \Lambda_0 \text{ w.r.t. } v \}.$$

Proof. Let us prove that

$$\left\{ \Delta_G^{(v)}(g) \cdot \Lambda_0 \otimes \mathbb{Q} \mid g \in G \right\} \subseteq \{ \Lambda_0' \otimes \mathbb{Q} \mid \exists \text{a vertex } V \in \mathcal{V}(\mathcal{G}) \text{ labelled } (\Lambda_0', v) \}.$$

Let $g \in G$. Let us consider a cycle $e_1, \dots, e_r$ based at v in $\mathscr{H}$ and elements $g_i \in G_{\mathbf{s}(e_i)}$ for every i and $g_{r+1} \in G_v$ such that

$$g = g_1 \cdot s_{e_1} \cdot g_2 \cdots s_{e_r} \cdot g_{r+1}$$

(where $s_{e_i} = t_{e_i}$ if $e_i \notin \mathscr{T}$ and 1 otherwise). Denoting by $(\mathrm{A}_i, \mathrm{B}_i)$ the label of e_i for every $i \in [\![1,r]\!]$, one has:

$$\Delta_G^{(v)}(g) = \mathrm{A}_1 \mathrm{B}_1^{-1} \dots \mathrm{A}_r \mathrm{B}_r^{-1}. \tag{5.7}$$

Let us consider an edge path $E_1, \dots, E_r$ based at V_0 and labelled $\overline{e_r}, \dots, \overline{e_1}$ in $\mathcal{G}$. For every $i \in [\![1,r]\!]$, let $(\Lambda_i, \mathbf{t}(e_i))$ be the label of $\mathbf{t}(E_i)$. By the Transfer Equation 3, we get

$$\mathrm{B}_{r-i}^{-1} \Lambda_i \cap \mathbb{Z}^d = \mathrm{A}_{r-i}^{-1} \Lambda_{i+1} \cap \mathbb{Z}^d$$

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for every $i \in [\![0,r-1]\!]$, which implies that

$$\mathrm{B}_{r-i}^{-1}\Lambda_i \otimes \mathbb{Q} = \mathrm{A}_{r-i}^{-1}\Lambda_{i+1} \otimes \mathbb{Q}.$$

Thus

$$\Lambda_r \otimes \mathbb{Q} = \mathrm{A}_1\mathrm{B}_1^{-1} \dots \mathrm{A}_r\mathrm{B}_r^{-1}\Lambda_0 \otimes \mathbb{Q}, \tag{5.8}$$

which implies, with Equation (5.7), that $\Lambda_0' := \Lambda_r$ satisfies $\Lambda_0' \otimes \mathbb{Q} = \Delta_G^{(v)}(g) \cdot \Lambda_0 \otimes \mathbb{Q}$.

If there exists a vertex $V \in \mathcal{V}(\mathcal{G})$ labelled (Λ_0', v), then Λ_0' is $\mathscr{H}$-equivalent to Λ_0 w.r.t. v by connectedness of $\mathcal{G}$. This proves that

$$\begin{aligned}&\{\Lambda_0' \otimes \mathbb{Q} \mid \exists \text{a vertex } V \in \mathcal{V}(\mathcal{G}) \text{ labelled } (\Lambda_0', v)\}\\ &\subseteq \{\Lambda_0' \otimes \mathbb{Q} \mid \Lambda_0' \text{ is } \mathscr{H}\text{-equivalent to } \Lambda_0 \text{ w.r.t. } v\}.\end{aligned}$$

Finally, let us prove that

$$\{\Lambda_0' \otimes \mathbb{Q} \mid \Lambda_0' \text{ is } \mathscr{H}\text{-equivalent to } \Lambda_0 \text{ w.r.t. } v\} \subseteq \left\{\Delta_G^{(v)}(g) \cdot \Lambda_0 \otimes \mathbb{Q} \mid g \in G\right\}.$$

Let Λ_0' be a subgroup that is $\mathscr{H}$-equivalent to Λ_0 w.r.t. v. Let us consider an $\mathscr{H}$-path $E_1, \dots, E_r$ labelled $e_1, \dots, e_r$ whose source is labelled (Λ_0, v) and whose target is labelled (Λ_0', v). For $i \in [\![1,r]\!]$, let us denote by $(\Gamma_i, \mathbf{s}(e_i))$ the label of $\mathbf{s}(E_i)$ (hence $\Gamma_1 = \Lambda_0$). By the Transfer Equation 3, denoting by $(\mathrm{A}_i, \mathrm{B}_i)$ the label of e_i in $\mathscr{H}$, we get, as before,

$$\Lambda_0' \otimes \mathbb{Q} = \mathrm{B}_r\mathrm{A}_r^{-1} \dots \mathrm{B}_1\mathrm{A}_1^{-1}\Lambda_0 \otimes \mathbb{Q}.$$

Thus, denoting by $g = s_{e_1} \cdots s_{e_r}$ (where $s_{e_i} = t_{e_i}$ if $e_i \notin \mathscr{T}$ and 1 otherwise), one has:

$$\Lambda_0' \otimes \mathbb{Q} = \Delta_G^{(v)}(g)^{-1} \cdot \Lambda_0 \otimes \mathbb{Q},$$

which concludes the proof of the lemma. □

Proof of Corollary 5.5. Let H be a subgroup of G whose graph of groups is finite. Let us show that H has finite index under the assumptions of the corollary. Let $\mathcal{G}$ be the $\mathscr{H}$-graph of H. Let us denote by (Λ_0, v) the label of a vertex of $\mathcal{G}$. By Lemma 5.6, the orbit of $\Lambda_0 \otimes \mathbb{Q}$ under the action of G *via* $\Delta_G^{(v)}$ is finite. The stabilizer

$$\{g \in G \mid \Delta_G^{(v)}(g) \cdot \Lambda_0 \otimes \mathbb{Q} = \Lambda_0 \otimes \mathbb{Q}\}$$

has finite index in G. This implies that

- either $\Lambda_0 \otimes \mathbb{Q} = \mathbb{Q}^d$, *i.e.* Λ_0 has finite index in $\mathbb{Z}^d$;
- or $\Lambda_0 = 0$.

Let us assume by contradiction that $\Lambda_0 = \{0\}$. By commensurability of the vertex stabilizers, this is equivalent to

$$H \cap G_w = \{1\} \quad \text{for every vertex } w \in \mathcal{V}(\mathscr{H})$$

(or equivalently, all the labels of the vertices of $\mathcal{G}$ are trivial).

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As the image of the homomorphism $\left|\det\circ\Delta_G^{(v)}\right|$ is non-trivial, there exists a (reduced) cycle of edges $e_1,\dots,e_n$ in $\mathscr{H}$ based at v and satisfying

$$\left|\frac{\prod_{i=1}^n \det(\mathrm{A}_i)}{\prod_{i=1}^n \det(\mathrm{B}_i)}\right| \neq 1.$$

Let n_i be the number of vertices of $\mathcal{G}$ labelled $\mathbf{s}(e_i)$. Every vertex labelled $\mathbf{s}(e_i)$ (*resp.* $\mathbf{t}(e_i)$) has $|\det(\mathrm{A}_i)|$ (*resp.* $|\det(\mathrm{B}_i)|$) outgoing (*resp.* incoming) edges labelled e_i. Hence the number of edges labelled e_i in $\mathcal{G}$ is

$$n_i|\det(\mathrm{A}_i)| = n_{i+1}|\det(\mathrm{B}_i)|.$$

Combining these equalities for $i = 1,\dots,n$, we get

$$\prod_{i=1}^n |\det(\mathrm{A}_i)| = \prod_{i=1}^n |\det(\mathrm{B}_i)|,$$

hence a contradiction. Thus, Λ_0 has finite index in $G_{\mathbf{s}(e_1)}$. By commensurability of the vertex stabilizers, this implies that $H\cap G_w$ has finite index in G_w for every $w\in\mathcal{V}(\mathscr{H})$, which implies that $H\backslash G$ is finite (because $\mathcal{G}$ is also finite).

Using Theorem 5.2, we finally deduce the equality

$$\mathcal{K}(G) = \mathrm{Sub}_{[\infty]}(G).$$

□

Remark 5.9. If G is a non-unimodular GBS group such that every non-trivial element of the image of the linear modular homomorphism $\Delta_G^{(v)}$ is $\mathbb{Q}$-irreducible, then the assumptions of Corollary 5.5 are satisfied. Let us explain why. As G is non-unimodular, there exists $g\in G$ such that $\left|\det\left(\Delta_G^{(v)}(g)\right)\right| \neq 1$. In particular, $\Delta_G^{(v)}(g)$ has infinite order. Now assume by contradiction that there exists a non-trivial proper vector subspace V of $\mathbb{Q}^d$ that is stabilized by a finite-index subgroup Γ of $\Delta_G^{(v)}(G)$. As Γ has finite index, there exists some $n\in\mathbb{N}^*$ such that $\Delta_G^{(v)}(g)^n \in \Gamma$. Consequently, V is stabilized by the non-trivial element $\Delta_G^{(v)}(g)^n$ of $\mathrm{Im}\left(\Delta_G^{(v)}\right)$, which contradicts the assumption made on G.

6. A dynamical partition of the perfect kernel

The goal of this section is to extend the decomposition of the perfect kernel obtained in [CGLMS25] and in [Bon26] for non-amenable GBS groups of rank 1.

6.1. Case where G is not a semidirect product $\mathbb{Z}^d\rtimes\mathbb{F}_r$

In this section, we assume that G is not a semidirect product $\mathbb{Z}^d\rtimes\mathbb{F}_r$, *i.e.* $\mathscr{H}$ does not consist of a single vertex with a collection of loops all of whose labels are in $\mathrm{GL}_d(\mathbb{Z})$. This allows us to make use of the equivalence relation on $\mathrm{Sub}(\mathbb{Z}^d)$ defined in Section 4.

Let us fix a vertex $v\in\mathcal{V}(\mathscr{H})$. We identify G_v with $\mathbb{Z}^d$. Let us denote by $\simeq$ the $\mathscr{H}$-equivalence relation with respect to v and by $\pi_v : \mathrm{Sub}(\mathbb{Z}^d)\to\mathrm{Sub}(\mathbb{Z}^d)/\simeq$ the canonical projection. Notice that the rank is constant on each fibre of π_v. We have the following statement:

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Proposition 6.1. *The set* $\mathrm{Sub}(\mathbb{Z}^d)/\simeq$ *is countably infinite.*

Proof. As $\mathrm{Sub}(\mathbb{Z}^d)$ is countable, the set $\mathrm{Sub}(\mathbb{Z}^d)/\simeq$ is also countable. Let us show that it is infinite. Let us define the finite subset $\mathcal{P}_{\mathscr{H}}$ of prime numbers

$$\mathcal{P}_{\mathscr{H}} = \{p \in \mathcal{P} \mid \text{there exists an edge } e \in \mathcal{E}(\mathscr{H}), p \mid \det(\mathrm{M}_{e,\mathbf{t}})\}.$$

Let us define

$$\delta : \begin{array}{ccc} \mathcal{L}(\mathbb{Z}^d) & \to & \mathbb{Z} \\ \Lambda & \mapsto & \prod_{p \notin \mathcal{P}_{\mathscr{H}}} p^{|\det(\Lambda)|_p} \end{array}.$$

The image of δ is exactly the set of integers that are divisible by no element of $\mathcal{P}_{\mathscr{H}}$, hence is infinite. Let us show that δ is constant on the fibres of π_v, *i.e.* that we have a factorization

$$\begin{array}{ccc} \mathcal{L}(\mathbb{Z}^d) & \xrightarrow{\delta} & \delta(\mathbb{Z}) \\ & \searrow^{\pi_v} & \uparrow \overline{\delta} \\ & & \mathcal{L}(\mathbb{Z}^d)/\simeq \end{array},$$

which will imply that $\mathcal{L}(\mathbb{Z}^d)/\simeq$ is infinite (hence $\mathrm{Sub}(\mathbb{Z}^d)/\simeq$ is infinite).

By a straightforward induction on the length of an $\mathscr{H}$-path, it suffices to prove that for any $\mathscr{H}$-edge labelled $e \in \mathcal{E}(\mathscr{H})$ with source labelled $(\Lambda_0, \mathbf{s}(e))$ and target labelled $(\Lambda_1, \mathbf{t}(e))$, one has $\delta(\Lambda_0) = \delta(\Lambda_1)$. Let E be such an edge. By the Transfer Equation 3, denoting by $(\mathrm{A},\mathrm{B}) = (\mathrm{M}_{e,\mathbf{s}}, \mathrm{M}_{e,\mathbf{t}})$, one has

$$\mathrm{A}^{-1}\Lambda_0 \cap \mathbb{Z}^d = \mathrm{B}^{-1}\Lambda_1 \cap \mathbb{Z}^d.$$

Let $p \in \mathcal{P} \setminus \mathcal{P}_{\mathscr{H}}$. We have

$$\frac{\det(\Lambda_0 \cap \mathrm{A}\mathbb{Z}^d)}{\det(A)} = \frac{\det(\Lambda_1 \cap \mathrm{B}\mathbb{Z}^d)}{\det(B)},$$

hence, as $p \nmid \det(\mathrm{A})$ and $p \nmid \det(\mathrm{B})$:

$$|\det(\Lambda_0 \cap \mathrm{A}\mathbb{Z}^d)|_p = |\det(\Lambda_1 \cap \mathrm{B}\mathbb{Z}^d)|_p. \tag{6.2}$$

From $\det(\mathrm{A})\Lambda_0 \leq \Lambda_0 \cap \mathrm{A}\mathbb{Z}^d \leq \Lambda_0$, we get

$$\det(\Lambda_0) \mid \det(\Lambda_0 \cap \mathrm{A}\mathbb{Z}^d) \mid \det(\Lambda_0)(\det(\mathrm{A}))^d.$$

Thus, as $p \nmid \det(\mathrm{A})$:

$$|\det(\Lambda_0)|_p = |\det(\Lambda_0 \cap \mathrm{A}\mathbb{Z}^d)|_p.$$

Likewise

$$|\det(\Lambda_1)|_p = |\det(\Lambda_1 \cap \mathrm{B}\mathbb{Z}^d)|_p,$$

so by Equation 6.2 we finally get

$$|\det(\Lambda_0)|_p = |\det(\Lambda_1)|_p.$$

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As this is true for any $p \notin \mathcal{P}_{\mathscr{H}}$, this implies that

$$\delta(\Lambda_0) = \delta(\Lambda_1)$$

as required. □

Let us define the **$\mathscr{H}$-phenotype with respect to** v as the following function:

$$\mathrm{Ph}_{\mathscr{H},v} : \begin{array}{ccc} \mathrm{Sub}(G) & \to & \mathrm{Sub}(\mathbb{Z}^d)/\simeq \\ H & \mapsto & \pi_v(H \cap G_v) \end{array}.$$

Proposition 6.3. *The $\mathscr{H}$-phenotype $\mathrm{Ph}_{\mathscr{H},v}$ is surjective and invariant under conjugation by any element of G.*

Proof. The surjectivity of $\mathrm{Ph}_{\mathscr{H},v}$ results from the surjectivity of π_v and of the function $\begin{array}{ccc} \mathrm{Sub}(G) & \to & \mathrm{Sub}(G_v) \\ H & \mapsto & H \cap G_v \end{array}$.

Let H be a subgroup of G and $g \in G$. Let $\Lambda_0 := H \cap G_v \leq \mathbb{Z}^d$ and $\Lambda_1 := gHg^{-1} \cap G_v \leq \mathbb{Z}^d$. Let $\mathcal{G}$ be the $\mathscr{H}$-graph of H. By definition, there exist two vertices labelled (Λ_0, v) and (Λ_1, v) in $\mathcal{G}$. Hence, by connectedness of $\mathcal{G}$, one has $\pi_v(\Lambda_0) = \pi_v(\Lambda_1)$. □

As the group G_v acts non-cocompactly on the Bass-Serre tree $\mathcal{T}$, any subgroup of G_v lies in the perfect kernel $\mathcal{K}(G)$ by Theorem 5.2. In particular, the restriction $\begin{array}{ccc} \mathcal{K}(G) & \to & \mathrm{Sub}(G_v)/\simeq \\ H & \mapsto & \pi_v(H \cap G_v) \end{array}$ remains surjective and invariant under conjugation.

This function leads to a dynamical partition of the perfect kernel

$$\mathcal{K}(G) = \bigsqcup_{\Lambda \leq \mathbb{Z}^d} \mathcal{K}(G) \cap \mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda)). \tag{6.4}$$

By the previous remark and Proposition 6.1, this partition is infinite countable. Now we are able to prove Theorem 3 which gives a description of the dynamics induced on each piece of the aforementioned decomposition:

Theorem 6.5. *For any $\Lambda_0 \leq \mathbb{Z}^d$, there exists a dense orbit in $\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda_0)) \cap \mathcal{K}(G)$.*

Proof. Let $\Lambda \leq \mathbb{Z}^d$ and let $(H_i)_{i\in\mathbb{N}} \in \left(\mathcal{K}(G) \cap \mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda))\right)^{\mathbb{N}}$ be the sequence of finitely generated subgroups lying in $\mathcal{K}(G) \cap \mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda))$. For $i \in \mathbb{N}$, let α_i be the saturated $\mathscr{H}$-preaction associated to H_i and let $\mathcal{G}_i$ be the $\mathscr{H}$-graph of α_i. Let β_i be a sub-$\mathscr{H}$-preaction of α_i whose stabilizer is H_i and whose $\mathscr{H}$-graph K_i is finite (legit, because H_i is finitely generated). As $H_i \in \mathcal{K}(G)$, Theorem 5.2 implies that there exists a vertex $V_i \in \mathcal{V}(K_i)$ labelled (Λ_i, v_i) which is non-saturated relative to some edge $e_i \in \mathcal{E}(\mathscr{H})$. Up to extending β_i, one can assume that $v_i = v$ and that there exists another vertex W_i labelled (Γ_i, v) which is non-saturated relatively to some edge f_i. In particular, the subgroups $(\Lambda_i)_{i\in\mathbb{N}}, (\Gamma_i)_{i\in\mathbb{N}}$ are pairwise $\mathscr{H}$-equivalent w.r.t. v, so by Lemma 4.2, there exists an $\mathscr{H}$-path $E_{1,i}, \dots, E_{r_i,i}$ labelled $f_i, \dots, \overline{e_{i+1}}$ that connects W_i to V_{i+1} for every $i \in \mathbb{N}$. Denoting by $\mathcal{G}$ the resulting $\mathscr{H}$-graph, Lemma 3.14 delivers an $\mathscr{H}$-graph $\mathcal{F}$

- that contains $(K_i)_{i\in\mathbb{N}}$ as disjoint subgraphs;
- such that the quotient $\mathcal{F}/\bigsqcup_{i\in\mathbb{N}} K_i$ is a forest.

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Hence, by Lemma 3.10, there exists a saturated $\mathscr{H}$-preaction β that extends β_i for every $i \in \mathbb{N}$ and whose $\mathscr{H}$-graph is $\mathcal{F}$. This proves that the conjugacy class of the subgroup of G that is associated with β is dense in $\mathcal{K}(G) \cap \mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda))$. □

Now we study the topology of the pieces of the partition

$$\mathrm{Sub}(G) = \bigsqcup_{\Lambda \leq \mathbb{Z}^d} \mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda)).$$

Proposition 6.6. *For any $\Lambda_0 \leq \mathbb{Z}^d$:*

1. *if Λ_0 has finite index in $\mathbb{Z}^d$, then the fibre $\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda_0))$ is open;*
2. *$\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda_0))$ is an F_σ;*
3. *$\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda_0))$ is closed if and only if $\{\Lambda_1 \leq \mathbb{Z}^d \mid \Lambda_1 \simeq \Lambda_0\}$ is finite.*

Remark 6.7. In particular, $\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\{0\}))$ is closed.

Remark 6.8. If the image of the modular homomorphism is trivial, then the third item of Proposition 6.6 together with Lemma 5.6 implies that all pieces are closed.

Proof of Proposition 6.6. Notice that we have

$$\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda_0)) = \bigcup_{\Lambda_1 \simeq \Lambda_0} \{H \leq G \mid H \cap G_v = \Lambda_1\}.$$

By Lemma 2.1, this is an F_σ as a countable union of closed subsets of $\mathrm{Sub}(G)$. Hence, we get the second point.

If Λ_0 has finite index in $\mathbb{Z}^d$, this is an open subset of $\mathrm{Sub}(G)$ as a union of open sets by Lemma 2.1. This proves the first point.

If $\{\Lambda_1 \leq \mathbb{Z}^d \mid \Lambda_1 \simeq \Lambda_0\}$ is finite, then $\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda_0))$ is closed as a finite union of closed subsets by Lemma 2.1. Otherwise, there exists a sequence of subgroups $(\Lambda_n)_{n\in\mathbb{N}} \in (\pi_v^{-1}(\pi_v(\Lambda_0)))^{\mathbb{N}}$ that converges to a subgroup $\Lambda \leq \mathbb{Z}^d$ whose rank is strictly less than $\mathrm{rk}(\Lambda_0)$. By a straightforward induction using Corollary 4.3, there exists an $\mathscr{H}$-path $E_1, \dots, E_n, \dots$ such that: for every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $\mathbf{s}(E_{k_n})$ is labelled (Λ_n, v). Thus, by Lemma 3.14, there exists a saturated $\mathscr{H}$-graph that is a forest that contains $E_1, \dots, E_n, \dots$ as a sub-$\mathscr{H}$-graph. Hence by Lemma 3.10, there exist a subgroup $H \leq G$ and elements $g_n \in G$ such that $g_n H g_n^{-1} \cap G_v = \Lambda_n$ for every $n \in \mathbb{N}$. Up to extracting, the sequence $g_n H g_n^{-1}$ converges to a subgroup $K \leq G$ that satisfies $K \cap G_v = \Lambda$. In particular, as the rank is constant on the fibres of π_v, one has $K \notin \mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda))$, which proves that $\mathrm{Ph}^{-1}_{\mathscr{H},v}(\pi_v(\Lambda_0))$ is not closed. □

Remark 6.9. The definition of the equivalence relation $\simeq$ still makes sense if the graph $\mathscr{H}$ is infinite and the same proof extends to the class of groups that act (not necessarily cocompactly) on an oriented tree with vertex and edge stabilizers isomorphic to $\mathbb{Z}^d$. In the case of a non-cocompact action on the Bass-Serre tree, we thus obtain a dynamical partition of the whole set of subgroups by Remark 5.4.

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6.2. Case where $G = \mathbb{Z}^d \rtimes \mathbb{F}_r$

Now we assume that $\mathscr{H}$ consists of a collection of r loops $e_1,\dots,e_r$ based at a single vertex v such that, for every $i \in [\![1,r]\!]$, the label $(\mathrm{A}_i,\mathrm{B}_i)$ of e_i satisfies $\mathrm{A}_i \in \mathrm{GL}_d(\mathbb{Z})$ and $\mathrm{B}_i \in \mathrm{GL}_d(\mathbb{Z})$. Denoting by

$$\mathrm{P}_i := \mathrm{A}_i\mathrm{B}_i^{-1},$$

the group G is isomorphic to the semidirect product of $G_v = \mathbb{Z}^d$ with the free group $\mathbb{F}_r = \langle a_1,\dots,a_r\rangle$ of rank r, where the generator a_i acts on $\mathbb{Z}^d$ by multiplication by P_i. Let us denote by $\rho : \mathbb{F}_r \to \mathrm{GL}_d(\mathbb{Z})$ the homomorphism that satisfies $\rho(a_i) = \mathrm{P}_i$ for every $i \in [\![1,r]\!]$ and by $\Gamma = \rho(\mathbb{F}_r)$. Observe that this $\mathbb{F}_r$-action induces an $\mathbb{F}_r$-action on $\mathrm{Sub}(G_v)$ defined as follows: for any subgroup $\Lambda \leq G_v$ and any $\gamma \in \mathbb{F}_r$:

$$\begin{aligned}\gamma \cdot \Lambda &= \rho(\gamma)\Lambda \\ &= (u,\gamma)\Lambda(u,\gamma)^{-1} \ \forall u \in \mathbb{Z}^d\end{aligned}$$

(when identifying G_v with $G_v \times \{1\}$ in $G_v \rtimes \mathbb{F}_r$).

Remark 6.10. If Λ is a finite-index subgroup of G_v, then for every $\gamma \in \mathbb{F}_r$, one has $|\det(\rho(\gamma)\Lambda)| = |\det(\Lambda)|$. As there exist only finitely many lattices of $\mathbb{Z}^d$ of a given determinant, the orbit $\mathbb{F}_r \cdot \Lambda = \{\rho(\gamma)\Lambda \mid \gamma \in \mathbb{F}_r\}$ is finite.

Let us denote by $\pi : \mathbb{Z}^d \rtimes \mathbb{F}_r \to \mathbb{F}_r$ the projection.

Our goal is to decompose the perfect kernel of G into countably many pieces on which the action by conjugation contains a dense orbit. Let us recall that (by Theorem 5.2), one has

$$\mathcal{K}(G) = \{H \leq G \mid \pi(H) \in \mathrm{Sub}_{[\infty]}(\mathbb{F}_r)\}.$$

Notice that, as the subgroup G_v is normal, denoting by $Conj(G_v)$ the set of classes for the action of G on $\mathrm{Sub}(G_v)$ by conjugation, the following partition

$$\mathrm{Sub}(G) = \bigsqcup_{\mathscr{C} \in Conj(G_v)} \{H \leq G \mid H \cap G_v \in \mathscr{C}\} \tag{6.11}$$

is G-invariant. Denoting by $\mathscr{C} = \{\rho(\gamma)\Lambda_0, \gamma \in \mathbb{F}_r\}$ for some $\Lambda_0 \leq G_v$, we get that

$$\{H \leq G \mid H \cap G_v \in \mathscr{C}\} = \bigcup_{\gamma \in \mathbb{F}_r} \{H \leq G \mid H \cap G_v = \rho(\gamma)\Lambda_0\}.$$

Notice that $|\det|$ is constant on $\mathscr{C}$ for every $\mathscr{C} \in Conj(G_v)$. In particular, there are infinitely many pieces in the decomposition (6.11).

This partition leads to a G-invariant partition of the perfect kernel into countably many pieces

$$\mathcal{K}(G) = \bigsqcup_{\mathscr{C} \in Conj(G_v)} \{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C}\}.$$

Notice that this is exactly the decomposition (6.4) we obtained in the previous case of a GBS group G which is not $\mathbb{Z}^d$-by-free: two subgroups Λ_0, Λ_1 of G_v can arise as the

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labels of two vertices of some connected $\mathscr{H}$-graph if and only if there exists some $\gamma \in \mathbb{F}_r$ such that $\rho(\gamma)\Lambda_0 = \Lambda_1$, or equivalently, if and only if Λ_0 and Λ_1 belong to the same orbit under the G-conjugation. However, the proof of Theorem 6.5 does not extend to our new setting, because the relation $\simeq$ need not be transitive any more. This difficulty turns out to be a real obstruction: the conjugation action need not be topologically transitive on each piece in our new setting. This comes from the fact that the skeleton of the $\mathscr{H}$-graph of a subgroup of G is related to its intersection with the vertex group G_v as follows:

Lemma 6.12. *Let H be a subgroup of G. Then $\pi(H) \leq \mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v)$.*

Proof. Let $\gamma \in \pi(H)$. There exists $u \in \mathbb{Z}^d$ such that $(u,\gamma) \in H$. For any $(w,1) \in H \cap G_v$, one has

$$\begin{aligned}(u,\gamma)(w,1)(u,\gamma)^{-1} &= (\rho(\gamma)(w),1)\\ &\in H\end{aligned}$$

which implies that $\rho(\gamma)(w) \in H \cap G_v$. Consequently, $\rho(\gamma)(H \cap G_v) = H \cap G_v$; thus, $\gamma \in \mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v)$. Hence, $\pi(H) \leq \mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v)$. □

Let us fix $\mathscr{C} \in Conj(G_v)$. Lemma 6.12 allows us to decompose

$$\begin{aligned}\mathcal{P}_{\mathscr{C}} :&= \{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C}\}\\ &= \{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C} \ \text{ and } \ \pi(H) \in \mathcal{K}(\mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v))\}\\ &\bigsqcup \{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C} \ \text{ and } \ \pi(H) \notin \mathcal{K}(\mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v))\},\end{aligned}$$

each of these two pieces being invariant under G-conjugation. Notice that the second piece

$$\mathcal{D}_{\mathscr{C}} = \{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C} \ \text{ and } \ \pi(H) \notin \mathcal{K}(\mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v))\}$$

of this last decomposition is always countable and open for the induced topology on $\mathcal{P}_{\mathscr{C}}$.

Remark 6.13. More precisely, two cases can occur:

- If $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ is infinitely generated or $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ has finite index in $\mathbb{F}_r$ (for some, equivalently for all $\Lambda_0 \in \mathscr{C}$), then $\mathcal{D}_{\mathscr{C}}$ is empty;
- Otherwise, it consists of

$$\{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C} \ \text{ and } \ \pi(H) \notin \mathrm{Sub}_{[\infty]}(\mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v))\}$$

if $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ is not infinite cyclic, and of $\mathcal{P}_{\mathscr{C}}$ otherwise.

Lemma 6.14. *For every $\mathscr{C} \in Conj(G_v)$, there exists a dense orbit in*

$$\mathcal{P}_{\mathscr{C}} \setminus \mathcal{D}_{\mathscr{C}} = \{H \in \mathcal{K}(G) \mid H \cap G_v \in \mathscr{C} \ \text{ and } \ \pi(H) \in \mathcal{K}(\mathrm{Stab}_{\mathbb{F}_r}(H \cap G_v))\}.$$

To prove this lemma, we will use the formalism of $\mathscr{H}$-graphs. Notice that in this context, the $\mathscr{H}$-graph $\mathcal{G}$ of a subgroup $H \leq G$ will be uniquely determined by $\pi(H)$ and $H \cap G_v$.

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It is the Schreier graph of the subgroup $\pi(H) \leq \mathbb{F}_r$ (with respect to the generating set $\{a_1,\dots,a_r\}$) whose labels are defined as follows: if V_0 is a vertex labelled $\Lambda_0 := H \cap G_v$, and V is any other vertex of $\mathcal{G}$, then, denoting by $E_1,\dots,E_s$ an edge path labelled $f_1,\dots,f_s$ that connects V_0 to V, the Transfer Equation 3 tells us that the label of V is the subgroup $\left(\mathrm{M}_{f_s,\mathbf{t}}\mathrm{M}_{f_s,\mathbf{s}}^{-1}\dots\mathrm{M}_{f_1,\mathbf{t}}\mathrm{M}_{f_1,\mathbf{s}}^{-1}\right)\Lambda_0$ of $\mathbb{Z}^d$. In particular, the set of labels of the vertices of $\mathcal{G}$ is exactly $\{\rho(\gamma)\Lambda_0, \gamma \in \mathbb{F}_r\}$.

Proof. Let $\mathscr{C} \in Conj(G_v)$ and let $\Lambda_0 \in \mathscr{C}$. Let us denote by $\Gamma_0 := \mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$. Let $(H_i)_{i\in\mathbb{N}^*} \in (\mathcal{P}_{\mathscr{C}} \setminus \mathcal{D}_{\mathscr{C}})^{\mathbb{N}}$ be the sequence of finitely generated subgroups such that $H_i \cap G_v = \Lambda_0$ for every $i \in \mathbb{N}^*$. For every $i \in \mathbb{N}^*$, let α_i be the saturated $\mathscr{H}$-preaction associated to H_i, and let $\mathcal{G}_i$ be its $\mathscr{H}$-graph (based at a vertex V_i). Let β_i be a sub-$\mathscr{H}$-preaction of α_i whose stabilizer is H_i and whose $\mathscr{H}$-graph K_i is a finite subgraph of $\mathcal{G}_i$ that contains V_i (legit, because H_i is finitely generated). Let S_i be the graph obtained by forgetting the labels of the vertices of $\mathcal{G}_i$, pointed at the vertex V_i. The graph S_i is the Schreier graph of $\pi(H_i)$ with respect to the generating set $\{a_1,\dots,a_r\}$ of $\mathbb{F}_r$. As $\pi(H_i)$ is a subgroup of Γ_0, denoting by S_0 the Schreier graph of Γ_0, the graph S_i is in fact a covering of the graph S_0 (for every $i \in \mathbb{N}^*$). As $\pi(H_i)$ belongs to $\mathcal{K}(\Gamma_0)$, one has the following dichotomy:

- either Γ_0 is not finitely generated;
- or Γ_0 is finitely generated and the covering map $S_i \to S_0$ has infinite degree for every $i \in \mathbb{N}^*$.

Let us define F_i as the subgraph of S_i obtained by forgetting the labels of the subgraph K_i of $\mathcal{G}_i$. In both cases, Remark 2.3 provides a covering map $S \to S_0$ such that S contains $(F_i)_{i\in\mathbb{N}^*}$ as disjoint subgraphs and such that the quotient $S/\bigsqcup_{i\in\mathbb{N}^*} F_i$ is a tree. This covering corresponds to an infinite-index subgroup $\Gamma \in \mathrm{Sub}_{[\infty]}(\Gamma_0)$. After labelling the vertex V_1 by the subgroup Λ_0 (and all the other vertices of S so that the Transfer Equation 3 is satisfied), we obtain an infinite $\mathscr{H}$-graph $\mathcal{F}$:

- that contains $(K_i)_{i\in\mathbb{N}^*}$ as disjoint subgraphs;
- such that the quotient $\mathcal{F}/\bigsqcup_{i\in\mathbb{N}^*} K_i$ is a tree.

Thus, by Lemma 3.10, there exists a saturated $\mathscr{H}$-preaction γ on a pointed countable set (X,x) that extends β_i for every $i \in \mathbb{N}^*$ and whose $\mathscr{H}$-graph $\mathcal{F}$ is infinite. Denoting by L the stabilizer of γ, as the $\mathscr{H}$-graph of L contains a vertex labelled Λ_0, one moreover has $L \cap G_v \in \mathscr{C}$. Thus, $L \in \mathcal{P}_{\mathscr{C}}$. As the covering map $S \to S_0$ has infinite degree, one also has $\pi(L) \in \mathrm{Sub}_{[\infty]}(\mathbb{F}_r) \cap \mathcal{K}(\mathrm{Stab}_{\mathbb{F}_r}(L \cap G_v))$, thus $L \in \mathcal{P}_{\mathscr{C}} \setminus \mathcal{D}_{\mathscr{C}}$. The fact that γ extends all the $\mathscr{H}$-preactions β_i implies that the conjugacy class of L accumulates on all the finitely generated subgroups of $\mathcal{P}_{\mathscr{C}} \setminus \mathcal{D}_{\mathscr{C}}$ whose intersection with G_v is Λ_0. For any other element K of $\mathcal{P}_{\mathscr{C}} \setminus \mathcal{D}_{\mathscr{C}}$, there exists a conjugate gKg^{-1} whose intersection with G_v is Λ_0. Writing gKg^{-1} as the limit of an increasing sequence of finitely generated subgroups whose intersection with G_v is Λ_0, we finally get that gKg^{-1} is the limit of a sequence of conjugates of L. Thus, K is itself the limit of a sequence of conjugates of L, so the orbit of L under the conjugation action is dense in $\mathcal{P}_{\mathscr{C}} \setminus \mathcal{D}_{\mathscr{C}}$. □

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Hence we obtain the following decomposition of $\mathcal{K}(G)$:

Theorem 6.15. *There exists a G-invariant countable partition*

$$\mathcal{K}(G) = \bigsqcup_{\mathscr{C} \in Conj(G_v)} \mathcal{P}_{\mathscr{C}}$$

into F_σ-subsets of $\mathcal{K}(G)$, and, for every $\mathscr{C} \in Conj(G_v)$, a countable G-invariant open subset $\mathcal{D}_{\mathscr{C}} \subseteq \mathcal{P}_{\mathscr{C}}$ (for the induced topology on $\mathcal{P}_{\mathscr{C}}$) such that there exists a dense orbit in $\mathcal{P}_{\mathscr{C}} \setminus \mathcal{D}_{\mathscr{C}}$. Moreover, denoting by $\mathscr{C} = \{\rho(\gamma)\Lambda_0, \gamma \in \mathbb{F}_r\}$ for some $\Lambda_0 \leq \mathbb{Z}^d$:

1. *if Λ_0 has finite index in $\mathbb{Z}^d$, then $\mathcal{P}_{\mathscr{C}}$ is a clopen set;*
2. *$\mathcal{P}_{\mathscr{C}}$ is closed if and only if $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ has finite index in $\mathbb{F}_r$;*
3. *if $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ either is infinitely generated or has finite index in $\mathbb{F}_r$, then $\mathcal{D}_{\mathscr{C}} = \emptyset$.*

Proof. The existence of a dense orbit in $\mathcal{P}_{\mathscr{C}} \setminus \mathcal{D}_{\mathscr{C}}$ is provided by Lemma 6.14. The fact that $\mathcal{P}_{\mathscr{C}}$ is an F_σ results from Lemma 2.1.

If Λ_0 has finite index in $\mathbb{Z}^d$, then $\mathscr{C}$ is finite (because it is a set of finite-index subgroups of prescribed finite index). Thus, $\mathcal{P}_{\mathscr{C}} = \bigsqcup_{C \in \mathscr{C}} \{H \in \mathcal{K}(G) \mid H \cap G_v = C\}$ is a finite union of clopen sets by Lemma 2.1, so is itself clopen. This proves the first item.

Now let us turn to the proof of the second item. Let $\Lambda_0 \leq \mathbb{Z}^d$ and let $\mathscr{C}$ be the G-conjugacy class of Λ_0. The fact that $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ has finite index in $\mathbb{F}_r$ is equivalent to the finiteness of the orbit $\{\rho(g)\Lambda_0, g \in G\}$. In particular, if $\mathrm{Stab}_{\mathbb{F}_r}(\Lambda_0)$ has finite index in $\mathbb{F}_r$, then

$$\mathcal{P}_{\mathscr{C}} = \bigcup_{g \in \mathbb{F}_r} \{H \in \mathcal{K}(G) \mid H \cap G_v = \rho(g)\Lambda_0\}$$

is closed as a finite union of closed sets by Lemma 2.1. Conversely, let us assume that the set $\{\rho(g)\Lambda_0, g \in G\}$ is infinite. In particular, there exists a sequence $(g_n)_{n \in \mathbb{N}} \in \mathbb{F}_r^{\mathbb{N}}$ such that $\rho(g_n)\Lambda_0$ converges to a subgroup $\Lambda \leq G_v$ of rank strictly less than the one of Λ_0. In particular, $\rho(g_n)\Lambda_0 \in \mathcal{P}_{\mathscr{C}}$ for every n, but $\Lambda \notin \mathcal{P}_{\mathscr{C}}$ which implies that $\mathcal{P}_{\mathscr{C}}$ is not closed.

The third item results from Remark 6.13. □

Remark 6.16. Again, we did not use the cocompactness of the action of G on its Bass-Serre tree. In other words, Theorem 6.15 extends to the case where $r = \infty$.

To conclude, we give an explicit example where the set $\mathcal{D}_{\mathscr{C}}$ is non-empty, and where the conjugation action has no dense orbit for the induced topology on $\mathcal{P}_{\mathscr{C}}$:

Example 6.17. Let P_1 and P_2 be two matrices that generate a free subgroup of $\mathrm{SL}_2(\mathbb{Z})$; for instance, $\mathrm{P}_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\mathrm{P}_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Let G be the semidirect product $\mathbb{Z}^2 \rtimes \mathbb{F}_2$, where, denoting by $\{a_1, a_2\}$ a basis of $\mathbb{F}_2$, the generator a_i acts by multiplication by P_i on $\mathbb{Z}^2$.

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Let $\Lambda_0 = \begin{pmatrix}1\\0\end{pmatrix}\mathbb{Z}$. Then, any matrix of $\mathrm{SL}_2(\mathbb{Z})$ stabilizing the subgroup Λ_0 of $\mathbb{Z}^2$ is a power of $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$. As the homomorphism $\rho : \begin{array}{ccc}\mathbb{F}_2 & \to & \mathrm{SL}_2(\mathbb{Z})\\ a_1 & \mapsto & \mathrm{P}_1\\ a_2 & \mapsto & \mathrm{P}_2\end{array}$ is injective, the stabilizer $\mathrm{Stab}_{\mathbb{F}_2}(\Lambda_0)$ is also infinite cyclic (in fact, it is the subgroup generated by a_1). In particular, its perfect kernel is empty, so, denoting by $\mathscr{C} := \mathbb{F}_2 \cdot \Lambda_0$ the G-conjugacy class of Λ_0, one has

$$\mathcal{D}_{\mathscr{C}} = \mathcal{P}_{\mathscr{C}}.$$

Let us show that there is no dense orbit for the conjugation action $G \curvearrowright \mathcal{P}_{\mathscr{C}}$. For any $v \in \mathbb{Z}^2$, let us consider the following invariant subset

$$U_v = \{H \in \mathcal{P}_{\mathscr{C}} \mid \exists g \in G, g(v,a_1)g^{-1} \in H\}.$$

The set U_v is open as a union of clopen sets:

$$U_v = \bigcup_{g\in G} \{H \in \mathcal{P}_{\mathscr{C}} \mid g(v,a_1)g^{-1} \in H\}.$$

Claim 6.18. *For every $v,w \in \mathbb{Z}^2$ such that $v-w \notin \Lambda_0$, one has $U_v \cap U_w = \emptyset$.*

As the quotient $\mathbb{Z}^2/\Lambda_0$ is infinite, this will imply the existence of an infinite countable set of pairwise disjoint G-invariant open subsets, making the existence of a dense orbit impossible.

Proof of the claim. By contraposition, let $v,w \in \mathbb{Z}^2$ be such that $U_v \cap U_w \neq \emptyset$. Let $H \in \mathcal{P}_{\mathscr{C}}$. Up to conjugating H, there exists $g \in G$ such that

- $(v,a_1) \in H$;
- $g(w,a_1)g^{-1} \in H$.

As $H \in \mathcal{P}_{\mathscr{C}}$, there exists $\gamma \in \mathbb{F}_2$ such that $H \cap \mathbb{Z}^2 = \rho(\gamma)\Lambda_0$. Thus, $\mathrm{Stab}_{\mathbb{F}_2}(H \cap \mathbb{Z}^2) = \gamma\langle a_1\rangle\gamma^{-1}$. Denoting by $g = (u_0,\gamma_0)$, one deduces that $a_1, \gamma_0 a_1 \gamma_0^{-1} \in \gamma\langle a_1\rangle\gamma^{-1}$. This forces γ and γ_0 to belong to $\langle a_1\rangle$. In particular, $H \cap \mathbb{Z}^2 = \Lambda_0$. Let $k \in \mathbb{Z}$ be such that $\gamma_0 = a_1^k$. One has

$$\begin{aligned}
g(w,a_1)g^{-1}(v,a_1)^{-1} &= \left(u_0,a_1^k\right)(w,a_1)\left(u_0,a_1^k\right)^{-1}(v,a_1)^{-1}\\
&= \left(u_0+\rho(a_1)^k w, a_1^{k+1}\right)\left(-\rho(a_1)^{-k}u_0, a_1^{-k}\right)(v,a_1)^{-1}\\
&= \left(u_0+\rho(a_1)^k w - \rho(a_1)u_0, a_1\right)\left(-\rho(a_1)^{-1}v, a_1^{-1}\right)\\
&= \left(u_0 - \mathrm{P}_1 u_0 + \mathrm{P}_1^k w - v, 1\right)\\
&\in \left(H \cap \mathbb{Z}^2\right) \times \{1\}\\
&= \Lambda_0 \times \{1\}.
\end{aligned}$$

Thus, $u_0 - \mathrm{P}_1 u_0 + \mathrm{P}_1^k w - v \in \Lambda_0$. Notice that $(I_2 - \mathrm{P}_1)\mathbb{Z}^2 \subseteq \Lambda_0$. Thus,

$$u_0 - \mathrm{P}_1 u_0 \in \Lambda_0,$$

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and

$$\begin{aligned} \mathrm{P}_1^k w &= w + \sum_{i=0}^{k-1} \mathrm{P}_1^i (\mathrm{P}_1 w - w) \\ &\in w + \Lambda_0, \end{aligned}$$

which implies that $w - v \in \Lambda_0$. □

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