Student Problem Corner

Students up to the age of 19 are invited to send solutions to either of the following problems to Stan Dolan, 126A Harpenden Road, St. Albans, Herts AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is, therefore, not necessary to submit solutions to both. Solutions should arrive by September 20th 2016. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award may be made. Please print your own name clearly! The names of all successful solvers will be published in the November 2016 edition.

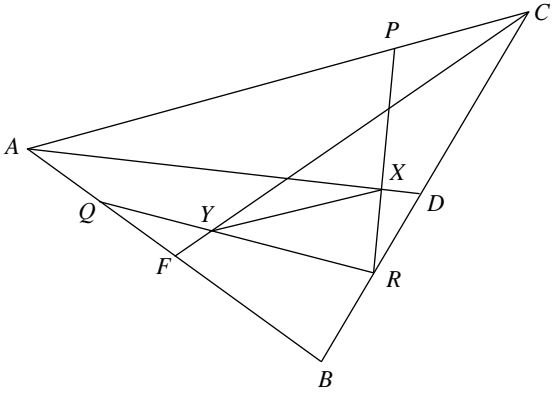
Problem 2016.3 (Nick Lord)

Find all whole numbers n such that

$$\frac{a_1^n + a_2^n + \ldots + a_n^n}{n} \geqslant a_1 a_2 \ldots a_n$$

for all real numbers $a_1, a_2, \ldots a_n$ with $a_1 + a_2 + \ldots + a_n \geqslant 0$.

Problem 2016.4 (Geoff Strickland)



ABC is a triangle. Points P, Q and R lie on CA, AB and BC, respectively, so that

$$CP : PA = AQ : QB = BR : RC.$$

Points F and D are the midpoints of AB and BC, respectively. AD meets PR at X and CF meets QR at Y.

Prove that XY is parallel to CA.

https://doi.org/10.1017/mag.2016.83 Published online by Cambridge University Press

Solutions to 2016.1 and 2016.2

Problem 2016.1 was solved by Ed Turnbull (Bishop Wordsworth's School). Problem 2016.2 was solved by James Cochrane (Bishop Wordsworth's School). Both problems were solved by Andrew Ng (Dulwich College) and Victor Wong (Charterhouse).

Problem 2016.1 (Chris Starr)

The four rods, AB, BC, CD and DE, are of equal length and are hinged at B, C and D. They lie on a flat surface in such a way that A touches CD and E touches AB.

Given that triangle ADE is isosceles, find $\angle ADE$.

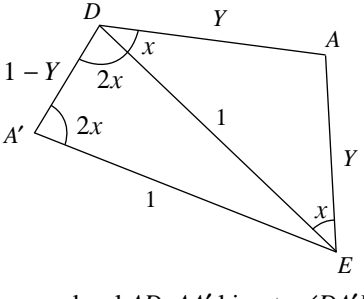
Solution

Andrew and Victor both attempted trigonometrical methods to find $\angle ADE = x$ but were forced to complete their solutions using numerical approximations. However, as pointed out by Geoff Strickland, one can obtain the equation

$$4\cos^3 x - 3\cos x = -\frac{1}{2}.$$

Then, recognising the formula for $\cos 3x$, $x = 40°$.

Ed let $AB = 1$ and $AE = Y$ and then obtained a very neat geometrical solution by forming a cyclic quadrilateral $ADA'E$, with EDA' congruent to BCA, as shown.



Considering angles on chord AD, AA' bisects $\angle DA'E$.

Let D' be the image of D under reflection in AA'. Then D' lies on $A'E$ and

$$AD' = D'E = EA = Y.$$

Therefore triangle $AD'E$ is equilateral. Then $\angle AED' = 60°$ and $\angle ADA' = 120°$ i.e. $x = 40°$.

https://doi.org/10.1017/mag.2016.83 Published online by Cambridge University Press

Problem 2016.2 (Paul Stephenson)

Consider this truncated pyramid of numbers. The k th row starts with k 1s. Each subsequent number is then the sum of the k previous numbers. The second row is therefore the first five Fibonacci numbers.

			1	1	1			
		1	1	2	3	5		
	1	1	1	3	5	9	17	
1	1	1	1	4	7	13	25	49
.	.	.	.	.	.	.	.	.

(a) What do you notice about the differences between vertical entries? Explain your conclusion.

(b) Find a formula for the i th number in the j th row.

Solution

(a) All respondents noted that the differences are zero in the left hand columns and are successive powers of 2 in the right hand columns. As Victor pointed out, a good way of proving this is to simply prove the general formula in part (b) first!

(b) The formula is

$$\begin{array}{ll} 1 & \text{for } 1 \leqslant j \\ 1 + (j-1)2^{i-j-1} & \text{for } j < i \leqslant 2j+1. \end{array}$$

The responses to this part from James, Andrew and Victor differed in terms of the degree of rigour shown in establishing this result. Victor gave a full proof, using mathematical induction. Combining this with an idea from James's solution, the crucial part of this proof can be written out very simply, as follows.

Let $F(i)$ be the i th number in the j th row. Then $F(i+j)$ is the sum of j 1s and so is equal to $1 + (j-1)2^0 = j$, as given by the formula. Assume the formula holds for $F(n+j)$ for some integer n, where $1 \leqslant n \leqslant j$. Then, we can combine the two formulae

$$\begin{aligned} F(n+j) &= F(n) + \ldots + F(n-1+j) \\ F(n+1+j) &= F(n+1) + \ldots + F(n+j) \end{aligned}$$

to obtain

$$\begin{aligned} F(n+1+j) &= 2F(n+j) - F(n) \\ &= 2\left(1 + (j-1)2^{n-1}\right) - 1 \\ &= 1 + (j-1)2^n. \end{aligned}$$

As required, the formula therefore holds for $F(n+1+j)$.

The first prize of £25 is awarded to Ed Turnbull. The second prize of £20 is awarded to Victor Wong.

10.1017/mag.2016.83 STAN DOLAN

https://doi.org/10.1017/mag.2016.83 Published online by Cambridge University Press