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Recurrences in multidimensional arithmetic sequences

BY S. HEMMER AND P. C. HEMMER
Blussuvoll skole and University of Trondheim

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1. *Introduction*

Let unity and the irrational components r_i of a d-dimensional vector $\mathbf{r}$ be linearly independent, and consider the integers n for which

$$\{nr_i\} \leqslant \phi_i, \quad 1 \leqslant i \leqslant d, \tag{1}$$

where $\{x\}$ denotes the fractional part of x. It is a trivial restriction to assume that r_i and ϕ_i lie between 0 and 1. This note is concerned with the question: what are the gaps between the successive n?

We write $\{n\mathbf{r}\}$ for the vector whose components are $\{nr_i\}$. We also let Φ be the interval defined by $0 \leqslant x_i \leqslant \phi_i$ $(i = 1, 2, \ldots, d)$, in d-dimensional Euclidean space. Letting the natural numbers represent a time development, the differences $n_{k+1} - n_k$ between successive integers fulfilling (1) are *recurrence times* for the event that $\{n\mathbf{r}\}$ returns to Φ.

The set of all $\{n\mathbf{r}\}$ is dense in Φ and by Weyl's theorem (4) uniformly distributed so that

$$\lim_{k\to\infty} n_k/k = \prod_{i=1}^{d} \phi_i^{-1}, \tag{2}$$

the natural definition of the *average* recurrence time.

The $d = 1$ case is well known ((2) and (3)). The result is surprisingly simple: At most *three* different recurrence times occur, for any r_1 and ϕ_1, and for $\phi_1 < \frac{1}{2}$ precisely three, of which one is the sum of the two others.

For $d > 1$ apparently little is known. Davidon and Hemmer (1) recorded that for the two-dimensional example $\mathbf{r} = (\sqrt{2}, \sqrt{3})$ and $\phi_1 = \phi_2 = \frac{1}{10}$ there were ten different recurrence times, *viz.* 41, 75, 116, 157, 198, 239, 302, 343, 541 and 582, and noted that one does not even know if the number of distinct recurrence times has an upper bound which is independent of $\mathbf{r}$ and Φ.

In the present note we prove in § 2 that for $d > 1$ no absolute upper bound for the number of distinct recurrence times exists. There is, of course, always a *lower* bound $l(d)$ for this number. With the restriction $\phi_i < \frac{1}{2}$, we show in § 3 that

$$l(2) \geqslant 5; \quad l(d) \geqslant 2^d, \tag{3}$$

and conjecture that $l(2) = 7$.

2. *Non-boundedness for* $d > 1$

A recurrence time is the time interval for one of the iterated vectors $\{n\mathbf{r}\}$ to return to Φ. We may consequently associate a recurrence time $t(\mathbf{x})$ with any vector $\mathbf{x}$ of Φ as the least positive integer with $\{\mathbf{x} + t(\mathbf{x})\,\mathbf{r}\} \in \Phi$.

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THEOREM 1. *For a given interval Φ of dimensionality $d > 1$ one can always select a vector* $\mathbf{r}$ *so that the number of distinct recurrence times is arbitrarily large.*

Proof. With integer $N > \max(\phi_i^{-1})$, choose a vector $\mathbf{r}$ with components r_i satisfying

$$\frac{1-\phi_i}{N} < r_i < \frac{1-\phi_i}{N-1}.$$

There lies no restriction in assuming ϕ_1 the largest among the ϕ_i's, i.e. $r_1 \leqslant r_i$. It will suffice to consider recurrence times for $\mathbf{x} = (x_1, \phi_2, \phi_3, \ldots, \phi_d)$ varying along an edge of Φ. The recurrence times $t(\mathbf{x})$ in these cases are at least N, since

$$\phi_2 < \phi_2 + tr_2 < 1, \tag{4}$$

for all positive integers $t \leqslant N-1$.

The recurrence time is precisely N for

$$1 - Nr_1 \leqslant x_1 \leqslant \phi_1, \tag{5}$$

since

$$1 \leqslant x_1 + Nr_1 < 1 + (1-\phi_1)/(N-1) < 1 + \phi_1, \tag{6}$$

and for $i > 1$

$$1 \leqslant \phi_i + Nr_i < 1 + \phi_i. \tag{7}$$

For $k = N+1, N+2, \ldots, k_0$, with $k_0 = [r_1^{-1}]$, the integer part of r_i^{-1}, the intervals

$$1 - kr_1 < x_1 < 1 - (k-1)r_1 \tag{8}$$

are disjoint and belong to the edge of Φ. One verifies that for *any* of these k-values

$$1 \leqslant \phi_i + Nr_i < \phi_i + kr_i \leqslant \phi_i + r_i[r_1^{-1}] \leqslant 1 + \phi_i \quad (\forall i > 1). \tag{9}$$

For an x_1 satisfying (8) we have

$$1 < x_1 + kr_1 < 1 + r_1 < 1 + \phi_1. \tag{10}$$

From these remarks it follows that the recurrence time for the interval (8) is precisely k.

The number of distinct recurrence times for this case is thus bounded below by $1 + k_0 - N$, and

$$1 + k_0 - N = [r_1^{-1}] - N + 1 > \left[\frac{N-1}{1-\phi_1}\right] - N + 1 \geqslant \frac{N-1}{1-\phi_1}\phi_1 \tag{11}$$

can be made as large as desired since N is arbitrary.

3. *A lower bound*

THEOREM 2. *In d dimensions the number of distinct recurrence times has the lower bound $l(d)$, with*

$$l(2) \geqslant 5; \quad l(d) \geqslant 2^d$$

for any vector $\mathbf{r}$ *and any interval Φ with edges $\phi_i < \frac{1}{2}$ $(i = 1, 2, \ldots, d)$.*

Proof. First note that two corners of Φ cannot have the same recurrence time. For at least one direction i the coordinates of the two corners in question are different, equal to 0 (corner at $\boldsymbol{\alpha}$) and ϕ_i (corner at $\boldsymbol{\beta}$), respectively. Since

$$\phi_i < \phi_i + \{t(\boldsymbol{\alpha})r_i\} \leqslant 2\phi_i < 1, \tag{12}$$

$\boldsymbol{\beta}$ can *not* have the recurrence time $t(\boldsymbol{\alpha})$.

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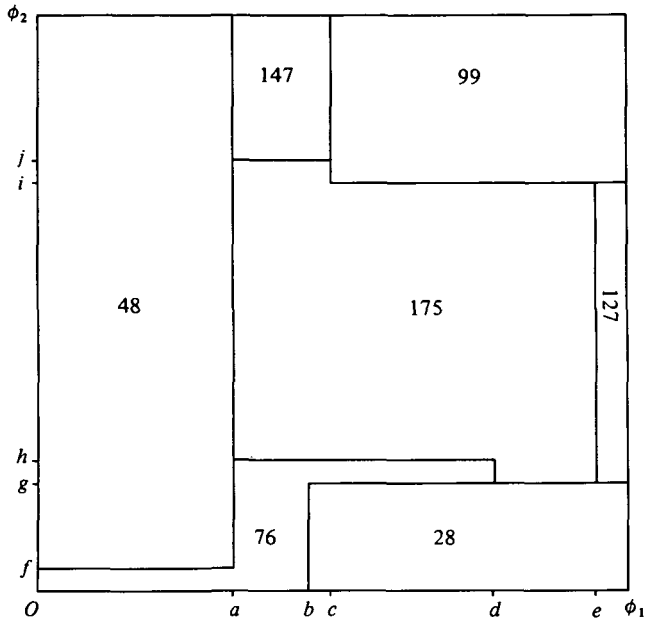


Fig. 1. The two-dimensional interval $\phi_1 = \phi_2 = \frac{1}{10}$ partitioned into domains of distinct recurrence times for the vector $\mathbf{r} = (\sqrt{13}, \sqrt{7})$. Here $a = 173{\cdot}1 - 48\sqrt{13}$, $b = 101 - 28\sqrt{13}$, $c = 357 - 99\sqrt{13}$, $d = 274{\cdot}1 - 76\sqrt{13}$, $e = 458 - 127\sqrt{13}$, $f = 127 - 48\sqrt{7}$, $g = 74{\cdot}1 - 28\sqrt{7}$, $h = 201{\cdot}1 - 76\sqrt{7}$, $i = 262 - 99\sqrt{7}$, and $j = 389 - 147\sqrt{7}$.

This implies that the 2^d recurrence times of the corners are all different.

Let

$$\boldsymbol{\delta}(\mathbf{x}) = \{\mathbf{x} + t(\mathbf{x})\,\mathbf{r}\} - \mathbf{x} \tag{13}$$

denote the displacement of $\mathbf{x}$ at the recurrence, and let the domain $\mathscr{D}(\mathbf{x}) \subset \Phi$ consist of all vectors $\mathbf{y} \in \Phi$ such that

$$\{\mathbf{y} + t(\mathbf{x})\,\mathbf{r}\} \in \Phi. \tag{14}$$

In other words

$$\mathscr{D}(\mathbf{x}) = \{\mathbf{y} \in \Phi\colon \mathbf{y} + \boldsymbol{\delta}(\mathbf{x}) \in \Phi\}. \tag{15}$$

A pair of diagonally opposite corners $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ of Φ can be characterized by the property that $\delta_i(\boldsymbol{\alpha}) \cdot \delta_i(\boldsymbol{\beta}) < 0$, $\forall i$. Consider such a pair and assume $t(\boldsymbol{\alpha}) > t(\boldsymbol{\beta})$, with no loss of generality. We will now show that $\mathscr{D}(\boldsymbol{\alpha})$ and $\mathscr{D}(\boldsymbol{\beta})$ do not overlap. Consider the vector $\mathbf{z} = \boldsymbol{\alpha} - \boldsymbol{\delta}(\boldsymbol{\beta})$. Clearly $\mathbf{z} \in \mathscr{D}(\boldsymbol{\beta})$ since

$$\mathbf{z} + \boldsymbol{\delta}(\boldsymbol{\beta}) = \boldsymbol{\alpha} \in \Phi, \tag{16}$$

and, since $\mathbf{z}$ is displaced during the time $t(\boldsymbol{\beta})$ to a corner of Φ, $\mathbf{z}$ itself must be located at a corner of $\mathscr{D}(\boldsymbol{\beta})$, the corner diagonally opposite to $\boldsymbol{\beta}$. Thus $\mathbf{z} \in \mathscr{D}(\boldsymbol{\alpha}) \cap \mathscr{D}(\boldsymbol{\beta})$ if this intersection is not empty. It would lead to a contradiction, however, if $\mathbf{z} \in \mathscr{D}(\boldsymbol{\alpha})$ since this implies

$$\boldsymbol{\alpha} + \boldsymbol{\delta}(\boldsymbol{\alpha}) - \boldsymbol{\delta}(\boldsymbol{\beta}) = \mathbf{z} + \boldsymbol{\delta}(\boldsymbol{\alpha}) \in \Phi. \tag{17}$$

$\boldsymbol{\alpha} + \boldsymbol{\delta}(\boldsymbol{\alpha}) - \boldsymbol{\delta}(\boldsymbol{\beta}) \in \Phi$ is another way of stating that $\boldsymbol{\alpha}$ is back in Φ after a time $t(\boldsymbol{\alpha}) - t(\boldsymbol{\beta})$ contrary to the original assumption that $t(\boldsymbol{\alpha})$ was the recurrence time of $\boldsymbol{\alpha}$.

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Hence the domains $\mathscr{D}(\alpha)$ and $\mathscr{D}(\beta)$ of every pair α, β of diagonally opposite corners of Φ do not overlap. In one and two dimensions this is only possible if at least one additional (central) domain $\mathscr{D}(\mu)$ exists, with a recurrence time $t(\mu)$ different from any of the recurrence times of the corners of Φ. Hence $l(2) \geqslant 5$.

That $l(2) \leqslant 7$ can be seen by a specific example. Take $\mathbf{r} = (\sqrt{13}, \sqrt{7})$ and $\phi_1 = \phi_2 = \frac{1}{10}$. The domains of the seven distinct recurrence times 28, 48, 76, 99, 127, 147 and 175 are shown in Fig. 1.

On the basis of empirical data we conjecture that

$$l(2) = 7. \tag{18}$$

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