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CATEGORICAL HARMONY AND PATH INDUCTION

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Abstract. This paper responds to recent work in the philosophy of Homotopy Type Theory by James Ladyman and Stuart Presnell. They consider one of the rules for identity, path induction, and justify it along 'pre-mathematical' lines. I give an alternate justification based on the philosophical framework of inferentialism. Accordingly, I construct a notion of harmony that allows the inferentialist to say when a connective or concept is meaning-bearing and this conception unifies most of the prominent conceptions of harmony through category theory. This categorical harmony is stated in terms of adjoints and says that any concept definable by iterated adjoints from general categorical operations is harmonious. Moreover, it has been shown that identity in a categorical setting is determined by an adjoint in the relevant way. Furthermore, path induction as a rule comes from this definition. Thus we arrive at an account of how path induction, as a rule of inference governing identity, can be justified on mathematically motivated grounds.

§1. The challenge to path induction. In a recent paper [13], the authors consider the question of justification for the identity rules in Homotopy Type Theory (HoTT). They conclude that the justification for the identity rule called *path induction* in HoTT is too mathematical to take part in a foundational justification. Whereas the justification in the HoTT Book [34] is explicitly homotopical, the authors propose a method of justification that is 'pre-mathematical'. Accordingly, they supply their own justification for the identity rules that is intended not to rest on higher mathematical notions like homotopy.

I think this challenge is appropriate. Path induction is surely a subtle and powerful rule in HoTT and its nature might need both explanation and justification. But I do not think that the justification given in [13] is sufficient. To meet the challenge in an alternate way, the paper has three major theses:

1. A mathematically motivated *inferentialism* should be preferred to the pre-mathematical requirement.
2. We can justify constants and connectives, like identity, by appeal to a notion of inferential *harmony* which can be specified in terms of adjoints from category theory.
3. Identity is harmonious in this sense and path induction, its elimination rule in HoTT, is likewise justified.

§2. Inferentialism. Inferentialism, in our context, is the thesis that the meaning of the logical concepts (e.g., conjunction, conditional, negation, identity) is determined by the rules governing them (cf. [21, 27, 32, 33], and Part II of [22]). Although I believe this holds

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for mathematical concepts more broadly—namely 'informal' mathematical concepts—considering only the logical case gives us very precise rules to consider. These rules are most commonly the rules from natural deduction in the style of Gentzen.[1] Using these rules, we can focus questions of meaning and, I argue, justification of rules of inference. More specifically, we ask the question whether the rules of inference are at least *meaning-bearing*. The way the inferentialist understands when rules are *meaning-bearing* relates to the information needed in order to fix the meaning of the connective.

We can take either the introduction rules or the elimination rules for a connective to be constitutive of its meaning [7, p. 396]. The former view is called *verificationism* and the latter *pragmatism*. These are not to be confused with the empirical verificationisms of the Vienna Circle (though see [16]) or the pragmatism of Peirce and James. Two prominent inferentialists, Dummett and Martin-Löf, chose verificationism for their inferentialism, but Dummett notes that the two conceptions are really not that different. This separation of inferentialisms isn't particularly helpful in my view since connectives like disjunction and the existential quantifier are best understood through their elimination rules and connectives like conjunction and the universal quantifier are best understood through their introduction rules. This will correspond to the fact that disjunction and existentials are left adjoints and conjunctions and universals are right adjoints (see §4). However, for simplicity, we will only consider the verificationist perspective when discussing the philosophical view of inferentialism.

If, for example, we take the meaning of conjunction to be completely determined by the introduction rule

$$\frac{A \qquad B}{A \wedge B}$$

then we can detach the justification for the rules from the pre-systematic notions of conjunction. That is, once we have the appropriate introduction rule, we need not concern ourselves with the thing they were trying to represent when considering the elimination rule; we have other criteria at that stage. We need to only find an appropriate elimination rule (or a set of elimination rules) to make sure the inferential role taken by a conjunction is *meaning-bearing*. The way to 'derive' the elimination rule can either be informally done [17] or formally so [26] but the intuition goes like this: to get a verification of a conjunction $A \wedge B$, one needs a verification of each conjunct A and B.[2] So the elimination rules must only allow one to derive at most each individual conjunct. Thus we give two rules that take the conjunction $A \wedge B$ and derive either A or B from it. That this is the correct way to derive the elimination rules is shown by *harmony* which is discussed in the next section.

The above is an example of Martin-Löf's *meaning explanations* (for the case of conjunction) and they are key to the intuitionistic nature of his type theory.[3] Similarly, Dummett used his verificationism to support a preference for intuitionistic logic, for instance in [8].

[1] Hilbert-style systems were also meant to capture the meaning of the concepts used in mathematical inference (e.g., disjunction, conditional) [4] since the development there is so historically tied to that of Gentzen's.

[2] The word "verification" is not synonymous with "proof" in either Dummett's or Martin-Löf's writing. Instead it may be understood as a certain kind of proof; a canonical choice that is compositional in the sense that it refers only to components of the complex formulae. So although we may prove conjunctions using different means, we take the verification to be a special form of proof using only the stipulated introduction rule.

[3] It should be noted here that the formal framework of HoTT is based heavily on Martin-Löf Type Theory.

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To coherently talk about this sort of logical inferentialism we need a unified account of what an inferential role is.[4] After all, if we would like to say the meaning of a connective is determined fully and uniquely by the inference rules governing that connective, we need a way to talk about these inference rules in a systematic way. One such way is meta-logic, a method used by many of the authors who endorse this inferentialism [24, 27, 32]. In this methodological framework, we consider natural deduction rules, possibly in sequent style, and consider their relations to each other to determine harmony or other inferential features (e.g., uniqueness of connectives, conservative extension). My account does not use these tools at its core; it instead uses category theory.

This shift to category theory is motivated by noting certain connections to the inferentialist philosophy, demonstrating how category theory fits in with the philosophical framework. First, in the practice of category theory, we use naturally inferentialist language to talk about the objects of a category. The universal characterization of limits and colimits, for instance, render these important constructions in terms of their role in the structure of arrows. If we read the arrows as deductions and the product as conjunction, we get a structural definition of conjunction by referring only to when there exist inferences with a conjunction as a result.[5] But this is precisely a verificationist reading of conjunction. Similar readings can be given to the other connectives.

The second feature of category theory that indicates a match with an inferentialist philosophy of logic is the Yoneda Lemma. The full build up of the Lemma would take us too far afield of this section, but I give here a way to interpret this central fact about categories. We can soundly interpret arrows into an object A as 'generalized elements' of A and considering all such generalized objects of A uniquely characterizes A. Perhaps the best philosophical gloss of the Yoneda Lemma is due to Urs Schrieber [9]:

> One way to look at it is this: for $\mathbb{C}$ a category, one wants to look at presheaves on $\mathbb{C}$ as being "generalized objects modeled on $\mathbb{C}$" in the sense that these are objects that have a sensible rule for how to map objects of $\mathbb{C}$ into them. You can "probe" them by test objects in $\mathbb{C}$. For that interpretation to be consistent, it must be true that some X in $\mathbb{C}$ regarded as just an object of $\mathbb{C}$ or regarded as a generalized object is the same thing. Otherwise it is inconsistent to say that presheaves on $\mathbb{C}$ are generalized objects on $\mathbb{C}$.
>
> The Yoneda lemma ensures precisely that this is the case.

This is all to say something, a bit subtly, about how objects are characterized by arrows. It is also worth noting that the Yoneda Lemma has two versions: contravariant and covariant. So we can either look at all the ways to probe the object by looking at arrows to it or, to stretch the metaphor, we can look at all the emissions of the object by looking at arrows from it. In either case, we get a full picture of the object in question.

This gives us a picture, though only a very brief sketch, of how category theory unifies conceptions of inferentialism, giving us a mathematical framework in which to systematically implement the philosophical. Moreover, the fact that inferentialism has such deep connections to category theory suggests that inferentialism has grounding (conceptual, not historical) in mathematics and can be considered a mathematically motivated choice of

[4] I take the phrase "inferential role" to indicate the place a complex formula has in the inferential (or deductive) structure.

[5] These definitions by universal property match almost exactly with the structural definitions of logical connectives by Koslow in [11].

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philosophy—though that is not to say that inferentialists will generally appeal to mathematical methods. However, the power of category theory to systematize notions of inferentialism will find its most compelling expression in terms of how it unifies the notions of harmony: notions to which we now turn.

§3. Harmony. To justify path induction, we will show that the identity rules in HoTT are meaning-bearing. This is established through a notion of *harmony* that ensures the rules are balanced and conservative with respect to provability. What this means is the task we take up now: characterizing harmony. In the end, we will settle on a notion of harmony based on the categorical notion of *adjoint*. But the idea of harmony has been around for a long time in various guises. The impetus comes from a note by Prior [25] in which a connective "tonk" is introduced as a sort of counterexample or warning to certain philosophical approaches to logic. With the following rules:

$$\frac{A}{A \; tonk \; B}\ \text{tonk-I} \quad \text{and} \quad \frac{A \; tonk \; B}{B}\ \text{tonk-E}$$

we can derive $A \vdash B$ for any A and B we like. The question being addressed in the ensuing discussion by Belnap [3], Stevenson [29], and others is whether we need to have a meaning in mind before we introduce a new connective by the rules governing it. If we are allowed to introduce whatever connective we like, then tonk is acceptable.

The view is that the problem with *tonk* is that although the introduction and elimination rules, considered separately, are fine, making them about the same connective is where we get into trouble; it is the interaction of rules that harmony considers. After all, tonk-I is the same as $\vee$-I and tonk-E is the same as $\wedge$-E, two seemingly fine rules, when about their respective distinct connectives. Though tonk may not incite the same worry it may have when it was first discussed, there are three basic conceptions of harmony that have persisted. We will outline these conceptions before turning to the categorical unification.

There are three major types of harmony:

1. Inversion Principle
2. Strength / Weakness
3. Conservative Extension

I will refer to these as *inversion harmony, inferential harmony,* and *conservative harmony*, respectively. Now, we describe and define each.

3.1. Inversion harmony. Inversion harmony is due to Prawitz [24] and is proof-theoretic. It is based on the *inversion principle* that Prawitz defines as follows:

> Let α be an application of an elimination rule that has B as a consequence. Then, deductions that satisfy the sufficient condition [...] for deriving the major premiss of α, when combined with deductions of the minor premisses of α (if any), already 'contain' a deduction of B; the deduction of B is thus obtainable directly from the given deductions without the addition of α. [24, p. 33]

We take the case of the conditional $\rightarrow$ to illustrate the inversion principle. Say we have a derivation of B that looks like this:

$$\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \rightarrow B}\ \rightarrow\text{I} \qquad \begin{array}{c}\mathcal{D}\\ A\end{array}}{B}\ \rightarrow\text{E}$$

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Since we have a proof $\mathcal{D}$ of A, we can replace the assumption $[A]$ by the proof of A. This allows us to transform the proof to

$$\begin{array}{c} \mathcal{D} \\ A \\ \vdots \\ B \end{array}$$

which doesn't use the $\to$E used in the first derivation (i.e., "without the addition of α"). This is what Prawitz means that there was already a proof of B 'contained' in our original one that didn't need (the last two applications of) the rules for implication. This process is one of *local reduction*, where we create a proof that does not utilize the steps α.

The inversion principle has a converse, or dual notion. As given above, the fact that it was $\to$I followed by $\to$E and not the reverse was important. If we call the above procedure "local reduction", the reverse would be a local expansion. Again taking as our example the conditional, we can expand proofs of the form

$$\begin{array}{c} \mathcal{D} \\ A \to B \end{array}$$

to proofs of the form

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \to B\end{array} \qquad [A]^u}{B}\,{\to}\mathrm{E}}{A \to B}\,{\to}\mathrm{I},\,\mathrm{u}\,.$$

This time we created a longer proof that introduced two unnecessary steps, $\to$E then $\to$I. This direction, instead of being called "harmony" is usually called "stability" [8, 28]. Stability is not as often required as harmony is, but the characterization in terms of adjoints later will give us a categorical analogue of stability as well.

The fact that *tonk* fails inversion harmony can easily be seen by attempting to locally reduce the detour

$$\dfrac{\dfrac{A}{A\ tonk\ B}}{B}\,.$$

There is no proof of B 'contained' in the proof without using *tonk*, unless of course $A = B$.

Local reduction and expansion of derivations contribute to Prawitz's normalization theorem of natural deduction. But we also need to deal with the nondeterministic decision of when to end a hypothetical derivation allowed by $\vee$-E, $\exists$-E, and $\bot$-E. For example, eliminating a disjunction $A \vee B$ relies on two sub-derivations of some conclusion C from A and B respectively. The same conclusion can be proved with the application of $\vee$E occurring 'later' or 'earlier' in the proof. Thus we need *permutation* conversions to ensure that no matter when you apply these elimination rules, we can arrive at a normal proof. The permutation conversion for disjunction is as follows:

$$\dfrac{\begin{array}{ccc} & [A] & [B] \\ & \mathcal{D}_1 & \mathcal{D}_2 \\ A \vee B & C & C \end{array}}{\begin{array}{c} C \\ \mathcal{E} \\ D \end{array}}\,{\vee}\mathrm{E} \quad \mapsto \quad \dfrac{\begin{array}{ccc} & [A] & [B] \\ & \mathcal{D}_1 & \mathcal{D}_2 \\ & C & C \\ & \mathcal{E} & \mathcal{E} \\ A \vee B & D & D \end{array}}{D}\,{\vee}\mathrm{E}$$

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And though normalization is not the focus of this paper, it is worth noting that permutations will turn out to be the naturalness aspect of adjoints.[6] With local reduction, expansion, and permutations, Prawitz can create an equivalence relation on natural deduction derivations with unique normal forms as representatives of their equivalence class. Though the equality of proofs is an interesting topic, we will not go further into it here.

An important take on inversion harmony is found in type theory. The propositions as types interpretation allows a quite direct way to think of harmony and stability in type-theoretic contexts. In their discussion of constructive modal logics, Pfenning and Davies in [23] use different terms, but the idea is similar, if a bit broader:

Local soundness: The elimination rules cannot be too strong. No matter how we apply elimination rules to the result of an introduction rule we cannot gain any new information. We demonstrate this by showing that we can find a more direct proof of the conclusion of the elimination which does not first introduce and then eliminate the connective in question. This is witnessed by a *local reduction* of the given introduction and the subsequent elimination.

Local completeness: The elimination rules are not too weak. There has to always be a way to apply elimination rules so that we can reconstitute a proof of the original proposition from the results by applying introduction rules. This is witnessed by a *local expansion* of an arbitrary given derivation into some eliminations followed by some introductions.

Notice that these descriptions even include an information-based analysis of harmony. Indeed, this language also preempts the inferential harmony which we now turn to.

3.2. Inferential harmony. Though the various conceptions of harmony use language that refers to logical strength, as in the description of local soundness and completeness, inferential harmony makes it the central feature. Through several revisions [30, 31] in response to tonk-like counterexamples, Tennant created a characterization of harmony that was split up into a strength condition and weakness condition [32]. For simplicity, Tennant frames inferential harmony in terms of a binary infix connective λ:

Strength: $A\lambda B$ is the strongest conclusion possible under conditions described by λ-I. Moreover, in order to show this

i. one needs to exploit all the conditions described by λ-I;
ii. one needs to make full use of λ-E; but
iii. one may not make any use of λ-I.

Weakness: $A\lambda B$ is the weakest major premise under the conditions described by λ-E. Moreover, in order to show this

i. one needs to exploit all the conditions described by λ-E;
ii. one needs to make full use of λ-I; but
iii. one may not make any use of λ-E.

One notable thing about this formulation is in its constraints on *how* to show the strength and weakness of $A\lambda B$. Accordingly, the proof of the two sides of inferential harmony, strength and weakness, must not fall afoul of the six given conditions.

The notions of strength and weakness, in turn, are analyzed in terms of inferential roles. The definitions are as follows:

[6] I owe this insight to Ed Morehouse in discussions and in [20, p. 16].

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> P is the *strongest proposition with property* F if P has property F and any proposition with property F is a logical consequence of P.
>
> P is the *weakest propositions with property* F if P has property F and is a logical consequence of any proposition with property F.

This too will be handled uniformly by the categorical interpretation of harmony, since adjoints have a universal property. However, the proof constraints are less part of the problem of harmony and more a part of the system in which Tennant works, and so we will not preserve that feature when we move to category theoretic discussions.

An example will help clarify inferential harmony. We will show that implication $\rightarrow$ is inferentially harmonious.

Strength: $A \rightarrow B$ is the strongest proposition implied by Γ whenever $\Gamma \cup \{A\}$ entails B.

Proof. Clearly $A \rightarrow B$ has the property in question. That is, by $\rightarrow$E,

$$A \rightarrow B, A \vdash B. \tag{1}$$

Now suppose there is a θ that is implied by Γ whenever $\Gamma \cup \{A\}$ implies B. Given 1, $A \rightarrow B$ fits the role of Γ, giving us the desired

$$A \rightarrow B \vdash \theta. \qquad \square$$

Note that we used $\rightarrow$E fully in 1 but nowhere appealed to $\rightarrow$I.

Weakness: $A \rightarrow B$ is the weakest proposition that with A implies B.

Proof. Suppose θ is a proposition that with A implies B. By $\rightarrow$I, we get

$$\dfrac{\dfrac{\theta \qquad [A]^u}{B}}{A \rightarrow B}\ {\rightarrow}\text{I,u},$$

which shows that θ implies $A \rightarrow B$ as desired. $\square$

In this proof, we never appealed to $\rightarrow$E nor did we use any less than the full $\rightarrow$I rule.

The requirement to *fully* use either λE or λI only arises when we have more than one I-principle or E-principle. For illustration, we show that $A \wedge B$ satisfies the Strength requirement in inferential harmony.

CLAIM 3.1. *$A \wedge B$ is the strongest proposition implied by the combination of A and B.*

Proof. Let the pair of propositions A and B imply θ. Then we derive θ from $A \wedge B$:

$$\dfrac{\dfrac{A \wedge B}{A}\ \wedge\text{E1} \qquad \dfrac{A \wedge B}{B}\ \wedge\text{E2}}{\theta}$$

Note that we use both elimination rules of conjunction; tonkish examples arise when we do not require using all the E-principles [32]. Note also that no appeal was made to the introduction rule. $\square$

3.3. Conservative harmony. If we say that the I-principles and E-principles of a connective λ are conservatively harmonious, we are ascribing a property to the pair

$$((\lambda\text{-I}, \lambda\text{-E}), S)$$

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where S is a logical system including (λ-I, λ-E) and possibly other connectives and their rules. So conservative harmony is only used with respect to a logical system S, where the two previous conceptions focused only on the I-principles and E-principles of one connective at a time. Conservative harmony is satisfied if S is a conservative extension of $S\backslash(\lambda\text{-I}, \lambda\text{-E})$, the system S without the rules of inference governing λ. This sort of conservative extension is not one of the theory, nor one of the language, per se. We may think of this as a *systematic* conservative extension, introducing new rules of inference. Though the change occurs in the deductive machinery, the definition of *systematic* conservativity is straightforward. Suppose we have languages $\mathcal{L}$ and $\mathcal{L}'$ with their respective deductive systems S and S', with $\mathcal{L}' = \mathcal{L} \cup \{\lambda\}$ for a new operator λ and $S' = S \cup \{(\lambda\text{-I}, \lambda\text{-E})\}$. Then for any formula A of $\mathcal{L}$, A is a theorem in S' only if A is a theorem in S. In this case, we'd say that (λ-I, λ-E) is conservatively harmonious with respect to the system S.

Conservative harmony is usually credited to Nuel Belnap's [3] response to Prior's Tonk. There is a further requirement that Belnap imposes for harmony: uniqueness. This is a crucial requirement for inferentialists as well. Although not necessarily tied to conservative harmony, uniqueness is usually included and so we include it here as well.

To say what uniqueness means, let σ be a connective with rules of inference $\mathcal{R}$. Then we say that $\mathcal{R}$ ensures the uniqueness of σ when for any new connective σ', we can show that when we replace σ with σ' in all rules of $\mathcal{R}$, the following is derivable

$$\sigma(x_1, \ldots, x_n) \leftrightarrow \sigma'(x_1, \ldots, x_n)\,. \tag{2}$$

In other words, if σ and σ' have the same rules, modulo symbol choice, then the connectives are logically equivalent. Of course, if this did not hold, then we have denied inferentialism in logic, for two concepts would have the same rules governing them and yet be somehow distinguishable in the system. Moreover, as illustrated in the example below, this uniqueness condition requires some sort of inverse behavior between σ-I and σ'-E which amounts to inverse behavior between σ-I and σ-E since they're structurally identical.

For an illustration, say we would like to show that the rules of inference governing conjunction ensure the uniqueness of $\wedge$. To show this, suppose that there is a connective $\wedge^*$ with the following rules of inference:

$$\frac{A \qquad B}{A \wedge^* B}\,\wedge^*\text{I}$$

$$\frac{A \wedge^* B}{A}\,\wedge^*\text{E} \qquad \frac{A \wedge^* B}{B}\,\wedge^*\text{E}$$

Given these rules, which are just the rules for conjunction with $\wedge^*$ replacing $\wedge$, we can show the necessary biconditional (2) via the following derivations:

$$\dfrac{\dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B}}{A \wedge^* B}\,\wedge^*\text{I} \qquad\qquad \dfrac{\dfrac{A \wedge^* B}{A} \qquad \dfrac{A \wedge^* B}{B}}{A \wedge B}\,\wedge\text{I}$$

In the end, I will not endorse conservative harmony, but I will describe how uniqueness is satisfied in the categorical setting.

§4. Categorical harmony. Several authors have noted that inversion harmony is easily represented for logical connectives by their adjoint definitions in category theory [10,18,20]. However, inversion harmony is not the only sort of harmony that we can capture

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with the notion of adjointness in categorical contexts. My definition of categorical harmony largely agrees with Maruyama's in [18] and so it is worth going over the presentation there.

First, it is worth mentioning that the goals of this paper are not the same as those in [18]. There, harmony is expressly used to give us a way to judge the degrees of paradoxicality for Russell's paradox and Prior's tonk connective. It is shown that both fail categorical harmony but in different ways, thus leading to a way to distinguish these pathologies. I want to use categorical harmony for two alternate goals: to unify the major notions of harmony as given in the above taxonomy and to address questions of inference rule legitimacy in recent foundational work, especially in HoTT.

In the following, we heavily rely on the notion of adjoint.

DEFINITION 4.1. *Let $\mathbb{C}$ and $\mathbb{D}$ be categories and $L : \mathbb{C} \to \mathbb{D}$ and $R : \mathbb{D} \to \mathbb{C}$ be functors. Then L is* left adjoint *to R, and R is* right adjoint *to L, denoted $L \dashv R$, if any of the following equivalent conditions hold*:[7]

Natural isomorphism: *There exists a natural isomorphism*

$$\theta : \mathrm{Hom}_{\mathbb{D}}(L(-), -) \simeq \mathrm{Hom}_{\mathbb{C}}(-, R(-)),$$

which is to say that for any objects $C \in \mathbb{C}$ and $D \in \mathbb{D}$, we have a bijection between the hom sets

$$\frac{\mathrm{Hom}_{\mathbb{D}}(L(C), D)}{\mathrm{Hom}_{\mathbb{C}}(C, R(D))}\,\theta(C, D)$$

such that for any arrows $f : C' \to C$ and $g : D \to D'$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{D}}(L(C), D) & \xrightarrow{\theta(C,D)} & \mathrm{Hom}_{\mathbb{C}}(C, R(D)) \\ \Big\downarrow{\scriptstyle \mathrm{Hom}(L(f),g)} & & \Big\downarrow{\scriptstyle \mathrm{Hom}(f,R(g))} \\ \mathrm{Hom}_{\mathbb{D}}(L(C'), D') & \xrightarrow[\theta(C',D')]{} & \mathrm{Hom}_{\mathbb{C}}(C', R(D')) \end{array}$$

Universal property of unit: *There is a natural transformation, called the* unit *of the adjunction $L \dashv R$,*

$$\eta : 1_{\mathbb{C}} \to R \circ L$$

that has the following universal property:

For any $C \in \mathbb{C}$, $D \in \mathbb{D}$, and $f : C \to R(D)$, there exists a unique $g : L(C) \to D$ such that

$$f = R(g) \circ \eta_C.$$

Universal property of counit: *There is a natural transformation, called the* counit *of the adjunction $L \dashv R$,*

$$\varepsilon : L \circ R \to 1_{\mathbb{D}}$$

that has the following universal property:

For any $C \in \mathbb{C}$, $D \in \mathbb{D}$, and $g : L(C) \to D$, there exists a unique $f : C \to R(D)$ such that

$$g = \varepsilon_D \circ L(f)\,.$$

[7] These definitions are taken from [2].

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To characterize harmony via adjoints, we first distinguish the *universally definable operations*, a type of functor, as those that are definable in the general language of category theory. Two examples would be the diagonal $\Delta : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ and the functor $\mathbf{1} : \mathbb{C} \to \mathbb{1}$ that takes all objects in $\mathbb{C}$ to the unique object $*$ in the category $\mathbb{1}$ and all arrows to the identity arrow on $*$. We then proceed in an iterated way to construct the harmonious operations by taking adjoints to the universally definable operations and adjoints to any of these newly acquired operations and so on. We can think of it inductively, namely, that operations defined by adjunctions to the universally definable operations are harmonious and if F is harmonious then operations defined by adjunction to F are also harmonious. This allows us to already declare that $\wedge$, $\vee$, $\top$, and $\bot$ are harmonious,[8] since, from [14],

$$\vee \dashv \Delta \dashv \wedge$$

and

$$\bot \dashv \mathbf{1} \dashv \top.$$

Now, since $\wedge$ is harmoniously produced, we can say that fixed-antecedent conditionals are also harmonious since we have the adjunction

$$A \wedge (-) \dashv A \implies (-).$$

And as a sharpening of the analogy between universal quantifiers and conjunctions on the one hand and existential quantifiers and disjunctions on the other, we point out that they have the same 'parity': both universal quantifiers and conjunctions are right adjoints and both existential and disjunctions are left adjoints. The functor with respect to which the quantifiers are adjoint is analogous to the diagonal functor. Instead of doubling up objects $X \to \langle X, X \rangle$, we have a weakening functor π that add hypotheses to contexts (like in a sequent calculi). Then, although there are many details left out like closure conditions on the category in question (cf. [1, §2.5]), we have

$$\forall \dashv \pi \dashv \exists.$$

And since weakening is surely a universally definable operation, we have the quantifiers as harmonious, which is rarely even asked for while simulataneously giving mathematical clarity to the analogy with conjunction and disjunction.

But there are two things about this iterative conception of logical operations that need to be explicitly stated:

1. We must define new operations by a *single* adjunction.
2. A logical constant must be defined as an adjoint of an already existing operation, namely not to itself.

Now we justify the claim that categorical harmony unifies the other conceptions.

4.0.1. Inversion harmony. Given these characterizations of adjoints, we give categorical semantics to (intuitionistic) natural deduction. Each of the usual connectives can be

[8] Although we use the logical symbols here, these are usually called product, coproduct, terminal object, and initial object, respectively. They do not always exist in a category, but for our purposes, we are assuming they do. For instance, the functor denoted by $\wedge$ is the product functor $(-) \times (-) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ that takes two objects to their binary product. This is the categorical equivalent to conjunction. Similarly, the other logical notions defined by adjunction are defined as functors, since adjunction is defined in terms of functors on categories.

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defined as an adjoint to a functor. First, let us define the category that will simplify the exposition of harmony, following [20].

Let L be a set of well-formed formulae that is closed under some inductive term constructors and let $\vdash_L$ be a preorder[9] on L.

DEFINITION 4.2. *A category of provability* $(L, \vdash_L) = \mathbb{L}$ *is defined by*

- *The objects are* $A, B, C, \ldots \in L$;
- *The arrows are derivability claims: there is an arrow* $p : A \to B$ *if and only if* $A \vdash_L B$.

Since any category of provability is a preorder, there is at most one arrow between any two formulae. This simplifies the hom set definition of adjunction, since each hom set is either empty or a singleton. This provides a sort of flattened version of adjunction that allows a formulation in terms of biconditionals. Let $\mathbb{L} = (L, \vdash_L)$ and $\mathbb{L}' = (L', \vdash_{L'})$ be categories of provability as just defined. Now suppose we have two functors $F : \mathbb{L} \to \mathbb{L}'$ and $G : \mathbb{L}' \to \mathbb{L}$. Then $F \dashv G$ when for $A \in \mathbb{L}$ and $B \in \mathbb{L}'$ we have the biconditional

$$\frac{F(A) \to B}{\overline{\overline{A \to G(B)}}}$$

We can now begin characterizing the usual connectives as adjoints in categories of provability. Define the diagonal $\Delta : \mathbb{L} \to \mathbb{L} \times \mathbb{L}$ on objects by $\Delta(A) = (A, A)$. The definition extends obviously to arrows.

DEFINITION 4.3. *Define conjunction as the right adjoint to the diagonal*:

$$\Delta \dashv \wedge.$$

This gives us the biconditional

$$\frac{(A, A) \to (B, C)}{\overline{\overline{A \to B \wedge C}}}$$

which amounts to

$$\frac{A \vdash_L B \qquad A \vdash_L C}{\overline{\overline{A \vdash_L B \wedge C}}}$$

which is a sequent description of $\wedge$I.

For $\wedge$E, we look to the counit. The counit gives us the biconditional

$$\frac{A \wedge B \vdash_L A \qquad A \wedge B \vdash_L B}{\overline{\overline{A \wedge B \vdash_L A \wedge B}}}.$$

And since $\vdash_L$ is reflexive, we have an arrow on the bottom. Thus we get our E-principles $A \wedge B \vdash_L A$ and $A \wedge B \vdash_L B$.

We have local reduction by the universal property of the counit. That is, two proofs of the form

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ A\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ B\end{matrix}}{A \wedge B}}{A} \quad \text{and} \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ A\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ B\end{matrix}}{A \wedge B}}{B}$$

[9] A preorder is a reflexive and transitive relation. So we have that for $A, B, C \in L$, both that $A \vdash_L A$ and if $A \vdash_L B$ and $B \vdash_L C$ then $A \vdash_L C$.

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are uniquely associated to two proofs of the form

$$\begin{matrix}\mathcal{D}_1\\ A\end{matrix} \quad \text{and} \quad \begin{matrix}\mathcal{D}_2\\ B\end{matrix}.$$

To see this, we take $\mathcal{D}_1 : \Gamma \to A$ and $\mathcal{D}_2 : \Gamma \to B$ creating an arrow $(\mathcal{D}_1, \mathcal{D}_2) : \Delta(\Gamma) \to (A, B)$. By the universal property of the counit, this gives us a unique arrow $f : \Gamma \to A \wedge B$ such that

$$(\mathcal{D}_1, \mathcal{D}_2) = \varepsilon_{(A,B)} \circ \Delta(f).$$

And since the counit is given by the pair of 'projections' $(p_1, p_2) : A \wedge B \to (A, B)$ and $f = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$, we get

$$(\mathcal{D}_1, \mathcal{D}_2) = (p_1, p_2) \circ \Delta\langle \mathcal{D}_1, \mathcal{D}_2 \rangle,$$

giving us the local reduction. In the other direction of the counit, we get that

$$id = \langle p_1, p_2 \rangle$$

which allows us to expand

$$A \wedge B \qquad \text{to} \qquad \dfrac{\dfrac{A \wedge B}{A}\, p_1 \qquad \dfrac{A \wedge B}{B}\, p_2}{A \wedge B}\, \wedge\text{I}$$

where we understand $\wedge$I as the deduction given by the adjunction when we have an arrow to both A and B. The expansion is really only repeating the fact that the conjunction is the Cartesian product in the category.

We take disjunction as another example that will finish our discussion of the relation between categorical and inversion harmony. This case is illustrative for both the existence of permutations and the fact that duality in logic is represented by duality of the unit and counit. Where we used properties of the counit in the case of conjunction, for disjunction, as a left adjoint, we will exploit properties of the unit in establishing inversion harmony.[10]

DEFINITION 4.4. *Define disjunction as the left adjoint to the diagonal*:

$$\vee \dashv \Delta.$$

This gives us the ('upside down') biconditional

$$\dfrac{(A, B) \to (C, C)}{A \vee B \to C}$$

which amounts to

$$\dfrac{A \vdash_L C \qquad B \vdash_L C}{A \vee B \vdash_L C}$$

which is a sequent description of $\vee$E.[11]

For $\vee$I, we look to the unit. The unit gives us the biconditional

[10] For the cases of $\top$, $\bot$, $\to$, $\forall$, $\exists$, see [20]. The method for showing right adjoints satisfy inversion harmony is generalizable from the conjunction case and the left adjoints are dual to these in the sense to follow.

[11] Morehouse points out in [20, pp. 64–66] that in the categorical setting, we have to be careful about context internalization and that this isn't quite the right $\vee$E. Since we're not sticking exactly to Gentzen's sequent calculus, it is a minor point.

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$$\frac{A \vdash_L A \vee B \qquad B \vdash_L A \vee B}{A \vee B \vdash_L A \vee B}.$$

And since $\vdash_L$ is reflexive, we have an arrow on the bottom. Thus we get our I-principles $A \vdash_L A \vee B$ and $B \vdash_L A \vee B$.

The unit for this adjunction is the pair (inl, inr) corresponding to the two disjunction introductions. Take the derivations

$$\dfrac{\dfrac{A}{A \vee B} \quad \begin{matrix} A \\ \mathcal{D}_1 \\ C \end{matrix} \quad \begin{matrix} B \\ \mathcal{D}_2 \\ C \end{matrix}}{C} \quad \text{and} \quad \dfrac{\dfrac{B}{A \vee B} \quad \begin{matrix} A \\ \mathcal{D}_1 \\ C \end{matrix} \quad \begin{matrix} B \\ \mathcal{D}_2 \\ C \end{matrix}}{C}.$$

Dual to the case of conjunction, we use the universal property of the unit to derive

$$(\mathcal{D}_1, \mathcal{D}_2) = \Delta[\mathcal{D}_1, \mathcal{D}_2] \circ (inl, inr)$$

Which allows us to reduce the two above derivations to

$$\begin{matrix} A \\ \mathcal{D}_1 \\ C \end{matrix} \quad \text{and} \quad \begin{matrix} B \\ \mathcal{D}_2 \\ C \end{matrix}$$

as desired. The naturality of adjoints, in this case the codomain part, gives us

$$[\mathcal{D}_1, \mathcal{D}_2] \circ \mathcal{E} = [\mathcal{D}_1 \circ \mathcal{E}, \mathcal{D}_2 \circ \mathcal{E}].$$

This allows us to get the permutation rules we desire, providing a way to move later derivations $\mathcal{E}$ into the minor branch of a deduction:

$$\dfrac{A \vee B \quad \begin{matrix} A \\ \mathcal{D}_1 \\ C \end{matrix} \quad \begin{matrix} B \\ \mathcal{D}_2 \\ C \end{matrix}}{\begin{matrix} C \\ \mathcal{E} \\ D \end{matrix}} \vee\text{E} \quad \mapsto \quad \dfrac{A \vee B \quad \begin{matrix} A \\ \mathcal{D}_1 \\ C \\ \mathcal{E} \\ D \end{matrix} \quad \begin{matrix} B \\ \mathcal{D}_2 \\ C \\ \mathcal{E} \\ D \end{matrix}}{D} \vee\text{E}.$$

Finally, using the biconditional from the unit, we get that

$$id = [inr, inl]$$

which gives us the expansion from

$$A \vee B \qquad \text{to} \qquad \dfrac{A \vee B \quad \dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B}}{A \vee B}.$$

4.0.2. Inferential harmony. The strength and weakness conditions in inferential harmony are satisfied by the universal properties of the connectives. Indeed, the structuralist logic from [11] exemplifies this approach to logic via universal definitions of the connectives. Since universal properties are closely related to adjoints, we immediately see that since $\vee, \wedge, \rightarrow, \bot, \top, \forall, \exists$ are defined by adjoints, they have the desired properties.

For example, continuing the case of conjunction and disjunction, we note that these are represented by the Cartesian product and coproduct respectively. Being defined by adjoints, the product and coproduct functors satisfy the strength and weakness conditions:

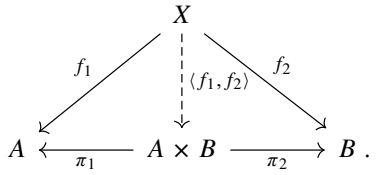
1. The conjunction $A \wedge B$ is the weakest proposition that implies both A and B.
2. The conjunction $A \wedge B$ is the strongest proposition that is implied by the pair A, B.

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3. The disjunction $A \vee B$ is the weakest proposition that implies some C when both A and B imply C.
4. The disjunction $A \vee B$ is the strongest proposition that is implied by either A or B.

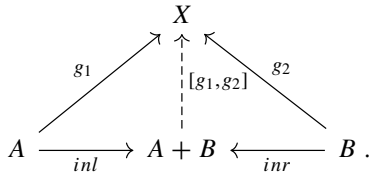
We need only state the universal properties of the respective adjoints to get these facts:

The Cartesian product $A \times B$ comes with two projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ such that for any $f_1 : X \to A$ and $f_2 : X \to B$, there exists a unique arrow $\langle f_1, f_2 \rangle : X \to A \times B$ such that the following diagram commutes:



Thus (1) is achieved. Further, if we take some X that is implied by both A and B, i.e., arrows $a : A \to X$ and $b : B \to X$, then we merely compose with the projections to get two arrows $(a \circ \pi_1), (b \circ \pi_2) : A \times B \to X$, giving us (2).

The coproduct $A + B$ comes with two injections $inl : A \to A + B$ and $inr : B \to A + B$ such that for any $g_1 : A \to X$ and $g_2 : B \to X$, there exists a unique arrow $[g_1, g_2] : A + B \to X$ such that the following diagram commutes:



Dual to conjunction, this gives us (4) from above. To get (3), we take a proposition X that implies C whenever both A and B imply C. But then using $A + B$ as C, we get that $X \to A + B$ since we already have the required arrows $inl : A \to A + B$ and $inr : B \to A + B$.

Now we must consider the 'way we proved' the above facts, since inferential harmony restricts our proof methods, but as we'll see, this part can be dispensed with in categorical harmony. Let's take the case of conjunction's weakness to show how the above follows Tennant's description of harmony from §3.2:

i. We exploited all the conditions of $\wedge$E because we considered both projections, i.e., the two eliminators for products/conjunctions.
ii. We made full use of $\wedge$I by getting an arrow $X \to A \times B$ through the two arrows $X \to A$ and $X \to B$.
iii. We didn't use $\wedge$E in the sense that we didn't use the fact that the counit of the adjunction gives us the two arrows π_1 and π_2.

This last requirement, however, is not quite right. The difference really is in the framework that we are using; where inferential harmony uses natural deduction and some sort of constrained informal proof, we use categorical deduction. I am inclined to think that the framework used in [32] is responsible for the many conditions on how we establish harmony. Categorical harmony has no such complication. In some sense, everything comes from the adjunction and so cannot really be separated into what 'we use' and 'what we do

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not use'. So although we abandon this small feature of inferential harmony, I take it to be an unburdening of harmony and a positive feature of categorical harmony that it need not deal with aspects that are difficult to establish. We do not lose the ability to distinguish between the sorts of connectives that are avoided in [32].

Therefore, we have captured all the (important) parts of inferential harmony with categorical harmony.

4.0.3. Conservative harmony. As alluded to earlier, I will not be capturing conservative harmony in its full sense with categorical harmony. Conservative extensions aren't really part of the categorical framework and I contend that this is not a shortcoming. The most important aspect of Belnap's harmony for the inferentialist is not the conservativity, but the uniqueness of connectives in the sense described in §3.3 since uniqueness warrants the claim that the connectives have unique inferential roles.

This inferentially important point is given in the categorical setting as the uniqueness of adjoints up to isomorphism:

THEOREM 4.5. *Given a functor* $F : \mathbb{C} \to \mathbb{D}$ *with right adjoints* $U, V : \mathbb{D} \to \mathbb{C}$,

$$F \dashv U \quad \text{and} \quad F \dashv V$$

we then have $U \cong V$. *Likewise for left adjoints.*

Which we can read in the following way:

> A connective λ defined by adjunction is unique in the sense that anything also defined by that adjoint is logically equivalent to λ.

This ensures inferential uniqueness, up to logical equivalence, for all those connectives defined by adjunction.

4.1. Inferentialism again. The inferentialism laid out in §2 focused on how introduction rules may satisfy the, admittedly somewhat vague, requirement of being *meaning-bearing*. This is referred to as a *verificationist* perspective. As Dummett points out, we could also take the elimination rules to be constitutive of the meaning of the connective. He calls this inferentialist position *pragmatism*. He notes in [8], that the two formulations are equivalent but he picks verificationism for his exposition; Martin-Löf does as well.

This distinction is not entirely helpful, in my view. It requires picking a uniform mode of fixing meaning, independent of the connective in question and we are told that verificationism and pragmatism are equivalent, but as far as I can make out, no argument is given for this equivalence.

Categorical harmony, however, gives us a way to improve upon both of these aspects of inferentialism. For the first, namely the supposition of there being only one way to fix meaning, we instead distinguish connectives by whether they are defined by left or by right adjoints to universally definable operations. To my mind, the introduction rules for disjunction are not nearly as central to disjunction as the elimination rules are. Indeed, argument by cases (the elimination rule) seems to be much more plausibly constitutive of the meaning of disjunction. We see that reflected in categorical harmony by the fact that disjunction is defined as a left adjoint. Similarly, the nonspecificity of the existential quantifier—by that I am referring to the fact that the existential quantifier merely asserts the existence of *some* object satisfying the formula—is captured by the elimination rule. This is reflected again in the fact that existentials are defined as left adjoints. We can rephrase this nice analogy between positive connectives (i.e., $\vee, \exists, \bot$) and left adjoints by referring back to the Yoneda Lemma since this tells us that we can characterize the connective by what we might deduce from it.

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Similarly, right adjoints capture the negative connectives, namely conjunction and the universal quantifier. Likewise, the covariant Yoneda Lemma gives us the inferentialist reading of these connectives.

Categorical harmony also gives a sense in which verification and pragmatism are 'equivalent': they are both instances of adjunctions. Although this is not, we suspect, what Dummett had in mind, it allows us to consider the two inferentialisms as part of the same idea, namely definable by adjoints to universally definable operations. This unity allows us to dispense with the dichotomy, in terms of choosing which inferentialism to use, of verificationism and pragmatism while at the same time allowing us to see the interesting relationship between the two in a clearer light.

§5. Justifying path induction. With this robust notion of categorical harmony, we can now come back and say something about path induction. The situation in HoTT is a bit more complex than in Gentzen-style natural deduction. Accordingly, we generalize from the pre-order category to a *hyperdoctrine* (see Lawvere's work in [14,15]) that allows proof-relevance. I will not reproduce all of the details—which can be found in [35]—but the main point is that *identity in a hyperdoctrine is determined by adjunction*.

From the categorical details, we can give a characterization of identity in terms of its inferential role:

> Identity is that relation that implies all and only reflexive relations.

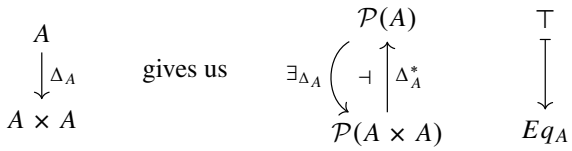
This is a universal property and a definition by an adjoint to a general (hyperdoctrinal) categorical operation of diagonals and terminal objects. Thus identity in a hyperdoctrine satisfies categorical harmony.

Let us give a very brief description of identity in the hyperdoctrine, though we leave out some technical details since it would require describing hyperdoctrines in full detail. Equality predicates are defined via adjoints to simple functors. The equality predicate on a type A which we write as Eq_A will be defined via a left adjoint to the substitution along the diagonal. In general, the diagonal $\Delta_A : A \to A \times A$ gives rise to the the substitution $\Delta_A^* : \mathcal{P}(A \times A) \to \mathcal{P}(A)$ between predicates over $A \times A$ and A respectively. We call Δ_A^* the *contraction functor* and takes predicates over two argument in A to predicates just over A. We define the equality attribute (relation or predicate) on $A \times A$ to be left adjoint to Δ_A^* applied to the terminal object of $\mathcal{P}(A)$ which we call $\top$ (consider this to be the 'always true predicate'). In hyperdoctrines, we denote the left adjoint with an existential symbol.

We define the equality relation to be $\exists_{\Delta_A}$ applied to this always true predicate. That is,

$$Eq_A := \exists_{\Delta_A}(\top).$$

For clarity, the following picture shows us how Eq_A is derived given the diagonal morphism:



Putting this in our familiar inference rule format, we get the following biconditional for any relation $R \in \mathcal{P}(A \times A)$:

$$\frac{Eq_A \to R}{\top \to \Delta_A^*(R)}.$$

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This can be roughly read as "Equality on A implies the relation R if and only if ..." and so we see how this gives an inferentially specified sufficient and necessary condition for the identity relation. Adding variables and replacing notation by more familiar ones, we get:

$$\frac{Eq_A(x, y) \vdash R(x, y)}{\vdash R(x, x)}$$

which justifies our characterization of identity's inferential role.

Furthermore, this definition of identity allows path induction as a rule[12] and so we now have two theses:

1. Identity is a concept determined by adjoint and so has a meaning-bearing inferential role, being categorically harmonious;
2. The inferential role of identity in these hyperdoctrines indeed provides a justification for path induction.

To reiterate, it may help to outline our justification of path induction. We constructed a notion of harmony that allows the logical inferentialist to say when a connective or concept is meaning-bearing and this conception unified most of the prominent conceptions of harmony. This categorical harmony was stated in terms of adjoints and so any concept definable by iterated adjoints from general categorical operations was harmonious. It has been shown that identity in a categorical setting is determined by an adjoint [15]. Furthermore, path induction as a rule comes from this definition. Thus we arrive at an account of how path induction, as a rule of inference governing identity, can be justified.

§6. Challenge reconsidered. In this section, I will examine the justification for path induction given in [13]. Path induction is the following rule of inference in HoTT:

$$\frac{x:A, y:A, p:\mathtt{Id}_A(x, y) \vdash C(x, y, p)\,\mathtt{Type} \qquad x:A \vdash t(x):C(x, x, \mathtt{refl}_x)}{x:A, y:A, p:\mathtt{Id}_A(x, y) \vdash \mathtt{J}(t; x, y, p):C(x, y, p)}$$

where A is a type. This allows us to construct a term $\mathtt{J}$ of a type C dependent on an identification of two terms of A as long as we've given a term of that type for the trivial identification.[13] This identity rule is motivated by the fact that the type theory is dependent and intensional. So we don't want reflection of 'external' identity in 'internal' identity, but something that still acts like a congruence. This rule seems quite different than the usual rule for identity elimination for some predicate P:

$$\frac{P(x) \qquad x = y}{P(y)}$$

This, among other aspects of the identity type in HoTT, is cause enough to examine whether the rule is properly justified.

In [12], we get a deeper understanding of why this is of foundational concern. If the justification for path induction requires the homotopy interpretation given in [34], then how can the putative foundation of HoTT be *autonomous*? After all, homotopy is part of mathematics and it seems circular to allow it into the justification of a foundation of mathematics. The authors explicitly cite the 'pre-mathematical' requirement given in [19]

[12] The proof that path induction is derivable from the definition of identity in a hyperdoctrine can be found in the appendix to [35]. The proof there is due to Steve Awodey.

[13] The word "identification" refers to terms of the identity type. In the above formulation, the trivial identification is $\mathtt{refl}_x$.

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when they say that a justification of the rules of inference must be given "a grounding or basis in pre-mathematical ideas" [12, p. 3]. The pre-mathematical requirement amounts to requiring that a justification of, say, path induction is independent of mathematical knowledge, experience, observation, or history of mathematics. I believe that this pre-mathematical requirement is too strong: it ignores the value of mathematical experience and practice.

The strictness of this requirement can be illustrated by considering how axiomatic set theory would fare. It may be admitted for the sake of argument that pairing, extensionality, and even powerset would be justifiable without any mathematical experience whatsoever. It seems unlikely, however, that the axioms of infinity, replacement, or choice could be so justified. After all, the development of our most trusted foundations comes after many years of trial and error, of testing the axioms to see what kind of mathematics it gives us. It is thus problematic to require that foundations be derived entirely from our pre-mathematical notions; mathematics is complex and the design of a foundational system has to take into consideration the framework it intends to capture. The requirement may amount to a restriction to naive foundations.

Of course, there is the response that although we have historically grounded foundations in what we have learned in mathematics, we can still require that there be a different, purer justification that does not require such mathematical experience. That is, we can avoid the historically 'accidental' justification and achieve real justification by appealing only to pre-mathematical notions. This response makes it clear, however, the extreme measures we must go to satisfy the criterion. We arrive at a foundation like axiomatic set theory, being motivated by problems with the antimonies, consistency, flourishing philosophies of mathematics at the turn of the 20th century, and methodological questions, and are then required to forget all that to really justify the end result. The burden in this case seems to be clearly on the defender of the pre-mathematical requirement to tell us why going back to square one is needed or even desirable.

Now let us look at how path induction is to be justified in this framework, according to the argument in [13]. In lieu of using homotopy, the justification rests on two principles: *substitution salva veritate* and the *uniqueness principle for identity types*. The former is called *transport* in the HoTT literature and matches the standard identity elimination rule that allows us to *transport* terms of one type to another:[14]

$$\frac{p : \mathtt{Id}_A(x, y) \qquad q : P(x)}{J(p)(q) : P(y)}$$

This reflects our pre-mathematical notion that identical terms should be substitutable in all cases without affecting 'truth-value'.

The second principle is not as straightforward. The intuition behind it states that the identity type is really only inhabited by the trivial self-identifications, *up to some other identity type*. If we simply said that the identity type is only inhabited by the trivial identifications, the type theory would become extensional, thus losing a crucial feature of HoTT. So we have to say something more careful. Define the dependent sum

$$\mathtt{E}_a :\equiv \sum_{x:A} \mathtt{Id}_A(a, x).$$

So $\mathtt{E}_a$ is the "singleton type" [6] that consists of pairs (b, p) where $b : A$ and $p : \mathtt{Id}_A(a, b)$. The uniqueness principle for identity types then states that the following type is always

[14] Here I leave out the other terms that J is dependent on to increase readability.

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inhabited:

$$\prod_{(b,p):\mathtt{E}_a} \mathtt{Id}_{\mathtt{E}_a}((a, \mathtt{refl}_a), (b, p)).$$

This principle states that each pair in $\mathtt{E}_a$ is propositionally identical to $(a, \mathtt{refl}_a)$. In the HoTT literature, this is called contractibility of the identity type.

These two principles together are sufficient for path induction, as shown in [5]. And this concludes the pre-mathematical justification of path induction. However, on the face of it, it seems not to be very pre-mathematical, especially given the second principle. How are we to justify, without mathematical knowledge, the claim that identity types should be inhabited by terms identical to the trivial identification *up to propositional identity over* $\mathtt{E}_a$? Although I think this is patently not pre-mathematical, I will not dwell on the point here. So instead of analyzing an admissible splitting-up of the rule, we should analyze whether the rule works with others that govern the same concept, in our case identity. It is also worth noting that with this expanded notion of harmony, we are not concerned with notions of 'logicality' as an avenue of justification. We do not need to claim that harmony is a sufficient condition for an operation or relation to be logical in order to say that it does justificatory work. Indeed, instead of considering the logicality of identity as a relation, we have considered the *universal* character of the identity type, which gives us insight into why the identity rules work the way they do. This way, we avoid the pre-mathematical requirement and instead use the inferentialist conception of justification. Is the elimination rule harmonious with its introduction rule? Yes, path induction is determined by the adjoint which defines identity.

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