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Equilibrium stability for non-uniformly hyperbolic systems

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Abstract. We prove that for a wide family of non-uniformly hyperbolic maps and hyperbolic potentials we have equilibrium stability, i.e. the equilibrium states depend continuously on the dynamics and the potential. For this we deduce that the topological pressure is continuous as a function of the dynamics and the potential. We also prove the existence of finitely many ergodic equilibrium states for non-uniformly hyperbolic skew products and hyperbolic Hölder continuous potentials. Finally, we show that these equilibrium states vary continuously in the weak* topology within such systems.

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1. *Introduction*

The study of probability measures, which remain invariant under the action of a dynamical system, provides relevant information about the topological behavior of its orbits. For instance, Poincaré's recurrence theorem states that the orbit of almost every point with respect to any invariant probability measure returns arbitrarily close to its initial state. When the dynamics has more than one invariant probability measure, an efficient way to choose an interesting one is to select those maximizing the free energy of the system, which are called equilibrium states.

More formally, given a continuous map $T : X \to X$ defined on a compact metric space X and a continuous potential $\phi : X \to \mathbb{R}$, we say that a T-invariant probability measure μ on the Borel sets of X is an *equilibrium state* for (T, ϕ) if it satisfies the following variational principle:

$$h_\mu(T) + \int \phi \, d\mu = \sup_{\eta \in \mathbb{P}_T(X)} \left\{ h_\eta(T) + \int \phi \, d\eta \right\},$$

where $\mathbb{P}_T(M)$ denotes the set of T-invariant probability measures on the Borel sets of X, which we always consider endowed with weak* topology.

The theory of equilibrium states was initiated by Sinai, Ruelle, and Bowen (SRB) in the 1970s through the application of techniques and results from statistical mechanics to smooth dynamical systems. In the pioneering work [**29**], Sinai studied the problem of existence and finiteness of equilibrium state for Anosov diffeomorphisms and Hölder continuous potentials. This strategy was extended by Ruelle and Bowen in [**8**, **25**, **26**] to the theory of uniformly hyperbolic (Axiom A) dynamical systems.

In the non-uniformly hyperbolic setting in dimensions greater than one, several advances were obtained by Sarig (see [**27**, **28**]) and Buzzi [**11**], who studied countable Markov shifts, and Buzzi, Paccaut, and Schmitt [**10**], who studied piecewise expanding maps. Arbieto, Matheus, and Oliveira [**6**], Oliveira and Viana [**20**], and Varandas and Viana [**30**] studied certain classes of non-uniformly expanding maps.

The problem of existence and finiteness of equilibrium states for partially hyperbolic systems has been fraught with greater challenges. In [**9**], Buzzi *et al* obtained the uniqueness of the maximal entropy measure for partially hyperbolic maps derived from Anosov. Climenhaga, Fisher, and Thompson in [**14**, **15**] addressed the question of the existence and uniqueness of equilibrium states for Bonatti–Viana diffeomorphisms and Mañé diffeomorphisms for suitable classes of potentials. Castro and Nascimento in [**12**] showed the uniqueness of the maximal entropy measure for partially hyperbolic attractors semi-conjugated to non-uniformly expanding maps. For a family of partially hyperbolic horseshoes introduced by Díaz *et al* in [**17**], the existence of equilibrium states for any continuous potential was proved by Leplaideur, Oliveira, and Rios in [**19**]. Later, Rios and Siqueira [**24**] proved the uniqueness of equilibrium states for a class of Hölder continuous potentials with small variation, which do not depend on the stable direction. Recently, Ramos and Siqueira [**22**] extended this result to a broader class of Hölder continuous potentials.

Once we have established the existence and finiteness of equilibrium states, a natural question that arises is *how do the equilibrium states vary with the underlying dynamics*

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and the potential? Actually, one of the main goals in the field of dynamical systems is to understand how the behavior of the system changes under perturbations of the underlying dynamics. The concept of *structural stability* was introduced by Andronov and Potryagin [**4**] and states that the whole orbit structure remains unchanged under small perturbations. Nevertheless, structural stability has been proved to be a concept that is too strong in the sense that many relevant models are not structurally stable, whereas some of the properties of their dynamics remain unchanged after small perturbations. Motivated by this, Alves and Viana introduced in [**3**] the notion of *statistical stability*, which expresses the persistence of statistical properties in terms of the continuity of the physical measure. In the aforementioned work, Alves and Viana considered a robust class of maps with non-uniform expansion and proved that the referent (unique) SRB measure varies continuously with the dynamics in the L^1-norm.

In the present work, we introduce the concept of *equilibrium stability*, meaning the persistence of the equilibrium states under small perturbations of the underlying dynamics and the potentials. In [**30**], Varandas and Viana proved statistical stability for a class of non-uniformly expanding local homeomorphisms on compact manifolds. Moreover, they show that if the topological pressure function depends continuously on such systems, then equilibrium stability holds for Hölder continuous potentials without a very large variation. The subsequent works [**7**, **13**] by Bomfim, Castro, and Varandas studied the continuity and even differentiability of several thermodynamical quantities for certain classes of non-uniformly expanding dynamical systems and potentials with small variation. Here we consider a family of non-uniformly expanding maps and hyperbolic potentials and prove that the topological pressure varies continuously within this family. This enables us to derive that this family is equilibrium stable.

We also consider skew products over non-uniformly expanding maps with uniform contraction on the fiber. For this kind of system, Ramos and Viana in [**23**] showed the existence of finitely many ergodic equilibrium states for hyperbolic potentials that do not depend on the stable direction. Here we give a sufficient condition on the fiber dynamic to enlarge the class of potentials for which there exists finiteness. We use the strategy of Ramos and Siqueira [**22**] to prove that the independence of the stable direction required in [**23**] is not necessary. We prove that given a hyperbolic Hölder continuous potential we can construct a homologous potential that is still hyperbolic and Hölder continuous, but does not depend on the stable direction. This allows us to prove the finiteness of equilibrium states. Finally, we obtain equilibrium stability for a family of skew products and hyperbolic potentials. We believe that these results can be an important step towards understanding the more difficult class of partially hyperbolic diffeomorphisms with a non-uniformly expanding central direction and a uniformly contracting direction.

This work is organized as follows. In §2 we define the equilibrium stability and precisely state our main results. In §3 we recall the definition of relative pressure and some classical results in functional analysis. In §4 we show the continuity of the topological pressure as a function of the base dynamics and the potential. From this result we derive the stability of equilibrium states associated to non-uniformly expanding maps and hyperbolic potentials. In §5 we prove the existence of finitely many ergodic equilibrium states and we extend the equilibrium stability obtained for the base dynamics to the skew product. We also prove

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the continuity of the topological pressure function. Finally, in §6 we describe some classes of examples which satisfy our results.

2. *Statement of results*

Let M be a compact Riemannian manifold and $\mathcal{F}$ a family of C^1 local diffeomorphisms $f : M \to M$. Given $\alpha > 0$, consider $C^\alpha(M)$ the space of Hölder continuous potentials $\phi : M \to \mathbb{R}$ endowed with the seminorm

$$|\varphi|_\alpha = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^\alpha}$$

and the norm

$$\|\varphi\|_\alpha = \|\varphi\|_0 + |\varphi|_\alpha,$$

where $\| \quad \|_0$ denotes the sup norm in $C^0(M)$. We shall always consider $\mathcal{F} \times C^\alpha(M)$ endowed with the product topology. Assume that $\mathcal{H}$ is a subset of $\mathcal{F} \times C^\alpha(M)$ such that each $(f, \phi) \in \mathcal{H}$ has a unique equilibrium state $\mu_{f,\phi}$. We say that $\mathcal{H}$ is *equilibrium stable* if the function assigning to each $(f, \phi) \in \mathcal{H}$ of its unique equilibrium state $\mu_{f,\phi} \in \mathbb{P}_f(M)$ is continuous.

2.1. *Non-uniform expansion.*

Given $c > 0$, define $\Sigma_c(f)$ as the set of points $x \in M$ where f is *non-uniformly expanding*, i.e.

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq -c. \tag{1}$$

We say that $\phi : M \to \mathbb{R}$ continuous is a *c-hyperbolic potential for* f if the topological pressure of ϕ (with respect to f) is equal to the relative pressure of ϕ on the set $\Sigma_c(f)$; for the definition of topological pressure relative to a set, see §3. Proposition 3.1 gives, in particular, that the set of c-hyperbolic potentials for f is an open subset of $C^0(M)$. The existence of only finitely many ergodic equilibrium states for (f, ϕ) with c-hyperbolic ϕ was established in [**23**, Theorem 2]. In addition, under the assumption that

$$\{f^{-n}(x)\}_{n \geq 0} \quad \text{is dense in } M \text{ for all } x \in M, \tag{$*$}$$

the uniqueness of the equilibrium state is also established. Clearly, condition $(*)$ holds whenever f is *strongly topologically mixing*, i.e. if for every open set $U \subset M$ there is some $N \in \mathbb{N}$ such that $f^N(U) = M$. Given $c > 0$, define

$$\mathcal{H}_c = \{(f, \phi) \in \mathcal{F} \times C^\alpha(M) : \ \phi \text{ is } c\text{-hyperbolic and } (*) \text{ holds for } f\}.$$

In our first result we obtain the continuous dependence of the equilibrium states over the elements in the family $\mathcal{H}_c$.

THEOREM A. *The family $\mathcal{H}_c$ is equilibrium stable.*

As a byproduct of our method to prove Theorem A, we obtain, in Corollary 4.3, the continuity of the topological pressure within the family $\mathcal{H}_c$.

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2.2. *Uniformly expanding maps.* Theorem A has a simple but interesting consequence for uniformly expanding maps. Letting $\mathcal{E} \subset \mathcal{F}$ be the set of uniformly expanding maps, for each $f \in \mathcal{E}$ we have $\Sigma_c(f) = M$ for suitable (locally constant) choices of $c > 0$ and the norm in M. Moreover, it is well-known from the classical theory that each $(f, \phi) \in \mathcal{E} \times C^\alpha(M)$ has a unique equilibrium state $\mu_{f,\phi}$ and $(*)$ holds for all $f \in \mathcal{E}$. The next result is then an immediate consequence of Theorem A.

COROLLARY B. *We have that* $\mathcal{E} \times C^\alpha(M) \ni (f, \phi) \longmapsto \mu_{f,\phi} \in \mathbb{P}_f(M)$ *is continuous.*

Although not explicitly stated anywhere in the literature, to the best of the authors' knowledge, we believe that Corollary B can probably be considered a folklore result by experts in the field.

2.3. *Skew-product maps.* In the second part of this work, we study skew products over non-uniformly expanding maps. Let N be a compact metric space with a distance d_N and let $g : M \times N \to N$ be a continuous map, which is a uniform contraction on N, i.e. there exists $0 < \lambda < 1$ such that for all $x \in M$ and all $y_1, y_2 \in N$ we have

$$d_N(g(x, y_1), g(x, y_2)) \leq \lambda d_N(y_1, y_2). \tag{2}$$

We assume that there exists some $y_0 \in N$ such that $g(x, y_0) = y_0$ for every $x \in M$. We define a family $\mathcal{S}$ of skew-product maps $F : M \times N \to M \times N$ such that

$$F(x, y) = (f(x), g(x, y))$$

for all $(x, y) \in M \times N$, where the *base dynamics* $f : M \to M$ belongs to $\mathcal{F}$ and the fiber dynamics $g : M \times N \to N$ satisfies (2). For notational simplicity, we shall denote the base dynamics of F by b_F.

Given $c > 0$ and $F \in \mathcal{S}$ we say that a continuous function $\phi : M \times N \to \mathbb{R}$ is a *c-hyperbolic potential for F* if the topological pressure of the system (F, ϕ) is equal to the relative pressure on the set $\Sigma_c(f) \times N$. Our second main result states the finiteness of equilibrium states for skew products with respect to hyperbolic potentials.

THEOREM C. *Let $F \in \mathcal{S}$ and $\phi : M \times N \to \mathbb{R}$ be a Hölder continuous and c-hyperbolic potential for F, for some $c > 0$. Then there exists a finite number of ergodic equilibrium states for (F, ϕ).*

We point out that if the fiber N can be decomposed as a finitely union $N = N_1 \cup \cdots \cup N_n$ of pairwise disjoint compact sets $N_1, \ldots, N_n$, then the condition $g(x, y_0) = y_0$ for all $x \in M$ can be replaced by $g_i(x, y_i) = y_i$ for all $x \in M$ and some $y_i \in N_i$, $i = 1, \ldots, n$. In fact, because $M \times N$ is a product space and $N = N_1 \cup \cdots \cup N_n$, we may define n fiber dynamics $g_i : M \times N_i \to N_i$ by $g_i(x, y) = g(x, y)$ when $y \in N_i$, for each $i = 1, \ldots, n$. See Example 6.3.

The conditions on the fiber dynamics allow us to enlarge the class of potentials considered in [**23**, Theorem 3], where the independence of the stable direction on the potential was required.

Given $c > 0$, consider

$$\mathcal{G}_c = \{(F, \phi) \in \mathcal{S} \times C^\alpha(M \times N) : \ \phi \text{ is } c\text{-hyperbolic and } (*) \text{ holds for } b_F\}.$$

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It easily follows from Lemma 5.3 below that each $F \in \mathcal{S}$ with $(*)$ holding for b_F necessarily has a unique equilibrium state. Our next result establishes the continuity of such equilibria within this family.

THEOREM D. *The set $\mathcal{G}_c$ is equilibrium stable.*

3. *Preliminaries*

To make this work self-contained, we present here some basic definitions and results that will be used in the following sections. We start by defining relative pressure through the notion of dynamic balls. This is the approach from dimension theory and is very useful in computing the topological pressure of non-compact sets.

3.1. *Topological pressure.* Let X be a compact metric space. Consider $T : X \to X$ and $\phi : X \to \mathbb{R}$ both continuous. Given $\delta > 0$, $n \in \mathbb{N}$ and $x \in X$, consider the *dynamic ball*

$$B_\delta(x, n) = \{y \in X : d(T^j(x), T^j(y)) < \delta, \text{ for } 0 \le j \le n\}.$$

Consider for each $N \in \mathbb{N}$

$$\mathcal{F}_N = \{B_\delta(x, n) : x \in X \text{ and } n \ge N\}.$$

Given $\Lambda \subset X$ denote by $\mathcal{F}_N(\Lambda)$ the finite or countable families of elements in $\mathcal{F}_N$ that cover Λ. Define for $n \in \mathbb{N}$

$$S_n\phi(x) = \phi(x) + \phi(T(x)) + \cdots + \phi(T^{n-1}(x)).$$

and

$$R_{n,\delta}\phi(x) = \sup_{y \in B_\delta(x,n)} S_n\phi(y).$$

Given a T-invariant set $\Lambda \subset X$, not necessarily compact, define for each $\gamma > 0$

$$m_T(\phi, \Lambda, \delta, N, \gamma) = \inf_{\mathcal{U} \subset \mathcal{F}_N(\Lambda)} \left\{ \sum_{B_\delta(x,n) \in \mathcal{U}} e^{-\gamma n + R_{n,\delta}\phi(x)} \right\}.$$

Define

$$m_T(\phi, \Lambda, \delta, \gamma) = \lim_{N \to +\infty} m_T(\phi, \Lambda, \delta, N, \gamma),$$

and

$$P_T(\phi, \Lambda, \delta) = \inf\{\gamma > 0 \mid m_T(\phi, \Lambda, \delta, \gamma) = 0\}.$$

Finally, define the *relative pressure* of ϕ on Λ as

$$P_T(\phi, \Lambda) = \lim_{\delta \to 0} P_T(\phi, \Lambda, \delta).$$

The *topological pressure of* ϕ is, by definition, $P_T(\phi, X)$ and it satisfies

$$P_T(\phi) = \sup\{P_T(\phi, \Lambda), P_f(\phi, \Lambda^c)\}, \tag{3}$$

where Λ^c denotes the complement of the set Λ on M. We refer the reader to [**21**] for the proof of (3) and for additional properties of the pressure. See also [**32**, Theorem 4.1] for a proof of the fact that

$$P_T(\phi) = \sup_{\eta \in \mathbb{P}_T(X)} \left\{ h_\eta(T) + \int \phi \, d\eta \right\}. \tag{4}$$

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Consider $C^0(X)$ the space of continuous functions from X to $\mathbb{R}$, endowed with the supremum norm.

PROPOSITION 3.1. *Let $\Lambda \subset X$ be a T-invariant set and $\phi \in C^0(X)$. If $P_T(\phi, \Lambda^c) < P_T(\phi, \Lambda) = P_T(\phi)$, then there exists $\zeta > 0$ such that for each $\psi \in C^0(X)$ with $\|\psi - \phi\| < \zeta$ we have $P_T(\psi, \Lambda^c) < P_T(\psi, \Lambda) = P_T(\psi)$.*

Proof. Let $\phi \in C^0(X)$ satisfy $P_T(\phi, \Lambda^c) < P_T(\phi, \Lambda) = P_T(\phi)$. Fix $\varepsilon_\phi > 0$ such that $P_T(\phi, \Lambda^c) < P_T(\phi) - \varepsilon_\phi$. Recalling that the topological pressure depends continuously on the potential $\phi \in C^0(X)$ (see e.g. [**31**, Proposition 10.3.6]), given $0 < \varepsilon < \varepsilon_\phi$ we can find $\zeta > 0$ such that for any $\psi \in C^0(X)$ with $\|\psi - \phi\| < \zeta$ we have $P_T(\psi) \geq P_T(\phi) - \varepsilon$. For every $\gamma \in \mathbb{R}$ we have

$$\begin{aligned} m_T(\psi, \Lambda^c, \delta, N, \gamma) &= \inf_{\mathcal{U} \subset \mathcal{F}_N(\Lambda^c)} \left\{ \sum_{B_\delta(x,n) \in \mathcal{U}} e^{-\gamma n + R_{n,\delta}\psi(x)} \right\} \\ &\leq \inf_{\mathcal{U} \subset \mathcal{F}_N(\Lambda^c)} \left\{ \sum_{B_\delta(x,n) \in \mathcal{U}} e^{-\gamma n + \zeta n + R_{n,\delta}\phi(x)} \right\} \\ &= m_T(\phi, \Lambda^c, \delta, N, \gamma - \zeta), \end{aligned}$$

and, therefore,

$$P_T(\psi, \Lambda^c) \leq P_T(\phi, \Lambda^c) - \zeta < P_T(\phi) - \varepsilon_\phi - \zeta \leq P_T(\phi) - \varepsilon \leq P_T(\psi).$$

Hence, for any $\psi \in C^0(X)$ with $\|\psi - \phi\| < \zeta$ we have

$$P_T(\psi, \Lambda^c) < P_T(\psi, \Lambda) = P_T(\psi),$$

which gives the desired conclusion. □

Now we introduce the definition of hyperbolic potentials. Let M be a compact Riemannian manifold and let $f : M \to M$ be a local C^1 diffeomorphism. Consider the set $\Sigma_c(f)$ as in §2.1. We say that $\phi : M \to \mathbb{R}$ continuous is a *c-hyperbolic potential for f* if

$$P_f(\phi, (\Sigma_c(f))^c) < P_f(\phi, \Sigma_c(f)) = P_f(\phi).$$

Proposition 3.1 gives, in particular, that the hyperbolic potentials form an open class in the C^0-topology.

3.2. *Cones.* We will also use a few tools from classic functional analysis that we present here. Let E be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We say that a closed and convex set $\{0\} \neq \mathcal{C} \subset E$ is a *cone* in E if:

(i) for all $\lambda \geq 0 : \lambda\mathcal{C} \subset \mathcal{C}$;

(ii) $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$.

A cone $\mathcal{C}$ defines a partial order in E through the relation

$$x \leq y \iff y - x \in \mathcal{C}.$$

We say that a cone $\mathcal{C} \subset E$ is *normal* if

$$\text{there exists } \gamma \in \mathbb{R} : 0 \leq x \leq y \Longrightarrow \|x\| \leq \gamma \|y\|.$$

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Assume that E is a Banach space partially ordered by a normal cone $\mathcal{C}$ with non-empty interior and $T : E \to E$ is a bounded linear operator. We say that T is *positive* (with respect to $\mathcal{C}$) if $T(\mathcal{C}) \subset \mathcal{C}$. Note that if T is positive, then $T(x) \leq T(y)$ whenever $x \leq y$. The *spectral radius* of T is defined by

$$r(T) = \lim_{n\to\infty} \sqrt[n]{\| T^n \|}.$$

The dual space of E is defined by

$$E^* = \{x^* : E \to \mathbb{K} \mid x^* \text{ is linear and bounded}\}$$

and the dual operator of T is $T^* : E^* \to E^*$ defined for all $x^* \in E^*$ and all $y \in E$ by

$$T^*(x^*)(y) = x^*(T(y)).$$

Given $x^* \in E^* \backslash \{0\}$ and $c \in \mathbb{R}$, we define the real hyperplane

$$H = H(x^*, c) = \{y \in E \mid \mathcal{R}x^*(y) = c\},$$

where $\mathcal{R}a$ denotes the real part of $a \in \mathbb{C}$. We say that H *separates* the sets E_1, $E_2 \subset E$ if

$$\sup_{y\in E_1} \mathcal{R}x^*(y) \leq c \leq \inf_{z\in E_2} \mathcal{R}x^*(z).$$

Given a cone $\mathcal{C} \subset E$, we say that $x^* \in E^*$ is a positive functional (with respect to $\mathcal{C}$) if $\mathcal{R}x^*(w) \geq 0$ for all $w \in C$. The dual cone $\mathcal{C}^*$ is the cone given by all positive functionals.

A classical result due to Mazur can be stated as follows: *let E be a Banach space and $C \subset E$ a convex set with non-empty interior. If $C_1 \neq \emptyset$ is a convex set such that $C_1 \cap \operatorname{int}(C) = \emptyset$, then there exists a closed real hyperplane that separates C and C_1*. For a proof, see e.g. [**16**, Proposition 7.12].

We end this section with a useful corollary from Mazur's theorem. This is somewhat of a folklore result, but for the sake of completeness we present its proof here.

LEMMA 3.2. *Let E be a Banach space partially ordered by a normal cone $\mathcal{C}$ with non-empty interior. Let T be a positive-bounded operator on E. Then the spectral radius of T is an eigenvalue of the adjoint T^*, associated to an eigenfunctional $x^* \in \mathcal{C}^*$.*

Proof. Defining $E_0 = \{r(T)x - Tx : x \in E\}$, we start by showing that $E_0 \cap \operatorname{int}(\mathcal{C}) = \emptyset$. Assume by contradiction that $E_0 \cap \operatorname{int}(\mathcal{C}) \neq \emptyset$. This means that there exist $x \in E$ and $y \in \operatorname{int}(\mathcal{C})$ such that $y = r(T)x - Tx$. Thus, there exists $\delta > 0$ such that $y - \delta(-x) \in \operatorname{int}(\mathcal{C}) \subset \mathcal{C}$, and so $y \geq \delta(-x)$. Using that $T(-x) = r(T)(-x) + y$, we easily deduce that

$$T(-x) \geq (r(T) + \delta)(-x).$$

By linearity and positivity of T, we obtain for all $n \in \mathbb{N}$

$$T^n(-x) \geq (r(T) + \delta)^n(-x).$$

On the other hand, since we are assuming the cone $\mathcal{C}$ with non-empty interior, there must be $a \in \mathcal{C}$ and $\delta' > 0$ such that $a + \delta'(-x) \geq 0$. Taking $K = 1/\delta'$ we obtain $Ka - x \geq 0$ and

$$T^n(-x) + (r(T) + \delta)^n(Ka) \geq (r(T) + \delta)^n(-x + Ka) \geq 0.$$

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Since $\mathcal{C}$ is a normal cone, by definition there is $\gamma > 0$ such that

$$\gamma \|T^n(-x) + [r(T) + \delta]^n Ka\| \geq \|[r(T) + \delta]^n(-x + Ka)\|.$$

It easily follows that for all odd $n \in \mathbb{N}$ we have

$$\sqrt[n]{\gamma \| - x\|}\sqrt[n]{\|T^n\|} \geq [r(T) + \delta]\sqrt[n]{\|Ka - x\| - \gamma \|Ka\|},$$

taking a limit in n we obtain $r(T) \geq r(T) + \delta$, which is absurd. Hence, we must have $E_0 \cap \operatorname{int}(\mathcal{C}) = \emptyset$.

Now since E_0 and $\mathcal{C}$ are convex sets (E_0 is actually a vectorial subspace of E), Mazur's theorem guarantees the existence of a closed real hyperplane separating them. This means that there exist $c \in \mathbb{R}$ and a non-trivial continuous linear functional $x^* \in E^*$ such that

$$\sup_{x \in E_0} \mathcal{R}x^*(x) \leq c \leq \inf_{y \in \mathcal{C}} \mathcal{R}x^*(y).$$

Since 0 belongs to both E_0 and $\mathcal{C}$ we must have $c = 0$. Moreover, as E_0 is a vectorial subspace of E we necessarily have $x^*(x) = 0$ for all $x \in E_0$. This implies that for all $x \in E$ we have

$$x^*(r(T)x - Tx) = 0,$$

which is equivalent to

$$T^*x^*(x) = r(T)x^*(x).$$

This gives, in particular, that the spectral radius of T is an eigenvalue of T^*. Moreover, since $\mathcal{R}x^*(y) \geq 0$ for all $y \in \mathcal{C}$, and so we have $x^* \in \mathcal{C}^*$. □

3.3. *Hyperbolic times.* Consider $f \in \mathcal{F}$ and $\Sigma_c(f)$ as in §2.1. In order to explore the non-uniform expansion on the set $\Sigma_c(f)$ we use the following concept introduced in [**1**] and generalized in [**2**]. We say that n is a *hyperbolic time* for x if

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \leq e^{-ck/2} \quad \text{for all } 1 \leq k < n.$$

Observe that as we are assuming maps with no critical/singular sets, the definition of hyperbolic times given in [**2**, Definition 5.1] reduces to the definition we present here. Condition (1) of non-uniform expansion is enough to guarantee the existence of infinitely many hyperbolic times for the points in $\Sigma_c(f)$. For a proof of the next result see [**2**, Lemma 5.4].

LEMMA 3.3. *Each $x \in \Sigma_c(f)$ has infinitely many hyperbolic times.*

The next result shows that the iterates of a map at hyperbolic times behave locally as uniformly expanding maps. We refer the reader to [**2**, Lemma 5.2] for the proof of the first item and [**2**, Corollary 5.3] for the second one, with ϕ playing the role of $\log|\det Df|$ in the latter case.

LEMMA 3.4. *There exist $K, \delta_1 > 0$ such that for every $0 < \varepsilon \leq \delta_1$ and every hyperbolic time n for $x \in M$ the dynamic ball $B_\varepsilon(x, n)$ is mapped diffeomorphically under f^n onto the ball $B(f^n(x), \varepsilon)$. Moreover, for all Hölder continuous potentials ϕ and all $y, z \in B_\varepsilon(x, n)$ we have*

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(i) $d(f^{n-j}(y), f^{n-j}(z)) \le e^{-cj/4} d(f^n(y), f^n(z))$ *for each* $1 \le j \le n$;
(ii) $1/K \le e^{S_n\phi(y) - S_n\phi(z)} \le K$.

Remark 3.5. The constant $\delta_1 > 0$ in the previous lemma can be taken uniform in a neighborhood of $f \in \mathcal{F}$. Actually, δ_1 is obtained as the conclusion of the Claim in the proof of [**2**, Lemma 5.2] with $\sigma = e^{-c/2}$, where the only requirement is that

$$\|Df(y)^{-1}\| \le e^{c/4} \|Df(f^{n-j}(x))^{-1}\|, \tag{5}$$

for any $1 \le j < n$ and any y in the ball of radius $2\delta_1 e^{-cj/4}$ around $f^{n-j}(x)$. In our situation, where f is a local diffeomorphism on a compact manifold, we can clearly choose $\delta_1 > 0$ sufficiently small so that

$$\|Df(y)^{-1}\| \le e^{c/4} \|Df(z)^{-1}\|$$

whenever $d(y, z) < 2\delta_1$. This choice of δ_1 is naturally sufficient for (5) to hold and can be taken uniform in a neighborhood of f in the C^1 topology.

3.4. *The reference measure.* Let $C^0(M)$ be the space of continuous functions $\phi : M \to \mathbb{R}$ endowed with the sup norm. Fix $c > 0$ and define for $(f, \phi) \in \mathcal{H}_c$ the *transfer operator*

$$\mathcal{L}_{f,\phi} : C^0(M) \to C^0(M),$$

associating to each $\varphi \in C^0(M)$ the continuous function $\mathcal{L}_{f,\phi}(\varphi) : M \to \mathbb{R}$ defined by

$$\mathcal{L}_{f,\phi}\varphi(x) = \sum_{y \in f^{-1}(x)} e^{\phi(y)} \varphi(y).$$

We clearly have that $\mathcal{L}_{f,\phi}$ is a bounded linear operator. Moreover, considering $C^0(M)$ ordered by the cone $\mathcal{C}$ of non-negative functions, we have that $\mathcal{L}_{f,\phi}$ is a positive operator. For each $n \in \mathbb{N}$, we have

$$\mathcal{L}^n_{f,\phi}\varphi(x) = \sum_{y \in f^{-n}(x)} e^{S_n\phi(y)} \varphi(y),$$

where $S_n\phi$ denotes the Birkhoff sum

$$S_n\phi(x) = \sum_{j=0}^{n-1} \phi(f^j(x)).$$

Using that $\|\mathcal{L}^n_{f,\phi}\| = \|\mathcal{L}^n_{f,\phi} 1\|$ for all $n \ge 1$, we can easily deduce that the spectral radius $\lambda_{f,\phi}$ of $\mathcal{L}_{f,\phi}$ satisfies

$$\deg(f) e^{\inf \phi} \le \lambda_{f,\phi} \le \deg(f) e^{\sup \phi}. \tag{6}$$

As we are considering $\mathcal{L}_{f,\phi} : C^0(M) \to C^0(M)$, by the Riesz–Markov theorem, we may think of its dual operator $\mathcal{L}^*_{f,\phi} : \mathbb{P}(M) \to \mathbb{P}(M)$. Moreover, we have

$$\int \varphi \, d\mathcal{L}^*_{f,\phi}\eta = \int \mathcal{L}_{f,\phi}(\varphi) \, d\eta,$$

for every $\varphi \in C^0(M)$ and every $\eta \in \mathbb{P}(M)$. It follows from Lemma 3.2 that for each $(f, \phi) \in \mathcal{H}_c$ there exists a probability measure $\nu_{f,\phi}$ satisfying

$$\mathcal{L}^*_{f,\phi}\nu_{f,\phi} = \lambda_{f,\phi}\nu_{f,\phi},$$

where $\lambda_{f,\phi}$ is the spectral radius of $\mathcal{L}_{f,\phi}$.

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The next result gives a *Gibbs property* for the measure $\nu_{f,\phi}$ at hyperbolic times. Our proof follows closely the proof of [**23**, Proposition 5.2], where similar conclusions hold for the spectral radius $\lambda_{f,\phi}$ and the respective eigenmeasure $\nu_{f,\phi}$.

LEMMA 3.6. *Given $(f, \phi) \in \mathcal{H}_c$ and assuming $\mathcal{L}^*_{f,\phi}\nu = \lambda\nu$ for some $\lambda \in \mathbb{R}$ and $\nu \in \mathbb{P}(M)$, then* $\operatorname{supp}(\nu) = M$. *Moreover, for all $\varepsilon \le \delta_1$ there exists $C = C(\varepsilon) > 0$ such that if n is a hyperbolic time for $x \in M$, then*

$$C^{-1} \le \frac{\nu(B_\varepsilon(x, n))}{\exp(S_n\phi(y) - n\log\lambda)} \le C$$

for all $y \in B_\varepsilon(x, n)$.

Proof. Since ν is an eigenmeasure associated to λ, it follows that if A is a Borel set such that $f|_A$ is injective, then for any sequence $\{\psi_s\}_s$ in $C^0(M)$ that converges to the characteristic function χ_A of A for ν almost every point we have

$$\lambda\nu(e^{-\phi}\psi_s) = \mathcal{L}^*_\phi\nu(e^{-\phi}\psi_s) = \int_M \mathcal{L}_\phi(e^{-\phi}\psi_s)\,d\nu \xrightarrow{s\to\infty} \nu(f(A)).$$

As the first member converges to $\int_A \lambda e^{-\phi}\,d\nu$, we conclude that

$$\nu(f(A)) = \int_A \lambda e^{-\phi}\,d\nu.$$

In particular, if $f^k|_A$ is injective it follows that

$$\nu(f^k(A)) = \int_A \lambda^k e^{-S_k\phi}\,d\nu.$$

Let $U \subset M$ be an arbitrary open set. From the density of the pre-orbit $\{f^{-n}(x)\}_{n\ge0}$ for every point $x \in M$ we have the inclusion $M \subset \bigcup_{k\in\mathbb{N}} f^k(U)$. Since each f^k is a local homeomorphism we can decompose U into subsets $V_i(k) \subset U$ such that $f^k|_{V_i(k)}$ is injective. Using the fact that ν is an eigenmeasure, we deduce that

$$\begin{aligned} 1 = \nu(M) \le \sum_k \nu(f^k(U)) &\le \sum_k \sum_i \int_{V_i(k)} \lambda^k e^{-S_k\phi(x)}\,d\nu \\ &\le \sum_k \lambda^k \sum_i \sup_{x\in V_i(k)} (e^{S_k\phi(x)})\nu(V_i(k)). \end{aligned}$$

Hence, there exists some $V_i(k) \subset U$ such that $\nu(U) \ge \nu(V_i(k)) > 0$. Thus, we have $\operatorname{supp}(\nu) = M$.

Consider now $x \in M$ and $n \in \mathbb{N}$ a hyperbolic time for x. Since f^n maps the hyperbolic dynamic ball $B_\varepsilon(x, n)$ homeomorphically onto the ball $B(f^n(x), \varepsilon)$, it follows that there exists a uniform constant $\gamma_\varepsilon > 0$ depending on the radius ε of the ball such that

$$\gamma_\varepsilon \le \nu(B(f^n(x), \varepsilon)) = \nu(f^n(B_\varepsilon(x, n))) = \int_{B_\varepsilon(x,n)} \lambda^n e^{-S_n\phi(z)}\,d\nu \le 1.$$

Using the Hölder continuity of the potential ϕ, we apply the distortion control on hyperbolic times (Lemma 3.4) to obtain

$$\begin{aligned} \gamma_\varepsilon \le \int_{B_\varepsilon(x,n)} \lambda^n e^{-S_n\phi(y)} \frac{e^{-S_n\phi(z)}}{e^{-S_n\phi(y)}}\,d\nu &\le K e^{-S_n\phi(y)+n\log\lambda}\nu(B_\varepsilon(x, n)) \\ &\le K \int_{B_\varepsilon(x,n)} \lambda^n e^{-S_n\phi(y)}\,d\nu \le K. \end{aligned}$$

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Hence, there exists $C = C(\varepsilon) > 0$ such that

$$C^{-1} \leq \frac{\nu(B_\varepsilon(x, n))}{\exp(S_n\phi(y) - n\log\lambda)} \leq C$$

for all $y \in B_\varepsilon(x, n)$. □

LEMMA 3.7. *For each $(f, \phi) \in \mathcal{H}_c$ we have $\lambda_{f,\phi} = e^{P_f(\phi)}$ and this is the only real eigenvalue of $\mathcal{L}^*_{f,\phi}$.*

Proof. The fact that $\lambda_{f,\phi} = e^{P_f(\phi)}$ has been proved in [**23**, Proposition 5.1]. Assume now that $\mathcal{L}^*_{f,\phi}\nu = \lambda\nu$ for some $\lambda \in \mathbb{R}$ and $\nu \in \mathbb{P}(M)$. As every point $x \in \Sigma_c(f)$ has infinitely many hyperbolic times, for $\varepsilon > 0$ small we can fix $N > 1$ sufficiently large such that

$$\Sigma_c(f) \subset \bigcup_{n\geq N} \bigcup_{x\in H_n} B_\varepsilon(x, n),$$

where H_n denotes the set of points that have n as a hyperbolic time. Using the fact that each $f^n(B_\varepsilon(x, n))$ is the ball $B(f^n(x), \varepsilon)$ in M and Besicovitch's covering lemma, it is not difficult to see that there exists a countable family $F_n \subset H_n$ such that every point $x \in H_n$ is covered by at most $d = \dim(M)$ dynamical balls $B_\varepsilon(x, n)$ with $x \in F_n$. Hence,

$$\mathcal{F}_N = \{B_\varepsilon(x, n) : x \in F_n \text{ and } n \geq N\}$$

is a countable open covering of $\Sigma_c(f)$ by hyperbolic dynamic balls with a diameter less than $\varepsilon > 0$.

Take now any $\gamma > \log\lambda$. Recalling the definition of relative pressure given in §3 we apply Lemma 3.6 to each element in $\mathcal{F}_N$ to deduce that for some $\widetilde{C} = \widetilde{C}(\varepsilon) > 0$ we have

$$\sum_{B_\varepsilon(x,n)\in\mathcal{F}_N} e^{-n\log\lambda + R_{n,\varepsilon}\phi(x)} \leq \widetilde{C}.$$

Hence,

$$\sum_{B_\varepsilon(x,n)\in\mathcal{F}_N} e^{-\gamma n + R_{n,\varepsilon}\phi(x)} \leq \widetilde{C} \sum_{n\geq N} e^{-(\gamma-\log\lambda)n} \leq \widetilde{C} e^{-(\gamma-\log\lambda)N}.$$

Taking the limit in N we obtain

$$m_f(\phi, \Sigma_c(f), \varepsilon, \gamma) = \lim_{N\to+\infty} m_f(\phi, \Sigma_c(f), \varepsilon, N, \gamma) = 0.$$

As this holds for arbitrary $\varepsilon > 0$ and $\gamma > \log\lambda$ we necessarily have

$$P_f(\phi, \Sigma_c(f)) \leq \log\lambda.$$

Now, since ϕ is a hyperbolic potential and $\lambda_{f,\phi}$ is the spectral radius of $\mathcal{L}^*_{f,\phi}$ we obtain

$$\log\lambda \leq \log\lambda_{f,\phi} = P_f(\phi) = P_f(\phi, \Sigma_c(f)) \leq \log\lambda,$$

thus having proved the result. □

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4. *Equilibrium stability*

In this section we finish the proof of Theorem A. Let $\mathcal{B}(C^\alpha(M))$ be the space of bounded linear maps from $C^\alpha(M)$ to $C^\alpha(M)$, considering the norm $\| \quad \|_\alpha$ in the first space and $\| \quad \|_0$ in the second space. As we have the functions f and ϕ Hölder continuous for $(f, \phi) \in \mathcal{H}_c$, one easily sees that

$$\mathcal{L}_{f,\phi}(C^\alpha(M)) \subset C^\alpha(M).$$

We introduce a map

$$\Gamma : \mathcal{H}_c \longrightarrow \mathcal{B}(C^\alpha(M)),$$

assigning to each $(f, \phi) \in \mathcal{H}_c$ the restriction of $\mathcal{L}_{f,\phi}$ to $C^\alpha(M)$. In the next result, we shall deduce the continuity of Γ, which obviously implies the weaker pointwise convergence: for any $\psi \in C^\alpha(M)$ we have $\mathcal{L}_{f_n,\phi_n}(\psi)$ converging to $\mathcal{L}_{f,\phi}(\psi)$ in the C^0-norm. This pointwise convergence is actually what we need for what comes next. However, as the proof of this statement is essentially the same as the weaker pointwise convergence, we leave this stronger version here.

LEMMA 4.1. *We have that* Γ *is continuous.*

Proof. Given $(f, \phi) \in \mathcal{H}_c$, let $(f_n, \phi_n)_n$ be an arbitrary sequence in $\mathcal{H}_c$ converging to (f, ϕ). For each $x \in M$ and $i = 1, \ldots, \deg(f)$ denote by y_i the preimage of x under f. Since $\deg(f_n) = \deg(f)$ for all $n \in \mathbb{N}$, it follows that there exists a unique point $y_{i,n}$, preimage of x under f_n, in a neighborhood of y_i for every $i = 1, \ldots, \deg(f)$. We point out that the sequence $(y_{i,n})_n$ converges to y_i when n goes to infinity. In fact, let z be an accumulation point of $(y_{i,n})_n$. From the uniform convergence of $(f_n)_n$ to f we have that the sequence $(f_n(y_{i,n}))_n$ converges to $f(z)$. Since $f_n(y_{i,n}) = x$ for all n, we conclude that $z = y_i$.

By definition of the operator norm in $\mathcal{B}(C^\alpha(M))$, we have

$$\begin{aligned}
\|\mathcal{L}_{f,\phi} - \mathcal{L}_{f_n,\phi_n}\| &= \sup_{\|\varphi\|_\alpha \le 1} \|\mathcal{L}_{f,\phi}(\varphi) - \mathcal{L}_{f_n,\phi_n}(\varphi)\|_0 \\
&= \sup_{\|\varphi\|_\alpha \le 1} \sup_{x \in M} \sum_{i=1}^{\deg(f)} |e^{\phi(y_i)}\varphi(y_i) - e^{\phi_n(y_{i,n})}\varphi(y_{i,n})| \\
&\le \sup_{\|\varphi\|_\alpha \le 1} \sup_{x \in M} \sum_{i=1}^{\deg(f)} |\varphi(y_i)||e^{\phi(y_i)} - e^{\phi_n(y_{i,n})}| \\
&\quad + \sup_{\|\varphi\|_\alpha \le 1} \sup_{x \in M} \sum_{i=1}^{\deg(f)} |e^{\phi_n(y_{i,n})}||\varphi(y_i) - \varphi(y_{i,n})|.
\end{aligned}$$

Since $(y_{i,n})_n$ converges to y_i and $(\phi_n)_n$ converges uniformly to ϕ we have that each term in the last inequality converges uniformly to zero for all $\varphi \in C^1(M)$ with $\|\varphi\|_\alpha \le 1$. □

Let now $(f_n, \phi_n)_n$ be a sequence in $\mathcal{H}_c$ converging to $(f, \phi) \in \mathcal{H}_c$. For notational simplicity, for each $n \ge 0$ let λ_n be the spectral radius of $\mathcal{L}_{f_n,\phi_n}$ and λ be the spectral radius of $\mathcal{L}_{f,\phi}$. In the next result we use, in particular, the continuity of Γ to prove the convergence of the spectral radius.

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LEMMA 4.2. *The sequence $(\lambda_n)_n$ converges to λ.*

Proof. By (6) we have, for all $n \geq 1$,

$$\deg(f_n)e^{\inf \phi_n} \leq \lambda_n \leq \deg(f_n)e^{\sup \phi_n}.$$

In particular, the convergence of (f_n, ϕ_n) to (f, ϕ) implies that the sequence $(\lambda_n)_n$ admits some accumulation point $\bar{\lambda} \in \mathbb{R}$ satisfying

$$\deg(f)e^{\inf \phi} \leq \bar{\lambda} \leq \deg(f)e^{\sup \phi}.$$

Now we prove that $\bar{\lambda}$ coincides with λ. By Lemma 3.7 it is enough to show that $\bar{\lambda}$ is an eigenvalue of $\mathcal{L}^*_{f,\phi}$. Taking subsequences, if necessary, we may assume that there is $\nu \in \mathbb{P}(M)$ such that

$$\nu_n \xrightarrow{w*} \nu \quad \text{and} \quad \lambda_n \longrightarrow \bar{\lambda}.$$

Let us show that $\mathcal{L}^*_{f,\phi}(\nu) = \bar{\lambda}\nu$. Since $C^\alpha(M)$ is dense in $C^0(M)$, it is enough to see that

$$\mathcal{L}^*_{f,\phi}(\nu)(\psi) = \bar{\lambda}\nu(\psi) \quad \text{for all } \psi \in C^\alpha(M).$$

Using the continuity of Γ and ν, and the fact that $\nu_n \to \nu$, we may deduce for each $\psi \in C^\alpha(M)$

$$\begin{aligned}
\mathcal{L}^*_{f,\phi}(\nu)(\psi) &= \nu(\mathcal{L}_{f,\phi}(\psi)) \\
&= \nu\Big(\lim_{n\to\infty} \mathcal{L}_{f_n,\phi_n}(\psi)\Big) \\
&= \lim_{n\to\infty} \nu(\mathcal{L}_{f_n,\phi_n}(\psi)) \\
&= \lim_{n\to\infty} \nu_n(\mathcal{L}_{f_n,\phi_n}(\psi)) \\
&= \lim_{n\to\infty} \mathcal{L}^*_{f_n,\phi_n}(\nu_n)(\psi) \\
&= \lim_{n\to+\infty} \lambda_n \nu_n(\psi) \\
&= \bar{\lambda}\nu(\psi).
\end{aligned}$$

As Lemma 3.7 assures that λ is the only eigenvalue of $\mathcal{L}^*_{f,\phi}$ we necessarily have $\bar{\lambda} = \lambda$. □

Recalling that $P_f(\phi) = \log \lambda$, as a consequence of the previous result we obtain the continuity of the topological pressure on the set $\mathcal{H}_c$.

COROLLARY 4.3. *The function*

$$\begin{array}{ccc} \mathcal{H}_c & \longrightarrow & \mathbb{R} \\ (f, \phi) & \longmapsto & P_f(\phi) \end{array}$$

is continuous.

Now we are able to prove the equilibrium stability of the family $\mathcal{H}_c$. Consider, as before, $(f_n, \phi_n)_n$ a sequence in $\mathcal{H}_c$ converging to $(f, \phi) \in \mathcal{H}_c$. For each $n \in \mathbb{N}$, let μ_n be an equilibrium state for (f_n, ϕ_n). We are going to show that any accumulation point μ_0 of the sequence $(\mu_n)_n$ is an equilibrium state for (f, ϕ). We start with the invariance.

LEMMA 4.4. *The accumulation point μ_0 is an f-invariant measure.*

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Proof. Since each μ_n is f_n-invariant, for any continuous function $\varphi : M \to \mathbb{R}$ we have

$$\int \varphi \circ f_n \, d\mu_n = \int \varphi \, d\mu_n \longrightarrow \int \varphi \, d\mu_0 \quad \text{when } n \to +\infty.$$

Hence, to verify the f-invariance of μ_0 it suffices to prove that

$$\int \varphi \circ f_n \, d\mu_n \longrightarrow \int \varphi \circ f \, d\mu_0 \quad \text{when } n \to +\infty.$$

For each $n \in \mathbb{N}$, we may write the inequality

$$\begin{aligned}\left| \int \varphi \circ f_n \, d\mu_n - \int \varphi \circ f \, d\mu_0 \right| \leq & \left| \int \varphi \circ f_n \, d\mu_n - \int \varphi \circ f \, d\mu_n \right| \\ & + \left| \int \varphi \circ f \, d\mu_n - \int \varphi \circ f \, d\mu_0 \right|.\end{aligned}$$

Combining the convergence of f_n to f and the fact that μ_0 is an accumulation point of the sequence $(\mu_n)_n$, we have that each term in the sum above is close to zero for n sufficiently large. This implies that μ_0 is f-invariant. □

Now take $\delta_1 > 0$ as Lemma 3.4. Consider $\mathcal{P}$ a finite partition of M with a diameter smaller than $\delta_1/2$ with $\mu_0(\partial\mathcal{P}) = 0$. For all $n \geq 0$ and all $m \geq 1$ define the partition $\mathcal{P}_m^n$ by

$$\mathcal{P}_m^n = \{P_m^n = P_{i_0} \cap \cdots \cap f_n^{-(m-1)}(P_{i_{m-1}}); \, P_{i_j} \in \mathcal{P}, \, 0 \leq j \leq m-1\}.$$

Given $x \in M$, define also $P_m^n(x)$ as the element $P_m^n \in \mathcal{P}_m^n$ such that $x \in P_m^n$. Note that, by definition, the sequence $\{P_m^n(x)\}_{m \geq 1}$ is non-increasing in m, meaning that

$$P_{m+1}^n(x) \subset P_m^n(x) \quad \text{for all } m \geq 1.$$

LEMMA 4.5. *For all $x \in \Sigma_c(f_n)$ and all $n \geq 0$ the diameter of $P_m^n(x)$ goes to zero when m goes to infinity.*

Proof. It follows from Lemma 3.4 that if $m = m(x, n)$ is a hyperbolic time for $x \in \Sigma_c(f_n)$, then $\operatorname{diam}(P_m^n(x)) \leq e^{-cm}\delta_1$. Since every point $x \in \Sigma_c(f_n)$ has infinitely many hyperbolic times we conclude that $\operatorname{diam}(P_m^n(x)) \to 0$ when m goes to infinity for every $x \in \Sigma_c(f_n)$ and all $n \geq 0$. □

Now we are ready to finish the proof of Theorem A. By [**23**, Lemma 4.3] we have $\mu_n(\Sigma_c(f_n)) = 1$ for all $n \geq 1$. Then Lemma 4.5 gives that $\mathcal{P}$ is a generating partition for all (f_n, μ_n). As a straightforward application of [**5**, Theorem 11] and Kolmogorov–Sinai's theorem we have

$$\limsup_{n \to \infty} h_{\mu_n}(f_n) \leq h_{\mu_0}(f). \tag{7}$$

Using the continuity of the topological pressure (Corollary 4.3) and the fact that μ_n is an equilibrium state for (f_n, ϕ_n), we obtain

$$P_f(\phi) = \lim_{n \to +\infty} P_{f_n}(\phi_n) = \lim_{n \to +\infty} \left(h_{\mu_n}(f_n) + \int \phi_n \, d\mu_n \right) \leq h_{\mu_0}(f) + \int \phi \, d\mu_0.$$

This clearly implies that μ_0 is an equilibrium state for (f, ϕ). In the last inequality we have used that μ_n converges to μ in the weak* topology and converges ϕ_n to ϕ in the C^0-norm.

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5. *Skew products*

In this section we prove the existence of finitely many ergodic equilibrium states for skew products with respect to hyperbolic Hölder continuous potentials and their stability, namely Theorems C and D. We point out that Theorem C enlarges the class of potentials considered in [**23**], where it is also required that the potential does not depend on the stable direction. Here we prove that this condition is not necessary.

First we extend the definition of hyperbolic potentials for skew products. Let $F : M \times N \to M \times N$ be the skew product defined in §2. Recall that a continuous function $\phi : M \times N \to \mathbb{R}$ is a *c-hyperbolic potential for* F if

$$P_F(\phi, (\Sigma_c(f))^c \times N) < P_F(\phi, \Sigma_c(f) \times N) = P_F(\phi),$$

where $\Sigma_c(f)$ as in §2.1. In the next result we show that for every Hölder continuous potential, which is hyperbolic for F, it is possible to construct a potential homologous to it that does not depend on the stable direction and remains hyperbolic for F. We say that two potentials $\bar{\phi}, \phi : M \times N \to \mathbb{R}$ are *homologous* if there is a continuous function $u : M \times N \to \mathbb{R}$ such that $\bar{\phi} = \phi - u + u \circ F$;

PROPOSITION 5.1. *Let $\phi : M \times N \to \mathbb{R}$ be a Hölder continuous potential. There exists a Hölder continuous potential $\bar{\phi} : M \times N \to \mathbb{R}$ not depending on the stable direction such that:*

(i) *$\bar{\phi}$ is homologous to ϕ;*
(ii) *if ϕ is c-hyperbolic, then $\bar{\phi}$ is c-hyperbolic;*
(iii) *$P_F(\bar{\phi}) = P_F(\phi)$;*
(iv) *(F, ϕ) and $(F, \bar{\phi})$ have the same equilibrium states.*

Proof. Let $y_0 \in N$ be the fixed point of the fiber dynamics. Define a function $u : M \times N \to \mathbb{R}$ by

$$u(x, y) = \sum_{j=0}^{\infty} (\phi \circ F^j(x, y) - \phi \circ F^j(x, y_0)).$$

Since (x, y) and (x, y_0) are in the same stable direction, for every $(x, y) \in M \times N$ and all $j > 0$ we have

$$d(F^j(x, y), F^j(x, y_0)) \le \lambda^j,$$

where λ is the contraction rate of the fiber dynamics (see §2). Using the last inequality and the θ-Hölder continuity of ϕ we obtain

$$\begin{aligned} u(x, y) &= \sum_{j=0}^{\infty} \phi \circ F^j(x, y) - \phi \circ F^j(x, y_0) \\ &\le C \sum_{j=0}^{\infty} d(F^j(x, y), F^j(x, y_0))^{\theta} \\ &\le C \sum_{j=0}^{\infty} \lambda^{\theta j}. \end{aligned}$$

Thus, it follows that u is well-defined and is a continuous function.

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Defining the potential $\bar{\phi} : M \times N \to \mathbb{R}$ by $\bar{\phi} := \phi - u + u \circ F$ we have that $\bar{\phi}$ is a continuous function homologous to ϕ. Moreover, we can write

$$\begin{aligned}\bar{\phi}(x, y) &= \phi(x, y) - u(x, y) + u \circ F(x, y)\\ &= \phi(x, y) - \sum_{j=0}^{\infty} \phi \circ F^j(x, y) - \phi \circ F^j(x, y_0)\\ &\quad + \sum_{j=0}^{\infty} \phi \circ F^{j+1}(x, y) - \phi \circ F^j(f(x), y_0)\\ &= \phi(x, y_0) + \sum_{j=0}^{\infty} \phi \circ F^{j+1}(x, y_0) - \phi \circ F^j(f(x), y_0)\end{aligned}$$

which implies that $\bar{\phi}$ does not depend on the stable direction.

Since we are assuming that $g(x, y_0) = y_0$ for every $x \in M$, from the last equality we obtain

$$\begin{aligned}\bar{\phi}(x, y) &= \phi(x, y_0) + \sum_{j=0}^{\infty} \phi \circ F^{j+1}(x, y_0) - \phi \circ F^j(f(x), y_0)\\ &= \phi(x, y_0) + \sum_{j=0}^{\infty} \phi \circ F^j(f(x), g_x(y_0)) - \phi \circ F^j(f(x), y_0)\\ &= \phi(x, y_0). \qquad (8)\end{aligned}$$

This gives, in particular, that $\bar{\phi}$ is Hölder continuous and is a c-hyperbolic potential if ϕ is c-hyperbolic.

The third item is a consequence of (4) and the fact that ϕ and $\bar{\phi}$ have the same integral with respect to any F-invariant probability measure. The fourth item is a consequence of the third. □

Let $\phi : M \times N \to \mathbb{R}$ be a c-hyperbolic Hölder continuous potential for F. According to Proposition 5.1 there exists a Hölder continuous potential $\bar{\phi} : M \times N \to \mathbb{R}$ that is homologous to ϕ and not depending on the stable direction. Fixing any point $z \in N$, the potential $\bar{\phi}$ induces a Hölder continuous potential $\varphi : M \to \mathbb{R}$ given by

$$\varphi(x) = \bar{\phi}(x, z). \qquad (9)$$

This means that considering $\pi : M \times N \to M$ the natural projection onto M defined by $\pi(x, y) = x$ for every $(x, y) \in M \times N$, we have

$$\bar{\phi} = \varphi \circ \pi. \qquad (10)$$

Moreover, π is a continuous semi-conjugacy between F and f. Using (4) it is straightforward to check that

$$P_f(\varphi) \leq P_F(\bar{\phi}). \qquad (11)$$

Now we are going to prove that there exists a bijection between the sets of equilibrium states for (F, ϕ) and for (f, φ). For this we will use the following result due to Ledrappier and Walters, the proof can be found in [**18**].

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THEOREM 5.2. (Ledrappier–Walters formula) *Let $\tilde{X}$, X be compact metric spaces and let $\tilde{T} : \tilde{X} \to \tilde{X}$, $T : X \to X$, $\tilde{\pi} : \tilde{X} \to X$ be continuous maps such that $\tilde{\pi}$ is surjective and $\tilde{\pi} \circ \tilde{T} = T \circ \tilde{\pi}$. Then*

$$\sup_{\tilde{\mu};\tilde{\pi}_*\tilde{\mu}=\mu} h_{\tilde{\mu}}(\tilde{T}) = h_\mu(T) + \int h_{top}(\tilde{T}, \tilde{\pi}^{-1}(y))\, d\mu(y).$$

LEMMA 5.3. *If $\mu \in \mathbb{P}_f(M)$ is ergodic, then there exists a unique ergodic measure $\tilde{\mu} \in \mathbb{P}_F(M \times N)$ such that $\mu = \pi_*\tilde{\mu}$. Moreover, μ is an equilibrium state for (f, φ) if and only if $\tilde{\mu}$ is an equilibrium state for (F, ϕ).*

Proof. Given $\mu \in \mathbb{P}_f(M)$ the existence of $\tilde{\mu} \in \mathbb{P}_F(M \times N)$ such that $\mu = \pi_*\tilde{\mu}$ is a classical result that we sketch as follows. Consider the functional

$$L(h \circ \pi) = \int h\, d\mu,$$

defined on the closed subspace of observables in $C^0(M \times N)$ that are constant on fibers of the skew product. By the Hahn–Banach theorem we can can extend L to $C^0(M \times N)$. Since L is positive and $L(1) = 1$ it can be identified with a probability measure $\nu \in \mathbb{P}(M \times N)$. Consider an accumulation point

$$\tilde{\mu} = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} (F^j)_*\nu.$$

Using the semi-conjugacy $f \circ \pi = \pi \circ F$ and the definition of L we have

$$\begin{aligned}\pi_*\tilde{\mu} = \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \pi_*(F^j)_*\nu &= \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} (f^j)_*\pi_*\nu \\ &= \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} (f^j \circ \pi)_*\nu \\ &= \lim_{k\to\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} (f^j)_*\mu = \mu.\end{aligned}$$

The uniqueness of $\tilde{\mu}$ is a consequence of the contraction on the fibers. In fact, suppose that $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are ergodic measures that satisfy

$$\pi_*\tilde{\mu}_1 = \mu = \pi_*\tilde{\mu}_2.$$

Let $B_{\tilde{\mu}_1}(F)$ and $B_{\tilde{\mu}_2}(F)$ be the ergodic basins of $\tilde{\mu}_1$ and $\tilde{\mu}_2$, respectively. From the uniform contraction on fibers and the ergodicity of $\tilde{\mu}_1$, $\tilde{\mu}_2$, there are Borel sets A_1, $A_2 \subset M$ with $A_1 \cap A_2 = \emptyset$ such that

$$B_{\tilde{\mu}_1}(F) = A_1 \times N \quad \text{and} \quad B_{\tilde{\mu}_2}(F) = A_2 \times N.$$

On the other hand, since μ is ergodic and the sets $\pi(B_{\tilde{\mu}_1}(F))$, $\pi(B_{\tilde{\mu}_2}(F))$ are f-invariant, it follows that

$$\mu(\pi(B_{\tilde{\mu}_1}(F))) = \mu(\pi(B_{\tilde{\mu}_2}(F))) = 1.$$

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Thus, we have

$$\pi(B_{\tilde{\mu}_1}(F)) \cap \pi(B_{\tilde{\mu}_2}(F)) \neq \emptyset$$

and, by the definition of π, we obtain that $A_1 \cap A_2 \neq \emptyset$. This gives $\tilde{\mu}_1 = \tilde{\mu}_2$.

Now we prove the second part of the lemma. First observe that the uniform contraction of g on the fibers gives $h_{\mathrm{top}}(F, \pi^{-1}(x)) = 0$ for every $x \in M$. Then, by the Ledrappier–Walters formula we obtain

$$h_{\tilde{\mu}}(F) = h_{\mu}(f) \tag{12}$$

for any $\mu \in \mathbb{P}_f(M)$ and $\tilde{\mu} \in \mathbb{P}_F(M \times N)$ such that $\pi_*\tilde{\mu} = \mu$. It follows from (4) and (11) that

$$\begin{aligned} P_f(\varphi) &\leq P_F(\bar{\phi}) \\ &= \sup_{\tilde{\eta}} \left\{ h_{\tilde{\eta}}(F) + \int \bar{\phi}\, d\tilde{\eta} \right\} \\ &\leq \sup_{\eta;\pi_*\tilde{\eta}=\eta} \left\{ h_{\eta}(f) + \int h_{\mathrm{top}}(F, \pi^{-1}(x))\, d\eta(x) + \int \varphi\, d\eta \right\} \\ &\leq P_f(\varphi). \end{aligned}$$

Thus, we must have

$$P_f(\varphi) = P_F(\bar{\phi}). \tag{13}$$

Assume now that μ is an ergodic equilibrium state for (f, φ). Using (4), (10), (12), (13), and the fact that $\pi_*\tilde{\mu} = \mu$, we obtain

$$P_F(\bar{\phi}) = P_f(\varphi) = h_{\mu}(f) + \int \varphi\, d\mu = h_{\tilde{\mu}}(F) + \int \bar{\phi}\, d\tilde{\mu}.$$

This shows that $\tilde{\mu}$ is an equilibrium state for $(F, \bar{\phi})$, and so an equilibrium state for (F, ϕ), by Proposition 5.1.

Finally, assume that $\tilde{\mu}$ is an equilibrium state for (F, ϕ). Using again (4), (10), (12), (13), and the fact that $\pi_*\tilde{\mu} = \mu$, we obtain

$$P_f(\varphi) = P_F(\bar{\phi}) = h_{\tilde{\mu}}(F) + \int \bar{\phi}\, d\tilde{\mu} = h_{\mu}(f) + \int \varphi\, d\mu,$$

thus finishing the proof of the result. □

LEMMA 5.4. *If ϕ is a c-hyperbolic potential for F, then φ is a c-hyperbolic potential for f.*

Proof. First observe that by Proposition 5.1, we have that $\bar{\phi} = \varphi \circ \pi$ is also a c-hyperbolic potential for F. As F contracts on fibers, for each $\delta > 0$ and each $(x, z) \in M \times N$ we have

$$B_{\delta}^{f}(x, n) = \pi(B_{\delta}^{F}((x, z), n)),$$

where the superscripts indicate the dynamics with respect to which we take the dynamic balls. Recalling the definition of relative topological pressure given in §3, for all $\delta > 0$,

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$n \in \mathbb{N}$ and $(x, z) \in M \times N$ we have

$$\begin{aligned} R_{n,\delta}\bar{\phi}(x, z) &= \sup_{(y,w)\in B_\delta^F((x,z),n)} S_n\bar{\phi}(y, w) \\ &= \sup_{(y,w)\in B_\delta^F((x,z),n)} S_n(\varphi \circ \pi)(y, w) \\ &= \sup_{y\in\pi(B_\delta^F((x,z),n))} S_n\varphi(y) \\ &= \sup_{y\in B_\delta^f(x,n)} S_n\varphi(y) \\ &= R_{n,\delta}\varphi(x). \end{aligned}$$

Hence, for each $N \in \mathbb{N}$ and $\gamma > 0$ we have

$$m_f(\varphi, (\Sigma_c(f))^c, \delta, N, j) = m_F(\bar{\phi}, (\Sigma_c(f))^c \times N, \delta, N, j)$$

and

$$m_f(\varphi, (\Sigma_c(f)), \delta, N, j) = m_F(\bar{\phi}, (\Sigma_c(f)) \times N, \delta, N, j).$$

This yields

$$P_f(\varphi, (\Sigma_c(f))^c) = P_F(\bar{\phi}, (\Sigma_c(f))^c \times N)$$

and

$$P_f(\varphi, \Sigma_c(f)) = P_F(\bar{\phi}, \Sigma_c(f) \times N).$$

Therefore,

$$P_f(\varphi, (\Sigma_c(f))^c) = P_F(\bar{\phi}, (\Sigma_c(f))^c \times N) < P_F(\bar{\phi}, \Sigma_c(f) \times N) = P_f(\varphi, \Sigma_c(f)),$$

and using (13), we obtain

$$P_f(\varphi, (\Sigma_c(f))^c) < P_f(\varphi, \Sigma_c(f)) = P_F(\bar{\phi}, \Sigma_c(f) \times N) = P_F(\bar{\phi}) = P_f(\varphi).$$

This shows that φ is a c-hyperbolic potential. □

The existence and finiteness of equilibrium states for the base dynamics f were obtained in [**23**]. Combining this fact with Lemma 5.3 we conclude that there exist finitely many ergodic equilibrium states for $(F, \bar{\phi})$. Since $\bar{\phi}$ is homologous to the initial potential ϕ we have completed the proof of Theorem C.

Now we are able to prove Theorem D. Given $(F_0, \phi_0) \in \mathcal{G}_c$, consider a sequence $(F_n, \phi_n)_n$ in $\mathcal{G}_c$ converging to (F_0, ϕ_0) in the C^1-topology. Let $\tilde{\mu}_n$ be an equilibrium state of (F_n, ϕ_n). We are going to show that every accumulation point $\tilde{\mu}$ of the sequence $(\tilde{\mu}_n)_n$ is an equilibrium state for (F_0, ϕ_0).

For each $n \geq 0$, let $\bar{\phi}_n$ be the potential associated to ϕ_n by Proposition 5.1, and let φ_n be the potential on M induced by $\bar{\phi}_n$. It follows from Lemma 5.4 that φ_n is c-hyperbolic (with respect to f), and this means that $(f_n, \varphi_n) \in \mathcal{H}_c$ for all $n \geq 0$. Moreover, using (8) and (9), we have that the C^1-convergence of (F_n, ϕ_n) to (F_0, ϕ_0) implies the C^1-convergence of (f_n, φ_n) to (f_0, φ_0).

For each $n \in \mathbb{N}$ consider $\mu_n = \pi_*\tilde{\mu}_n$. From Lemma 5.3 we have that μ_n is an equilibrium state for (f_n, φ_n). Since the projection π is continuous, if $\tilde{\mu}$ is an accumulation

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point of the sequence $(\tilde{\mu}_n)_n$, then $\mu = \pi_*\tilde{\mu}$ is an accumulation point of $(\mu_n)_n$. By the equilibrium stability given by Theorem A, we have that μ is an equilibrium state for (f_0, φ_0). Hence, applying Lemma 5.3 again we have that $\tilde{\mu}$ is an equilibrium state for (F_0, ϕ_0). This concludes the proof of Theorem D.

We finish this section by deducing the continuity of the topological pressure on the set $\mathcal{G}_c$.

COROLLARY 5.5. *The function*

$$\begin{array}{ccc} \mathcal{G}_c & \longrightarrow & \mathbb{R} \\ (F, \phi) & \longmapsto & P_F(\phi) \end{array}$$

is continuous.

Proof. As before, given $(F, \phi) \in \mathcal{G}_c$ consider the induced $(f, \varphi) \in \mathcal{H}_c$. Using (8) and (9) it follows that if (F, ϕ) varies continuously in $\mathcal{G}_c$, then (f, φ) also varies continuously in $\mathcal{H}_c$. Considering $\bar{\phi}$ given by Proposition 5.1, we have $P_f(\phi) = P_F(\bar{\phi})$. It follows from (13) that $P_f(\varphi) = P_F(\phi)$. Hence, by Corollary 4.3 we have $P_f(\varphi)$ varying continuously in the C^1-topology within $\mathcal{H}_c$, then $P_F(\phi)$ varies continuously within $\mathcal{G}_c$ as well. □

6. *Applications*

In this section we present some classes of systems that satisfy our results. We begin describing a robust class of non-uniformly expanding maps studied by Alves, Bonatti, and Viana [**2**], Arbieto, Matheus, and Oliveira [**6**], Oliveira and Viana [**20**], and Varandas and Viana [**30**].

Example 6.1. Let M be a compact manifold and let $f : M \to M$ be a C^1 local diffeomorphism. Fixing $\delta > 0$ small and $\sigma < 1$, consider a covering $\mathcal{P} = \{P_1, \ldots, P_q, P_{q+1}, \ldots, P_s\}$ of M by domains of injectivity for f and a region $A \subset M$ satisfying:
(H1) $\|Df^{-1}(x)\| \leq 1 + \delta$, for every $x \in A$;
(H2) $\|Df^{-1}(x)\| \leq \sigma$, for every $x \in M \backslash A$;
(H3) A can be covered by q elements of the partition $\mathcal{P}$ with $q < \deg(f)$.
The previous authors showed that there exists a constant $c > 0$ and a set $H \subset M$ such that for every $x \in H$ we have

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq -c.$$

Moreover, if $\phi : M \to \mathbb{R}$ is a Hölder continuous potential with *small variation*, i.e.

$$\sup \phi - \inf \phi < \log \deg(f) - \log q,$$

then the relative pressure $P(\phi, H)$ satisfies

$$P_f(\phi, H^c) < P_f(\phi, H) = P_f(\phi)$$

and, thus, ϕ is c-hyperbolic for f.

Let $\mathcal{F}$ be the class of C^1 local diffeomorphisms satisfying the conditions (H1)–(H3) and assume that for every $f \in \mathcal{F}$ and $x \in M$ the set $\{f^{-n}(x)\}_{n \geq 0}$ is dense in M. Consider the family

$$\mathcal{H} = \{(f, \phi) : f \in \mathcal{F} \text{ and } \phi : M \to \mathbb{R} \text{ is } C^1 \text{ with small variation}\}.$$

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Since the constant $c > 0$ is uniform in the class $\mathcal{F}$ we can apply Theorem A to conclude that the family $\mathcal{H}$ is equilibrium stable.

An interesting and more specific case of application of Theorem A is the family of intermittent maps described in the next example. This can be seen as a particular case of the previous example.

Example 6.2. Fix some positive constant $\alpha \in (0, 1)$ and define on S^1 the local diffeomorphism

$$f_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } 0 \le x \le \frac{1}{2}, \\ x - 2^\alpha (1 - x)^{1+\alpha} & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Since f_α is strongly mixing (for each open set U there exists $N \in \mathbb{N}$ such that $f_\alpha^N(U) = M$), then for every f_α and $x \in S^1$ the set $\{f_\alpha^{-n}(x)\}_{n\ge 0}$ is dense in S^1. Let $\mathcal{F}$ be the class of C^1 local diffeomorphisms $\{f_\alpha\}_{\alpha\in(0,1)}$ and consider the family

$$\mathcal{H} = \{(f, \phi) : f \in \mathcal{F} \text{ and } \phi : S^1 \to \mathbb{R} \text{ is } C^1 \text{ with small variation}\}.$$

Hence, the family $\mathcal{H}$ satisfies the hypotheses of Theorem A and, therefore, $\mathcal{H}$ is equilibrium stable.

Now we present a class of maps and potentials for which the techniques used to prove Theorems C and D can be applied. This is a family of partially hyperbolic horseshoes whose dynamics is given by a step skew product over a horseshoe. This class of maps was defined in [**17**] and has been studied in [**19**, **22**, **24**]. Here we obtain the equilibrium stability of this family.

Example 6.3. Consider the cube $R = [0, 1] \times [0, 1] \times [0, 1] \subset \mathbb{R}^3$ and the paralellepipeds

$$R_0 = [0, 1] \times [0, 1] \times [0, 1/6] \quad \text{and} \quad R_1 = [0, 1] \times [0, 1] \times [5/6, 1].$$

Consider a map defined for $(x, y, z) \in R_0$ as

$$F_0(x, y, z) = (\rho x, f(y), \beta z),$$

where $0 < \rho < 1/3$, $\beta > 6$, and

$$f(y) = \frac{1}{1 - (1 - 1/y)e^{-1}}.$$

Consider also a map defined for $(x, y, z) \in R_1$ as

$$F_1(x, y, z) = (\tfrac{3}{4} - \rho x, \sigma(1 - y), \beta_1(z - \tfrac{5}{6})),$$

where $0 < \sigma < 1/3$ and $3 < \beta_1 < 4$. We define the horseshoe map F on R as

$$F|_{R_0} = F_0, \quad F|_{R_1} = F_1,$$

with $R \backslash (R_0 \cup R_1)$ being mapped injectively outside R.

In [**17**], it was proved that the non-wandering set of F is partially hyperbolic when we consider fixed parameters satisfying the conditions above. In [**24**], the map F^{-1} was considered and it was proved that it can be written as a skew product whose base dynamics, denoted by G, is strongly topologically mixing and non-uniformly expanding. Moreover,

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the fiber dynamics is a uniform contraction. Consider Ω the maximal invariant set for F^{-1} on the cube R.

We say that a Hölder continuous potential $\phi : R_0 \cup R_1 \to \mathbb{R}$ has *small variation* if

$$\sup \phi - \inf \phi < \frac{\log \omega}{2}, \quad \text{where } \omega = \frac{1+\sqrt{5}}{2}. \tag{14}$$

Let $\mathcal{S}$ be the family of horseshoes described above and note that it depends on the parameters ρ, β, β_1, and σ. Considering the family

$$\tilde{\mathcal{G}} = \{(F^{-1}, \phi) : F \in \mathcal{S} \text{ and } \phi : \Omega \to \mathbb{R} \text{ is } C^1 \text{ and satisfies (14)}\}.$$

We are going to verify that $\tilde{\mathcal{G}}$ is equilibrium stable. Let $(F_n^{-1}, \phi_n)_n$ be a sequence in $\tilde{\mathcal{G}}$ converging to (F_0^{-1}, ϕ_0) in the C^1-topology. By Lemma 5.4, for each $n \geq 0$ the potential ϕ_n induces a potential φ_n that is hyperbolic with respect to the base dynamics G_n. Moreover, the C^1 convergence of (F_n^{-1}, ϕ_n) to (F_0^{-1}, ϕ_0) implies the C^1 convergence of (G_n, φ_n) to (G_0, φ_0).

Consider $\tilde{\mu}_n$ the equilibrium state for (F_n^{-1}, ϕ_n). From Lemma 5.3 we know that the push-forward $\mu_n = \pi_* \tilde{\mu}_n$ is an equilibrium state for (G_n, φ_n) for every $n \geq 1$. Hence, if $\tilde{\mu}$ is an accumulation point of the sequence $(\tilde{\mu}_n)_n$, then $\mu = \pi_* \tilde{\mu}$ is an accumulation point of $(\mu_n)_n$. Since the base dynamics G_n is strongly mixing and non-uniformly expanding and the potential φ_n is hyperbolic, it follows from Theorem A that μ is an equilibrium state for (G_0, φ_0). Hence, applying Lemma 5.3 again we have that $\tilde{\mu}$ is an equilibrium state for (F_0^{-1}, ϕ_0).

Now define the family

$$\mathcal{G} = \{(F, \phi) : F \in \mathcal{S} \text{ and } \phi : \Omega \to \mathbb{R} \text{ is } C^1 \text{ and satisfies (14)}\}.$$

Since each $F \in \mathcal{S}$ is a diffeomorphism, the set of equilibrium states for $(F, \phi) \in \mathcal{G}$ coincides with that for $(F^{-1}, \phi) \in \tilde{\mathcal{G}}$. Thus, $\mathcal{G}$ is equilibrium stable as well.

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